TECHNOLOGICAL IMPROVEMENT AND THE DECENTRALIZATION PENALTY IN A SIMPLE PRINCIPAL/AGENT MODEL

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We consider the organizer of a firm who compares a decentralized arrangement where divisions are granted total autonomy with an arrangement where perfect monitoring and policing guarantee that all divisions make the choices the organizer wants them to make. We ask: when does improvement in the divisions’ technology strengthen the case for decentralization and when does it weaken it? The question is difficult and it is natural to start with a stripped-down model, where there is just one division. In the decentralized mode, the organizer appoints a Principal who rewards a single autonomous Agent (the division manager). The Agent freely chooses an effort $x$. The effort need not be hidden. The Agent bears its cost. The firm then achieves the surplus $R(x) - t \cdot C(x)$, where $R$ is revenue, $t$ is a positive technology parameter known to both parties, and $t \cdot C(x)$ is the cost of the Agent’s effort. When technology improves, $t$ drops. The Agent receives a share of the revenue, namely $r \cdot R(x)$, were $0 < r < 1$. The Principal receives the residual $(1 - r) \cdot R(x)$. In the exogenous case the share $r$ is determined outside the model (perhaps by Principal/Agent bargaining). In the endogenous case the Principal, who knows how the Agent responds to every possible $r$ given the current $t$, chooses the $r$ which maximizes the residual revenue. The Decentralization Penalty for a given $t$ equals the maximal possible surplus for that $t$ — attained under perfect monitoring — minus the surplus achieved in the decentralized mode. It turns out that there are no simple conditions on $C$ and $R$ which imply that the Penalty grows (shrinks) when technology improves. Instead, we obtain a variety of results about relations between $t$, surplus, the Principal's "generosity" (the size of the Principal’s chosen share in the endogenous case), and “effectiveness” (the effect of a small rise in the share on the Agent’s effort).
1. Introduction

Does the case for decentralizing a firm get stronger or weaker when the technology used by one or more of its divisions improves? Consider the Organizer of the firm, who seeks a good balance between the cost of the divisions’ efforts and the revenue which those efforts yield. One way to achieve a good balance may be intrusive but perfect monitoring and policing, which fully reveals the chosen efforts and guarantees that they are those the Organizer prefers by punishing other choices. There are costs for perfect monitoring/policing. In particular, the Agents may find it unpleasant, and extra payments may be needed to induce their participation. On the other hand, the technology used in perfect monitoring/policing may improve, so that its total cost drops.

A better mode of organizing might be “decentralization”. Let the divisions be totally autonomous. Appoint a Principal who treats each division as an Agent. Each Agent freely chooses her effort and bears the effort’s cost. The Principal observes the realized revenue and rewards the Agents. Each Agent’s reward is a function of revenue and her net earnings are her reward minus the cost of her chosen effort. The reward functions the Principal chooses are acceptable to the Agents and are preferred by the Principal to other possible reward functions that are also acceptable to the Agents. The Principal pockets the residual revenue which is left over after the rewards have been paid. When an Agent’s technology improves, the cost of a given effort drops.

We are interested in surplus: revenue minus the sum of the Agents’ costs. High surplus is desirable from the welfare point of view. If the firm is a regulated monopoly, for example, then high surplus might be the regulator’s goal. The regulator compares decentralization with perfect monitoring/policing and favors the mode that achieves the higher surplus. Other policy makers might ask whether a given advance in monitoring technology, which lowers the cost of perfect monitoring/policing, is preferable, from the welfare point of view, to a given advance in the autonomous Agents’ technology. They would study surplus in both cases. For the monitoring advance, surplus is the highest attainable value of the excess of revenue over Agents’ costs (which perfect monitoring/policing guarantees) minus the lowered monitoring cost.

Alternatively surplus may be viewed as the firm’s profit. If the Organizer is also the firm’s owner then profit may be what he seeks to maximize. Suppose the Organizer/owner has chosen the decentralized mode. He employs a Principal and Agents and specifies their tasks. They may, in particular, be told to behave in the usual self-interested Principal/Agents manner. But the owner now takes all but a small fraction of the Principal’s residual. The owner also takes all but a small fraction of each Agent’s net earnings (which means that the owner pays all but a small fraction of the cost of the Agent’s chosen effort), provided the Agent’s small fraction is large enough so that the Agent wants to participate. Since the owner keeps all but a small fraction of profit, he wants profit to be maximal.

Perfect monitoring/policing guarantees that efforts are “first best”, i.e., surplus is maximal if we ignore the monitoring/policing cost. In the decentralized mode, on the other hand, we have a Penalty: the Agent’s efforts are not, in general, first-best and surplus is not maximal. Our central question is this: when the Agents’ technology improves — so the cost of any given
effort drops — does the Decentralization Penalty rise or fall? If it rises, then decentralization has become less attractive and perfect monitoring/policing may now be worth what it costs. If technology improvement significantly diminishes the Penalty, then perfect monitoring/policing becomes less attractive unless there is also a major advance in monitoring/policing technology.

Our central question is tricky for the following reason: as the Agents’ technology improves, maximal surplus rises (under weak assumptions). Maximal surplus is a “moving target”. Decentralized surplus also rises, under reasonable assumptions. But that does NOT mean, in general, that as technology improves, the rising decentralized surplus gets closer to the moving surplus target. Our question appears to be very rarely asked in the abundant Principal/Agent literature. The cost of an Agent’s effort appears in many papers and so does the welfare loss due to Agents’ second-best choices. But the effect of cost improvement on welfare loss seems to be widely neglected.

2. The model

There is a single effort variable $x$, chosen from a set $\Sigma \subseteq \mathbb{R}^+$ of possible positive efforts. The set $\Sigma$ may be finite or it may be a continuum. The effort $x$ generates a positive revenue $R(x)$, where $R$ is strictly increasing. The effort $x$ costs $t \cdot C(x)$, where $C$ is positive and strictly increasing. A drop in $t$ occurs when technology improves (or there is a fall in the price of the inputs which effort requires). We shall consider the surplus at the effort $x$, denote $\tilde{W}(x,t)$. Thus

$$W(x,t) = R(x) - t \cdot C(x).$$

In the centralized mode perfect monitoring/policing guarantees that effort maximizes surplus. In the decentralized mode there is no direct monitoring. Instead there is a self-interested Principal and — to keep the model simple — a single self-interested Agent who freely chooses $x \in \Sigma$ and bears the cost $t \cdot C(x)$. The functions $R$ and $C$, and the technology parameter $t$, are known to both parties. The Principal observes the revenue $R(x)$. If $R$ is strictly increasing, then that observation also reveals the Agent’s chosen $x$. The Principal rewards the Agent, using a reward which is a function of the observed revenue. We study an extremely simple reward scheme, namely linear revenue sharing. The Principal pays the agent a share $r \in (0,1]$ of the revenue. If the Agent chooses the effort $x$, she earns $rR(x) - t \cdot C(x)$ and the net amount received by the Principal is the residual $(1 - r) \cdot R(x)$.\(^1\) The Agent chooses to exert the effort $\hat{x}(r,t)$, the smallest maximizer of $rR(x) - t \cdot C(x)$ on the set $\Sigma$.\(^2\) We denote the surplus when the share is $r$ by $W(r,t)$ (the tilde is deleted). So

$$W(r,t) \equiv \tilde{W}(\hat{x}(r,t),t) = R(\hat{x}(r,t)) - t \cdot C(\hat{x}(r,t)).$$

Note that if $r = 1$, then the Agent’s effort choice $\hat{x}(1,t)$ is surplus-maximizing. Thus

$$W(1,t) = \tilde{W}(\hat{x}(1,t),t) \text{ is the largest possible surplus}.$$

\(^1\)We will assume that for every $(r,t)$ there is an effort $x \in \Sigma$ such that the Agent’s gain $rR(t) - tC(x)$ is nonnegative, and that this is sufficient for the Agent to be willing to participate.

\(^2\)Our results would be essentially unchanged if we studied the largest maximizer instead.
In the centralized mode, perfect monitoring/policing insures that \(W(1, t)\) is achieved.

We shall study the decentralized mode by considering two cases. In the *exogenous* case, the reward share is determined outside the model. It might, for example, be the result of previous bargaining between Principal and Agent, or it might be prescribed by law. In the *endogenous* case, the Principal considers all the shares in the open interval \((0, 1)\) and chooses a share which maximizes \((1 - r) \cdot R(\hat{x}(r, t))\), the residual if the Agent uses the best-effort function \(\hat{x}\) when he responds to a given share. We let \(r^*(t)\) denote the maximizer which the Principal chooses. So in the endogenous case the Agent’s effort is \(\hat{x}(r^*(t), t)\) and surplus is \(\tilde{W}(\hat{x}(r^*(t), t), t) = W(r^*(t), t)\).

Our main concern, as noted in the Introduction, is the *Decentralization Penalty*. We shall often use *surplus gap* as an alternative term for Penalty. The Penalty, as we define it, is the *difference* between maximal surplus — which perfect monitoring and policing achieves — and the surplus achieved in the decentralized mode. An alternative definition of penalty would be the *ratio* of decentralized surplus to maximal surplus. Both the difference and the ratio are worth studying. The difference concept is natural if one knows the cost of perfect monitoring/policing. That cost drops when monitoring technology improves. If improvement in the Agent’s technology substantially raises (lowers) the surplus difference, then perfect monitoring/policing becomes more (less) attractive.

Our results about the Agent’s technology and the Penalty (defined as a difference) turn out to be surprisingly complex and varied. It appears technically difficult to obtain parallel results when the Penalty is a ratio, and there is no reason to believe that such parallel results would be less complex and less varied than the ones we obtain. In the Related Literature section below we comment on computer-science studies of the ratio (the “Price of Anarchy”). In our Concluding Remarks section we suggest the ratio as one of a number of ways of extending our results and we briefly point to some examples where looking at the ratio leads to results that contrast sharply with ours.

Our Decentralization Penalty is \(W(1, t) - W(r, t)\) in the exogenous case and \(W(1, t) - W(r^*(t), t)\) in the endogenous case. We shall see, in many examples, that if \(t\) drops, then both the first term \(W(1, t)\) and the second term \(W(r, t)\) (or \(W(r^*(t), t)\)) rise. Then, if we want to find the effect of a drop in \(t\) on the Penalty, we face the challenging question noted in the Introduction: *which of the two terms rises faster when \(t\) drops?*

In addition to the effect of a drop in \(t\) on the Penalty (the surplus gap), we study the following questions:

- In each case — exogenous and endogenous — what is the effect of technological improvement — i.e., a drop in \(t\) — on the Agent’s effort? Does the Agent always work harder when technology improves?

- In both cases, is technological improvement always “good news” for the Agent? For the Principal? Is it always good news from the welfare point of view?

- We study the *effectiveness* of a share increase — the increase in effort due to a small increase in the share. Does effectiveness rise or fall when \(t\) drops? That is an important question in both cases.

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In both cases we consider the effort gap — the amount by which effort under decentralization falls short of surplus-maximizing effort. That gap is $\hat{x}(1,t) - \hat{x}(r,t)$ in the exogenous case and $\hat{x}(1,t) - \hat{x}(r^*(t),t)$ in the endogenous case. Does the effort gap rise or fall when technology improves? We can again give this question a “moving target” interpretation. As technology improves, think of first-best effort as a moving target. Under weak assumptions it rises when technology improves ($t$ drops). Decentralized effort may also rise, but that does not mean that it gets closer to the moving target.

When does the surplus gap track the effort gap? When is it the case that as technology improves, decentralized surplus gets closer to the moving maximal-surplus target if and only if decentralized effort gets closer to the moving first-best-effort target? In the exogenous case, when do $\hat{x}(1,t) - \hat{x}(r,t)$ and $W(1,t) - W(r,t)$ move in the same direction when $t$ changes? In the endogenous case, when do $\hat{x}(1,t) - \hat{x}(r^*(t),t)$ and $W(1,t) - W(r^*(t),t)$ move in the same direction when $t$ changes? When do we have opposite directions rather than tracking?

In the endogenous case, does the Principal become more or less generous when technology improves? (Does the share $r^*(t)$ rise or fall when $t$ drops?).

In many examples the “tracking” question is difficult, especially in the endogenous case. We study it by considering its relation to effectiveness and generosity. Each of these may increase or decrease when there is a drop in $t$, so there are four possible combinations. We shall examine the tracking question for each of those combinations.

3. Related literature.

Our problem is a moral hazard problem where effort need not be hidden, there is no uncertainty, and there is no informational asymmetry. To make a connection between our problem and standard moral-hazard results where there is uncertainty but again no informational asymmetry, consider the simplest framework, adapted to fit our problem. The Agent has two effort choices, $x_L$ and $x_H$, where $0 < x_L < x_H$. The lower effort costs the Agent $t \cdot C_L$, while the higher effort costs $t \cdot C_H$, where $t > 0$ is our technology parameter, known to both parties. Let the outcome of an effort be a random variable. It is one of two revenues $R_L$ and $R_H$, where $0 < R_L < R_H$. The effort $x_L$ yields $R_H$ with probability $q$ and $R_L$ with probability $1 - q$. The effort $x_H$ yields $R_H$ with probability $p$ and $R_L$ with probability $1 - p$, where $p > q$. If the Agent declines to participate, he receives a reservation amount which we normalize to be zero. The Principal proposes a contract, which the Agent accepts. Under the contract, the Agent receives $w_L$ from the Principal if revenue turns out to be $R_L$ and $w_H$ if revenue turns out to be $R_H$, where $w_H > w_L$ and $w_L$ may be negative. Suppose that both parties are risk-neutral and there is no liability constraint on $|w_L|$. Then among all contracts acceptable to the Agent, the Principal’s favorite one induces the Agent to make the choice that maximizes average surplus.\(^3\) Let our Decentralization Penalty be the difference between highest attainable (“first best”) average surplus and average

\(^3\)See, for example, Chapter 4 in Laffont and Martimort (2002).
surplus under the Principal’s favorite contract. With no liability constraint, the Penalty is zero, whatever $t$ may be. What we have called the “effort gap” is also zero (the Agent’s choice is “first best”). When there is a liability constraint, the Agent might not have the assets to cover $w_L$ and hence the Principal’s menu of contracts narrows, but not so much that the linear sharing of our model becomes the only candidate. For the chosen contract does the Penalty rise or fall when technology improves ($t$ drops)? The question has been neglected but it is a natural one for future research.

As noted in the Introduction, a great many papers, starting with the earliest ones, use a framework that allows an Agent’s effort to have a cost. The Agent has a utility function on her actions and rewards. In many papers Agent utility for the action $a$ and the reward $y$ takes the form $V(y) - g(a)$. The action $a$ might be effort and $g(a)$ could be its cost. Welfare loss also appears very early in the moral-hazard literature. But papers whose main concern is the relation between effort cost and welfare loss are scarce.

Yet the question has strong motivation. The ongoing IT revolution, for example, calls for better understanding of one important case: the sole Agent is an information-gatherer who sends a signal about the current state to the Principal. Revenue depends on a choice that the Principal makes when the signal is received, and more effort by the Agent yields a more useful signal. Improvement in IT lowers the cost of any given effort. Does that strengthen or weaken the case for giving the information-gatherer autonomy? (That might be done by outsourcing, where an external independent enterprise does the information gathering). The question is studied in Marschak, Shanthikumar, and Zhou (2017).

If we allow more than one Agent, then parts of the large literature on the design of organizations become relevant. The designer has a goal, say surplus (profit) maximization, and can choose between a structure where a single member commands the choices made by all the others, and a structure where everyone is autonomous. The latter structure might be modeled as a game. A rather small piece of the design literature studies the communication and computation costs of each structure and the trade-off between those costs and some measure of gross performance (e.g., gross expected surplus, before the costs are subtracted). The problem is far more complex than the one we consider here and the results remain scarce and specialized.

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4Among the early papers where this occurs are Holmstrom (1979), (1982) and Grossman and Hart (1983).

5Ross (1973) finds conditions under which the solution to the Principal’s problem maximizes welfare (as measured by the sum of Agent’s utility and Principal’s utility) and notes that these conditions are very strong.

6In that paper the Principal chooses a production quantity in response to the Agent’s signals, and the question asked is whether average production quantity rises or falls when the information-gatherer’s technology improves. If improved technology indeed causes average production quantity to rise, and if a rise in quantity implies a rise in surplus, then when technology improves it becomes more attractive to give the information gatherer autonomy. In Courtney and Marschak (2009) a “sharing game” is studied. There are $n$ autonomous players. Each chooses an effort and bears its cost, and the $n$ choices determine a revenue which all share. An equilibrium of the game may be efficient (surplus is maximized), shirking (the chosen efforts are less than the efficient ones), or squandering (the efforts are greater than the efficient ones). It is shown, under differentiability assumptions, that if we are in a squandering equilibrium, if the players’ costs shift downward, and if the new equilibrium is again a squandering one, then the Decentralization Penalty at the new equilibrium (the amount by which surplus falls short of its maximum) is less than the Penalty at the old one. That need not be true for shirking equilibria.
Finally, we note a related strand of research by computer scientists, which they label “the price of anarchy”. Typically the object of study is a game and the price of anarchy is the ratio of the payoff sum in the socially “worst-case” equilibrium to the highest attainable sum. A simple example is a symmetric two-player Prisoners’ Dilemma, where the price of anarchy is the sum of payoffs in Nash equilibrium divided by the sum of payoffs when the players cooperate. A variety of social situations are studied from this point of view. Many of these studies develop bounds on the price of anarchy. Several of the papers consider a Principal/Agent setting. In our Conclusion we shall make some observations about what might happen to our central question if we redefined the Decentralization Penalty so that it becomes a ratio rather than a difference.

4. Plan of the remainder of the paper.

In Section 5 we examine five examples where the set of possible efforts and the set of possible values of $t$ are not finite. The examples provide a preview of our general results. In Section 6 we develop basic results, presented in two theorems. They do not require differentiability, so finite examples are covered. The results concern the exogenous case in the first theorem and the endogenous case in the second. Section 7 presents two exogenous-case theorems which require differentiability. Section 8 presents two endogenous-case theorems which again require differentiability. Section 9 considers the shape of the function which relates $r$ to the Principal’s gain. A concave shape implies a proposition about the negotiation set when the two parties bargain about the size of $r$. Section 10 provides concluding remarks about extensions and variations of the model.

5. Some examples

Our model, simple as it is, turns out to have quite diverse results. To illustrate the diversity, we now consider five examples. In all five of them the effort set and the set of possible values of the technology parameter $t$ are continua and calculus methods are used to study them. In the simplest finite example, on the other hand, there would be just two values of $t$ and two values of effort. We can construct simple finite examples yielding a variety of answers to the questions we just listed, but they are not presented here. In each of our five non-finite examples we provide some statements that we shall subsequently generalize.

5.1 A “Classic monopoly” example where marginal revenue falls and marginal cost is flat.

For convenient reference we shall call this our Classic monopoly example — or, for brevity, our Classic example. It is suggested by the introductory monopoly diagram in the typical text, where marginal revenue drops and marginal cost is flat or rising. We may think of the Principal as a monopolist who delegates the choice of product quantity to the Agent. Quantity will be our “effort”. At effort $x$, price is $A - Bx$, where $A > 0, B > 0$. Cost is $t \cdot C(x) = tx$ and revenue is

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9 One of them concerns optimal versus “selfish” routing in transportation networks (see Roughgarden (2005)). Others are found in Nissan, Roughgarden, Tardos, and Vazirani (eds.) (2007).

\[ R(x) = Ax - Bx^2. \] Marginal revenue becomes negative at \( x = \frac{A}{2B} \). To keep price and marginal revenue positive, our set of possible efforts will be

\[ \Sigma = \left( 0, \frac{A}{2B} \right). \]

We consider a set \( \Gamma \) of pairs \((r, t)\), where (i) every \( r \in (0, 1) \) belongs to one and only one pair, and (ii) for every pair \((r, t) \in \Gamma \) there is a positive effort \( \hat{x}(r, t) \) which belongs to \( \Sigma \) and is the unique maximizer of \( r \cdot R(x) - t \cdot C(x) \). That is the case for

\[ \Gamma \equiv \{(r, t) : 0 < r < 1; 0 < t < Ar \} \]

and

\[ \hat{x}(r, t) = \frac{A}{2B} - \frac{t}{2Br}. \]

For a given \( t \), the Agent’s best response to the share \( r \) — if \((r, t) \in \Gamma \) — is the effort \( \hat{x}(r, t) \), which belongs to \( \Sigma \).

In every example that we study we will specify a similar set \( \Gamma \), having the properties (i) and (ii). Note that in the Classic example, \( \Gamma \) is the interior of a triangle. In a diagram with \( r \) on the horizontal axis and \( t \) on the vertical axis the triangle has vertices at \((0,0), (1,0) \) and \((1, A)\). In other examples \( \Gamma \) might be a rectangle, as in the example which follows (in 5.2). In still other examples one of the boundaries of \( \Gamma \) might have curvature. We shall let \( \hat{\Gamma} \) denote the set of values of \( t \) which we consider. Thus

\[ \hat{\Gamma} \equiv \{ t : (r, t) \in \Gamma \text{ for some } r \in (0,1) \}. \]

In our Classic example, \( \hat{\Gamma} = (0, A) \).

Now for every \((r, t) \in \Gamma \) consider the derivative of \( \hat{x} \) with respect to \( r \), the derivative with respect to \( t \), and the cross derivative. They will be denoted by \( \hat{x}_r, \hat{x}_t \) and \( \hat{x}_{rt} \). We have the following results. Some of them will be generalized to wider classes of examples.

- \( \hat{x}_r(r, t) = \frac{1}{2B^2} \), which is positive. For a given \( t \), increasing the share evokes more effort. We shall show\(^{11}\) that in any example, finite or nonfinite, increasing the share never evokes less effort.

- \( \hat{x}_t(r, t) = -\frac{1}{2Br} \), which is negative. When \( r \) is fixed and technology improves (when \( t \) drops), the Agent works harder. We shall show\(^{12}\) that in any example, finite or nonfinite, the Agent never works less when \( t \) drops.

- We have \( \hat{x}_{rt}(r, t) = \frac{1}{2B} \cdot \frac{1}{r^2} > 0 \). So technology improvement (a drop in \( t \)) diminishes effectiveness (the effort increase evoked by a small rise in \( r \)).

- When the Agent uses the best effort \( \hat{x}(r, t) \), he receives \( r \cdot R(\hat{x}(r, t)) - t \cdot \hat{x}(r, t) \). In our Classic example, the derivative of that expression with respect to \( t \) turns out to be negative.\(^{13}\) So in the

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\(^{11}\)In Part (a) of Theorem 1.

\(^{12}\)In Part (b) of Theorem 1.

\(^{13}\)The derivative is \( \hat{x}_t(r, t) \cdot [rR'(\hat{x}(r, t)) - t \cdot C'(\hat{x}(r, t))] - C(\hat{x}(r, t)) \). That is negative, since \( 0 < r < 1 \) and \( \hat{x}(r, t) \) satisfies the first-order condition \( 0 = rR' - tC' \).
exogenous case, technology improvement is good news for the Agent. We shall provide\textsuperscript{14} a simple proof that this statement holds in any example, finite or nonfinite.

- We find that

\[
\text{surplus} = W(r, t) = R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t)) = \frac{1}{4B^2r^2} \cdot [(Ar - t) \cdot (BAr + Bt - 2Br)].
\]

The derivative with respect to $t$ of the expression in square brackets is

\[
-2BAr^2 - 2Bt + 4Br.
\]

Our requirement that $t < Ar$ implies that this is negative.\textsuperscript{15} Thus, for a fixed $r < 1$, decentralized exogenous-case surplus rises when technology improves ($t$ drops). We shall see\textsuperscript{16} that this always holds, for both finite and infinite message sets, as long as $R$ and $C$ are strictly increasing.

- For all $t \in \tilde{\Gamma}$, we have\textsuperscript{17} $W_t(1, t) < 0$. Maximal surplus\textsuperscript{18} rises when technology improves ($t$ drops). As we shall see\textsuperscript{19}, a trivial argument shows that this always holds in both finite and non-finite examples.

- We have $W_{rt}(r, t) = \frac{(1 - r) \cdot t}{Br^3} > 0$. So $W_{rt}(r, t)$ and $\hat{x}_{rt}(r, t)$ have the same sign. That implies, as we shall see, that the exogenous Decentralization Penalty (surplus gap) $W(1, t) - W(r, t)$ and the exogenous effort gap $\hat{x}(1, t) - \hat{x}(r, t)$ move in the same direction when technology improves, i.e., the exogenous surplus gap tracks the exogenous effort gap. There are finite examples where that is not the case. But we shall show\textsuperscript{20} that if $R$ and $C$ are thrice differentiable then it must be the case, because, as we shall prove, $W_{rt} \cdot \hat{x}_{rt} \geq 0$.

We now turn to the endogenous case. The Principal chooses a best share $r$, but excludes $r = 1$, which would give the Agent all of the revenue. To study the consequence of choosing $r = 0$, we would have to specify how the Agent responds to $r = 0$. The natural answer is zero effort, but there would then be some technical difficulties in our analysis of certain examples. So, as already noted, we confine attention to the case where the Principal chooses a share in the open interval $(0, 1)$ and the Agent’s response is a positive effort. We will show\textsuperscript{21} that under simple conditions (which are satisfied in the Classic example), the Principal’s gain $(1 - r) \cdot (R(\hat{x}(r, t)))$
is positive for all \( r \in (0, 1) \) and is concave on \((0, 1)\). That implies that there is a share in \((0, 1)\) which solves the first-order condition

\[
0 = \frac{d}{dr} [(1 - r) \cdot R(\hat{x}(r, t))] = -R(\hat{x}(r, t)) + (1 - r) \cdot R'(\hat{x}(r, t)) \cdot \hat{x}_r(r, t)
\]

and maximizes the Principal’s gain on the set \((0, 1)\). That share is our \( r^*(t) \in (0, 1) \). In our Classic example the Principal’s first-order condition turns out to be the cubic equation

\[
0 = A^2 r^3 + rt^2 - 2t^2.
\]

When we graph the implicit function \( r^*(t) \), we obtain the following figure for the case \( A = 2, B = 3 \):

![FIGURE 1 HERE](image1.png)

The graph reveals that \( r^* \) is increasing in \( t \). The share-choosing Principal becomes less generous when technology improves. But we can establish this fact — for a very wide class of examples that includes the Classic example — without any graphing. We shall show\(^{22}\) that it must hold whenever \( R'' < 0 \) and \( \hat{x}_{rt} < 0 \) (as in the Classic example). Thus a drop in \( t \) makes the Principal less generous if it makes effectiveness drop and in addition the Agent’s increased effort (in response to the lower \( t \)) makes marginal revenue drop.

Next consider the endogenous effort \( \hat{x}(r^*(t), t) \). Figure 2 shows that in the Classic example the endogenous effort rises when technology improves (\( t \) drops). We shall show\(^{23}\) that this must happen, for the endogenous case, in every example, finite or nonfinite.

![FIGURE 2 HERE](image2.png)

The endogenous decentralization Penalty (surplus gap) is \( W(1, t) - W(r^*(t), t) \). That can be graphed as a function of \( t \), even without an explicit expression for \( r^*(t) \). We do so in Figure 3 for the Classic example. We see that for large \( t \) further technical improvement (a further drop in \( t \)) raises the Penalty, but for small \( t \) further improvement lowers it.

![FIGURE 3 HERE](image3.png)

We now turn to the tracking question. For the Classic example, Figure 4 shows both the Penalty (surplus gap) and the effort gap \( \hat{x}(1, t) - \hat{x}(r^*(t), t) \). When \( t \) increases each gap first rises and then falls and for each \( t \) the gaps move in the same direction, so we indeed have tracking.

![FIGURE 4 HERE](image4.png)

5.2 A “Cubic-revenue” example, where, just as in the Classic example, technology improvement diminishes effectiveness and generosity, but now we do not have tracking.

\(^{22}\)In Theorem 6.

\(^{23}\)In Part (b) of Theorem 2.
Fig. 1 graph of $r^*(t)$ for the Classic example with $A = 2, B = 3$
Fig. 2  graph of $\hat{x}(r \ast (t), t)$ for the Classic example with $A = 2, B = 3$
Figure 3: $W(1, t)$ and $W(r^*(t), t)$ for the Classic case, with $A = 2, B = 3$
The two gaps: surplus gap \( W(1, t) - W(r^*(t), t) \) and effort gap \( \hat{x}(1, t) - \hat{x}(r^*(t), t) \) for the Classic case, with \( A = 2, B = 3 \)
In this example:

\[ R(x) = x^3 - x^2, \quad C(x) = x, \]

and the set of possible \((r, t)\) pairs is \(\Gamma = \{(r, t) : r \in (0, 1); t \leq r\}\), so the set of possible values of \(t\) is \(\tilde{\Gamma} = (0, 1)\). We find — just as in the Classic example — that for all \((r, t)\) in \(\Gamma\), effectiveness diminishes when \(t\) drops, i.e., \(\hat{x}_{rt}(r, t) > 0\).24 Now consider \(r^*(t)\), the implicit function which solves the Principal’s first-order equation for \(t \in (0, 1)\). That function turns out to satisfy a cumbersome polynomial equation.25 Figure 5 graphs the implicit function \(r^*\).

Just as in the Classic example, \(r^*\) rises at every possible value of \(t\) (all \(t \in (0, 1)\)). Finally, we plot the surplus gap and the effort gap.

We find that for \(t\) in the interval \((.48, .63)\), the effort gap rises but the surplus gap falls. Unlike the Classic example, we do not have tracking.

5.3 A “Price-taker” example where marginal cost rises and marginal revenue is flat

In this example we may think of the Principal as a price-taker (with price equal to one). He delegates quantity choice to the Agent and the Agent’s cost function is quadratic. “Price-taker” is a convenient label for this example. The example is defined as follows:

\[ \hat{x}(r, t) = \frac{1}{\sqrt{3}} \cdot \left(1 - \frac{t}{r}\right)^{1/2}. \]

Next we obtain

\[ \hat{x}_r(r, t) = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} \cdot \frac{d}{dr} \left[ \left(1 - \frac{t}{r}\right)^{1/2}\right] = \frac{1}{4\sqrt{3}} \cdot \left(1 - \frac{t}{r}\right)^{-1/2} \cdot \frac{t}{r^2}. \]

We then have:

\[ 4\sqrt{3} \cdot \hat{x}_{rt}(r, t) = \frac{d}{dt} \left[ \left(1 - \frac{t}{r}\right)^{-1/2} \cdot \frac{t}{r^2}\right] \]

\[ = \left(1 - \frac{t}{r}\right)^{1/2} \cdot \left(\frac{1}{r^2}\right) + t \cdot \frac{1}{r^2} \cdot \left(-\frac{1}{2}\right) \cdot \frac{d}{dt} \left[ \left(1 - \frac{t}{r}\right)^{-1/2}\right]. \]

But

\[ \frac{d}{dt} \left[ \left(1 - \frac{t}{r}\right)^{-1/2}\right] = -\frac{1}{2} \cdot \left(1 - \frac{t}{r}\right)^{-3/2} \cdot \frac{t}{r^2} < 0. \]

So we indeed have \(\hat{x}_{rt}(r, t) > 0\).

The equation is

\[ 0 = 16[r^*(t)]^6 - 12t^2 - 27t^4 \cdot [r^*(t)]^4 - 4t^3 + 108t^4 \cdot [r^*(t)]^3 - 162t^4 \cdot [r^*(t)]^2 + 108t^4 \cdot r^*(t) - 27t^4. \]
Fig. 5. Graph of $r^*(t)$ in the Cubic-revenue example.
Fig. 6 Surplus and effort gaps in the Cubic-revenue example. For $t \in (0.48, 0.63)$, effort gap rises but surplus gap falls.
• The set of possible efforts is $\Sigma = IR^+$.
• $R(x) = x$.
• $C(x) = \frac{1}{2}(x - 1)^2$.
• The set of possible pairs $(r, t)$ is the rectangle $\Gamma = \{(r, t) : 0 < r < 1; 0 < t < 1\}$.

We find the following:
• Given $r$, the Agent chooses $\hat{x}(r, t) = \frac{r}{t} + 1$. We then have $\hat{x}_{rt}(r, t) = \frac{-1}{t^2} < 0$. Effectiveness rises when technology improves (when $t$ drops).

• When the Agent uses the best effort $\hat{x}(r, t)$, he receives $r + \frac{r^2}{2t}$. The derivative with respect to $t$ is $-\frac{r^2}{2t^2} < 0$. So, just as in the Classic example technology improvement is good news for the Agent in the exogenous case. As we already noted, we will provide a simple (calculus-free) proof that this “good news” statement holds, for the exogenous case, in any example, finite or nonfinite.

• Exogenous surplus is $R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t)) = \frac{r}{t} + 1 - \frac{r^2}{2t}$.
• Surplus-maximizing effort is $\frac{1}{t} + 1$ and maximal surplus is $1 + \frac{1}{2t}$.

• The exogenous surplus gap (the Penalty) is $\frac{1}{2t} - \frac{r}{t} + \frac{r^2}{2t}$. Its derivative with respect to $t$ is negative. The exogenous effort gap is $\frac{1-r}{t}$, which also has a negative derivative. So the exogenous surplus gap tracks the exogenous effort gap, just as in the Classic example. As already noted, this will be proved to hold, in the exogenous case, for any differentiable example.

We now turn to the endogenous case. We find that:

• The solution to the Principal’s first order condition $0 = \frac{d}{dr}[R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t))]$ is $r^*(t) = \frac{1-t}{2}$. So we have $r^*(t) < 0$ at every possible $t$. (Recall that $t < 1$). So, in sharp contrast to the Classic monopoly example, the Principal becomes more generous when technology improves.

• We find that — just as in the exogenous case — a drop in $t$ is good news from the welfare point of view. This must be the case whenever — as in the Price-taker example and the Rising Marginals example which we consider next — the Principal becomes more generous (or stays just as generous) when $t$ drops.\[26\]

\[26\]Shown in Part (d) of Theorem 2.

\[27\]The argument is as follows: We have
\[
\frac{d}{dt} [R(\hat{x}(r^*(t), t)) - t \cdot C(\hat{x}(r^*(t), t))] = [\hat{x}_r \cdot r^* + \hat{x}_t] \cdot (R' - tC') - C.
\]

We have $R' - tC' > 0$ (because of the first-order condition $rR' - tC' = 0$, where $0 < r < 1$). Since $\hat{x}_t < 0$ and $r^* \leq 0$, we conclude that the derivative is negative, so we indeed have “good news” from the welfare point of view.
• The endogenous Penalty (the endogenous surplus gap) is $\frac{1}{4} + \frac{1}{8t} + \frac{t}{8}$. Its derivative with respect to $t$ is $\frac{1}{8t^2} \cdot (t^2 - 1)$, which is negative, since $t < 1$. The Penalty rises when technology improves. Again, note the contrast with the Classic example, where the Penalty drops when technology improves, once $t$ has dropped below a critical value.

• The endogenous effort gap is $\frac{1+t}{t} - \frac{r^*(t)+t}{t} = \frac{1}{2t} + \frac{1}{2}$. That also has a negative derivative.

• So the endogenous effort gap tracks the exogenous surplus gap. But that is NOT implied, as we shall see, by the fact that $\dot{x}_{rt} < 0$ and $r^*'(t) < 0$.

5.4 A “Cubic-cost” example, where, just as in the Price-taker example, technology improvement increases both effectiveness and generosity, but now we have “opposite directions” rather than tracking.

In this example

$$R(x) = \frac{1}{2}x^2$$

and

$$C(x) = \frac{1}{3}x^3 + \frac{a}{2}x^2 - \epsilon x,$$

where $\epsilon > 0$ and $a > 0$. The numbers $a, \epsilon$ and the set $\Sigma$ of possible efforts will be chosen as we proceed. The triple $(a, \epsilon, \Sigma)$ will have the property that $C(x) > 0$ for all $x \in \Sigma$.

The Agent’s first-order condition for given $r, t$ is

$$rx = t \cdot (x^2 + ax - \epsilon).$$

This is solved by

$$\hat{x}(r, t) = \sqrt{\left(\frac{a}{t} - \frac{r}{t}\right)^2 + 4\epsilon - \left(\frac{a}{t} - \frac{r}{t}\right)} > 0.$$ 

Note that

$$\hat{x}_r = \frac{1}{2} \cdot \left(1 - \frac{1}{2} \left[\left(\frac{a}{t} - \frac{r}{t}\right)^2 + 4\epsilon\right]^{-1/2} \cdot \frac{2}{t} \cdot \left(\frac{a}{t} - \frac{r}{t}\right)\right)$$

$$= \frac{1}{2t} \cdot \left(1 - \frac{a - \frac{r}{t}}{\sqrt{\left(\frac{a}{t} - \frac{r}{t}\right)^2 + 4\epsilon}}\right)$$

$$= \frac{1}{2t} \cdot \sqrt{\left(\frac{a}{t} - \frac{r}{t}\right)^2 + 4\epsilon - \left(\frac{a}{t} - \frac{r}{t}\right)} \cdot \frac{\sqrt{\left(\frac{a}{t} - \frac{r}{t}\right)^2 + 4\epsilon}}{\sqrt{\left(\frac{a}{t} - \frac{r}{t}\right)^2 + 4\epsilon}}.$$
Our set $\Gamma$ of possible $(r, t)$ pairs will be

$$\Gamma = \left\{(r, t) ; t \in \left(\frac{1}{a}, \frac{2}{\sqrt{a^2 + 4\epsilon}}\right) ; 0 < r < 1\right\}.$$

Now assume that

- $t \geq \frac{1}{a}$
- $\epsilon < \frac{3}{4}a^2$.

Then $\frac{1}{a} < \frac{2}{\sqrt{a^2 + 4\epsilon}}$, so $\Gamma$ is not empty. Moreover $a - r/t \geq 0$ for all $r \in (0, 1)$. Under these assumptions we have $\hat{x}_{rt}(r, t) < 0$. Effectiveness increases when technology improves. \(^\text{28}\)

In the endogenous case the first-order condition satisfied by the Principal’s chosen share $r^*$ satisfies the first-order condition

$$r = 1 - \frac{R}{R\hat{x}_r} = 1 - \frac{1}{2} \frac{x^2}{\hat{x}_r} = 1 - \frac{\hat{x}}{2\hat{x}_r} = 1 - \frac{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon - (a - \frac{r}{t})}}{1 \sqrt{(a - \frac{r}{t})^2 + 4\epsilon - (a - \frac{r}{t})}^{\frac{1}{2}}} \cdot$$

So, if $r^*(t)$ is a maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$, it satisfies

\[(+) \quad r^*(t) = 1 - \frac{t \sqrt{(a - \frac{r^*(t)}{t})^2 + 4\epsilon}}{2} \cdot\]

\(^\text{28}\)Consider the fraction

$$\hat{x}_r = \frac{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon - (a - \frac{r}{t})}}{2t \cdot \sqrt{(a - \frac{r}{t})^2 + 4\epsilon}} \cdot$$

Multiply numerator and denominator by $\sqrt{(a - \frac{r}{t})^2 + 4\epsilon + (a - \frac{r}{t})}$. The new fraction simplifies to

$$\frac{4\epsilon}{2t \cdot \left[(a - \frac{r}{t})^2 + \sqrt{(a - \frac{r}{t})^2 + 4\epsilon}\right]} \cdot$$

Since $a > \frac{r}{t}$, the denominator is strictly increasing in $t$. Hence the whole fraction is strictly decreasing in $t$. So, as claimed, we have $\hat{x}_{rt}(r, t) < 0$.  

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We can show the following for every $t$ in our set $\tilde{\Gamma} = \left( \frac{1}{a}, \frac{2}{\sqrt{a^2 + 4\epsilon}} \right)$ of possible values of $t$:

1. There is a unique value of $r$, denoted $r^*(t)$, such that for every $t \in \tilde{\Gamma}$, $r^*(t)$ satisfies the first-order condition and hence $r^*(t)$ is the unique maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$ on the interval $(0, 1)$.

2. $r^*(t) < 0$ (the Principal becomes more generous when technology improves).

Now consider the case where $a = 1$ and $\epsilon = 0.6$. That meets our requirement $\epsilon < \frac{3}{4} a^2$. Define our set of possible efforts to be $\Sigma = (1, \infty]$. Then $\tilde{\Gamma} = (1, 1.084)$ and $C(x) > 0$ for every $x \in \Sigma$. Figure 7 graphs the surplus gap, which falls when technology improves ($t$ drops). Figure 8 graphs the effort gap, which rises when technology improves. We have “opposite directions” rather than tracking.

5.5 An example where marginal revenue rises but marginal cost rises faster.

It will be convenient to call this the “Rising Marginals” example. We have:

- The set of possible efforts is $\Sigma = \mathbb{R}^+$.
- $R(x) = x^a, C(x) = x^b, 0 < a < b$.

29 We first prove (2). To do so, we use a very general result, which holds even without differentiability. It will be proved below, in Part (a) of Theorem 2. It says that if $t_H > t_L$, then $\frac{r^*(t_H)}{t_H} \leq \frac{r^*(t_L)}{t_L}$, where $r^*(t)$ is any maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$ on $(0, 1)$. So when $t$ rises, the right side of (+) falls, i.e., $r^*(t) < 0$.

Next rewrite the first-order condition satisfied by the Principal’s chosen share as

$$0 = f(r, t) \equiv 1 - r - \frac{\sqrt{(at - r)^2 + 4\epsilon t^2}}{2}.$$ 

Then $f(0, t) < 0$, i.e., $2 > t\sqrt{a + 4\epsilon}$ for all of the possible values of $t$. That is the case since

$$2 = \frac{2}{\sqrt{a^2 + 4\epsilon}} \cdot \frac{2}{\sqrt{a^2 + 4\epsilon}}$$

and $t < \frac{2}{\sqrt{a^2 + 4\epsilon}}$. We also have $f(1, t) < 0$. Hence, by the Intermediate Value Theorem, for any $t \in \left( 0, \frac{2}{\sqrt{a^2 + 4\epsilon}} \right)$, we have $f(r^*(t), t) = 0$ for some $r^*(t) \in (0, 1)$. Moreover, since $r^*(t) < 0$, that $r^*(t)$ is the only solution to $f(r, t) = 0$ in $(0, 1)$. That establishes (1).
Fig. 7: Surplus gap in the Cubic-cost example.

\[ W(1, t) - W(r^*(t), t) \]
Fig. 8: Effort gap in the Cubic-cost example.

\[ x(1, t) - x(r^*(t), t) \]
• The set of possible pairs \((r, t)\) is \(\Gamma = \{(r, t) : 0 < r < 1; t > 0\}\).

We obtain the following:

• \(\hat{x}(r, t) = (\frac{tb}{a})^{1/(a-b)}\).

• \(\hat{x}_{rt}(r, t) = -\frac{1}{(a-b)^2} \cdot t^{1/(a-b)-1} \cdot \left(\frac{b}{a}\right)^{1/(a-b)} \cdot r^{1/(b-a)-1}\). That is negative. So when technology improves, effectiveness increases.

• In the endogenous case the Principal chooses the share \(r^*(t) = \frac{a}{b}\). The Principal's generosity remains unchanged when technology changes.\(^{30}\)

• Just as in the Price-taker example, Improvement in technology is good news for the share-choosing Principal. That is the case because \(r^* = 0\).

• Even though we have an explicit expression for \(r^*\), computing the derivative of endogenous effort gap (Penalty) with respect to \(t\) and the derivative of endogenous surplus gap with respect to \(t\) is cumbersome. It turns out that both are negative. So the endogenous surplus gap tracks the endogenous effort gap. This is true as well in the Classic and Price-taker examples (but not in the Cubic-revenue example). The fact that it is true in the Classic and Price-taker examples does not follow from the signs of \(\hat{x}_{rt}\) and \(r^*\) in those examples. In contrast, we shall show\(^{31}\) that, in the endogenous case, the surplus gap tracks the effort gap whenever (as in the Rising Marginals example) \(\hat{x}_{rt} < 0\) and \(r^* \geq 0\).

6. Basic results that do not require differentiability.

We now state two theorems which are proved without requiring differentiability of \(R\) or \(C\). Theorem 1 concerns the exogenous case and Theorem 2 concerns the endogenous case. Both theorems hold for all examples in which \(R\) and \(C\) are strictly increasing. An example is defined

\[ r = 1 - \frac{R(\hat{x}(r, t))}{R'(\hat{x}(r, t)) \cdot \hat{x}_{r}(r, t)}. \]

In the example we obtain:

\[ \frac{R(x)}{R'(x)} = \frac{x}{a}, \]

\[ \hat{x}_{r} = \left(\frac{tb}{a}\right)^{1/(a-b)} \cdot \frac{1}{b - a} \cdot r^{1/(b-a)}, \]

and

\[ \frac{\hat{x}}{\hat{x}_{r}} = r \cdot (b - a). \]

So the first-order condition is \(r = 1 - \frac{r_{-1/(b-a)}}{a}\). That is solved by \(r^* = \frac{a}{b}\).

\(^{30}\)The first-order condition satisfied by \(r^*\) can be written

\[ r = 1 - \frac{R(\hat{x}(r, t))}{R'(\hat{x}(r, t)) \cdot \hat{x}_{r}(r, t)}. \]

In the example we obtain:

\[ \frac{R(x)}{R'(x)} = \frac{x}{a}, \]

\[ \hat{x}_{r} = \left(\frac{tb}{a}\right)^{1/(a-b)} \cdot \frac{1}{b - a} \cdot r^{1/(b-a)}, \]

and

\[ \frac{\hat{x}}{\hat{x}_{r}} = r \cdot (b - a). \]

So the first-order condition is \(r = 1 - \frac{r_{-1/(b-a)}}{a}\). That is solved by \(r^* = \frac{a}{b}\).

\(^{31}\)In Part (b) of Theorem 5.
by a set $\Sigma$ of possible positive efforts, the functions $R$ and $C$, and a set $\Gamma$ of possible pairs $(r, t)$. Recall that for every $r \in (0, 1)$, $\Gamma$ contains some pair $(r, t)$. The set of values of $t$ such that $(r, t) \in \Gamma$ for some $r$ is again denoted $\hat{\Gamma}$. Recall that the function $\hat{x}$ has the property that for all $(r, t) \in \Gamma$, we have $\hat{x}(r, t) \in \Sigma$ and $r \cdot R(\hat{x}(r, t)) - tC(\hat{x}(r, t)) \geq r \cdot R(x) - tC(x)$ for all $x \in \Sigma$.

The exogenous-case Theorem 1 has eight parts. Part (a) says that the Agent never works less when the share $r$ rises (while remaining less than one), and strictly prefers the higher share. Part (b) says that the Agent never works less hard when $t$ drops (technology improves) Part (c) says that the surplus-maximizing effort cannot fall when $t$ drops. Part (d) says that maximal surplus must rise when $t$ drops. Part (e) says that a drop in $t$ is never bad news for the Principal and Part (f) says that it must be good news for the Agent. Part (g) says that a drop in $t$ is never bad news from the welfare point of view. Part (h) says that a rise in the share $r$ is never bad news from the welfare point of view and is good news if and only if the Agent’s effort changes after the drop. Thus, in the exogenous case, it is in the “social” interest for the Principal to be more generous.

**Theorem 1**

Let $R$ and $C$ be strictly increasing on $\Sigma$. Then:

(a) $\hat{x}(r_H, t) \geq \hat{x}(r_L, t)$ and $R(\hat{x}(r_H, t)) - tC(\hat{x}(r_H, t)) > R(\hat{x}(r_L, t)) - tC(\hat{x}(r_L, t))$ whenever $(r_L, t) \in \Gamma, (r_H, t) \in \Gamma,$ and $0 < r_L < r_H < 1$.

(b) $\hat{x}(r, t_L) \geq \hat{x}(r, t_H)$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma,$ and $0 < t_L < t_H$.

(c) $\hat{x}(1, t_L) \geq \hat{x}(1, t_H)$ whenever $t_L, t_H \in \hat{\Gamma},$ and $0 < t_L < t_H$.

(d) $W(1, t_L) > W(1, t_H)$ whenever $t_L, t_H \in \hat{\Gamma}$ and $0 < t_L < t_H$.

(e) $(1 - r) \cdot R(\hat{x}(r, t_L)) \geq (1 - r) \cdot R(\hat{x}(r, t_H))$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma,$ and $0 < t_L < t_H$.

(f) $r R(\hat{x}(r, t_L)) - t LC(\hat{x}(r, t_L)) > r R(\hat{x}(r, t_H)) - t HC(\hat{x}(r, t_H))$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma,$ and $0 < t_L < t_H$.

(g) $W(r, t_L) > W(r, t_H)$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma,$ and $0 < t_L < t_H$.

(h) $W(r_H, t) \geq W(r_L, t)$ whenever $(r_H, t) \in \Gamma, (r_L, t) \in \Gamma,$ and $0 < r_L < r_H < 1.$ The inequality is strict if and only if $\hat{x}(r_H, t) \neq \hat{x}(r_L, t)$.

The proof of Theorem 1, like all the subsequent proofs, is found in the Appendix. In proving Parts (e),(f),(g), (h) we use the simple observation that when $t$ drops or $r$ rises, the Agent could continue to use the same effort as before the change. In proving Parts (a),(b),(c),(d) we use a basic proposition from monotone comparative statics.32

32See, for example, Sundaram (1996). The proposition concerns a function $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ which displays strictly increasing differences. [For such a function we have $h(u_H, v_H) - h(u_L, v_H) > h(u_H, v_L) - h(u_L, v_L)$ whenever $u_H > u_L, v_H > v_L$.] The proposition is as follows:
Theorem 2 concerns the endogenous case. Part (a) says that the ratio of the Principal’s chosen share to the technology parameter $t$ cannot fall when $t$ drops. But, as we have already seen in the examples, the chosen share itself may rise or fall or stay the same. Nevertheless Part (b) says that in the endogenous case the Agent never works less hard when $t$ drops. Part (c) says that in the endogenous case a drop in $t$ is never bad news for the Principal. (That is the endogenous counterpart of Part (h) of theorem 1). Part (d) says that in the endogenous case a drop in $t$ must be good news from the welfare point of view.

**Theorem 2**

Let $R$ and $C$ be strictly increasing on $\Sigma$. Let $r^*(t)$ denote a maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$ on the interval $(0, 1)$. Then

(a) $\frac{r^*(t_L)}{t_L} \geq \frac{r^*(t_H)}{t_H}$ whenever $t_L, t_H \in \hat{\Gamma}$ and $0 < t_L < t_H$.

(b) $\hat{x}(r^*(t_L), t_L) \geq \hat{x}(r^*(t_H), t_H)$ whenever $t_L, t_H \in \hat{\Gamma}$ and $0 < t_L < t_H$.

(c) $(1 - r^*(t_L)) \cdot R(\hat{x}(r^*(t_L), t_L)) \geq (1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_H))$ whenever $t_L, t_H \in \hat{\Gamma}$ and $0 < t_L < t_H$.

(d) $W(r^*(t_L), t_L) > W(r^*(t_H), t_H)$ whenever $t_L \in \hat{\Gamma}, t_H \in \hat{\Gamma}$, and $0 < t_L < t_H$.

While Part (c) of Theorem 2 tells us that in the endogenous case technical improvement can never be bad news for the Principal, the situation is different for the Agent. Figure 9 is a graph of the Agent’s endogenous-case net earnings $r^*(t) \cdot R(\hat{x}(r^*(t)), t) - t \cdot C(\hat{x}(r^*(t)), t)$ in the Classic example. Once $t$ has dropped to a critical value that is close to 0.5, a further drop is Bad news for the Agent. Informally: in the endogenous case, the Principal is never the enemy of technical progress but the Agent might be.

**FIGURE 9 HERE**

7. Two exogenous-case theorems which require differentiability.

7.1 Effectiveness and the effort gap move in the same direction when $t$ changes.

**Theorem 3**

Let $\Gamma$ be an open set in $\mathbb{R}^2^+$. Suppose that the functions $R$ and $C$ are thrice differentiable. Suppose that the following monotonicity condition is met:

we either have

$\hat{x}_{rt} > 0$ for all $(r, t) \in \Gamma$

or

$\hat{x}_{rt} < 0$ for all $(r, t) \in \Gamma$.

Suppose, in addition, $\hat{x}_t$ is continuous with respect to $r$ at all points in $(0, 1]$.

If a function $h(u, v)$ displays strictly increasing differences, and if $u_H$ maximizes $h(u, v_H)$ while $u_L$ maximizes $h(u, v_L)$, then $u_H \geq u_L$ if $v_H > v_L$. 

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The Agent’s net earnings for the Classic example with $A = 2$, $B = 3$

Fig. 9 The Agent’s net earnings for the Classic example with $A = 2$, $B = 3$
Then \( \hat{x}_{rt}(r, t) > 0 \) (< 0) at every \( (r, t) \in \Gamma \) if and only if

\[
\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] > (0 < 0) \text{ at every } (r, t) \in \Gamma.
\]

The assumptions of the theorem are met in all the examples we have presented. Note that the pair \( (r^*(t), t) \) belongs to \( \Gamma \), so the theorem applies, in particular, to \( \hat{x}_{rt}(r^*(t), t) \) and the endogenous effort gap \( \hat{x}(1, t) - \hat{x}(r, t) \). The proof (in the Appendix) is very simple.

### 7.2. A theorem about the effort gap and the surplus gap.

We first formally define exogenous tracking in examples where \( R \) and \( C \) are twice differentiable on the effort set \( \Sigma \) and, for fixed \( r \in (0, 1] \), the Agent’s effort choice \( \hat{x}(r, t) \in \Sigma \) solves the first-order condition \( 0 = rR'(x) - tC'(x) \).

**Definition 1**

An example \( (R, C, \Gamma, \Sigma) \), with \( R \) and \( C \) thrice differentiable, has the exogenous tracking (opposite directions) property if

\[
\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] \cdot \frac{d}{dt} [W(1, t) - W(r, t)] > 0 \text{ (< 0) at all } (r, t) \in \Gamma.
\]

The next theorem concerns exogenous tracking in Interior examples. In an Interior example the Agent’s best effort is the unique solution to a first-order equation and the same is true for the Principal’s chosen share. Before providing the definition, we recall that for every \( (r, t) \in \Gamma \) we have \( 0 < r < 1 \). Recall also that \( \tilde{\Gamma} \) denotes the set of possible values of \( t \). (\( \tilde{\Gamma} = \{t : (r, t) \in \Gamma \text{ for some } r\} \)).

**Definition 2**

An example \( (\Sigma, R, C, \Gamma) \) is Interior if

- \( \Sigma \subseteq IR^+ \), and \( \Gamma \subseteq IR^2^+ \), are open sets.
- \( R, C \) are thrice differentiable on \( \Sigma \) and \( R' > 0, C' > 0 \).

\footnote{Note that we could state a more general definition, not requiring differentiability. There we would say that the example has the exogenous tracking property if we have

\[
\begin{align*}
\{ W(1, t_L) - W(r, t_L) &> W(1, t_H) - W(r, t_H) \} \\
\text{whenever } (r, t_L), (r, t_H) &\in \Gamma \text{ and } 0 < t_L < t_H
\end{align*}
\]

if and only if we also have

\[
\begin{align*}
\{ W(1, t_L) - W(r, t_L) &> W(1, t_H) - W(r, t_H) \} \\
\{ W(1, t_L) - W(r, t_L) &< W(1, t_H) - W(r, t_H) \} \text{ whenever } (r, t_L)(r, t_H) &\in \Gamma \text{ and } 0 < t_L < t_H.
\end{align*}
\]

For “opposite directions” the appropriate inequalities are reversed. Using this definition, one could explore the tracking question for finite examples.}
• There exists a twice differentiable function \( \hat{x} : (0, 1] \times \tilde{\Gamma} \to \Sigma \) such that for \( r \in (0, 1] \), \( \hat{x}(r, t) \) satisfies the first-order condition \( 0 = rR'(x) - tC'(x) \) and is the unique maximizer of \( rR(x) - tC(x) \) on \( \Sigma \).

• For every \( t \in \tilde{\Gamma} \), there exists a share \( r^*(t) \in (0, 1) \) which satisfies the first-order condition \( 0 = \frac{d}{dr} [(1 - r) \cdot R(\hat{x}(r, t))] \) and is the unique maximizer of \( (1 - r) \cdot R(\hat{x}(r, t)) \) on \( (0, 1) \).

All the examples we have discussed satisfy these conditions.\(^{34}\)

**Theorem 4**

An interior example has the exogenous tracking property if the effort set is \( \Sigma = (0, J) \), where \( J > 0 \), and the monotonicity condition of Theorem 2 holds (we either have \( \hat{x}_{rt} > 0 \) for all \( (r, t) \in \Gamma \) or \( \hat{x}_{rt} < 0 \) for all \( (r, t) \in \Gamma \)).

Straightforward calculation yields the following Corollary, proved (together with theorem 4) in the Appendix.

**Corollary**

The following hold for an interior example in which the monotonicity condition of Theorem 2 is satisfied, the effort set is \( \Sigma = (0, J) \) (where \( J > 0 \)), and \( \hat{x}_r(r, t) > 0, \hat{x}_t(r, t) < 0 \) for all \( (r, t) \in \Gamma \):

(i) the Decentralization Penalty (surplus gap) is decreasing in \( t \) (so the Penalty grows when technology improves) if at every effort \( x \in (0, J) \) we have \( R''(x) \geq 0, R'''(x) = C'''(x) = 0 \).

(ii) the Decentralization Penalty (surplus gap) is increasing in \( t \) (so the Penalty shrinks when technology improves) if at every effort \( x \in (0, J) \) we have \( R''(x) < 0, C''(x) = 0, R'''(x) \leq 0 \).

Even though we are in the relatively straightforward exogenous case, the Corollary’s sufficient conditions for the Penalty to grow (shrink) when technology improves are restrictive but simple. When we turn to the endogenous case, we find no similarly simple conditions on \( R \) and \( C \) which tell us, all by themselves, the direction in which the Penalty moves when technology improves.

**8. Endogenous-case results which require differentiability.**

We have seen that in the endogenous case there are examples where the Decentralization Penalty (surplus gap) rises when technology improves and there are examples where it falls. There are examples where we have “endogenous tracking” (surplus and effort gaps move in the same direction when \( t \) changes), but there other examples where that is not true. There is no endogenous analog of Theorem 4 in which we again have tracking under very general assumptions. Here is our tracking definition for the endogenous case.

**Definition 3**

\(^{34}\)Consider the condition we imposed in Theorem 3. We require that the function \( \hat{x}_t \) is continuous with respect to \( r \) at every \( r \in (0, 1] \). The third item in our Interior Example definition insures that this is indeed the case.
An interior example \((R, C, \Gamma, \Sigma)\) has the \textit{endogenous tracking (opposite directions)} property if

\[
\frac{d}{dt} [\hat{x}(1,t) - \hat{x}(r^*(t),t)] \cdot \frac{d}{dt} [W(1,t) - W(r^*(t),t)] > 0 \quad (< 0) \text{ at all } t \in \tilde{\Gamma} = \{ t : (r, t) \in \Gamma \text{ for some } r \}.
\]

An instructive way to bring order to the rich variety of endogenous results is to characterize the way that (a) effectiveness \(\hat{x}_{rt}(r,t)\) and (b) the Principal’s chosen share \(r^*\) move when \(t\) changes. For a drop in \(t\), we consider four combinations: (a) and (b) both rise; they both fall; (a) rises and (b) falls; (a) falls and (b) rises. In particular we find — in Theorem 5 — that when one rises and the other falls, then we indeed have endogenous tracking. In Theorem 6 we find that if \(R'' < 0\) then a drop in \(t\) cannot lead to more effectiveness and greater Principal’s generosity. Before proceeding to these theorems we present a four-box endogenous-case table which serves as a guide to those theorems and their relation to the examples we have considered.
The effect of improved technology (a drop in $t$) in four groups of interior examples

<table>
<thead>
<tr>
<th>When $t$ drops, effectiveness of a share increase falls</th>
<th>When $t$ drops, effectiveness of a share increase rises</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x}_{rt} &gt; 0$ and hence $\frac{d}{dt}[\hat{x}(1, t) - \hat{x}(r, t)] &gt; 0$</td>
<td>$\hat{x}_{rt} &lt; 0$ and hence $\frac{d}{dt}[\hat{x}(1, t) - \hat{x}(r, t)] &lt; 0$</td>
</tr>
</tbody>
</table>

When $t$ drops, effectiveness of a share increase rises, hence so does the exogenous effort gap (see Theorem 3).

When $t$ drops, effectiveness of a share increase falls, hence so does the exogenous effort gap (see Theorem 3).

<table>
<thead>
<tr>
<th>BOX 1</th>
<th>SEE “CLASSIC” AND “CUBIC-REVENUE” EXAMPLES. WE HAVE TRACKING IN THE CLASSIC EXAMPLE BUT IN THE CUBIC-REVENUE EXAMPLE WE HAVE “OPPOSITE DIRECTIONS” (IF THE SET OF POSSIBLE VALUES OF $t$ IS PROPERLY CHOSEN).</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>BOX 2</th>
<th>SEE “RISING MARGINALS” EXAMPLE. EVERY EXAMPLE THAT LIES IN THIS BOX HAS THE TRACKING PROPERTY. (See Theorem 5, Part (a)).</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>BOX 3</th>
<th>SEE “EXPLODING MARGINALS” EXAMPLE. EVERY EXAMPLE THAT LIES IN THIS BOX HAS THE TRACKING PROPERTY. (See Theorem 5, Part (b)). AN EXAMPLE WITH $R'' &lt; 0$ CANNOT BE IN THIS BOX. (See Theorem 6).</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>BOX 4</th>
<th>SEE THE “PRICE-TAKER” EXAMPLE, WHERE WE HAVE TRACKING AND THE “CUBIC-COST” EXAMPLE, WHERE WE HAVE “OPPOSITE DIRECTIONS”.</th>
</tr>
</thead>
</table>

The example in Box 3, which we call the “Exploding Marginals” example, has functions $R$ and $C$ such that $C'' > R''$ and both $C''$ and $R''$ are very large. It appears difficult to construct Box-3 examples where that is not the case.35 The Exploding Marginals example is as follows:

- $\Sigma = (0, 1)$.
- $\Gamma = \{(r, t) : 0 < r < 1; \frac{r}{t} \in (e, e^e)\}$ ($e$ is the base of the natural logarithms).
- $R(x) = e^{x^2}$.
- $C(x) = \int_0^x \left[2e^{ep} \cdot e^{p} \cdot p\right] dp$.

The proof that the Exploding Marginals example indeed lies in Box 3 is provided in the Appendix.

We now have Theorem 5, a two-part theorem, which concerns Box 2 and Box 3.

---

35 One can prove, for example, that if $R = \frac{1}{2}x^2$, so that $R'' = 1$, then we cannot be in Box 3.
Theorem 5

Consider an interior example \((\Sigma, \Gamma, R, C)\).

(a) Suppose the following holds:

\[
\text{for every } t \in \tilde{\Gamma} \text{ we have } r^*(t) \geq 0 \text{ and for every } (r, t) \in \Gamma \text{ we have } \hat{x}_{rt}(r, t) < 0.
\]

Then we have endogenous tracking.

(b) Suppose the following holds:

\[
\text{for every } t \in \tilde{\Gamma} \text{ we have } r^*(t) < 0 \text{ and for every } (r, t) \in \Gamma \text{ we have } \hat{x}_{rt}(r, t) > 0.
\]

Then we have endogenous tracking.

The next theorem does not directly concern the two gaps. But it implies that if marginal revenue is decreasing or constant \((R'' \leq 0)\) in an interior example and the Principal has a unique best share, then the example cannot be in Box 3.

Theorem 6

Suppose that in the interior example \((\Sigma, \Gamma, R, C)\) we have:

- \(R''(x) \leq 0\) at every \(x \in \Sigma\).
- \(\hat{x}_{rt}(r, t) \geq 0, \hat{x}_t(r, t) < 0\) and \(\hat{x}_r(r, t) > 0\) at every \((r, t) \in \Gamma\).
- \(r^*(t)\) is the unique maximizer of \((1 - r) \cdot R(\hat{x}(r, t))\) on \((0, 1)\),

Then \(r^*(t) \geq 0\) for all \(t \in \Gamma\).

It is difficult to give a clear intuition for Theorems 4 and 5. That is a little easier for Theorem 6, which says that if marginal revenue is decreasing, and effectiveness drops when technology improves, then when technology improves, the Principal does not become more generous \((r^*(t) \geq 0)\), i.e., we cannot be in Box 3. Intuitively one might say: when \(t\) drops, increasing the share above its previous level would damage the Principal, because the extra revenue due to extra effort has dropped (marginal revenue has declined) and at the same time the extra effort evoked by a share increase has dropped as well.

8.2 A summary: the effect of technical improvement on the Decentralization Penalty (surplus gap) in the endogenous case.

Return to our original puzzle: when does technical improvement lower the Penalty and when does it raise the Penalty? Here is a summary of what Theorems 5 and 6 have told us about the puzzle in the endogenous case.
Consider any interior example. If, in that example, technical improvement makes the Principal less generous or keeps his generosity unchanged, while at the same time it raises the effectiveness of a share increase \((r^* (t) \geq 0 \text{ and } \hat{x}_{rt}(r, t) < 0)\), then the improvement raises the Penalty or keeps it unchanged. If technical improvement makes the Principal more generous, while at the same time it decreases the effectiveness of a share increase or leaves it unchanged \((r^* (t) < 0 \text{ and } \hat{x}_{rt}(r, t) \geq 0)\) — which cannot happen if marginal revenue is nonincreasing — then the improvement lowers the Penalty or keeps it unchanged.

Unfortunately there are no simple conditions on \(R\) and \(C\), similar to those in the Corollary to the exogenous-case Theorem 4, which imply, all by themselves, that the Penalty rises (falls) when technology improves.

9. Finding the Principal’s best share for a given \(t\): when is the Principal’s gain a concave function of the share?

The function \(r^*(t)\) may be increasing on the set \(\Gamma\) of possible values of \(t\). It may also be decreasing or constant. We have discussed the implications of each case. But we have not yet studied, in a general way, the shape of the Principal’s gain as a function of \(r \in (0, 1)\) when \(t\) is fixed. The gain for fixed \(t\) is \((1 - r) \cdot R(\hat{x}(r, t))\). The graph of the non-negative values of \((1 - r) \cdot R(\hat{x}(r, t))\), with \(r\) on the horizontal axis, starts at zero and ends at zero. The graph coincides with the Principal’s gain curve except at \(r = 0\) and \(r = 1\), since the Principal confines attention to the open interval \((0, 1)\). It would be particularly helpful if the gain curve rises and then falls, achieving its maximum height at \(r^*(t)\). More generally, the curve could rise until \(r = r^*(t)\) and could then be flat for an interval before descending. Let us call such a gain curve single-peaked. As long as the gain is positive at some \(r \in (0, 1)\) the curve is single-peaked if it is concave on \((0, 1)\). The following theorem provides conditions under which the gain is indeed concave. The theorem has two parts. The first part does not require differentiability with respect to \(r\), but the second part does. Informally, the second part says that we have concavity if marginal revenue drops \((R'' < 0)\) and in addition the effectiveness of a share increase drops when the share increases \((\hat{x}_{rr} < 0)\).

**Theorem 7**

(a) If, for a fixed \(t\), \(R(\hat{x}(r, t))\) is concave on \((0, 1)\), then the Principal’s gain \((1 - r) \cdot R(\hat{x}(r, t))\) is also concave on \((0, 1)\).

(b) Consider an interior example \((\Sigma, \Gamma, R, C)\) where \(\Sigma = (0, J)\), with \(J > 0\). Then \(R\) is concave on \((0, J)\) if for all \(x \in (0, J)\) we have \(R''(x) < 0\), and for all \((r, t) \in \Gamma\) we have \(\hat{x}_{rr}(r, t) < 0\). If \(R''(x) < 0\), then a sufficient condition for \(\hat{x}_{rr} < 0\) is

\[
r \cdot R''(x) - t \cdot C''(x) \leq 0.
\]

Suppose that the Principal’s gain curve is indeed single-peaked, and suppose that the share the Principal uses is determined by bargaining between the Principal and the Agent. In the
interval between zero and the peak (the interval \((0, r^*(t))\)). The Agent strictly benefits from a rise in \(r\), as we established in Part ((a) of Theorem 1. The Principal prefers a higher \(r\) as well (since the gain curve is rising in the interval). But in the interval between the peak and zero (the interval \((r^*(t), 1)\)), where the gain curve is falling, the Agent prefers a higher \(r\) and the Principal prefers a lower \(r\). (If \(r^*(t)\) is followed by a flat interval, then the Principal is indifferent between shares in the flat interval, but is damaged by share increases beyond the flat interval). So the negotiation set, where bargaining occurs, is the interval \((r^*(t), 1)\). Now consider the outcome of the bargaining from the welfare point of view. Assume that \(R\) and \(C\) are differentiable and that \(\hat{x}(r, t)\) is the solution to the first-order equation 0 = \(r \cdot R(x) - t \cdot C(x)\). Then we know, from Part (h) of Theorem 1, that for a fixed \(t\), welfare increases when the exogenous share increases. So, informally speaking, increasing the Agent’s bargaining strength increases welfare. A formal model of the bargaining process is needed to make “bargaining strength” precise. Such a model might also reveal the welfare implications of the fact that if \(r^*\) is increasing in \(t\), then the negotiation interval \((0, r^*(t))\) shrinks when \(t\) drops.


Recall our central question: does technical improvement strengthen the case for full Agent autonomy or does it weaken it so much that perfect monitoring and policing has now become attractive? One might have reasonably hoped for a straightforward answer since our revenue-sharing Principal/Agent model is so simple. Specifically one might have hoped that a natural condition like rising marginal cost and falling marginal revenue unambiguously implies that the Decentralization Penalty rises (or falls) when technology improves. Instead we have found that there is no easy answer to our central question. On the other hand, we have found a rich array of other results. One of them is that in both the exogenous case and the endogenous case, an advance in technology increases welfare. Another is that an advance in technology causes the Agent to work harder. That is obvious in the exogenous case, since the Agent benefits from the advance even if he continues to use his previous effort. It is not obvious in the endogenous case.

Other interesting results for the challenging endogenous case concern the tracking question. If the effort gap always moves in the same direction as the surplus gap (the Penalty), then to see whether a technical advance has strengthened or weakened the case for autonomy, it suffices to observe (but not police) the Agent’s effort before and after the advance and to compare it with first-best effort. We saw that two key properties of an example are the sign of \(r^*\) and the sign of \(\hat{x}_{rt}\). We must have tracking if \(r^* \geq 0, \hat{x}_{rt} > 0\) or \(r^* < 0, \hat{x}_{rt} > 0\) — if a drop in \(t\) decreases generosity (or leaves it unchanged) and increases the effectiveness of a share increase in eliciting higher effort, or the drop increases generosity and decreases effectiveness. For the other combinations of the two signs, we may have tracking but we may also have “opposite directions”.

Can we obtain an easier answer to our central question if we vary or complicate the model? There are many ways to do so. Here are a few of them.

- **Change the definition of “Decentralization Penalty”**. As we have already noted, one could let the Penalty be the ratio \(W(\hat{x}(r, t)) / W(\hat{x}(1, t))\) (or, in the endogenous case, \(W(\hat{x}(r^*(t), t)) / W(\hat{x}(1, t))\)), rather than the difference, which we have been considering. Our central question becomes
technically harder and preliminary exercises suggest that it again has no simple answer. It appears, again, that there are no simple conditions on $R$ and $C$ implying that the redefined Penalty rises or falls when $t$ drops. In the Classic example, for instance, we again find (in the endogenous case) that when $t$ rises, the redefined Penalty first rises and then falls. Moreover, we can easily find propositions which are reversed when we move from the difference definition of Penalty to the ratio definition.

- **Introduce uncertainty about revenue for a given effort.** Here we rejoin the standard moral-hazard literature briefly reviewed in our Related Literature section. Revenue depends on effort and on a random variable whose distribution is known to both parties.

- **Replace linear sharing by a more complicated reward scheme.** In the nonfinite case, let the Principal offer the agent a reward of $\rho(R)$ if revenue turns out to be $R$. Can we find a non-linear function $\rho$ for which we get a simple answer to our central question? If so, does the Agent find the function $\rho$ acceptable, and does the Principal prefer it to other functions which the Agent accepts? In the finite case, can we find a reward for every possible revenue such that the vector of rewards is accepted by the Agent, is preferred by the Principal, and implies that the Penalty falls (rises) when $t$ drops?

\[ \text{In the exogenous case, with } r \text{ fixed, consider the derivative of Penalty with respect to } t. \text{ For the difference definition we have} \]
\[ \frac{d}{dt} [W(1, t) - W(r, t)] = W_t(1, t) - W_t(r, t), \]
\[ \text{which is negative if} \]
\[ (+) \quad W_t(1, t) < W_t(r, t). \]
\[ \text{For the ratio definition we have} \]
\[ \frac{d}{dt} \left[ \frac{W(r, t)}{W(1, t)} \right] = \left[ \frac{1}{W(1, t)} \right]^2 \cdot [W(1, t) \cdot W_t(r, t) - W(r, t) \cdot W_t(1, t)]. \]
\[ \text{But that is positive if } (+) \text{ holds, since we know that } W(1, t) \geq W(r, t). \text{ If } (+) \text{ fails to hold, then whether the two derivatives have opposite signs remains open. We have to look at the functions } R \text{ and } C. \]

Now consider the endogenous case. We have
\[ \frac{d}{dt} [W(1, t) - W(r^*(t), t)] = W_t(1, t) - [W_r \cdot r^* + W_t]. \]
(Here $W_r, W_t$ are abbreviations for $W_r(r^*(t), t)$ and $W_t(r^*(t), t)$). We know from Part (d) of Theorem 1 that $W_t \leq 0$. Hence the derivative for the difference definition is negative if
\[ (++) \quad W_r \cdot r^* + W_t > 0. \]
On the other hand, for the ratio definition we have
\[ \frac{d}{dt} \left[ \frac{W(r^*(t), t)}{W(1, t)} \right] = \left[ \frac{1}{W(1, t)} \right]^2 \cdot \left( W_r \cdot r^* + W_t \right) \cdot [W(1, t) - W(r^*(t), t) \cdot W_t(1, t)]. \]
Since $W_t \leq 0$, the whole expression is positive if $(++)$ holds.
• **Introduce uncertainty about the technology parameter** *t*. This variation is trivial if Principal and Agent are risk-neutral and if *W* becomes an expected value. Simply replace *t*, by its expected value, say ¯*t*. If we abandon risk neutrality, then it is conceivable that there are specific probability distributions of *t*, and specific utility functions for the Principal and the Agent under which we get a simple answer to our central question.

• **Introduce informational asymmetry.** Let *t* be a random variable observed by only one of the two parties. The other knows the probability distribution of *t*. If it is the Agent who observes *t*, then his best effort ˆ(*x*(*r*, *t*)) is a random variable for the Principal and so is the revenue *R*(ˆ(*x*(*r*, *t*))). Even if the Principal is risk-neutral and we retain our linear sharing scheme, it appears difficult to find simple conditions on *R*, *C* which imply that the Penalty rises (or falls) when technology improves. That is especially true for the endogenous case.

• **Many agents.** In the easiest case there are two Agents, the parameter *t* is known to all three parties, both Agents have the same function *C*, and we retain linear sharing. The realized revenue *R* is a function of *t* and the Agents’ efforts. The Principal chooses two shares whose sum must lie between zero and one. Then for every given *t* we have a three-player game. Each Agent chooses an effort *x* (*t*) and the Principal chooses the two shares, *r* *t* 1, *r* *t* 2. Agent *i*’s payoff is *r* *t* *i* · *R*(*x* *t* 1, *x* *t* 2) − *t* · *C*(*x* *t* *i* ) and the Principal’s payoff is (1 − *r* *t* 1 − *r* *t* 2) · *R*(*x* *t* 1, *x* *t* 2). Suppose that for every *t* the game has a pure-strategy equilibrium where the Principal chooses (˜*r* *t* 1, ˜*r* *t* 2) and Agent *i* chooses the effort ˜*x* *t* *i*, and suppose that for a given *t*, surplus is maximized by the efforts (¯*x* *t* 1, ¯*x* *t* 2). The Decentralization Penalty at the equilibrium is

\[
R(\bar{x}_{t1}, \bar{x}_{t2}) - t \cdot C(\bar{x}_{t1}) - t \cdot C(\bar{x}_{t2}) - R(\tilde{x}_{t1}, \tilde{x}_{t2}) - t \cdot C(\tilde{x}_{t1}) - t \cdot C(\tilde{x}_{t2})
\]

When *t* drops, does the equilibrium Penalty rise or fall?

It was natural to start with our stripped-down model, where we already saw the unexpected challenges posed by our central question. The question of the effect of improved technology on the merits of alternative modes of organizing is well motivated but has seldom been the focus of previous research. The variations and extensions that we have noted, and numerous others, merit further attention.

**APPENDIX**

**Proof of Theorem 1**

In proving Parts (a), (b), (c), (d), we shall use the standard proposition from monotone comparative statics which we summarized in the text and state more completely here.

Consider sets *U* ∈ ℜ, *V* ∈ ℜ and a function *h* : *U* × *V* → ℜ. The two arguments of *h* are denoted *u*, *v*. The function *h* displays *strictly increasing differences in the variables* *u*, *v* if

\[ h(u_H, v_H) - h(u_L, v_H) > h(u_H, v_L) - h(u_L, v_L) \]

whenever *u* *H*, *u* *L*, *v* *H*, *v* *L* ∈ *U*, *v* *H*, *v* *L* ∈ *V*, *u* *H* > *u* *L*, and *v* *H* > *v* *L*.
Suppose that for every $v \in V$, the problem

\[
\begin{array}{l}
\text{maximize } h(u, v) \text{ subject to } u \in U \\
\end{array}
\]

(*) has at least one solution. Suppose also that $h$ satisfies strictly increasing differences in $u, v$. Consider $v_H, v_L \in V$ with $v_H > v_L$. Let $u_H$ be a maximizer of $h(u, v_H)$ on $U$ and let $u_L$ be a maximizer of $h(u, v_L)$ on $U$. Then $u_H \geq u_L$.

Note the following:

(a) If $h$ takes the form $h(u, v) = f(u, v) + g(u)$, then $h$ displays strictly increasing differences in $u, v$ if and only if $f$ displays strictly increasing differences in $u, v$.

(b) If $h$ takes the form $h(u, v) = f(u) \cdot g(v)$ and $f$ is strictly increasing while $g$ is nondecreasing, then $h$ displays strictly increasing differences in $u, v$.

(c) If $h$ takes the form $h(u, v) = u \cdot g(v)$ and $g$ is nondecreasing, then $h$ displays strictly increasing differences in $u, v$.

Proof of Part (a)

By (a), the function $r \cdot R(x) - tC(x)$, where $t$ is fixed, displays strictly increasing differences in $r, x$ if $r \cdot R(x)$ displays strictly increasing differences in $r, x$. But, in view of (γ), that is the case, since $R$ is nondecreasing. Since, for fixed $t$, the effort $\hat{x}(r, t)$ maximizes $r \cdot R(x) - tC(x)$ on the effort set $\Sigma$, Proposition (*) implies $\hat{x}(r_H, t) \geq \hat{x}(r_L, t)$, as (a) asserts. Part (a) also asserts that the Agent strictly prefers the higher share. That is the case since $\hat{x}(r_H, t)$ is a maximizer of $r_H \cdot R(x) - t \cdot C(x)$, so we have

\[
r_H \cdot R(\hat{x}(r_H, t)) - t \cdot C(\hat{x}(r_H, t)) \geq r_H \cdot R(\hat{x}(r_L, t)) - t \cdot C(\hat{x}(r_L, t)) > r_H \cdot R(\hat{x}(r_L, t)) - t \cdot C(\hat{x}(r_L, t)).
\]

Proof of Part (b)

By (a), the function $r \cdot R(x) - tC(x)$, where $r \in (0, 1)$ is fixed, displays strictly increasing differences in $-t, x$ if $-t \cdot C(x)$ displays strictly increasing differences in $-t, x$. By (γ), that is the case, since $C$ is nondecreasing. Since, for fixed $r$, the effort $\hat{x}(r, t)$ maximizes $r \cdot R(x) - tC(x)$ on $\Sigma$, Proposition (*) implies $\hat{x}(r, t_L) \geq \hat{x}(r, t_H)$, as (b) asserts.

Proof of Part (c)

By (a) and (γ), $R(x) - t \cdot C(x)$ displays strictly in $-t, x$. The effort $\hat{x}(1, t)$ is a maximizer of $R(x) - t \cdot x$. Hence, by Proposition (*), $\hat{x}(1, t_L) \geq \hat{x}(1, t_H)$, as Part (c) asserts.

Proof of Part (d)

Recall that $W(1, t)$ is the maximal surplus for a given $t$. Using (c), and the fact that $R$ and $C$ are nondecreasing, we have

\[
W(1, t_L) = R(\hat{x}(1, t_L)) - t_L \cdot C(\hat{x}(1, t_L)) \geq R(\hat{x}(1, t_H)) - t_L \cdot C(\hat{x}(1, t_H)) \geq R(\hat{x}(1, t_H)) - t_H \cdot C(\hat{x}(1, t_H)) = W(1, t_H),
\]

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as (d) asserts.

**Proof of Part (e)**

This follows immediately from (b) and the fact that $R$ is nondecreasing.

**Proof of Part (f)**

Since $\hat{x}(r,t)$ is a maximizer of $R(x) - t \cdot C(x)$, we have

$$rR(\hat{x}(r,t_L)) - t_L \cdot C(\hat{x}(r,t_L)) \geq rR(\hat{x}(r,t_H)) - t_L \cdot C(\hat{x}(r,t_H)) > rR(\hat{x}(r,t_H)) - t_L \cdot C(\hat{x}(r,t_H)).$$

That implies (f).

**Proof of Part (g)**

Part (g) says:

$$W(r,t_L) > W(r,t_H) \text{ whenever } t_L, t_H \in \tilde{\Gamma} \text{ and } 0 < t_L < t_H.$$ 

The effort $\hat{x}(r,t_L)$ is a maximizer of $rR(x) - t_H \cdot C(x)$. Hence

$$r \cdot R(\hat{x}(r,t_L)) - t_L \cdot C(\hat{x}(r,t_L)) \geq r \cdot R(\hat{x}(r,t_H)) - t_L \cdot C(\hat{x}(r,t_H))$$

or

$$r \cdot [R(\hat{x}(r,t_L)) - R(\hat{x}(r,t_H))] \geq t_L \cdot [C(\hat{x}(r,t_L)) - C(\hat{x}(r,t_H))].$$

That implies — since $0 < r < 1$ — that

$$R(\hat{x}(r,t_L)) - R(\hat{x}(r,t_H)) > t_L \cdot [C(\hat{x}(r,t_L)) - C(\hat{x}(r,t_H))],$$

or

$$R(\hat{x}(r,t_L)) - t_L \cdot C(\hat{x}(r,t_L)) > R(\hat{x}(r,t_H)) - t_L \cdot C(\hat{x}(r,t_H))$$

and hence (since $t_H > t_L$)

$$R(\hat{x}(r,t_L)) - t_L \cdot C(\hat{x}(r,t_L)) > R(\hat{x}(t_H),t_H) - t_H \cdot C(\hat{x}(r,t_H))$$

The term on the left of the inequality is $W(r,t_L)$ and the term on the right is $W(r,t_H)$. That completes the proof of Part (g).

**Proof of Part (h):**

When the Agent’s share is $r_H$, he chooses an effort $\hat{x}(r_H,t)$ which satisfies

$$r_H R(\hat{x}(r_H,t)) - tC(\hat{x}(r_H,t)) \geq r_H R(\hat{x}(r_L,t)) - tC(\hat{x}(r_L,t)),$$

or equivalently

$$r_H \cdot [R(\hat{x}(r_H,t)) - R(\hat{x}(r_L,t))] \geq t \cdot [C(\hat{x}(r_H,t)) - C(\hat{x}(r_L,t))].$$

(1)
Part (a) of Theorem 1 tells us that \( \hat{x}(r_H, t) \geq \hat{x}(r_L, t) \). Since \( R \) is strictly increasing, that means that the left side of (1) is either positive or zero. First suppose that it is positive. Then, since \( r_H < 1 \), (1) implies that
\[
R(\hat{x}(r_H, t)) - R(\hat{x}(r_L, t)) > t \cdot [C(\hat{x}(r_H, t)) - C(\hat{x}(r_L, t))],
\]
or equivalently,
\[
R(\hat{x}(r_H, t)) - t \cdot C(\hat{x}(r_H, t)) > R(\hat{x}(r_L, t)) - t \cdot C(\hat{x}(r_L, t)),
\]
i.e.,
\[
W(r_H, t) > W(r_L, t).
\]
If \( \hat{x}(r_H, t) \neq \hat{x}(r_L, t) \), then, since \( R \) is strictly increasing, the left side of (1) is indeed positive, so (4) holds. If, on the other hand, \( \hat{x}(r_H, t) = \hat{x}(r_L, t) \), then both sides of (1) equal zero and (2),(3),(4) become equalities. So, as claimed, \( W(r_H, t) \geq W(r_L, t) \) and the inequality is strict if and only if \( \hat{x}(r_H, t) \neq \hat{x}(r_L, t) \).

That concludes the proof of Theorem 1.

\[\square\]

**Proof of Theorem 2**

**Proof of Part (a)**

This part concerns the ratio \( \frac{r}{t} \), which we shall denote \( \rho(t) \). For fixed \( t \), the set of possible ratios is \( (0, \frac{1}{t}] \). Given a ratio \( \rho \) and the share \( r = t \cdot \rho \), the Agent’s chosen effort \( \hat{x}(r, t) \) is the smallest maximizer of \( t \cdot \rho \cdot R(x) - tC(x) \) on the effort set \( \Sigma \). Let \( \phi(\rho) \) be a new symbol, denoting the effort the Agent chooses when the ratio is \( \rho \). Thus \( \phi(\rho(t)) = \hat{x}(t\rho(t), t) \).

We now claim that
\[
\phi(\rho_H) \geq \phi(\rho_L) \quad \text{whenever} \quad 0 < \rho_L < \rho_H.
\]
To see this, suppose that for a fixed \( r \) we have
\[
\rho_H = \frac{r}{t^*} > \frac{r}{t^{**}} = \rho_L.
\]
Then \( t^* < t^{**} \). So, by Part (b) of Theorem 1, the Agent works at least as hard at \( \rho_H \) as at \( \rho_L \), i.e., \( \phi(\rho_H) \geq \phi(\rho_L) \).

Using the symbols \( \rho, \phi \), we can now restate the Principal’s choice for a given \( t \). He chooses the ratio \( \rho^*(t) = \frac{r^*(t)}{t} \), where
\[
\rho^*(t) = \min \left\{ \arg\max_{\rho \in (0, \frac{1}{t})} M(\rho, -t) \right\},
\]
and
\[
M(\rho, -t) = (1 - tp) \cdot R(\phi(\rho)) = R(\phi(\rho)) - t \cdot \rho \cdot R(\phi(\rho)).
\]
In view of the statement (α) — which we showed in the proof of Theorem 1 to be implied by the monotone-comparative-statics Proposition (*) — the function $M$ has strictly increasing differences in $\rho, -t$ if the function $-t \cdot \phi \cdot R(\phi(\rho))$ has strictly increasing differences in $\rho, -t$. But that is the case (using statement (γ) in the proof of Theorem 1) since $R$ is nondecreasing, which implies (using (+)) that $R(\phi(\cdot))$ is also nondecreasing. Since $\rho^*(t)$ is a maximizer of $M(\rho, -t)$, the monotone-comparative-statics Proposition (*) then implies that

$$\frac{r^*(t_L)}{t_L} = \rho^*(t_L) \geq \rho^*(t_H) = \frac{r^*(t_H)}{t_H} \text{ whenever } 0 < t_L < t_H,$$

as Part (a) asserts.

**Proof of Part (b)**

We use the terminology just used in the proof of Part (a). Since $\phi\left(\frac{r^*(t)}{t}\right) = \hat{x}(r^*(t), t)$, we have, using (+), $\hat{x}(r^*(t_L), t_L) \geq \hat{x}(r^*(t_H), t_H)$, as (b) asserts.

**Proof of Part (c)**

Part (c) says:

$$(1 - r^*(t_L)) \cdot R(\hat{x}(r^*(t_L), t_L)) > (1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_H)) \text{ whenever } t_L, t_H \in \tilde{\Gamma} \text{ and } 0 < t_L < t_H.$$ 

When $t$ drops from $t_H$ to $t_L$, the Principal could continue to use the share $r^*(t_H)$. It suffices to show that if he does so, the Principal’s gain cannot be less than it was at $t = t_H$. Then, a fortiori, it cannot be less when he uses $r^*(t_L)$, which is his best share when $t = t_L$. Part ((b)) tells us that $\hat{x}(r^*(t_H), t_L) \geq \hat{x}(r^*(t_H), t_H)$. Since $R$ is nondecreasing, we have

$$(1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_L)) \geq (1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_H)),$$

as (c) asserts.

**Proof of Part (d)**

Part (d) says:

$$W(r^*(t_L), t_L) > W(r^*(t_H), t_H) \text{ whenever } t_L, t_H \in \tilde{\Gamma} \text{ and } 0 < t_L < t_H.$$ 

The effort $\hat{x}(r^*(t_L), t_L)$ is a maximizer of $r^*(t_L) \cdot R(x) - t_L \cdot C(x)$. Hence

$$r^*(t_L) \cdot R(\hat{x}(r^*(t_L), t_L)) - t_L \cdot C(\hat{x}(r^*(t_L), t_L)) \geq r^*(t_L) \cdot R(\hat{x}(r^*(t_H), t_L)) - t_L \cdot C(\hat{x}(r^*(t_H), t_L))$$

or

$$r^*(t_L) \cdot [R(\hat{x}(r^*(t_L), t_L)) - R(\hat{x}(r^*(t_H), t_L))] \geq t_L \cdot [C(\hat{x}(r^*(t_L), t_L)) - C(\hat{x}(r^*(t_H), t_L))].$$

That implies — since $0 < r^*(t_L) < 1$ — that

$$R(\hat{x}(r^*(t_L), t_L)) - R(\hat{x}(r^*(t_H), t_H)) > t_L \cdot [C(\hat{x}(r^*(t_L), t_L)) - C(\hat{x}(r^*(t_H), t_H))].$$
Proof of Theorem 3

That establishes \((d)\). We will now show that
\[
\hat{x}(r^*(t_L), t_L) < \hat{x}(r^*(t_H), t_H) \quad \text{and hence (since } t_H > t_L \text{)}
\]
\[
R(\hat{x}(r^*(t_L), t_L)) - t_L \cdot C(\hat{x}(r^*(t_L), t_L)) > R(\hat{x}(r^*(t_H), t_H)) - t_H \cdot C(\hat{x}(r^*(t_H), t_H))
\]
The term on the left of the inequality is \(W(r^*(t_L), t_L)\) and the term on the right is \(W(r^*(t_H), t_H)\).
That establishes \((d)\).

That concludes the proof of Theorem 2.

\(\square\)

Proof of Theorem 3

Recall that if \((r, t) \in \Gamma\), then \(0 < r < 1\). For \(\epsilon > 0\), \(1 - \epsilon > r\), and \(0 < r < 1\), we have
\[
\frac{d}{dt} [\hat{x}(1 - \epsilon, t) - \hat{x}(r, t)] = \hat{x}_t(1 - \epsilon, t) - \hat{x}_t(r, t).
\]
Now suppose that \(\hat{x}_t > 0\) at all \((r, t) \in \Gamma\). Then the difference on the right of the equality in \((+)\) is positive. But the same is true if \(\epsilon = 0\), since, by assumption, the function \(\hat{x}_t\) is continuous with respect to \(t\) at all \(r \in (0, 1]\). Hence \(\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] > 0\) at all \((r, t) \in \Gamma\).

An analogous argument shows that if \(\hat{x}_t < 0\) at all \((r, t) \in \Gamma\), then \(\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] < 0\) at all \((r, t) \in \Gamma\).

\(\square\)

Proof of Theorem 4

An argument analogous to the proof of Theorem 3 tells us that
\[
\begin{cases}
\text{if } W_{rt}(r, t) > 0 \text{ at all } (r, t) \in \Gamma, \text{ then } \frac{d}{dt} [W(1, t) - W(r, t)] > 0 \text{ at all } (r, t) \in \Gamma; \\
\text{if } W_{rt}(r, t) < 0 \text{ at all } (r, t) \in \Gamma, \text{ then } \frac{d}{dt} [W(1, t) - W(r, t)] < 0 \text{ at all } (r, t) \in \Gamma.
\end{cases}
\]
We will now show that
\[
\hat{x}_{rt}(r, t) \cdot W_{rt}(r, t) > 0 \text{ at all } (r, t) \in \Gamma.
\]
Recall our assumption that we either have \(\hat{x}_{rt} > 0\) at all \((r, t) \in \Gamma\) or \(\hat{x}_{rt} < 0\) at all \((r, t) \in \Gamma\).
Using that assumption as well as Theorem 3 and (1) and (2), we see that:

- If \(\hat{x}_{rt}(r, t) > 0\) at all \((r, t) \in \Gamma\), then at all \((r, t) \in \Gamma\) we have \(\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] > 0\) and \(\frac{d}{dt} [W(1, t) - W(r, t)] > 0\).
- If \(\hat{x}_{rt}(r, t) < 0\) at all \((r, t) \in \Gamma\), then at all \((r, t) \in \Gamma\) we have \(\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] < 0\) and \(\frac{d}{dt} [W(1, t) - W(r, t)] < 0\).
So we indeed have exogenous tracking.

To establish (2), we shall use the fact that the cross partials $W_{rt}, W_{tr}$ are equal. Part (a) of Theorem 1 tells us that $\hat{x}_r(r, t) \geq 0$. Since $rR'(\hat{x}(r, t)) - tC'(\hat{x}(r, t)) = 0$ and $0 < r < 1$, we have

$$W_r(r, t) = \hat{x}_r(r, t) \cdot [R'(\hat{x}(r, t)) - tC'(\hat{x}(r, t))] \geq 0.$$  

Since, by assumption, $tC'(\hat{x}(r, t)) = rR'(\hat{x}(r, t))$, the equality in (3) can be rewritten

$$W_r(r, t) = (1 - r) \cdot R'(\hat{x}(r, t)) \cdot \hat{x}_r(r, t).$$  

Now differentiate both sides of (3) with respect to $t$. We have:

$$W_{rt}(r, t) = (1 - r) \cdot [R'(\hat{x}(r, t)) \cdot \hat{x}_{rt}(r, t) + \hat{x}_r(r, t) \cdot R''(\hat{x}(r, t)) \cdot \hat{x}_t(r, t)].$$

Next we differentiate $W$ in the reverse order: first with respect to $t$ and then with respect to $r$. We obtain the following, using condensed notation when convenient:

$$W_t(r, t) = R' \cdot \hat{x}_t - [t \cdot C' \cdot \hat{x}_t + C(\hat{x}(r, t))] = \hat{x}_t \cdot [R' - tC'] - C(\hat{x}(r, t)).$$

Differentiating the final expression with respect to $r$, we obtain:

$$W_{tr}(r, t) = \hat{x}_t \cdot [R'' \cdot \hat{x}_t - t \cdot C'' \cdot \hat{x}_r] + [R' - t \cdot C'] \cdot \hat{x}_{tr} - C' \cdot \hat{x}_r.$$  

We now rewrite (5) and (7). We identify separate terms so that cancellations in the equality $W_{rt} = W_{tr}$ can be easily detected. For (5) we obtain

$$W_{rt} = \underbrace{R' \cdot \hat{x}_t + \hat{x}_r \cdot R'' \cdot \hat{x}_t}_{1} - r \cdot R' \cdot \hat{x}_{rt} - r \cdot \hat{x}_r \cdot R'' \cdot \hat{x}_r.  
\underbrace{1}_{2}$$

For (7) we obtain

$$W_{tr} = \underbrace{\hat{x}_t \cdot R'' \cdot \hat{x}_t}_{2} - t \cdot C'' \cdot \hat{x}_r \cdot \hat{x}_t + \underbrace{R' \cdot \hat{x}_{tr} - t \cdot C' \cdot \hat{x}_t}_{1} - C' \cdot \hat{x}_r.  
\underbrace{1}_{2}$$

Deleting the terms $\boxed{1}$ and $\boxed{2}$ and multiplying both sides by $-1$, we can now write the equality $W_{rt} = W_{tr}$ as

$$r \cdot R'' \cdot \hat{x}_{rt} + r \cdot \hat{x}_r \cdot R'' \cdot \hat{x}_t = t \cdot C'' \cdot \hat{x}_r \cdot \hat{x}_t + t \cdot C' \cdot \hat{x}_{tr} + C' \cdot \hat{x}_r$$

---

1Thus $R, R', R'', C, C', C'', \hat{x}_t, \hat{x}_r, \hat{x}_{rt}, W_r, W_t, W_{rt}$ denote, respectively, $R(\hat{x}(r, t)), R'(\hat{x}(r, t))$, $\ldots, C(\hat{x}(r, t)))$, $\ldots, \hat{x}_t(r, t), \hat{x}_r(r, t), \hat{x}_{rt}(r, t), W_r(r, t), W_t(r, t), W_{rt}(r, t)$.  

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or
\[
\dot{x}_{rt} \cdot [rR' - tC'] + \dot{x}_r \dot{x}_t - [R'' - t \cdot C'] = C' \cdot \dot{x}_r.
\]

So

\begin{align}
(8) \quad C' \cdot \dot{x}_r &= \dot{x}_t \cdot \dot{x}_r \cdot (R'' - tC').
\end{align}

Now, using (8), we can rewrite (7) as
\[
W_{tr} = \dot{x}_t \cdot \dot{x}_r \cdot (R'' - tC') + \dot{x}_t \cdot \dot{x}_r \cdot \dot{x}_t = [R' - t \cdot C'] \cdot \dot{x}_{tr}.
\]

But \(R' - t \cdot C' > 0\). So \(W_{rt}\) has the same sign as \(\dot{x}_{rt}\). That establishes (2) and completes the proof.

Proof of Corollary to Theorem 4

Since we have exogenous tracking, the Decentralization Penalty (surplus gap) is decreasing in \(t\) if \(\dot{x}_{rt}(r, t) < 0\) at all \((r, t) \in \Gamma\), and is increasing in \(t\) if \(\dot{x}_{rt}(r, t) > 0\) at all \((r, t) \in \Gamma\). So Part (i) of the Corollary is proved if we establish that \(\dot{x}_{rt} < 0\), and Part (ii) is proved if we establish that \(\dot{x}_{rt} > 0\).

Proof of Part (i)

The first-order condition satisfied by \(\dot{x}(r, t)\) is, in abbreviated form (with arguments deleted):
\[
rR' - tC' = 0.
\]

Differentiating with respect to \(r\) on both sides:
\begin{align}
(+) \quad R' + rR'' \cdot \dot{x}_r - tC'' \cdot \dot{x}_r &= 0.
\end{align}

Differentiating with respect to \(t\) on both sides of (+):
\begin{align}
(++) \quad R'' \cdot \dot{x}_t + [rR'' - tC''] \cdot \dot{x}_{rt} + [rR''' \cdot \dot{x}_t - tC''' \cdot \dot{x}_t - C''] \cdot \dot{x}_r &= 0.
\end{align}

Since, by assumption, \(R''' = C''' = 0\), we obtain:
\begin{align}
(++++) \quad \frac{R' \cdot \dot{x}_t}{\dot{x}_r} \cdot \dot{x}_{rt} = R'' \cdot \dot{x}_t - C'' \cdot \dot{x}_r.
\end{align}

By assumption, \(\dot{x}_r > 0\) and \(\dot{x}_t < 0\). By assumption, \(R' > 0, C'' > 0\) and \(R'' > 0\). So the right side of (++++) is negative. That implies \(x_{rt} < 0\), which establishes part (i).

Proof of Part (ii)

In this part we assume \(R' > 0, R'' < 0, 0 = C'' = C'''\), \(R''' \leq 0\). We first establish that under these assumptions:
\begin{align}
(*) \quad \frac{R' \cdot \dot{x}_t}{\dot{x}_r} \cdot \dot{x}_{rt} = R'' \cdot \dot{x}_t + rR''' \cdot \dot{x}_r \cdot \dot{x}_t.
\end{align}
Statement (+) in the Part-(i) proof can now be written

\[ (** ) \hat{x}_r = \frac{-R'}{rR''}. \]

So the left side of (*) can be written

\[ R' \cdot \frac{-rR''}{R'} \cdot \hat{x}_{rt} = -rR'' \hat{x}_{rt}. \]

Statement (++) in the Part-(i) proof can now be written

\[ -rR'' \hat{x}_{rt} = R'' \hat{x}_t + rR''' \hat{x}_r \hat{x}_t. \]

So the left side of (*) indeed equals the right side. Under our Part-(ii) assumptions, we have (since \( \hat{x}_r > 0, \hat{x}_t < 0 \))

\[ R'' \hat{x}_t > 0, R''' \hat{x}_r \hat{x}_t \geq 0. \]

So the right side of (*) is positive. Since \( R' > 0, \hat{x}_r > 0, \) the equality (*) implies that \( \hat{x}_{rt} > 0 \), as required.

That concludes the proof of the Corollary. \( \square \)

**Proof of Theorem 5**

**Part (a)**

We have

\[ (1) \quad \frac{\partial}{\partial t} \left[ W(1, t) - W(r^*(t), t) \right] = W_t(1, t) - W_r(r^*(t), t) \cdot r^*(t) - W_t(r^*(t), t). \]

Our exogenous Theorem 4 tells us that for all \((r, t) \in \Gamma\) — including the pair \((r^*(t), t)\) — we have \( W_{rt}(r, t) \cdot \hat{x}_{rt}(r, t) > 0 \). That implies (since we assume that \( \hat{x}_{rt}(r, t) < 0 \) at all \((r, t) \in \Gamma\)) that \( W_{rt}(r^*(t), t) < 0 \). Now consider (1) the interval \([r^*(t), 1]\), (2) the function \( W_t \) on the possible shares, and (3) the derivative of that function, namely \( W_{rt} \). In an interior example the function \( \hat{x}_t \) is continuous with respect to \( t \) at all \( r \in [0, 1] \) and hence the same is true for the function \( \hat{x}_t = \frac{d}{dt} [R(\hat{x}(r, t) - tC(\hat{x}(r, t))] \). By the Mean Value Theorem, there exists \( r_0 \in (r^*(t), 1) \) such that

\[ W_t(1, t) - W_t(r^*(t), t) = W_{rt}(r_0, t) \cdot (1 - r^*(t)) < 0 \]

and hence

\[ (2) \quad W_t(1, t) - W_t(r^*(t), t) < 0. \]

Since we know, from Part (h) of Theorem 1, that \( W_r(r, t) \geq 0 \) at all \((r, t)\), including \((r^*(t), t)\), and since we assume \( r^*(t) \geq 0 \), we conclude, using (1) and (2), that

\[ (3) \quad \frac{\partial}{\partial t} \left[ W(1, t) - W(r^*(t), t) \right] < 0. \]

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We now turn to \( \frac{\partial}{\partial t} \left[ \hat{x}(1, t) - \hat{x}(r^*(t), t) \right] \). We have

\[
(4) \quad \frac{\partial}{\partial t} \left[ \hat{x}(1, t) - \hat{x}(r^*(t), t) \right] = \hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) - \hat{x}_r(r^*(t), t) \cdot r^\prime(t).
\]

Our assumption that \( \hat{x}_r(r^*(t), t) < 0 \) implies, using Theorem 3, that \( \hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) < 0 \).

Since we assume that \( r^\prime(t) > 0 \), and since we know from Part (a) of Theorem 1, that \( \hat{x}_r(r, t) \geq 0 \) at all \((r, t)\), we conclude that \( \frac{\partial}{\partial t} \left[ \hat{x}(1, t) - \hat{x}(r^*(t), t) \right] < 0 \). So we indeed have

\[
\frac{\partial}{\partial t} \left[ \hat{x}(1, t) - \hat{x}(r^*(t), t) \right] \cdot \frac{\partial}{\partial t} \left[ W(1, t) - W(r^*(t), t) \right] > 0.
\]

**Part (b)**

First consider again the three terms at the right of the equality (1). Since we now assume that \( \hat{x}_r(r, t) > 0 \) at all \((r, t) \in \Gamma\), Theorem 4 tells us that we now have \( W_{rt}(r, t) > 0 \). Hence — using the Mean-Value-Theorem argument again — we now have

\[
W_t(1, t) - W_t(r^*(t), t) < 0.
\]

Since we now assume \( r^\prime(t) < 0 \), we conclude (using (1)) that

\[
\frac{\partial}{\partial t} \left[ W(1, t) - W(r^*(t), t) \right] > 0.
\]

Now consider the three terms on the right of the equality in (4). Our assumption that \( \hat{x}_r(r^*(t), t) > 0 \) now implies, using Theorem 3, that \( \hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) > 0 \). Since we now assume that \( r^\prime(t) < 0 \), we now conclude, using (4), that \( \hat{x}_r(r, t) \geq 0 \) and hence \( \hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) < 0 \). So we again have

\[
\frac{\partial}{\partial t} \left[ \hat{x}(1, t) - \hat{x}(r^*(t), t) \right] \cdot \frac{\partial}{\partial t} \left[ W(1, t) - W(r^*(t), t) \right] > 0.
\]

\[\square\]

**Proof of Theorem 6**

The theorem concerns the Principal’s gain (residual revenue), which we now denote \( H(r, t) \). Thus

\[
H(r, t) \equiv (1 - r) \cdot R(\hat{x}(r, t)).
\]

We use the following fact about implicit functions: If we have \( f(\bar{u}, \bar{v}) = 0 \), where \( f \) is twice differentiable, then there exists a neighborhood of \((\bar{u}, \bar{v})\) and a twice differentiable function \( g \) such that for all \((u, v)\) in the neighborhood we have \( f(g(u), v) = 0 \) and, if \( f_v(g(\bar{u}), \bar{v}) \neq 0 \), we have

\[
g'(v) = -\frac{f_u(g(v), v)}{f_v(g(v), v)}.
\]

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To apply this, let \( r \) play the role of \( u \), let \( t \) play the role of \( v \), let \( H_r \) play the role of \( f \) and consider the first-order equation satisfied by \( r^*(t) \):

\[
H_r(r, t) = 0.
\]

Under our assumptions, \( H_r \) is differentiable with respect to \( r \) and \( t \) at every \((r, t) \in \Gamma\) and \( r^*(t) \) belongs to the open interval \((0, 1)\) and is the unique solution to \( H_r(r, t) = 0 \). The role of the twice differentiable function \( g \) is now played by \( r^* \).

Since \( r^*(t) \) is the unique interior maximizer of \( H(r, t) \) it satisfies the second-order condition

\[
H_{rr}(r, t) \leq 0.
\]

For every \( t \in \tilde{\Gamma} \), we have

\[
r^*(t) = \frac{-H_{rr}(r^*(t), t)}{H_{rt}(r^*(t), t)} \quad \text{if} \quad H_{rt}(r^*(t), t) \neq 0.
\]

In view of the second-order condition \((+)\), the numerator in this fraction is nonnegative. Hence, if \( H_{rt}(r^*(t), t) \geq 0 \), then

\[
(++) \quad r^*(t) > 0 \quad \text{if} \quad H_{rr}(r, t) < 0; \quad r^*(t) = 0 \quad \text{if} \quad H_{rr}(r, t) = 0.
\]

We now claim that \( H_{rt}(r^*(t), t) > 0 \). To see this, note that

\[
H_{rt} = (1 - r) \cdot [R' \cdot \hat{x}_t + \hat{x}_r \cdot R'' \cdot \hat{x}_t] - R' \cdot \hat{x}_t.
\]

By assumption we have \( \hat{x}_t < 0, \hat{x}_r > 0, R' > 0, R'' < 0 \), and \( \hat{x}_{rt} \geq 0 \). Hence \(-R' \cdot \hat{x}_t > 0\) and

\[
(1 - r) \cdot [R' \cdot \hat{x}_t + \hat{x}_r \cdot R'' \cdot \hat{x}_t] \geq 0.
\]

We conclude that \( H_{rt} > 0 \) at every \((r, t) \in \Gamma\). That implies, in view of \((+)\), that \( r^*(t) \geq 0 \) at every \( t \in \tilde{\Gamma} \), as the Theorem asserts. \( \square \)

**Proof of Theorem 7.**

**Part (a)**

We first establish the following general proposition:

Let \( h(x) = (1 - x) \cdot g(x) \). If \( g \) is increasing and concave on \((0, 1)\), then \( h \) is concave on \((0, 1)\)

Here is the proof:
Consider \( x_1 > 0 \) and \( x_2 < 1 \). Consider \( \lambda \in (0, 1) \) and let \( x_0 \) denote \( \lambda x_1 + (1 - \lambda) x_2 \). We shall prove that

\[
h(x_0) = (1 - x_0) \cdot g(x_0) \geq \lambda \cdot (1 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (1 - x_2) \cdot g(x_2).
\]

We have

\[
0 = \lambda \cdot [(x_0 - x_1) + (1 - \lambda) \cdot (x_0 - x_2)] \cdot g(x_2) \\
\geq \lambda \cdot (x_0 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (x_0 - x_2) \cdot g(x_2) \\
= \lambda \cdot (x_0 - 1 + 1 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (x_0 - 1 + 1 - x_2) \cdot g(x_2).
\]

Moving parts of the last expression to the left of the inequality we obtain:

\[
\lambda \cdot (1 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (1 - x_2) \cdot g(x_2) \leq \lambda \cdot (1 - x_0) \cdot g(x_1) + (1 - \lambda) \cdot (1 - x_0) \cdot g(x_2) \\
= (1 - x_0) \cdot [\lambda \cdot g(x_1) + (1 - \lambda) \cdot g(x_2)] \\
\leq (1 - x_0) \cdot g(x_0) \\
= h(x_0).
\]

The final inequality holds because \( g \) is concave.

Now apply this general proposition to our case, where \( r \) plays the role of “\( x \)” and \( R(\hat{x}(r, t)) \) plays the role of “\( g(x) \)”. That establishes Part (a).

Part (b)

\( R(\hat{x}(r, t)) \) is concave in \( r \) if its second derivative is negative, i.e.,

\[
R' \cdot \hat{x}_{rr} + \hat{x}_r \cdot R'' < 0.
\]

That is the case, as claimed, if \( R'' < 0 \) and \( \hat{x}_{rr} < 0 \). To check the claimed sufficient condition for \( \hat{x}_{rr} < 0 \), start by writing the first-order condition satisfied by \( \hat{x}(r, t) \):

\[
r \cdot R'(\hat{x}(r, t)) = t \cdot C'(\hat{x}(r, t)) = 0.
\]

Differentiating with respect to \( r \) we obtain

\[
\hat{x}_r = \frac{R'}{tC'' - rR''}.
\]

Since \( \hat{x}(r, t) \) is an interior maximizer, the denominator is negative. Differentiating both sides with respect to \( r \) we obtain:

\[
\hat{x}_{rr} = \left(\frac{1}{tC'' - rR''}\right)^2 \cdot [R'' \cdot (tC'' - rR'') + R' \cdot (rR''' - tC''')] + \frac{R' \cdot R''}{(tC'' - rR'')^2}.
\]
Since $tC'' - rR'' < 0$ and, by assumption, $R'' < 0$, we see, as claimed, that $\hat{x}_{rr} < 0$ if $rR''' - tC''' \leq 0$.

□

Proof that the “exploding marginals” example lies in Box 3 of the Interior Examples table.

We shall show this by examining the construction of the example. Recall that Box 3 requires that for every $(r, t) \in \Gamma$ we have $r^* (t) < 0$ and $\hat{x}_{rt}(r, t) \geq 0$.

As long as $R' \neq 0$, the first-order condition $r \cdot R' (\hat{x}(r, t)) = t \cdot C(\hat{x}(r, t))$ can be written

$$\frac{C'(\hat{x}(r, t))}{R'(\hat{x}(r, t))} = \frac{r}{t}.$$ 

That defines an implicit function $g$ which satisfies

$$\hat{x}(r, t) = g \left( \frac{r}{t} \right).$$

It will be convenient to let $S$ denote the ratio $\frac{r}{t}$. So

$$\hat{x}(r, t) = g(S) \text{ and } \frac{C'(g(S))}{R'(g(S))} = S.$$ 

We have:

$$\hat{x}_r = g' \cdot \frac{1}{t}; \quad \hat{x}_t = g' \cdot -\frac{r}{t^2}.$$ 

$$\hat{x}_{rt} = \hat{x}_{tr} = g'' \cdot -\frac{r}{t^3} - \frac{1}{t^2} \cdot g'.$$

(a) $$\hat{x}_{rt} > 0 \iff g'' \cdot \frac{r}{t} + g' < 0.$$ 

Since $r^*$ satisfies the first-order condition $0 = \frac{d}{dr} [(1 - r) \cdot R(\hat{x}(r, t))]$, we have

(b) $$1 - r^* = \frac{R}{R' \cdot \hat{x}_r} = \frac{R}{R'} \cdot \frac{t}{g'}.$$ 

Using (b), we obtain:

$$r^* (t) = -\Delta,$$ 

where

$$\Delta = \frac{d}{dt} \left[ \frac{R}{R'} \cdot \frac{t}{g'} \right] = \left[ \frac{d}{dt} \left( \frac{R}{R'} \right) \right] \cdot \frac{t}{g'} + \left[ \frac{d}{dt} \left( \frac{t}{g'} \right) \right] \cdot \frac{R}{R'}$$

$$= \frac{1}{(R')^2} \cdot \frac{t}{g'} \cdot ((R')^2 - R'' \cdot R) \cdot \hat{x}_t + \frac{R}{R'} \cdot \frac{g' - g'' \cdot t \cdot \frac{r}{t^2}}{(g')^2}$$
Since \( \hat{x}_t \cdot \frac{L}{g} = -S \) and \( \frac{\gamma}{t} = S \), we have:

\[
\Delta = -S \cdot \frac{(R')^2 - R'' \cdot R}{(R')^2} + R \cdot \frac{g' - g'' \cdot (-S)}{(g')^2}.
\]

To construct our example, we now let \( g(S) = \ln \ln S \). Then

\[
g' = \frac{1}{S \cdot \ln S}; \quad g'' = -\frac{(\ln S + 1)}{S^2 \cdot (\ln S)^2}.
\]

[We can then verify that the condition in (a) is satisfied, and hence \( \hat{x}_{rt} > 0 \), as Box 3 requires]. Hence

\[
g' + g'' \cdot S = \frac{1}{S \cdot \ln S} + S \cdot \left[ -\frac{(\ln S + 1)}{S^2 \cdot (\ln S)^2} \right] = \frac{\ln S - (\ln S + 1)}{S \cdot (\ln S)^2} = \frac{-1}{S \cdot (\ln S)^2}.
\]

Using (c),(d), and the fact that \( (g')^2 = \frac{1}{S^2 \cdot (\ln S)^2} \), we obtain

\[
\Delta = -S \cdot \frac{(R')^2 - R'' \cdot R}{(R')^2} - S \cdot \frac{R}{R'} = \frac{-S}{(R')^2} \cdot [(R')^2 - R'' \cdot R + RR'].
\]

Thus (recalling that \( r^{*'} = -\Delta \)) we have

\[
r^{*'} < 0 \iff (R')^2 - R'' \cdot R' + R \cdot R' < 0.
\]

To continue our construction, we now suppose that

\[
R = e^{kx^2}, \text{ where } k > 0.
\]

We now claim that

\[
(R')^2 - R'' \cdot R' + R \cdot R' = e^{2kx^2} \cdot (2kx - 2k).
\]

To show this we first note that

\[
R' = e^{kx^2} \cdot 2kx
\]

and

\[
R'' = e^{kx^2} \cdot 2k + 2kx \cdot e^{kx^2} \cdot 2kx.
\]

We then factor out the term \( e^{2kx^2} \) in writing the following expressions.

\[
R \cdot R' = e^{kx^2} \cdot e^{kx^2} \cdot 2kx = e^{2kx^2} \cdot 2kx.
\]

\[
(R')^2 = e^{2kx^2} \cdot 4k^2 x^2.
\]
\[ R''' \cdot R = 2k \cdot e^{kx^2} \cdot [1 + 2kx^2] \cdot e^{kx^2} = e^{2kx^2} \cdot [2k + 4k^2x^2]. \]

So
\[ (R')^2 - R'' \cdot R' + R \cdot R' = e^{2kx^2} \cdot [4k^2x^2 - 2k - 4k^2x^2 + 2kx] = e^{2kx^2} \cdot (2kx - 2k) \]

and (f) is verified.

So, in view of (e), we have
\[ r^* < 0 \] as Box 3 requires, if \( e^{2kx^2} \cdot (2kx - 2k) < 0. \)

But if \( e^{2kx^2} \cdot (2kx - 2k) < 0, \) then \( x < 1. \) So our set of available efforts will be \( \Sigma = [0, 1). \)

Summarizing, we have
\[
\begin{align*}
&\bullet R = e^{kx^2}. \\
&\bullet \hat{x}(r, t) = g(s) = \ln \ln s < 1 \text{ and hence } S < e^e.
\end{align*}
\]

We need to specify our set \( \Gamma \) of possible pairs \( (r, t). \) It will be the set
\[
\{(r, t) : 0 < r \leq 1; \frac{r}{t} \in (e, e^e)\}.
\]

It remains to specify the function \( C. \) We seek a function \( C \) with the following property:

for every \( S \) we have \( C'(g(S)) = S \cdot R'(g(S)). \)

That can be rewritten as:

for every \( S \) we have \( C'(g(S)) = g\left(g^{-1}(S)\right) \cdot R'(g(S)). \)

Now let \( M \) denote \( G(s). \) Since \( C(M) = \int C'(M) \, dM, \) we have:
\[
(gs) \quad C(M) = \int \left[g\left(g^{-1}(M)\right) \cdot R'(M)\right] dM.
\]

In our example
\[
\begin{align*}
&\bullet g(S) = \ln \ln S. \\
&\bullet \text{Hence, for any } M, \text{ we have } g^{-1}(M) = e^{e^M}. \\
&\bullet R(x) = e^{kx^2} \text{ and } R'(x) = e^{kx^2} \cdot 2kx^2.
\end{align*}
\]
Thus, in our example, the equality (g) becomes:

\[ C(M) = \int \left[ e^{e^M} \cdot e^{kM} \cdot 2kM \right] dM. \]

So for every \( x \) in our effort set \( \sigma = (0, 1] \) we have

\[ C(x) = \int_0^x \left[ e^{e^p} \cdot e^{kp} \cdot 2kp \right] dp. \]

To summarize, our Box 3 example is as follows.

- \( \Sigma = (0, \]. \)
- \( \Gamma = \{(r, t) : 0 < r \leq 1; \vec{r} \in (e, e^e)\}. \)
- \( R(x) = e^{kx^2}, \) where \( k > 0. \)
- \( C(x) = \int_0^x \left[ e^{e^p} \cdot e^{kp} \cdot 2kp \right] dp. \)

In the example provided in the text we have \( k = 1. \)

REFERENCES


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