Executive Exercise Explained: Patterns for Stock Options †

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Abstract

It is well documented that executives granted stock options tend to exercise early and in a few large transactions or "blocks". Standard risk-neutral valuation models cannot explain these patterns, and attempts to capture the exercise behavior of risk averse executives have been limited to the special case of one option. This paper solves for the optimal exercise behavior for a risk averse executive who is granted multiple stock options. We show that traditional utility-based models do not predict block exercise behavior. Rather, the risk averse executive exercises stock options individually at a sequence of increasing price thresholds. We give these thresholds in closed form.

When, in addition, the executive exerts costly effort to exercise options, he faces a trade-off between exercising little and often to maximize return, and exercising larger quantities on fewer occasions to minimize effort. We find that as costs increase, options are exercised on fewer dates. The costly exercise utility-based model generates block exercise behavior and yields new predictions. In particular, executives should begin by exercising large blocks of options, but the block sizes should become smaller over time.

Keywords: Stock options, compensation, risk aversion, incomplete markets, exercise, utility maximization

JEL Classification Numbers: C61, G11, G13, G30, J33

In this paper we model the exercise behavior of executives who are granted executive stock options (ESO's) as part of their compensation. It is well documented that executives receiving stock options exercise them well before their expiry date. For instance, in a study by Huddart and Lang (1996), the mean fraction of option life elapsed at the time of exercise varied from 0.26 to 0.79 over companies. More recently, in a dataset of nearly 4000 firms over the period 1996-2002, Bettis et al (2005) find ten year options were exercised a median of 4.25 years before expiry. Similar observations have been made by Aboody (1996), Hemmer et al (1996) and Carpenter (1998). It has also been observed empirically that executives tend to exercise options in a few large transactions. That is, exercise typically takes place in a small number of large "blocks". Huddart and Lang (1996) find that the median fraction of options exercised by an employee at one time varied from 0.18 to 0.72 over companies. Similarly, Aboody (1996) reports yearly mean percentages of options exercised over the life of five and ten year options, showing exercises are spread over the life of the options. This paper provides a model which is consistent with both of these empirical observations and, in addition, generates some new testable implications.

It has long been recognized that standard American options models (Black Scholes (1973), Merton (1973)) do not adequately capture the exercise behavior of executives, or indeed represent the value of options to the executive.[1] Models based on risk-neutrality or complete markets[2] conclude options should be exercised when the stock price reaches some threshold level. If the executive has many identical options, the conclusion is that all the options should be exercised at the *same* threshold. That is, they are exercised on one occasion under such a model. This prediction is clearly violated in practice.

[1] Rubinstein (1995) points out many features distinguishing ESO's from tradeable options.

[2] We refer to standard models for tradeable options (Black and Scholes (1973), Merton (1973)) as complete market models and reserve the term risk-neutral to refer to models where an agent prices via a linear utility function.

One reason advanced to explain observed early exercise behavior is that executives cannot trade the company stock and face unhedgeable risks by retaining their options. Risk aversion causes the executives to exercise their options early, mitigating this unhedgeable risk sooner. Previous research[3] has demonstrated that the executive's risk aversion results in early exercise relative to standard American options, and that the executive derives a lower value from the options than if he were in a complete market where risks could be perfectly hedged. Whilst this literature provides an explanation for early exercise of ESO's, the assumption is made (implicitly or explicitly) that the executive holds only one option.[4] That is, nothing can be said about the nature of option exercises over time.

In this paper, we investigate the behavior of a risk averse executive assuming he is granted multiple options and that he can choose to exercise these options over time. Although the executive cannot trade in the company stock and thus faces unhedgeable risks, we allow him to reduce this risk by taking a position in another risky asset which is correlated with the stock of the company. The executive chooses the quantities and times to exercise, as well as his position in the risky asset in order to maximize his expected utility of wealth. We ask the question: do such utility-based models produce predictions of exercise behavior which are consistent with

[3]Such models use a certainty equivalent option value corresponding to a given utility specification. The early literature assumes the executive has no non-option investments or outside portfolio choice and can simply invest in riskfree bonds. Contributions were made by Huddart (1994), Lambert et al (1991), Kulatilaka and Marcus (1994) and Hall and Murphy (2002). Carpenter (1998) allows for outside investments but does not optimize in the presence of the option, rather assuming that the executive invests in a Merton (1971) style portfolio. Models have been developed whereby the executive also invests into a risky portfolio of other assets and must decide on his optimal investment as well as optimal exercise time. Models with portfolio choice include those by Detemple and Sundaresan (1999), Jin (2002), Henderson (2005a) and Ingersoll (2006).

[4]Either it is explicitly stated that there is only one option or it is implicitly assumed because all options are exercised at the same time.

empirical observations ?

We show that risk aversion causes the executive to exercise options *individually* over time. Options are exercised at a sequence of increasing stock price thresholds. We give these thresholds in closed form as the solution to a set of recursive equations. The thresholds are decreasing with risk aversion and increasing with the correlation between the company stock and risky asset. In fact, the ability to invest in a correlated asset allows the executive to partially hedge the risk of the options and allows him to delay exercise. We show the impact of the additional portfolio choice is to effectively scale down his risk aversion. All options are exercised at thresholds which are lower than the single exercise threshold which holds in a complete market when the executive can hedge perfectly.

One of the main advantages of our continuous time formulation and closed-form solution is that we are able to show that risk aversion *alone* does not predict block-exercise. If the same question were asked in a binomial model (or a discrete approximation to a continuous time model), then block-exercise might emerge due to the discrete set of dates on which the executive could exercise options.[5] We show that in the continuous time limit, block exercise disappears.

We show risk aversion can still form the basis of an explanation of observed exercise behavior, if, in addition, we recognize that executives exert effort to make an exercise decision, and this effort is costly.[6] We model costly effort via a penalty which is incurred each time the executive exercises options, regardless of the number of options exercised. The risk averse executive will balance the benefit of exercising optimally with the desire to minimize costs by exercising less

[5] Jain and Subramanian (2004) and Grasselli (2005) observe in binomial models that risk aversion leads to block-exercise.

[6] For example, the executive may spend time assessing analysts' research on the company in order to decide on an exercise strategy. This example is more applicable to an employee rather than a CEO who would possess more information about his company.

frequently. We demonstrate the outcome of this trade-off.

The predictions of our utility-based model with costly exercise are as follows. We find that executives facing higher costs will exercise their options on a smaller number of occasions. Due to risk aversion, all options are exercised prior to the single complete market threshold. Taken together, these implications describe the behavior of executives which has been observed empirically. Further, we show as costs increase, the largest block size increases. Costs induce the executive to exercise on fewer dates and thus they exercise a larger proportion of their option grant on a single occasion. If the executive exercises options on more than one occasion, we find that the block size is non-increasing across the exercise dates. For example, an optimal strategy could be to exercise in proportions (40%, 30%, 20%, 10%) or (50%, 30%, 10%, 10%) but not (40%, 10%, 20%, 30%) or (10%, 20%, 20%, 50%). This conclusion is a testable implication of the model.

There are other factors contributing to the observed exercise patterns of ESO's which are not part of our model.[7] We mention in particular the common practice of vesting.[8] This is a period of time after the grant during which ESO's cannot be exercised. For example, a common vesting structure is 25% over four years, see Huddart and Lang (1996).[9] Under this structure, the options become exerciseable in tranches of 25% each year. Huddart and Lang (1996) observe that although exercises that correspond to vest dates are common, they do not account for all

[7]These include taxes, adverse private information known by the executive, liquidity needs of the executive, and termination from the company, see Carpenter (1998). Heath et al (1999) and Core and Guay (2001) also consider psychological reasons for early exercise and propose that executives exercise in response to stock price trends. Related work by Poteshman and Serbin (2003) finds irrational early exercises in CBOE traded options and relates these to high stock prices or returns.

[8]Blackout periods (see Reda et al (2005)) when options cannot be exercised due to events such as earnings announcements would have a similar effect.

[9]Kole (1997) provides one of the early descriptions of ESO's including an analysis of vesting structures.

the block-exercises in the data. Further, block exercise certainly occurs in proportions other than multiples of the (say 25%) vest parcel size.

Consider now the theoretical impact of vesting on exercise behavior. In either a risk-neutral or complete market situation, the executive would exercise only in multiples of 25%, depending on whether the single stock price threshold had been reached when each 25% parcel of options vested. In our model with risk aversion (but no costs to exercise) and with no vesting, we find the executive exercises options individually. If vesting were also included in this model, multiple options would be potentially exercised together (constituting a block), but this would *only* occur on the vest dates when tranches of options became available for exercise. Clearly, without costly exercise, neither vesting alone, nor risk aversion alone, nor vesting and risk aversion together can fully explain observed exercise patterns. Whilst vesting can provide a partial explanation for empirically observed block exercise behavior, it cannot provide the full story. By focusing on risk aversion and costly exercise in this paper, we show block exercise will occur even in the absence of vesting.

From a broader perspective, an understanding of executive's exercise behavior is important for a number of reasons. Following much debate, the American and International accounting boards require that options are recognized as a cost at the option grant date. They recommend exercise behavior be taken into account when computing costs for financial reporting. Our costly exercise utility-based model shows exercise patterns differ with risk aversion, hedging capabilities, costs of exercise, and with the volatility and expected return on the company stock. Secondly, an understanding of the factors affecting exercise could influence how companies grant stock options. Additionally, the incentives provided by stock options vary with exercise behavior.[10]

The paper is organized as follows. First we give a description of the model and our assump-

[10] A recent strand of the literature considers how a company should grant options in dynamic principal-agent models (see Carpenter (2000) and Cadinellas et al (2004)). However these models do not consider exercise timing.

tions. Section 2 presents the simplest version of the utility-based model where the risk averse executive selects the times at which to exercise his options. This model is extended in Section 3 to give the executive the opportunity to trade in another risky asset in order to partially hedge some of the risk he faces. The remainder of the paper develops the model of Section 3. Section 4 treats a simpler situation where the executive is restricted to choose only one date on which to exercise all of his options. The results of this model are a special case of the results of Section 3 and are used both as a point of comparison and in the solution of the model of Section 5. In Section 5, the executive exerts costly effort to exercise options. This leads to a trade-off between exercising little and often to maximize return and exercising larger quantities on fewer occasions to minimize effort. The model of Section 5 leads to block-like exercise behavior. Section 6 concludes.

1 The Model

Consider a risk averse executive[11] who is granted n call options on the stock of his company. Our objective in this paper is to describe how the executive optimally exercises these options. Each option has strike K and is American-style so that it can be exercised at any time. We assume for simplicity the options have no vesting period, or equivalently, are already vested.[12]

The company stock price V follows a geometric Brownian motion

$$\frac{dV}{V} = \nu dt + \eta dW \tag{1}$$

[11] We refer to any individual receiving stock options as part of their compensation, including employees, executives and CEO's.

[12] We comment on this assumption throughout the paper, see in particular the discussion in the Introduction. Aboody (1996) finds about 10% of options in his sample had no vesting period.

with expected return ν and volatility η.[13] We take dividends and interest rates to be zero.[14]

We will assume the executive cannot trade the stock V.[15] The executive faces unhedgeable risk as he is unable to transact in the company stock to undo the effect of the options. This puts the executive in the situation of an incomplete market. We assume the executive is risk averse, and has negative exponential utility denoted by

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x} \tag{2}$$

The executive selects exercise times $\tau^n \leq ... \leq \tau^1$ where τ^j denotes the exercise time when there are j options remaining. For example, τ^n is the first selected exercise time when the executive has all n options and is choosing when to exercise for the first time. Similarly, τ^1 is the exercise time of the last remaining option.[16] We allow for a maximum of n dates corresponding to exercising the options one-at-a-time.[17] Allowing for dates to coincide (so $\tau^k = \tau^{k-1}$) recognizes that it may be optimal to exercise multiple options at the same time.

Upon exercise of a single option, the executive pays K for the stock and immediately sells it for the current stock price, receiving a cash payout of the difference.[18] At each exercise time τ^j,

[13] In common with the existing literature, we are ignoring the potential impact of executives' effort on the stock price. Agrawal and Mandelker (1987), DeFusco et al (1990) and Lambert et al (1989) find evidence that executives compensated with options increase stock price variance and leverage of firms and reduce dividends to shareholders.

[14] This is equivalent to working with the discounted value of the stock price.

[15] In practice, executives cannot short sell stock as they are prohibited by Section 16-c of the Securities and Exchange Act (see Carpenter (1998)). Since unconstrained executives granted a large parcel of ESO's would not choose to be long the stock, we assume (without loss of generality) that the appropriate restriction is that executives cannot *trade* the stock (long or short). It is also consistent with ESO's as a link between compensation and company stock price (providing incentives to increase the stock price) that executives cannot hedge away all the risk of the options.

[16] Note if the kth from last option is never exercised then $\tau^k = \tau^{k-1} = ... = \tau^1 = \infty$.

[17] Options must be exercised in whole units.

[18] Huddart and Lang (1996) provide evidence of such cash exercises, possibly due to taxes on capital gains. Ofek

the executive receives the cash payoff $(V_{\tau^j} - K)^+$; $j = 1, ..., n$, per-option exercised.

Throughout the paper we will assume an infinite maturity.[19] We argue this is reasonable on a number of grounds. Firstly, most executive stock options have expiry of ten years and prices of ten year American options are not very different from those under an infinite maturity. Secondly, Huddart and Lang (1996) only find a weak relationship between time to maturity and option exercises in empirical work. Finally, the assumption ensures the model is tractable and allows us to obtain closed-form exercise thresholds. Other papers to exploit tractability afforded by an infinite maturity include Kadam et al (2005) and Sircar and Xiong (2005). Recent literature has proposed the shareholder cost of ESO's be estimated under the assumption that exercise takes place at some constant exogenous stock price level, see Hull and White (2004) (also Cvitanic et al (2005)). In fact, the only modeling assumption that will be consistent with such exercise behavior is an infinite horizon.

We will present the simplest version of the option exercise problem in the next section. Here, the risk averse executive chooses the optimal exercise times for his n options, and is unable to trade the company stock itself. In this first model, the executive is also not permitted to trade any other outside assets which are external to his option position. However, in Section 3 we allow the executive to trade another risky asset which proxies his outside portfolio choice problem. Since this risky asset is correlated with the company stock, it provides a partial hedge for the option risk.

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and Yermack (2000) also find most executives sell the shares acquired through option exercise. Note however, prior to May 1991, the SEC imposed a six-month trading restriction on stock acquired through option exercise.

[19]The equivalent finite maturity problem would generate a critical exercise surface which relates the time to expiry, stock price and number of unexercised options. This would require numerical solution and so we would lose the ability to easily distinguish the n individual threshold levels. This is a major advantage of the infinite maturity formulation.

2 The Optimal Exercise Policy of the Risk Averse Executive

In this section we will present the simplest version of this problem where the executive chooses how to exercise his n options, given he is risk-averse. The option grant represents an unhedgeable risk to the executive since he is not permitted to trade the company stock. In this section, the executive chooses when to exercise in isolation of any other assets he may hold. Recall he chooses n possible times $\tau^n \leq ... \leq \tau^1$ at which to exercise options, some of these times may be the same. Let

$$X_t = X_0 + \sum_{\tau^i \leq t} (V_{\tau^i} - K)^+ \tag{3}$$

be the executive's wealth at time t, comprising an initial amount plus cash received from option exercises up to (and including) that time. The executive's optimization problem at an intermediate time t and when $k \leq n$ options remain unexercised is to find

$$G^k(x, v) = \sup_{t \leq \tau^k \leq ... \leq \tau^1} \mathbb{E}_t[U(X_{\tau^1})|X_t = x, V_t = v] \tag{4}$$

where $U(x)$ is given in (2). Since the options have infinite maturity, the optimization in (4) does not depend upon current time. The value $G^n(x, v)$ represents the value the executive places today on the right to exercise the n options in the future.

We will show below that the executives' optimal behavior is to exercise the options at a sequence of constant threshold levels $\tilde{V}^j; j = 1, .., n$ which are obtained in recursive form in the following proposition. Proofs of all propositions are contained in an appendix.

Proposition 1 *Define* $\beta_0 = 1 - \frac{2\nu}{\eta^2}$. *The exercise times* $\tau^n \leq ... \leq \tau^1$ *are characterized as the first passage times of the stock price* V *to constant thresholds,* $\tilde{V}^j; j = 1, .., n$ *such that*

$$\tau^j = \inf\{t : V_t \geq \tilde{V}^j\}; \quad j = 1, ..., n.$$

Case 1: *If* $\beta_0 > 0$*, the constant exercise thresholds* $\tilde{V}^n, ..., \tilde{V}^1$ *solve*

$$C_{\gamma,\beta_0,K,\Gamma^{j-1}}(\tilde{V}^j) = 0; \quad j = 1, ..., n$$

where

$$C_{g,\xi,\kappa,G}(x) = x - \kappa - \frac{1}{g}\ln\left[1 + \frac{g}{\xi}(1 - Gx^{\xi})x\right] \tag{5}$$

and the constants $\Gamma^j = \Gamma^j(\gamma, \beta_0, K)$ *are given by*

$$\Gamma^0 = 0; \quad \Gamma^j = \left(\frac{1}{\tilde{V}^j}\right)^{\beta_0}\left(1 - e^{-\gamma(\tilde{V}^j - K)^+}(1 - \Gamma^{j-1}(\tilde{V}^j)^{\beta_0})\right); \quad j = 1, ..., n-1. \tag{6}$$

The time-independent functions $G^j(x, v)$ *for* $j = 1, ..., n$ *are given by*

$$G^j(x,v) = -\frac{1}{\gamma}e^{-\gamma x}\left[1 - \left(1 - e^{-\gamma(\tilde{V}^j - K)^+}(1 - \Gamma^{j-1}(\tilde{V}^j)^{\beta_0})\right)\left(\frac{v}{\tilde{V}^j}\right)^{\beta_0}\right]$$

Case 2: *If* $\beta_0 \leq 0$, $\tilde{V}^j = \infty$ *for* $j = 1, ..., n$ *and the executive waits indefinitely. It is never optimal to exercise an option.*

The upper panel of Figure 1 displays various exercise thresholds for $n = 10$ at-the-money options with strike $K = 1$. The three sets of thresholds correspond to different levels of risk aversion. We observe that for a fixed level of risk aversion, the executive should optimally exercise the ten options at ten distinct stock price thresholds, $\tilde{V}^{10}, ..., \tilde{V}^1$. In fact, we can show that these thresholds are ordered. This rules out the possibility that multiple options are exercised at one time.

Proposition 2 *For* $\gamma > 0$ *and* $\beta_0 > 0$, *the constant exercise thresholds defined in Proposition 1 (Case 1), satisfy*

$$\tilde{V}^n < ... < \tilde{V}^j < \tilde{V}^{j-1} < ... < \tilde{V}^1$$

The proposition and the figure show that the executive exercises each remaining option at a higher stock price threshold than the previous option. The risk averse executive exercises his options one-at-a-time at an increasing sequence of stock price thresholds. Observe from the figure that the sequence of thresholds is convex. When many options remain unexercised, risk aversion has the most impact and the next threshold is quite close to the last one. However, as fewer options remain, the executive's exposed position is smaller, and he is willing to wait longer between exercises.

We can also observe that for a fixed number i options remaining, a higher risk aversion results in a lower threshold $\tilde{V}^i$ at which the ith remaining option is exercised. The more risk averse the executive, the less tolerant he is of unhedgeable risk and the earlier he wants to exercise the option. This is exactly the explanation for early exercise proposed by Huddart (1994) (and many other authors). However, this established literature either assumes that the executive only has one option, or that the executive exercises all options on one date. Instead, as we have shown, the optimal exercise policy for a risk averse executive is in fact to exercise options individually.

Consider the case where the executive is risk-neutral so has linear utility. This is a variant on the standard perpetual American option of Merton (1973). Define $\tilde{V}_{\langle\gamma=0\rangle} = \frac{\beta_0}{\beta_0-1}K$ to be the single exercise threshold in this model.[20] The solid vertical line in the figure indicates the threshold $\tilde{V}_{\langle\gamma=0\rangle} = 3.5$. In this case, all n options are exercised at this single threshold. We see that all three sets of risk averse thresholds lie below the threshold $\tilde{V}_{\langle\gamma=0\rangle}$. Risk aversion causes the executive to exercise all options at lower thresholds (and thus earlier) than if he were risk-neutral.

We make a final observation that the single threshold $\tilde{V}_{\langle\gamma=0\rangle}$ is exactly that obtained in the limit as risk aversion approaches zero.

Corollary 3 *As* $\gamma \to 0$*,*

Case 1: *If* $\beta_0 > 1$*, for* $j = 1, ..., n$*,*

$$\lim_{\gamma \downarrow 0} \tilde{V}^j = \tilde{V}_{\langle\gamma=0\rangle} = \frac{\beta_0}{\beta_0 - 1}K$$

Case 2: *If* $\beta_0 \leq 1$*,* $\tilde{V}^j = \tilde{V}_{\langle\gamma=0\rangle} = \infty$ *for* $j = 1, ..., n$ *and the executive waits indefinitely.*

[20] The form of this threshold is obtained by standard arguments, see Merton (1973).

3 Simultaneous Determination of Optimal Exercise Policy and Portfolio Choice

It is unlikely that executives receiving stock options make their exercise decisions in isolation of the rest of their portfolio. That is, executives would take into consideration their other holdings when deciding whether to exercise some of their options grant. In this section, we extend the model of Section 2 to this more realistic case. We determine simultaneously the optimal exercise policy and optimal portfolio choice of the executive, assuming the executive has access to a risky asset P with dynamics

$$\frac{dP}{P} = \mu dt + \sigma dB \tag{7}$$

where μ is the expected return and σ the volatility. Denote by $\lambda = \mu/\sigma$, the asset's instantaneous Sharpe ratio. Let $dBdW = \rho dt$ so that the risky asset P is correlated with the company stock with $\rho \in [-1, 1]$. In practice, this risky asset is likely to be a stock in a similar industry, an industry index, or a stock index.

Allowing the executive to trade in the risky asset enables him to partially hedge the risk of his options. He holds a cash amount θ_s in P at time s. Let

$$X_t = X_0 + \int_0^t \theta_s \frac{dP}{P} + \sum_{\tau_i \le t} (V_{\tau_i} - K)^+ \tag{8}$$

be the executive's wealth at time t. Wealth comprises both the cash received from any option exercises up to (and including) that time, as well as the value of the position in the risky asset P. This generalizes the model of Section 2 where there was no asset P in which to trade, and wealth was generated only by exercising options, see (3).

The executive selects n exercise times $\tau^n \le ... \le \tau^1$ as well as holdings θ_s in the risky asset P. Recall it may be the case that some exercise times are the same so that multiple options are exercised together. His optimization problem at an intermediate time t, and with $k \le n$ options remaining unexercised is

$$H^k(t,x,v) = \sup_{t\le\tau^k\le\ldots\le\tau^1} \sup_{(\theta_s)_{t\le s<\tau^1}} \mathbb{E}_t[e^{-\zeta\tau^1}U(X_{\tau^1})|X_t = x, V_t = v] \tag{9}$$

where ζ is a subjective discount factor, and $U(x)$ is given in (2). We want to find $H^n(0,x,v) = G^n(x,v)$, the value to the executive today of having the n options to exercise in the future.

We now argue that in the above problem, the discount factor ζ cannot be arbitrary, and in fact, a particular choice of ζ must be taken in order not to bias conclusions about the exercise times. This choice arises from the underlying portfolio choice problem and the associated opportunity cost of delayed exercise. We want to remove any built-in incentives to exercise options early or late because of the opportunity to invest in the risky asset P.

In order to remove these incentives, we need the choice $\zeta = -\frac{1}{2}\lambda^2$. With this choice, an executive facing the underlying portfolio choice problem without the options would be indifferent over the choice of investment horizon. When the same executive also has options to exercise, his choice of exercise times is not biased by his opportunities to invest in P. The Appendix contains a formal justification of this choice.

We denote the resulting inter-temporal utility function $\tilde{U}(t,x)$, the *horizon-unbiased* exponential utility, where

$$\tilde{U}(t,x) = e^{\frac{1}{2}\lambda^2 t}U(x) = -\frac{1}{\gamma}e^{-\gamma x}e^{\frac{1}{2}\lambda^2 t} \tag{10}$$

The formulation $\tilde{U}(t,x)$ was first introduced in Henderson (2005b).

Note that if there was no portfolio choice in the model (as in Section 2), then effectively $\lambda \equiv 0$ and so the unbiased choice becomes $\zeta = 0$. In this case we recover $\tilde{U}(t,x) = U(x)$, the usual negative exponential utility used in Section 2.

Returning to the executive's optimization problem in (9), we solve for a sequence of constant exercise thresholds $\tilde{V}^j; j = 1, ..., n$ at which the executive optimally exercises his options.

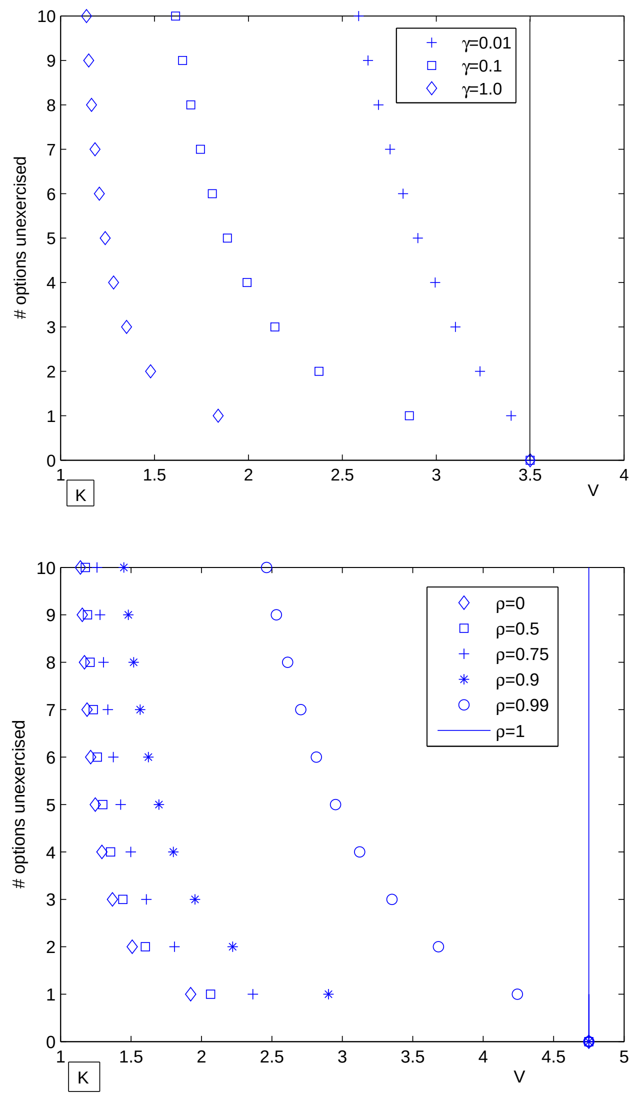

Figure 1: **Exercise thresholds:** Plots of exercise thresholds for an executive with $n = 10$ options with $v = K = 1$. The top panel gives thresholds for the model without outside portfolio choice given in Proposition 1. Thresholds for three values of risk aversion are plotted, for $\gamma = 0.01, 0.1$ and $\gamma = 1$. The solid line represents the threshold $\tilde{V}_{\langle\gamma=0\rangle}$ if the executive were risk-neutral. We take $\nu = -0.05, \eta = 0.5$ giving $\beta_0 = 1.4$. The lower panel gives thresholds for the model with outside portfolio choice given in Proposition 4. Risk aversion is fixed at $\gamma = 1$. The graph plots thresholds for various values of correlation. The threshold $\tilde{V}_{\langle\rho=1\rangle}$ corresponds to the choice $\rho = 1$. We take $\eta = 0.5$ and define $\alpha_\rho = \nu/\eta - \frac{\mu\rho}{\sigma}$ so that $\beta_\rho = 1 - \frac{2\alpha_\rho}{\eta}$. Fix $\alpha_\rho = -0.067$ giving $\beta_\rho = \beta_1 = 1.27$.

Similarly to the model without portfolio choice in Proposition 1, these thresholds are found in recursive form. The result follows below.

Proposition 4 *Define* $\beta_\rho = 1 - \frac{2(\nu - \mu\rho\eta/\sigma)}{\eta^2}$. *The exercise times* $\tau^n \leq ... \leq \tau^1$ *are characterized as the first passage times of stock price* V *to constant thresholds* $\tilde{V}^j; j = 1, .., n$ *such that*

$$\tau^j = \inf\{t : V_t \geq \tilde{V}^j\}; \quad j = 1, ..., n.$$

Case 1: *If* $\beta_\rho > 0$, *the constant exercise thresholds* $\tilde{V}^n, ..., \tilde{V}^1$ *solve*

$$C_{\gamma(1-\rho^2),\beta_\rho,K,\Lambda^{j-1}}(\tilde{V}^j) = 0; \quad j = 1, ..., n$$

where constants Λ^j *are given by* $\Lambda^0 = \Gamma^0 = 0$ *and*

$$\Lambda^j = \Gamma^j(\gamma(1-\rho^2), \beta_\rho, K); \quad j = 1, ..., n-1$$

where Γ^j *is given in (6). The time-independent functions* $G^j(x, v)$ *for* $j = 1, ..., n$ *are given by*

$$G^j(x, v) = -\frac{1}{\gamma} e^{-\gamma x} \left[1 - \left(1 - e^{-\gamma(1-\rho^2)(\tilde{V}^j - K)^+} (1 - \Lambda^{j-1}(\tilde{V}^j)^{\beta_\rho}) \right) \left(\frac{v}{\tilde{V}^j} \right)^{\beta_\rho} \right]^{\frac{1}{1-\rho^2}}$$

Case 2: *If* $\beta_\rho \leq 0$, $\tilde{V}^j = \infty$ *for* $j = 1, ..., n$ *and the executive waits indefinitely. It is never optimal to exercise an option.*

Note if we take $\rho = 0$ in the above we recover the results of Proposition 1 where there was no portfolio choice. The lower panel of Figure 1 gives exercise thresholds for $n = 10$ at-the-money options with $K = 1$ for different values of correlation ρ between the company stock and the asset P. Risk aversion is held fixed at $\gamma = 1.0$. Each set of points corresponds to a different level of correlation. For a given correlation, the executive should exercise the ten options at ten different stock price threshold levels. Again, we can show these thresholds are ordered, just as in the model without the opportunity to invest in the asset P.

Proposition 5 *For $\gamma(1-\rho^2) > 0$ and $\beta_\rho > 0$, the constant exercise thresholds defined in Proposition 4 (Case 1), satisfy*

$$\tilde{V}^n < ... < \tilde{V}^j < \tilde{V}^{j-1} < ... < \tilde{V}^1$$

The executive exercises his options one-at-a-time at an increasing sequence of stock price thresholds. For a fixed value of correlation, we observe the sequence of thresholds is convex. Correlation has most impact when more options are unexercised. As fewer options remain, the executive is exposed to less idiosyncratic risk and the distance between the thresholds increases.

We can also consider the executive's behavior if $\rho^2 = 1$ and so the risky asset P and the stock V are perfectly correlated. In this special case, the executive can perfectly hedge all the risk inherent in holding the options, and so faces a complete market. Denote by $\tilde{V}_{\langle\rho=1\rangle}$ the single exercise threshold in the complete market model. Standard arguments give $\tilde{V}_{\langle\rho=1\rangle} = \left(\frac{\beta_1}{\beta_1-1}\right)K$ where $\beta_1 = 1 - \frac{2(\nu-\mu\eta/\sigma)}{\eta^2}$. We observe that the single threshold $\tilde{V}_{\langle\rho=1\rangle}$ is obtained in the limit as correlation approaches one in the model with risk aversion.

Corollary 6 Perfect Hedging Case. *As $\rho \to 1$, $\beta_\rho \to \beta_1$ and*

Case 1: *If $\beta_\rho > 1$, the constant exercise thresholds $\tilde{V}^j$, $j = 1, ..., n$ satisfy*

$$\lim_{\rho\to 1} \tilde{V}^j = \tilde{V}_{\langle\rho=1\rangle} = \left(\frac{\beta_1}{\beta_1 - 1}\right)K$$

The value of the n options at time 0 to the executive is given by

$$G^n(x,v) = x + n(\tilde{V}_{\langle\rho=1\rangle} - K)\left(\frac{v}{\tilde{V}_{\langle\rho=1\rangle}}\right)^{\beta_1}$$

Case 2: *If $\beta_\rho \leq 1$, $\tilde{V}^j = \tilde{V}_{\langle\rho=1\rangle} = \infty$, for $j = 1, ..., n$, and the executive waits indefinitely.*

The solid line in the lower panel of Figure 1 represents the single threshold $\tilde{V}_{\langle\rho=1\rangle}$. All risk averse thresholds $\tilde{V}^n < ... < \tilde{V}^1$ lie below the threshold $\tilde{V}_{\langle\rho=1\rangle}$, as risk aversion causes the executive to exercise all options earlier than if he could fully hedge. For a fixed number i options remaining unexercised, a higher correlation results in a higher threshold $\tilde{V}^i$ at which the ith remaining option is exercised. The executive is exposed to less idiosyncratic risk as ρ increases, and is willing to exercise at a higher threshold. As correlation approaches one, the thresholds approach the single threshold $\tilde{V}_{\langle\rho=1\rangle}$. This is because the asset in which trading is performed is providing a better and better hedge as it becomes more similar to the stock of the company.

Observe from Proposition 4 that the risk averse executive's set of exercise thresholds solve a set of equations identical in form to those found in the model without investment opportunities of Proposition 1. In fact, if we fix $\beta_\rho = \beta_0$, the only difference in the model with investment opportunities is that the executive's risk aversion γ is scaled down by $1 - \rho^2$. This scaled risk aversion coefficient represents the effective risk aversion in the model with the risky asset P. The executive's opportunity to invest in a correlated stock means he is less exposed to the risk of the options. The more highly correlated the investment, the more risk he can hedge away and the more his risk aversion is scaled down. In fact, an executive with risk aversion γ and with the opportunity to invest in an asset with correlation ρ with his company stock chooses the same exercise thresholds as an executive without the investment opportunity, but with a (lower) risk aversion coefficient of $\gamma(1 - \rho^2)$. The implication is that for a given risk aversion level, γ, an executive with investment opportunities exercises each option at a higher threshold than the same executive without investment opportunities. The hedging opportunity allows the executive to delay exercising his options. Evidence of Hemmer at al (1996) is consistent with this observation, however, in their paper, hedging refers to the firm adjusting other compensation (in addition to stock options) to offset to some degree changes in the value of options. Their conclusion is that such a reduction in the risk to the executive does reduce the extent to which executives exercise early.

Figure 2 demonstrates how the risk averse executive's exercise thresholds are altered by the

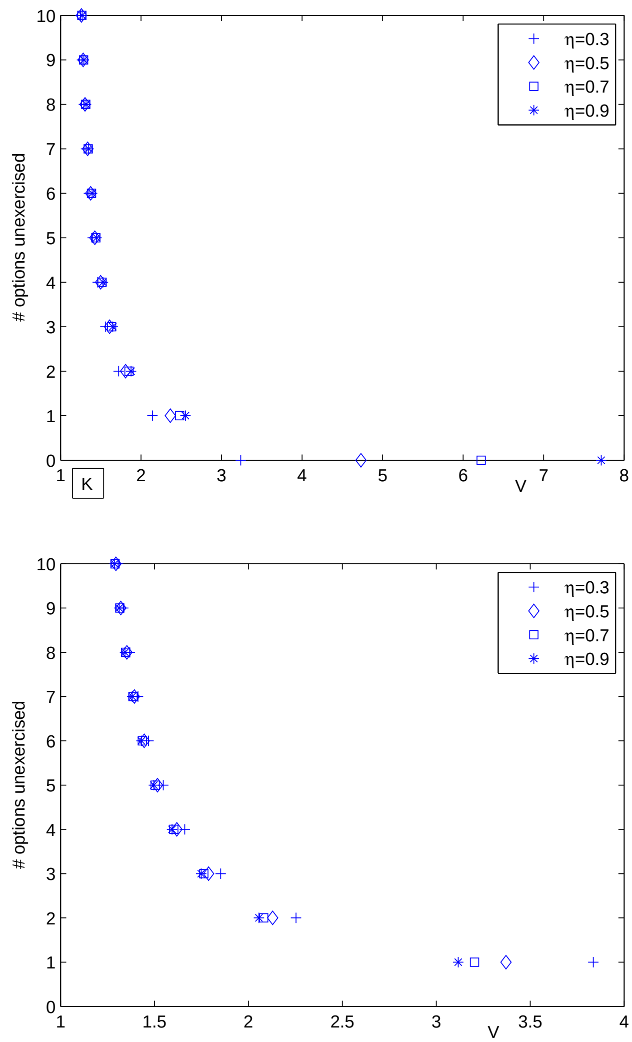

Figure 2: **Effect of Volatility:** Plots of exercise thresholds for an executive with $n = 10$ options with $v = K = 1$ in the model with portfolio choice given in Proposition 4. Risk aversion is fixed at $\gamma = 1$ in both panels. Both panels plot thresholds for various values of stock volatility, η. Both take $\rho = 0.75$. The upper panel takes $\alpha_\rho = -0.067$ giving values of $\beta_\rho > 1$. The thresholds marked along the x-axis correspond to $\rho = 1$ where the executive could hedge perfectly in a complete market. The lower panel takes $\alpha_\rho = 0.0476$ giving values of $\beta_\rho < 1$. In this case there are no corresponding perfect hedging thresholds since it is always optimal to wait indefinitely.

stock volatility, η. The executive's risk aversion is $\gamma = 1$ and the stock and risky hedge asset are correlated with $\rho = 0.75$. In the upper panel we take parameter values giving $\beta_\rho > 1$. The graph shows increases in volatility η raise the level of all ten individual exercise thresholds. On the x-axis we also plot the single threshold $\tilde{V}_{\langle \rho=1 \rangle}$ (for each value of volatility) where the executive would exercise if $\rho = 1$ and he could hedge perfectly. These are also increasing in volatility. In the lower panel we choose parameter values such that $\beta_\rho < 1$.[21] This time, in contrast, we see that increasing volatility results in lower exercise thresholds for each option.

We can reconcile these seemingly conflicting observations as follows. Stock volatility has two competing effects on the executive's exercise thresholds in utility-based models. The first effect occurs since option payoffs are convex. Volatility raises the (certainty equivalent) value the executive places on the option and correspondingly, increases the exercise threshold. However, higher volatility also imposes more idiosyncratic risk on the executive and this gives an incentive to exercise earlier at lower thresholds (and place a lower value on the options). In the parameter region such that $\beta_\rho > 1$, the convexity effect dominates. However, in the region where $\beta_\rho < 1$, the impact of risk aversion overwhelms convexity and the exercise thresholds are lowered with volatility. Empirically, there is evidence for this second case. Bettis et al (2005), Hemmer et al (1996) and Huddart and Lang (1994) all find that executives exercise earlier the greater the volatility of the firm's stock price.

We observe that the increase in volatility acts on all thresholds in the same direction. That is, all thresholds either increase or decrease. Additionally, we observe that the impact of a change in volatility is more pronounced when fewer options remain unexercised. This is linked to risk aversion having a larger effect on the thresholds when fewer options remain.

For the remainder of the paper we work with and build upon the model of this section. That

[21] Recall from Corollary 6 that in this case, the executive would wait indefinitely if $\rho = 1$ and he could perfectly hedge. Hence there are no thresholds on the lower panel corresponding to $\rho = 1$.

is, the executive has access to trading in the risky asset P and can use this to hedge some of the risk of his option position. We can easily recover the special case of no hedging opportunities by taking $\rho = 0$.

4 Optimal Exercise Decision with Restricted Exercise

In this section, we place a severe exercise restriction on the risk averse executive by allowing him to choose only one exercise threshold at which to exercise all n options. We will compare this restricted threshold to the executive's optimal (unrestricted) strategy described in Proposition 4 and also the single exercise threshold which would hold if he could hedge perfectly. Clearly forcing the executive to choose only one exercise time will lower the executive's expected utility as he has less choice than before. Additionally, the results of this section will be used in Section 5 when we introduce costly exercise.

We assume the executive holds $i \leq n$ options and must exercise all i options at the one time.[22] The executive chooses the exercise time τ_r^i and holdings θ_s in the risky asset P to maximize his expected utility of wealth at the exercise time.[23] The executive's wealth is given by

$$X_t = X_0 + \int_0^t \theta_s \frac{dP}{P} + i(V_{\tau_r^i} - K)^+ I_{\{\tau_r^i \leq t\}}.$$

At an intermediate time t, with i options remaining, the executive's optimization problem is to find

$$H_r^i(t, x, v) = \sup_{\tau_r^i} \sup_{(\theta_s)_{t \leq s < \tau_r^i}} \mathbb{E}_t[\tilde{U}(\tau_r^i, X_{\tau_r^i}) | X_t = x, V_t = v].$$

We want to find $H_r^i(0, x, v) = G_r^i(x, v)$ which is the value to the executive at time zero. The solution is given by the following result.

[22] We consider $i \leq n$ options as we potentially want to allow i to vary from one to n, although the choice where $i = n$ options is the most important for comparison purposes.

[23] The superscript on τ_r^i refers to the number of options i, whilst the subscript r denotes restricted exercise.

Proposition 7 *The restricted exercise time of i options, τ_r^i, is characterized as the first passage time of V to constant threshold $\tilde{V}_r^i$ such that*

$$\tau_r^i = \inf\{t : V_t \geq \tilde{V}_r^i\}$$

Case 1: *If $\beta_\rho > 0$, the constant exercise threshold $\tilde{V}_r^i$ solves*

$$C_{i\gamma(1-\rho^2),\beta_\rho,K,0}(\tilde{V}_r^i) = 0.$$

The time-independent function $G_r^i(x,v)$ is given by

$$G_r^i(x,v) = -\frac{1}{\gamma}e^{-\gamma x}\left[1 - \left(1 - e^{-i\gamma(1-\rho^2)(\tilde{V}_r^i - K)^+}\right)\left(\frac{v}{\tilde{V}_r^i}\right)^{\beta_\rho}\right]^{\frac{1}{1-\rho^2}}.$$

Case 2: *If $\beta_\rho \leq 0$, $\tilde{V}_r^i = \infty$ and the executive waits indefinitely.*

Figure 3 illustrates the results of Proposition 7. The plot depicts the exercise thresholds when there are up to ten options remaining unexercised. The two sets of markers in Figure 3 represent the thresholds $\tilde{V}^i$ at which the ith remaining option is exercised (marked with +) and the restricted thresholds $\tilde{V}_r^i$ at which all i options are exercised (marked with *).

We first observe that if there is only one option remaining to be exercised, then the thresholds are identical, $\tilde{V}^1 = \tilde{V}_r^1$. In this case, our restriction obviously makes no difference. The figure shows each restricted threshold is higher than the corresponding unrestricted threshold where one option is exercised. That is, if the executive has to exercise all ten options at once, he will do so at a higher stock price threshold than if he was just exercising his tenth remaining option. Each restricted threshold is somewhere between the first and last unrestricted thresholds so that the (single) restricted threshold is a (weighted) average of the unrestricted thresholds.

We can also consider the level of the restricted threshold for different numbers of options, i. We can prove the following.

Proposition 8 *For $\gamma(1-\rho^2) > 0$ and $\beta_\rho > 0$, the constant restricted exercise thresholds defined in Proposition 7 (Case 1), satisfy*

$$\tilde{V}_r^n < ... < \tilde{V}_r^{i+1} < \tilde{V}_r^i < ... < \tilde{V}_r^1$$

This ordering is evident in Figure 3. If there are fewer options to be exercised (at a single restricted threshold), the executive is exposed to less unhedgeable risk and waits longer to exercise the i options. The restricted thresholds are also convex. The difference between the thresholds gets larger as the number of options i gets smaller, again because of less risk exposure.

The restricted threshold levels are all still below the single threshold $\tilde{V}_{\langle\rho=1\rangle}$ which applied when the executive could perfectly hedge the option risk. This single threshold is given by the solid line in the figure. This difference is due to the risk aversion of the executive.

5 Optimal Exercise Decision with Costly Exercise

We incorporate costly effort into the executive's exercise decision and show it can explain the tendency of executives to exercise their options on a small number of occasions. Clearly, the act of exercising options involves some effort on the part of the executive. For example, the executive must inform his firm or broker of his intention to exercise at a particular time. He may also expend effort (in terms of the time spent) keeping track of the stock performance in order to make an informed exercise decision. In both of these situations, the executive has expended effort, which is costly. He may wish to minimize the effort he spends on making exercise decisions, possibly limiting the number of times he considers exercising options.

We incorporate costly effort into the model via a fixed cost c which is lost each time the executive exercises options, regardless of how many options are exercised. This reflects the fact that this cost represents the loss due to an executive's time being spent on making an exercise

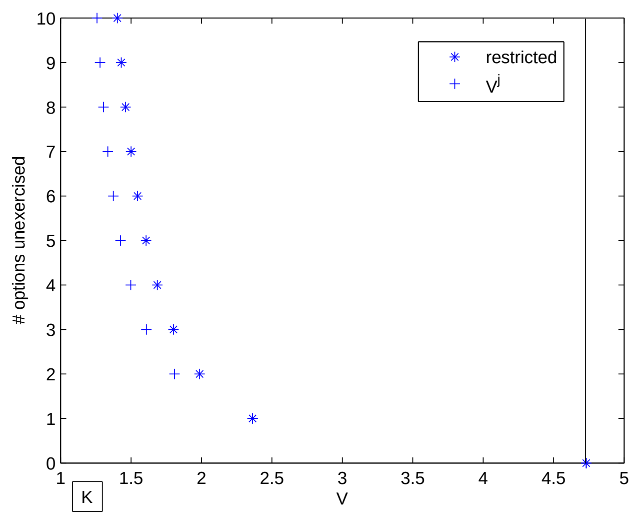

Figure 3: **Restricted exercise:** A comparison of exercise thresholds $\tilde{V}^j; j = 1, .., n$ for the risk averse executive with portfolio choice (Proposition 4) with the restricted exercise thresholds $\tilde{V}_r^i; i = 1, ..., n$ (Proposition 7). The threshold $\tilde{V}_{\langle\rho=1\rangle} = 4.75$ is indicated with the solid line. Parameters are: $n = 10$, $K = 1$, $\rho = 0.75$, $\gamma = 1$, $\eta = 0.5$, $\alpha_\rho = -0.067$ giving $\beta_\rho = 1.27$.

decision. As we will see in this section, the executive will balance the benefit of exercising optimally (as in Section 3) with the costs of exercise.

Consider an executive who is granted n options and is subject to costly exercise. The per-exercise cost is represented by a constant c.[24] Denote an exercise strategy of the n options by the vector of positive integers $q = (q_k, ..., q_1)$ where $\sum_{j=1}^{k} q_j = n$ and $q_j \geq 1; 1 \leq j \leq k$. The size of the final block exercised is q_1, more generally, the size of the jth block exercised is q_{k-j+1} and the strategy q represents the sequence of such block sizes. There are $1 \leq k \leq n$ exercise dates. We want to find the optimal such q. In the case $c = 0$, we are back in the setting of Proposition 4 where the optimal strategy was to take $k = n$ and $q_j = 1,\ j = 1, ..., n$. That is, the executive not subject to any exercise costs will exercise options one-at-a-time. In this section we find the optimal choice of strategy q for the situation with non-zero costs, c.

Each exercise strategy q will be associated with a set of exercise thresholds. If $k = 1$, so all options are exercised at one time, the threshold $\tilde{V}_c^{q_1}$ is the level at which q_1 options are exercised. In this case $q_1 = n$. If $k = 2$ and the executive exercises q_2 options on one date, and the remaining q_1 options on a subsequent date (where $q_1 + q_2 = n$), then $\tilde{V}_c^{q_2,q_1}$ denotes the (first) threshold at which q_2 options are exercised, and $\tilde{V}_c^{q_1}$ denotes the second threshold, where the remaining q_1 options are exercised. In general, $q_j;\quad j = 1, ..., k$ options are exercised the first time $\tau_c^{q_j,q_{j-1},...,q_1}$ that the stock price reaches threshold $\tilde{V}_c^{q_j,q_{j-1},...,q_1}$. For example, if the executive begins with $n = 3$ options, we may have $q = (1, 1, 1)$ with associated thresholds $\tilde{V}_c^{1,1,1}$ at which the first of three options is exercised, $\tilde{V}_c^{1,1}$ when the second is exercised and $\tilde{V}_c^{1}$ where the final option is exercised. Other strategies are $q = (1, 2)$, $q = (2, 1)$ or $q = (3)$. Note that options are only exercised if the stock price reaches the relevant threshold level. For example, under the strategy $q = (2, 1)$, two options are exercised if the stock price reaches threshold level

[24]This may be extended to the case where cost per exercise is made up of a fixed plus a proportional cost, $c + di$, for some constant d, where i is the number of options exercised.

$\tilde{V}_c^{2,1}$ and the third option is exercised if the stock price reaches $\tilde{V}_c^1$. If the stock price were never to reach threshold $\tilde{V}_c^1$, then the final option would not be exercised.

Before continuing further, we show how costly exercise impacts when the executive exercises multiple options on one date, so $k = 1$. Situations where $k > 1$, corresponding to more than one exercise date will build on the case $k = 1$. Assume there are l options to be exercised. The exercise cost c is paid when $q_1 = l$ options are exercised at some time τ_c^l, and reduces the option payoff to $l(V_{\tau_c^l} - K)^+ - c$. The options are only exercised if the payoff $l(V_{\tau_c^l} - K)^+$ exceeds c, so the effective payoff becomes $l(V_{\tau_c^l} - (K + c/l))^+$. In the presence of costs, the per-option strike is increased from K to $K + c/l$. This observation allows us to apply the results of the restricted exercise model of Proposition 7 with a modified strike to deduce the exercise threshold $\tilde{V}_c^l$ solves

$$C_{l\gamma(1-\rho^2),\beta_\rho,K+c/l,0}(\tilde{V}_c^l) = 0. \tag{11}$$

We now consider more than one exercise date, $k \geq 1$. We will build solutions via a recursive formula based on the number of exercise dates. Recall the notation $q = (q_k, ..., q_1)$ denotes an exercise strategy where $\sum_{i=1}^k q_i = n$. As was the case in Proposition 4, if $\beta_\rho \leq 0$ the executive waits indefinitely, so assume $\beta_\rho > 0$. Let $p = (p_j, ..., p_1)$ where $\sum_{i=1}^j p_i < n$. Let $\tau_c^p \equiv \tau_c^{p_j,...,p_1}$ be the first exercise time associated with the strategy p, at which p_j options are exercised. Let $r = (l, p)$ where $l + \sum_{i=1}^j p_i \leq n$.[25] The following proposition gives the first exercise threshold τ_c^r associated with the strategy r, (at which l options are exercised), and the value to the executive of following this strategy.

Proposition 9 Costly Exercise *Suppose $\beta_\rho > 0$, and suppose Ξ^p and $\tilde{V}_c^p$ are known. Then τ_c^r is characterized as the first passage time of V to constant threshold $\tilde{V}_c^r$ where $\tilde{V}_c^r \leq \tilde{V}_c^p$ satisfies*

$$C_{l\gamma(1-\rho^2),\beta_\rho,K+c/l,\Xi^p}(\tilde{V}_c^r) = 0$$

[25] If $j = 0$ then p is the empty vector $\emptyset$ and $r = (l)$. In this case, for the purposes of the following proposition, $\Xi^\emptyset = 0$ and $\tilde{V}_c^\emptyset = \infty$.

The constant Ξ^r *(necessary for the next inductive step) is given by*

$$\Xi^r = \Xi^{l,p} = \left(\frac{1}{\tilde{V}_c^r}\right)^{\beta_\rho} (1 - e^{-l\gamma(1-\rho^2)(\tilde{V}_c^r-(K+c/l))^+}(1 - \Xi^p(\tilde{V}_c^r)^{\beta_\rho}))$$

The value to the executive at time zero, given he has $l + \sum_{i=1}^{j} p_i$ *options and exercises them according to pattern* $r = (l, p_j, ..., p_1)$ *is*

$$G_c^r(x,v) = -\frac{1}{\gamma}e^{-\gamma x}\left[1 - (1 - e^{-l\gamma(1-\rho^2)(\tilde{V}_c^r-(K+c/l))^+}(1 - \Xi^p(\tilde{V}_c^r)^{\beta_\rho}))\left(\frac{v}{\tilde{V}_c^r}\right)^{\beta_\rho}\right]^{\frac{1}{1-\rho^2}}$$

We define the executive's optimal exercise strategy to be that $q = (q_k, ..., q_1)$ with $\sum_{i=1}^{k} q_i = n$ which maximizes value $G_c^q(x,v)$. We are interested in how this optimal strategy q varies with costs.

We now illustrate the results of Proposition 9. We first consider an executive receiving two options in the upper panel of Figure 4. It plots the value from

(i) exercising both options simultaneously and paying c once (the solid line), and

(ii) exercising the options one-at-a-time and paying c twice (the broken line).

Note in both situations (i) and (ii), options are only exercised if the relevant threshold is reached. In (ii) where the executive chooses to exercise the options individually, it could be the case that the stock price reaches the first threshold and one option is exercised but the second threshold is not reached. In that case, the executive only pays c once. Returning to the figure, these values are given as a function of the cost per exercise, c. Immediately, we see that the two lines cross at around $c = 0.08$. For lower costs, the executive should still choose to exercise the options one-at-a-time and pay the cost each time. Recall if there were no costs associated with exercise, the executive would optimally exercise the options individually. However, costs have a significant impact beyond $c = 0.08$, and higher costs mean the executive should exercise both options simultaneously. (Although costs also have an impact below $c = 0.08$ via altered thresholds and expected utility).

The lower panel of Figure 4 gives the value functions for $n = 3$ options. Possible strategies for the executive are to:

(i) exercise the three options simultaneously and pay c only once,

(ii) exercise all three options separately and pay c each time,

(iii) exercise one option first and then two later and pay c twice, or

(iv) exercise two options on a single date and the final one later, again paying c twice.

Again, for each of these scenarios, the exercise cost c is only paid if the stock price threshold is attained. The graph shows that three of these four possibilities occur. For very low cost levels, it is optimal to exercise each option separately and pay the cost each time. This is an exercise strategy of $q = (1, 1, 1)$. On the graph, this occurs roughly when $c < 0.02$. For c high (on the graph approximately $c > 0.19$), it is optimal to pay c only once and to exercise all three options on one date. However, for costs between these two extremes, there is a region where it is best to exercise two options on a single date and then the third option at a later date, so $q = (2, 1)$. It is never optimal to exercise in the other order, so $q = (1, 2)$ does not occur.

Since we are mainly concerned with the pattern of exercise strategies q as c varies, we plot in Figure 5 the breakpoints (in $\ln(c)$ units) between various optimal strategies for $n = 3, 5$ and 10 options. Consider first $n = 3$. The two diamonds on the graph give the values of $\ln(c)$ where the optimal strategy changes. For $\ln(c) < -3.9$ (or $c < 0.02$), the optimal exercise strategy is $q = (1, 1, 1)$. Between the two diamonds, the strategy $q = (2, 1)$ is optimal, and for $\ln(c) > -1.63$, $q = (3)$. This gives us the same information as was contained in Figure 4. However, now we can consider a larger number of options. For $n = 5$ options, the squares indicate the change points between different strategies. When $\ln(c) < -5.43$ (or $c < 0.0044$), the options are exercised one-at-a-time. As $\ln(c)$ increases towards zero, the optimal strategies are in turn $q = (2, 1, 1, 1)$, $q = (2, 2, 1)$, $q = (3, 1, 1)$, $q = (4, 1)$, and $q = (5)$. When $n = 10$ options.

the change from one strategy to another is marked by the crosses on Figure 5. When $\ln(c) < -7.27$, the options are exercised one-at-a-time so $q = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. As $\ln(c)$ increases towards zero, the optimal strategies in order are: $q = (2, 1, 1, 1, 1, 1, 1, 1, 1)$, $q = (2, 2, 2, 1, 1, 1, 1)$, $q = (3, 2, 2, 1, 1, 1)$, $q = (3, 3, 2, 1, 1)$, $q = (4, 2, 2, 1, 1)$, $q = (4, 3, 2, 1)$, $q = (5, 3, 1, 1)$, $q = (6, 3, 1)$, $q = (7, 2, 1)$, $q = (7, 3)$, $q = (8, 2)$, $q = (9, 1)$, and $q = (10)$.

The results of Figures 4 and 5 show that risk averse executives optimally exercise options in a small number of blocks once the exercise costs are large enough. This modeling implication is consistent with the empirical evidence of Huddart and Lang (1996) who find that the mean fraction of options exercised by an employee at one time varied from 0.18 to 0.72. They also plot exercise frequency as a function of elapsed life and the percentage of options granted, and note that "...the distribution across exercise percentages suggests employees typically exercise options in large blocks" (p21). Whilst the vesting structure of the options appeared to play a role in block exercise, it does not explain the cumulative exercise patterns for each individual company. For example, for the largest company in their study, only around 20% of options had been exercised after four years, by which time all options had fully vested. After the options are fully vested, the remaining options appear to be exercised according to a block-like structure over their remaining life. This pattern can be observed for each of the seven companies in their sample.[26]

We can make a number of further observations from Figure 5. First, we find the largest block size increases as the cost c increases. That is, as the cost per exercise increases, the executive is tempted to exercise more options in a single block to save on costs. A related observation is that the number of blocks decreases with c. The executive exercises options on fewer occasions

[26] They consider an eighth company (labeled company H) where it is observed that employees wait until the five year expiry date to exercise. However this company has an internally generated stock price which is predictable and was increasing over the five year period.

as costs rise.

It is to be expected that as costs rise, exercise takes place in fewer and larger blocks. However, less obvious is the finding that for a fixed value of c, if it is optimal to exercise across more than one date, the block size across the series of dates is always non-increasing. For example, for the situation with $n = 3$ options, it was never optimal to use the strategy $(1, 2)$. Indeed, for $n = 10$ options, only a small subset of possible strategies actually appear as optimal ones. This is an implication of the risk aversion of the executive who faces exercise costs. Whether block sizes decrease across exercise dates in practice is a testable implication of the model.[27]

6 Conclusions

Our emphasis in this paper is on providing a model which explains observed features in ESO exercise patterns such as early exercise, and the tendency of executives to exercise large quantities of options on a small number of occasions. We first show that risk averse executives who receive a grant of n identical options will optimally exercise them individually at an increasing sequence of stock price thresholds. All options will be exercised prior to the single threshold at which an executive who was able to fully hedge the option risk would exercise all n options. This implies that utility-based ESO models do not predict block exercise. Our conclusions held true both in a model where the executive was only permitted to invest in riskless bonds, and a richer model where he could also invest in a risky asset which was correlated with the company stock price.

If, in addition we recognize that executives exert effort to exercise options, and this effort is costly, then the executive will face a trade-off between exercising little and often (incurring

[27] Vesting may alter this conclusion when there are multiple vest dates. Consider a situation where more options are about to vest and the stock price has already passed a number of thresholds during the vesting period. The executive will exercise a block of options on the vest date which may exceed the previous quantity exercised.

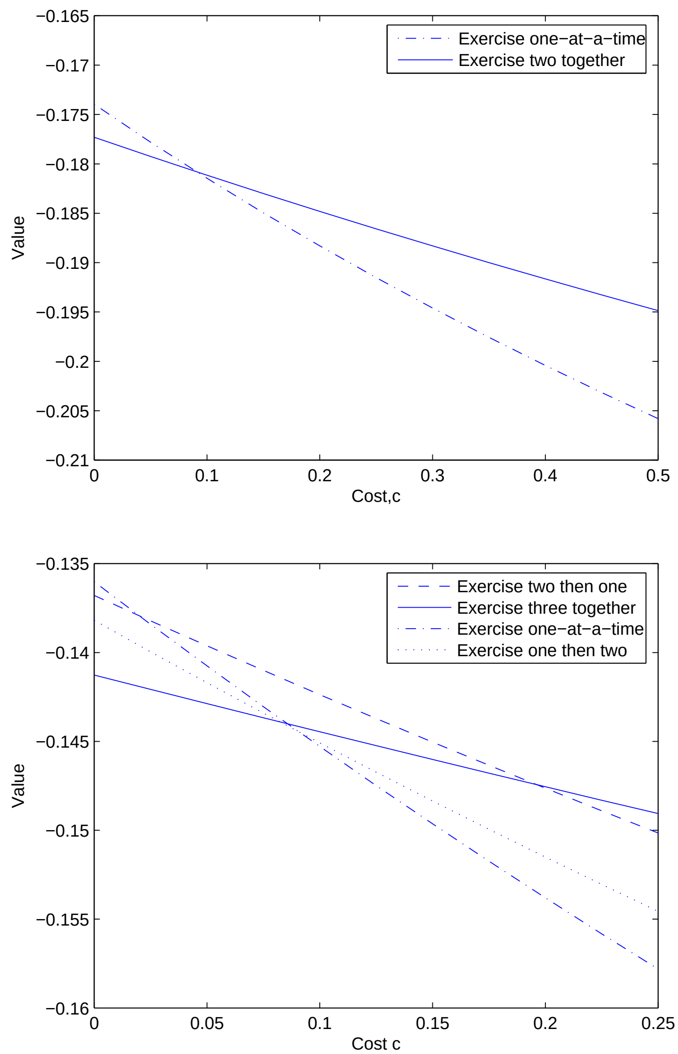

Figure 4: **Costly exercise:** A comparison of the value functions for alternative exercise strategies q, when there is a cost per exercise of c. Value functions for each q are plotted as a function of c. The top panel takes $n = 2$ options and compares the value from exercising one-at-a-time ($q = (1, 1)$) to that from exercising both together ($q = (2)$). The lower panel is for $n = 3$ options and compares the four possibilities to exercise the options: $q = (3), q = (2, 1), q = (1, 2), q = (1, 1, 1)$. Parameters are: $K = 1$, $v = 1$, $\rho = 0.75$, $\gamma = 1$, $\nu = 0.05, \eta = 0.5$, $\mu = 0.05, \sigma = 0.3$ giving $\alpha_\rho = -0.025$ and $\beta_\rho = 1.1$.

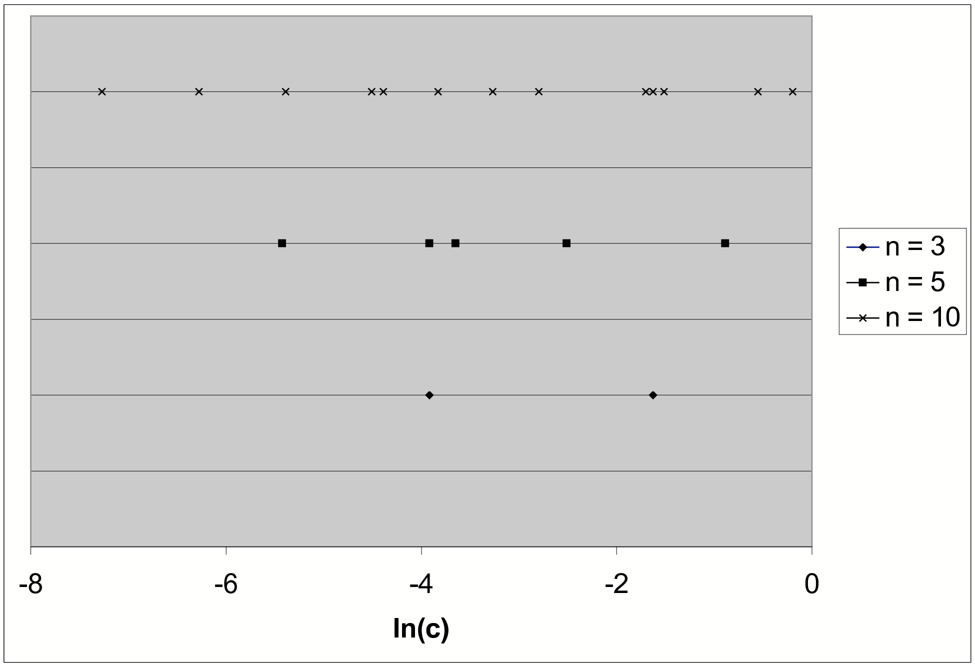

Figure 5: **Costly Exercise:** Optimal costly exercise strategies for $n = 3, 5$ and 10 options. The markers indicate breakpoints in $\ln(c)$ units between various optimal exercise strategies. The three sets of markers correspond to $n = 3$ (lowest), $n = 5$ and $n = 10$ (highest) options. For $n = 3$, the diamonds indicate switches from strategies $q = (1, 1, 1)$ to $q = (2, 1)$ to $q = (3)$ as $\ln(c)$ increases. For $n = 5$ options, the squares indicate the switches from strategies $q = (1, 1, 1, 1, 1)$, $q = (2, 1, 1, 1)$, $q = (2, 2, 1)$, $q = (3, 1, 1)$, $q = (4, 1)$ to $q = (5)$, as $\ln(c)$ increases. For $n = 10$ options. the change from one strategy to another is marked by the crosses. As $\ln(c)$ increases, the optimal strategies in order are: $q = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $q = (2, 1, 1, 1, 1, 1, 1, 1, 1)$, $q = (2, 2, 2, 1, 1, 1, 1)$, $q = (3, 2, 2, 1, 1, 1)$, $q = (3, 3, 2, 1, 1)$, $q = (4, 2, 2, 1, 1)$, $q = (4, 3, 2, 1)$, $q = (5, 3, 1, 1)$, $q = (6, 3, 1)$, $q = (7, 2, 1)$, $q = (7, 3)$, $q = (8, 2)$, $q = (9, 1)$, and $q = (10)$. Parameters are: $K = 1$, $v = 1$, $\rho = 0.75$, $\gamma = 1$, $\nu = 0.05$, $\eta = 0.5$, $\mu = 0.05$, $\sigma = 0.3$ giving $\alpha_\rho = -0.025$ and $\beta_\rho = 1.1$.

costs), and exercising less frequently to minimize costs. We showed that as costs increase, the executive exercises on a smaller number of occasions and thus exercises a larger proportion of the grant on a single date. The introduction of costly exercise into our utility-based model generates exercise behavior consistent with observed behavior. In addition, the model also gave the new testable prediction that executives should begin by exercising large blocks of options, but the block sizes should become smaller over time. Finally, we argued that although vesting could be incorporated into our model, in the absence of costly exercise, it would generate block-exercises only on the vest dates themselves. Therefore vesting alone cannot fully explain observed exercise patterns. Rather, we have shown that in the absence of vesting, the fact that executives are risk averse and exert effort to exercise is sufficient to generate realistic patterns.

Our work also has a number of implications for related areas of ESO research. Carpenter (1998) and Bettis et al (2005) have tested traditional utility-based ESO models on exercise data. These papers assume that the executive exercises all options on only one occasion.[28] However, as we demonstrate, this assumption is not consistent with optimal behavior under a utility-based model and executives exercise options over a number of dates. Another strand of the literature is focused on obtaining a computationally tractable approximation for the cost of ESO's to the company. Recently, Hull and White (2004) (see also Cvitanic et al (2005)) have proposed a cost model whereby executives exercise at a single constant exogenous barrier level.[29] The barrier is expressed as a constant multiple of the strike price. Our model shows that risk averse executives who face costly exercise will in fact choose a number of exercise thresholds. The number of thresholds depends upon the tradeoff between risk aversion and costs of exercise. An interesting

[28]They use a binomial style model and assume $n = 1$ option or equivalently, that all options are exercised at one time.

[29]Other authors including Rubinstein (1995), Cuny and Jorion (1995) and Carpenter (1998) previously used exogenous exercise to obtain an ESO cost.

extension would be to explore the implications of our model of optimal exercise behavior on the cost to shareholders and compare to the aforementioned tractable approximations.

7 Appendix

This appendix contains proofs of the results in the main text.

Horizon-unbiased Utilities: A Derivation of (10)

Consider the underlying portfolio choice problem faced by the executive in the absence of options. We will demonstrate that under the choice of utility function in (10)

$$\tilde{U}(t,x) = -\frac{1}{\gamma} e^{-\gamma x} e^{\frac{1}{2}\lambda^2 t}, \tag{A1}$$

the solution to this portfolio choice problem does not depend on the choice of horizon. Equivalently, we show that the discount factor ζ in (9) must be chosen to be $\zeta = -\frac{1}{2}\lambda^2$ for there to be no bias from the underlying portfolio choice problem. We would like this to be true so that when the executive faces the portfolio choice and option exercise problem, he does not have a built-in incentive to exercise options early or late.

Consider the portfolio choice problem over a given horizon τ

$$\sup_{(\psi_u)_{t \le u \le \tau}} \mathbb{E}[e^{-\zeta\tau} U(\hat{X}_\tau) | \hat{X}_t = x] \tag{A2}$$

where $\hat{X}_t$ is given by

$$\hat{X}_t = \hat{X}_0 + \int_0^t \psi_s \frac{dP}{P} \tag{A3}$$

and represents wealth from investing cash amount ψ in the risky asset P (and riskless bonds paying zero rate of interest). We will show that with $\zeta = -\frac{1}{2}\lambda^2$ and thus $\tilde{U}$ as in (A1), the solution to problem (A2) does not depend on the time horizon τ. In particular, if the executive is allowed to choose τ, then there is no way he can take advantage of this choice.

First we consider the portfolio choice problem in (A2) for fixed time horizons, T. For a fixed horizon T and the above choice of utility function, the problem in (A2) becomes

$$M^T(t,x) = \sup_{(\psi_u)_{t \le u \le T}} \mathbb{E}\left[-\frac{1}{\gamma} e^{-\gamma \hat{X}_T} e^{-\zeta T} \middle| \hat{X}_t = x \right]. \tag{A4}$$

The choice $\zeta = 0$ corresponds to the standard portfolio choice problem of Merton (1971) with exponential utility where utility of terminal wealth is maximized, and the investor can only invest in risk-free bonds (zero rate of interest) or a single risky asset. The solution to that standard problem depends on the time remaining until the terminal horizon. Our *horizon-unbiased* exponential utility $\tilde{U}(t, x)$ is chosen precisely to remove that dependence. We now solve (A4) using the same Hamilton-Jacobi-Bellman approach as used for the Merton (1971) problem. Using (A3), the HJB equation is given as

$$\sup_{\psi} \left\{ \dot{M}^T + M_x^T \psi \mu + \frac{1}{2} M_{xx}^T \psi^2 \sigma^2 \right\} = 0 \tag{A5}$$

with boundary condition $M^T(T, x) = -\frac{1}{\gamma} e^{-\gamma x} e^{-\zeta T}$.

Performing the maximization over ψ gives

$$\psi_t^* = -\frac{M_x^T \lambda}{M_{xx}^T \sigma} = -\frac{M_x^T \mu}{M_{xx}^T \sigma^2} \tag{A6}$$

and substitution into (A5) results in

$$\dot{M}^T - \frac{1}{2} \frac{\lambda^2 (M_x^T)^2}{M_{xx}^T} = 0.$$

The solution with the given boundary condition can be verified (by substitution) to be

$$M^T(t, x) = -\frac{1}{\gamma} e^{-\gamma x} e^{-\frac{1}{2}\lambda^2 (T-t)} e^{-\zeta T}.$$

We mentioned the choice $\zeta = 0$ gives the standard Merton (1971) solution $M^T(t, x) = -\frac{1}{\gamma} e^{-\gamma x} e^{-\frac{1}{2}\lambda^2 (T-t)}$, which depends on $T - t$. In this case the value function is larger, the greater the time remaining to the investment horizon T. Given the choice, an investor would prefer to choose T as large as possible.

The only choice of ζ which removes the dependence on the horizon T is $\zeta = -\frac{1}{2}\lambda^2$. In this case the solution becomes $M^T(t, x) = -\frac{1}{\gamma} e^{-\gamma x} e^{\frac{1}{2}\lambda^2 t}$. Under this choice, note further that

$M^T(t,x) = \tilde{U}(t,x)$, where $\tilde{U}$ is given in (A1). Thus we have shown

$$\tilde{U}(t,x) = \sup_{(\psi_u)_{t\le u\le T}} \mathbb{E}[\tilde{U}(T,\hat{X}_T)|\hat{X}_t = x] = \sup_T \sup_{(\psi_u)_{t\le u\le T}} \mathbb{E}[\tilde{U}(T,\hat{X}_T)|\hat{X}_t = x]$$

where the last equality holds since the solution did not depend on choice of horizon T.

We now extend this to random horizons, τ, by showing $\tilde{U}(t,\hat{X}_t)$ is a super-martingale in general and a martingale for the optimal ψ (recall $\tilde{U} \le 0$). It then follows that

$$\tilde{U}(t,x) = \sup_{(\psi_u)_{t\le u\le \tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)|\hat{X}_t = x] = \sup_\tau \sup_{(\psi_u)_{t\le u\le \tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)|\hat{X}_t = x]$$

where the second equality holds as the solution $\tilde{U}(t,x)$ does not depend on the random horizon τ.

Applying Itô's formula gives

$$d\tilde{U}(t,\hat{X}_t) = \frac{\tilde{U}(t,\hat{X}_t)}{2}\left[\lambda - \gamma\psi_t\sigma\right]^2 dt - \gamma\psi_t\sigma\tilde{U}(t,\hat{X}_t)dB_t$$

so $\tilde{U}(t,\hat{X}_t)$ is a super-martingale for any ψ. Integrating gives

$$\tilde{U}(\tau,\hat{X}_\tau) = \tilde{U}(t,\hat{X}_t) + \int_t^\tau \frac{\tilde{U}(s,\hat{X}_s)}{2}\left[\lambda - \gamma\psi_s\sigma\right]^2 ds - \int_t^\tau \gamma\psi_s\sigma\tilde{U}(s,\hat{X}_s)dB_s$$

so $\mathbb{E}\tilde{U}(\tau,\hat{X}_\tau) \le \tilde{U}(t,\hat{X}_t)$ for any ψ and using (A6),

$$\sup_{(\psi_u)_{t\le u\le \tau}} \mathbb{E}[\tilde{U}(\tau,\hat{X}_\tau)] = \tilde{U}(t,\hat{X}_t).$$

Hence $\tilde{U}(t,\hat{X}_t)$ is a super-martingale in general and a martingale for the optimal ψ.

Proof of Proposition 4:

The executive's optimization problem at an intermediate time t, and with $i \le n$ options remaining unexercised is to find

$$H^i(t,x,v) = \sup_{t\le\tau^i\le\ldots\le\tau^1} \sup_{(\theta_s)_{t\le s<\tau^1}} \mathbb{E}_t[\tilde{U}\left(\tau^1, X_{\tau^1}\right)|X_t = x, V_t = v]$$

where $\tilde{U}(t,x)$ is given in (10). Recall also that wealth X_{τ^1} includes the payoffs of all options exercised at times prior to and including date τ^1.

First define the time-independent function $G^i(x,v)$ via

$$G^i(x,v) = \sup_{t\leq\tau^i\leq\ldots\leq\tau^1} \sup_{(\theta_s)_{t\leq s<\tau^1}} \mathbb{E}_t \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2(\tau_1 - t) - \gamma X_{\tau^1}} \middle| X_t = x, V_t = v \right] \tag{A7}$$

Then

$$\begin{aligned} H^i(t,x,v) &= \sup_{t\leq\tau^i\leq\ldots\leq\tau^1} \sup_{(\theta_s)_{t\leq s<\tau^1}} \mathbb{E}_t \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2\tau_1 - \gamma X_{\tau^1}} \middle| X_t = x, V_t = v \right] \\ &= e^{\frac{1}{2}\lambda^2 t} \sup_{t\leq\tau^i\leq\ldots\leq\tau^1} \sup_{(\theta_s)_{t\leq s<\tau^1}} \mathbb{E}_t \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2(\tau_1 - t) - \gamma X_{\tau^1}} \middle| X_t = x, V_t = v \right] \\ &= e^{\frac{1}{2}\lambda^2 t} G^i(x,v) \end{aligned} \tag{A8}$$

and the problem reduces to finding $G^i(x,v)$ for $i \leq n$. Since $H^i(0,x,v) = G^i(x,v)$, the value today to the executive of the i remaining options is just $G^i(x,v)$ where $G^i(x,v)$ is a time-independent function.

We now proceed by backwards induction. We first solve the case where there is only one option remaining and obtain threshold $\tilde{V}^1$. We will label this section of the proof Part A. The problem with only one option is closely related to that studied in Henderson (2005b). The general form is proved by induction. We propose the form of the solution when $(i-1)$ options remain to be exercised, and then use this to solve for the case of i remaining unexercised options. The second, induction stage of the proof will be labeled Part B.

Part A: When there is only one option remaining unexercised, the value function $H^1(t,x,v)$ is given by

$$H^1(t,x,v) = \sup_{t\leq\tau^1} \sup_{\theta^1} \mathbb{E}_t \left[\tilde{U}\left(\tau^1, X_{\tau^1}\right) \middle| X_t = x, V_t = v \right]$$

where θ^1 denotes holdings in P between times t and τ^1. By (A8) and (A7), $H^1(t,x,v) =$

$e^{\frac{1}{2}\lambda^2 t}G^1(x,v)$ and

$$G^1(x,v) = \sup_{t\le\tau^1}\sup_{\theta^1}\mathbb{E}_t\left[-\frac{1}{\gamma}e^{\frac{1}{2}\lambda^2(\tau_1-t)-\gamma X_{\tau^1}}\middle| X_t = x, V_t = v\right].$$

By time homogeneity, we propose the exercise time as the first passage time of V to a constant threshold $\tilde{V}^1$ such that $\tau^1 = \inf\{t : V_t \ge \tilde{V}^1\}$. In the continuation region, $H^1(t,x,v) = e^{\frac{1}{2}\lambda^2 t}G^1(x,v)$ is a martingale under the optimal strategy θ^1, and the Bellman equation is given by

$$\frac{1}{2}\lambda^2 G^1 + \nu v G^1_v + \frac{1}{2}\eta^2 v^2 G^1_{vv} + \sup_{\theta^1}\left\{\theta^1\mu G^1_x + \frac{1}{2}\sigma^2(\theta^1)^2 G^1_{xx} + \theta^1\sigma\rho\eta v G^1_{xv}\right\} = 0 \tag{A9}$$

Optimizing over strategies gives

$$\theta^1 = \frac{-\lambda G^1_x - G^1_{xv}\rho\eta v}{\sigma G^1_{xx}}$$

and substituting back into the Bellman equation (A9) gives

$$0 = \frac{1}{2}\lambda^2 G^1 + \nu v G^1_v + \frac{1}{2}\eta^2 v^2 G^1_{vv} - \frac{1}{2}\frac{(\lambda G^1_x + \rho\eta v G^1_{xv})^2}{G^1_{xx}} \tag{A10}$$

with boundary condition $G^1(x,0) = -\frac{1}{\gamma}e^{-\gamma x}$. Value-matching at the exercise threshold gives

$$G^1(x,\tilde{V}^1) = -\frac{1}{\gamma}e^{-\gamma x}e^{-\gamma(\tilde{V}^1-K)^+}$$

and finally, smooth pasting gives $G^1_v(x,\tilde{V}^1) = e^{-\gamma x}e^{-\gamma(\tilde{V}^1-K)^+}$.

The problem is to solve (A10) subject to these three conditions. We first factor out wealth x by proposing $G^1(x,v) = -\frac{1}{\gamma}e^{-\gamma x}J^1(v)$ and setting $J^1(v) = (I^1)^{\frac{1}{1-\rho^2}}$, giving

$$0 = vI^1_v(\nu - \lambda\rho\eta) + \frac{1}{2}\eta^2 v^2 I^1_{vv}$$

with corresponding transformed conditions:

$$I^1(0) = 1 \tag{A11}$$

$$I^1(\tilde{V}^1) = e^{-\gamma(\tilde{V}^1-K)^+(1-\rho^2)} \tag{A12}$$

$$I^1_v(\tilde{V}^1)/I^1(\tilde{V}^1) = -\gamma(1-\rho^2)I_{\{\tilde{V}^1>K\}} \tag{A13}$$

Proposing a solution of the form $I^1(v) = Lv^{\psi}$ for some constant L results in the quadratic

$$\psi(\psi - 1)\eta^2/2 + \psi(\nu - \lambda\rho\eta) = 0$$

and we denote the non-zero root of the quadratic by β_ρ:

$$\beta_\rho = 1 - \frac{2(\nu - \lambda\rho\eta)}{\eta^2}.$$

The general solution is $I^1(v) = Lv^{\beta_\rho} + B$. Using (A11) gives $B = 1$. Now consider the remaining two conditions (A12) and (A13). If $\beta_\rho \leq 0$ then smooth pasting in (A13) fails and $L = 0$. In this case, the manager waits indefinitely. However, if $\beta_\rho > 0$, (A12) gives an expression for L and

$$I^1(v) = 1 - (1 - e^{-\gamma(1-\rho^2)(\tilde{V}^1 - K)^+}) \left(\frac{v}{\tilde{V}^1}\right)^{\beta_\rho}$$

Using (A13) gives the exercise threshold $\tilde{V}^1$ solves

$$\tilde{V}^1 - K = \frac{1}{\gamma(1-\rho^2)} \ln \left[1 + \frac{\gamma(1-\rho^2)}{\beta_\rho}\tilde{V}^1\right].$$

Finally, we obtain

$$G^1(x, v) = -\frac{1}{\gamma}e^{-\gamma x}\left[1 - (1 - e^{-\gamma(1-\rho^2)(\tilde{V}^1 - K)^+}) \left(\frac{v}{\tilde{V}^1}\right)^{\beta_\rho}\right]^{\frac{1}{1-\rho^2}}.$$

This completes the solution of the problem when only one option remains.

Part B: We will use the notation $\theta_s^i \equiv (\theta_s)_{\tau^{i+1} \leq s < \tau^i}$, where $\tau^{n+1} = 0$, to denote the holdings between exercise dates. Consider the problem when i options remain to be exercised. Observe that

$$
\begin{aligned}
H^i(t,x,v) &= \sup_{\tau^i \le \ldots \le \tau^1} \sup_{(\theta_s)_{t \le s < \tau^1}} \mathbb{E}_t \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2 \tau_1 - \gamma X_{\tau^1}} \middle| X_t = x, V_t = v \right] \\
&= \sup_{\tau^i \le \ldots \le \tau^1} \sup_{\theta^i,\ldots,\theta^1} \mathbb{E}_t \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2 \tau_1 - \gamma X_{\tau^1}} \middle| X_t = x, V_t = v \right] \\
&= \sup_{\tau^i \le \ldots \le \tau^1} \sup_{\theta^i,\ldots,\theta^1} \mathbb{E}_t \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2 \tau_1 - \gamma (X_t + \int_t^{\tau^1} \theta \frac{dP}{P} + \sum_{j=1}^{i} (V_{\tau^j} - K)^+)} \middle| X_t = x, V_t = v \right] \\
&= \sup_{\tau^i} \sup_{\theta^i} \mathbb{E}_t \Bigg[\sup_{\tau^{i-1} \le \ldots \le \tau^1} \sup_{\theta^{i-1},\ldots,\theta^1} \\
&\qquad \mathbb{E}_{\tau^i} \left[-\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2 \tau_1 - \gamma (X_{\tau^i} + \int_{\tau^i}^{\tau^1} \theta \frac{dP}{P} + \sum_{j=1}^{i-1} (V_{\tau^j} - K)^+)} \middle| X_{\tau^i}, V_{\tau^i} \right] \Bigg| X_t = x, V_t = v \Bigg] \\
&= \sup_{\tau^i} \sup_{\theta^i} \mathbb{E}_t \left[H^{i-1}(\tau^i, X_{\tau^i}, V_{\tau^i}) | X_t = x, V_t = v \right]
\end{aligned}
$$

Hence using (A8) we have

$$
G^i(x,v) = \sup_{\tau^i} \sup_{\theta^i} \mathbb{E} \left[e^{\frac{1}{2}\lambda^2 \tau^i} G^{i-1}(X_{\tau^i}, V_{\tau^i}) | X_0 = x, V_0 = v \right] \tag{A14}
$$

We now propose the form of the exercise thresholds $\tilde{V}^j$ and time independent functions $G^j(x,v)$ for all $j \le i-1$. Then we can use (A14) to solve for the value function $G^i(x,v)$ when i options remain. Suppose that for all $j \le i-1$,

$$
\tau^j = \inf\{t : V_t \ge \tilde{V}^j\}
$$

$$
G^j(x,v) = -\frac{1}{\gamma} e^{-\gamma x} \left[1 - (1 - e^{-\gamma(1-\rho^2)(\tilde{V}^j - K)^+} (1 - \Lambda^{j-1} (\tilde{V}^j)^{\beta_\rho})) \left(\frac{v}{\tilde{V}^j} \right)^{\beta_\rho} \right]^{\frac{1}{1-\rho^2}} ; v \le \tilde{V}_j \tag{A15}
$$

where Λ^j and $\tilde{V}^j$ are given in the statement of the proposition. Now we substitute (A15) into (A14) to solve for the function $G^i(x,v)$ when there are i options remaining. This gives $G^i(x,v) =$

$$
\begin{aligned}
&\sup_{\tau^i} \sup_{\theta^i} \mathbb{E} \left\{ -\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2 \tau_i} e^{-\gamma X_{\tau^i}} \left[1 - (1 - e^{-\gamma(1-\rho^2)(\tilde{V}^{i-1} - K)^+} (1 - \Lambda^{i-2} (\tilde{V}^{i-1})^{\beta_\rho})) \left(\frac{V_{\tau^i}}{\tilde{V}^{i-1}} \right)^{\beta_\rho} \right]^{\frac{1}{1-\rho^2}} \right\} \\
= \; &\sup_{\tau^i} \sup_{\theta^i} \mathbb{E} \left\{ -\frac{1}{\gamma} e^{\frac{1}{2}\lambda^2 \tau_i} e^{-\gamma (X_{\tau^i_-} + (V_{\tau^i} - K)^+)} \left[1 - \Lambda^{i-1} (V_{\tau^i})^{\beta_\rho} \right]^{\frac{1}{1-\rho^2}} \right\}.
\end{aligned}
$$

Again, by time homogeneity, we know the optimal τ^i is of the form $\tau^i = \inf\{t : V_t \ge \hat{V}^i\}$ for some constant $\hat{V}^i$. We now show $\hat{V}^i = \tilde{V}^i$ and that $G^i(x,v)$ is as stated in the proposition.

The ith remaining option is exercised if

$$G^i(x,v) = -\frac{1}{\gamma}e^{-\gamma(x+(v-K)^+)}[1-\Lambda^{i-1}v^{\beta_\rho}]^{\frac{1}{1-\rho^2}}$$

In the continuation region, $H^i(t,x,v) = e^{\frac{1}{2}\lambda^2 t}G^i(x,v)$ is a martingale under the optimal strategy θ^i and the (optimized) Bellman equation is:

$$0 = \frac{1}{2}\lambda^2 G^i + \nu v G^i_v + \frac{1}{2}\eta^2 v^2 G^i_{vv} - \frac{1}{2}\frac{(\lambda G^i_x + \rho\eta v G^i_{xv})^2}{G^i_{xx}} \tag{A16}$$

with boundary, value matching and smooth pasting conditions

$$G^i(x,0) = -\frac{1}{\gamma}e^{-\gamma x} \tag{A17}$$

$$G^i(x,\hat{V}^i) = -\frac{1}{\gamma}e^{-\gamma x}e^{-\gamma(\hat{V}^i-K)^+}[1-\Lambda^{i-1}(\hat{V}^i)^{\beta_\rho}]^{\frac{1}{1-\rho^2}} \tag{A18}$$

$$\begin{aligned} G^i_v(x,\hat{V}^i) = \frac{1}{\gamma}e^{-\gamma x}\Big[&\gamma(1-\Lambda^{i-1}(\hat{V}^i)^{\beta_\rho})^{\frac{1}{1-\rho^2}}e^{-\gamma(\hat{V}^i-K)^+} \\ &+ \frac{\beta_\rho\Lambda^{i-1}(\hat{V}^i)^{\beta_\rho-1}}{1-\rho^2}e^{-\gamma(\hat{V}^i-K)^+}[1-\Lambda^{i-1}(\hat{V}^i)^{\beta_\rho}]^{\frac{1}{1-\rho^2}-1}\Big] \end{aligned} \tag{A19}$$

Notice the Bellman equation (A16) is identical to (A10) in Part A, but we have a different set of conditions to satisfy. The same approach is used as in Part A, proposing solution $G^i(x,v) = -\frac{1}{\gamma}e^{-\gamma x}J^i(v)$ and $J^i(v) = (I^i)^{\frac{1}{1-\rho^2}}$. Again, we are led to a quadratic with non-zero root β_ρ and using (A17) gives solution $J^i(v) = Lv^{\beta_\rho} + 1$.

Again, the smooth pasting condition (A19) requires $\beta_\rho > 0$ for a solution. Provided $\beta_\rho > 0$, (A18) gives an expression for the constant L and

$$G^i(x,v) = -\frac{1}{\gamma}e^{-\gamma x}\left[1-\left(1-e^{-\gamma(1-\rho^2)(\hat{V}^i-K)^+}(1-\Lambda^{i-1}(\hat{V}^i)^{\beta_\rho})\right)\left(\frac{v}{\hat{V}^i}\right)^{\beta_\rho}\right]^{\frac{1}{1-\rho^2}}.$$

The smooth pasting condition (A19) now gives $\hat{V}^i$ solves

$$\hat{V}^i - K = \frac{1}{\gamma(1-\rho^2)}\ln\left(1+\frac{\gamma(1-\rho^2)}{\beta_\rho}(1-\Lambda^{i-1}(\hat{V}^i)^{\beta_\rho})\hat{V}^i\right)$$

and hence $\hat{V}^i = \tilde{V}^i$ and $G^i(x,v)$ is as stated in the proposition.

Proof of Proposition 1:

This can be proved as a special case of Proposition 4 by taking $\rho = 0$, $\lambda = 0$ in the proof. Note we recover $\tilde{U}(t,x) = U(x)$ in this case since there is no risky asset in which to invest. The proposition can also be proved directly using the same approach.

Proof of Corollary 3:

We take the limit as $\gamma \to 0$ in the equations of Proposition 1. As $\gamma \to 0$, $\Gamma^j \approx 0, j = 1, ..., n-1$. Then (5) gives for $j = 1, ..., n$,

$$(\tilde{V}^j - K) \approx \frac{1}{\gamma} \ln \left(1 + \frac{\gamma}{\beta_0} \tilde{V}^j\right) \approx \tilde{V}^j / \beta_0$$

giving $\lim_{\gamma \downarrow 0} \tilde{V}^j = \tilde{V}_{\langle \gamma = 0 \rangle}$, $j = 1, ..., n$.

Proof of Propositions 5 and 2:

We prove Proposition 5. Proposition 2 follows immediately by taking $\rho = 0$ giving $\Lambda^j = \Gamma^j(\gamma, \beta_0, K)$ in proof below.

We prove thresholds $\tilde{V}^j$ are decreasing in j, for $j = 1, ..., n$. Let $a = \gamma(1 - \rho^2) > 0$ and $\beta = \beta_\rho > 0$ to simplify the notation of the proof. From Proposition 4 we have that the thresholds $\tilde{V}^j$, $j = 1, ..., n$ solve

$$\tilde{V}^j - K = \frac{1}{a} \ln \left(1 + \frac{a}{\beta}(1 - \Lambda^{j-1}(\tilde{V}^j)^\beta)\tilde{V}^j\right) \tag{A20}$$

where

$$\Lambda^0 = 0, \quad \Lambda^j = \frac{1}{(\tilde{V}^j)^\beta}\left[1 - e^{-a(\tilde{V}^j - K)^+}(1 - \Lambda^{j-1}(\tilde{V}^j)^\beta)\right], \quad j = 1, ..., n-1 \tag{A21}$$

Define $A_j = \Lambda^{j-1}(\tilde{V}^j)^\beta$, $\;j = 2, ..., n$ so that $A_1 = 0$. Define also $E_j = \frac{1}{1-A_j}$, $\;j = 1, ..., n$ so that $E_1 = 1$. Since our interest is in solutions for which $\tilde{V}^j > K$, the R.H.S of (A20) is positive, and hence $A_j < 1$, $E_j > 1$ for all $j = 2, ..., n$.

Using these definitions, we rewrite (A20) as

$$\tilde{V}^j - K = \frac{1}{a} \ln \left(1 + \frac{a}{\beta} \frac{\tilde{V}^j}{E_j}\right) \tag{A22}$$

From (A22) we see for $j = 1, ..., n$, $\tilde{V}^j$ are decreasing in j if and only if E_j are increasing in j. (The function $f(x) = \frac{1}{a}\ln(1 + \frac{a}{\beta}\frac{\tilde{V}}{x})$ is decreasing in x for fixed threshold level $\tilde{V}$).

From (A21),

$$A_j = \left(\frac{\tilde{V}^j}{\tilde{V}^{j-1}}\right)^\beta \left\{1 - e^{-a(\tilde{V}^{j-1}-K)^+}(1 - A_{j-1})\right\}$$

and from (A20),

$$e^{-a(\tilde{V}^{j-1}-K)^+} = \frac{1}{1 + \frac{a}{\beta}(1 - A_{j-1})\tilde{V}^{j-1}}$$

Putting these together we obtain

$$\left(1 - \frac{1}{E_j}\right)(\tilde{V}^{j-1})^\beta = (\tilde{V}^j)^\beta\left(1 - \frac{1}{E_{j-1} + \frac{a}{\beta}\tilde{V}^{j-1}}\right). \tag{A23}$$

Now suppose $\tilde{V}^j \geq \tilde{V}^{j-1}$. Then from (A23),

$$1 - \frac{1}{E_j} \geq 1 - \frac{1}{E_{j-1} + \frac{a}{\beta}\tilde{V}^{j-1}}$$

and $E_j \geq E_{j-1} + \frac{a}{\beta}\tilde{V}^{j-1}$ so $E_j > E_{j-1}$. But from earlier, this implies $\tilde{V}^j < \tilde{V}^{j-1}$ which is a contradiction. Therefore $\tilde{V}^j < \tilde{V}^{j-1}$.

Proof of Proposition 7:

This is a straightforward modification of Part A of the proof of Proposition 4. The only difference is that here there are i options to be exercised at one time. The factor of i multiplies the option payoff throughout and results in the given solution.

Proof of Proposition 8:

From Proposition 7, we have the thresholds $\tilde{V}_r^i$ for $i = 1, ..., n$ solve

$$\tilde{V}_r^i - K = \frac{1}{ia}\ln\left(1 + \frac{ia}{\beta_\rho}\tilde{V}_r^i\right)$$

where $a = \gamma(1 - \rho^2)$. Differentiation of the above with respect to i gives $\frac{\partial \tilde{V}_r^i}{\partial i} < 0$ provided (i) $\beta_\rho + ia\tilde{V}_r^i > 1$ and, (ii)

$$\ln\left(1 + \frac{ia}{\beta_\rho}\tilde{V}_r^i\right) > 1 - \frac{\beta_\rho/a}{\beta_\rho/a + i\tilde{V}_r^i}.$$

Note for $x > 1$, $\ln(x) > 1 - 1/x$. Taking $x = 1 + \frac{ia}{\beta_\rho}\tilde{V}_r^i = \frac{\beta_\rho + ia\tilde{V}_r^i}{\beta_\rho}$ gives (ii). For (i), consider two cases. If $\beta_\rho \geq 1$, then (i) is immediate. Now consider $\beta_\rho < 1$. Let $\tilde{Y} = ia\tilde{V}_r^i$. Then $\tilde{Y}$ solves $\tilde{Y} - iaK = \ln(1 + \tilde{Y}/\beta_\rho)$. To prove (i) we need to show $\tilde{Y} > 1 - \beta_\rho$. This follows if $(1 - \beta_\rho) - iaK < \ln(1/\beta_\rho)$. It is sufficient that $(1 - \beta_\rho) < \ln(1/\beta_\rho)$ which we can show by taking $x = 1/\beta_\rho$ in the above.

Proof of Proposition 9

When $k = 1$, by definition $p = \emptyset$, and letting $q_1 = l$, it is immediate to verify that Proposition 9 gives the solution in (11) which was easily obtained via Proposition 7 with modified strike, $K + c/l$. When $k > 1$, we work backwards as in Proposition 4.

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