Financial Innovation in Segmented Markets

by

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1   Introduction

(To be written)

2   The Setup

We consider a two-period economy with uncertainty parametrized by finitely many states of the world, $s = 1, \ldots, S$. Assets are traded in several locations or "exchanges". One of these exchanges, labeled c, has complete markets. This is the exchange on which the underlying, or fundamental, assets are traded. The remaining exchanges, labeled $k = 1, \ldots, K$, may have (locally) incomplete markets.

Investor i on exchange ℓ ($\ell = c, k$) has endowments of $(\omega_0^{\ell,i}, \omega^{\ell,i}) \in \mathbb{R}_+ \times \mathbb{R}_+^S$, and preferences which allow a quasilinear quadratic representation

$$U^{\ell,i}(x_0, \{x_s\}) = x_0 + \sum_{s=1}^{S} \pi_s \left[x_s - \frac{1}{2}\beta^{\ell,i} x_s^2 \right],$$

where x_0 is consumption at date 0, x_s is consumption in state s at date 1, and π_s is the probability (common across agents) of state s. The coefficient $\beta^{\ell,i}$ is positive. Investors are price-taking and are assumed to be able to trade on their own exchange only. There are I^ℓ investors on exchange ℓ.

In addition there are N arbitrageurs who possess the trading technology which allows them to also trade across exchanges. For simplicity, we assume that arbitrageurs only care about time zero consumption. Arbitrageurs are imperfectly competitive.

Asset payoffs on exchange ℓ are given by a full rank payoff matrix R^ℓ of dimension $S \times J^\ell$. Since exchange c has complete markets, Cournot-Walras equilibria (a formal definition is given in the next section) of our economy converge to Walrasian equilibria with restricted participation (see Zigrand (2002)). But then it is without loss of generality to assume that $R^c = I_{S \times S}$. Arbitrageurs may trade all the assets on exchange c, and a subset of assets on the remaining exchanges. For convenience we partition the asset payoff matrix R^k as follows: $R^k = (R^{k_1} \quad R^{k_2})$, where R^{k_1} consists of the J^{k_1} "local" assets and R^{k_2} represents the remaining J^{k_2} "arbitraged" assets. Thus we have $J^{k_1} + J^{k_2} = J^k$.

3   Cournot-Walras Equilibria

Definition 1 *Given an asset structure $\{R^\ell\}$, a Cournot-Walras equilibrium (CWE) of the economy is an array of asset prices, portfolios, and arbitrageur supplies, $\{q^\ell, \theta^{\ell,i}, y^{\ell,n}\}$, $(\ell = c, k; \; k = 1, \ldots, K; \; i = 1, \ldots, I^\ell; \; n = 1, \ldots, N)$, such that*

1. *Investor optimization: $\theta^{\ell,i}$ solves*

$$\max_{\theta^{\ell,i} \in \mathbb{R}^{J_\ell}} x_0^{\ell,i} + \sum_s \pi_s \left[x_s^{\ell,i} - \frac{\beta^{\ell,i}}{2}(x_s^{\ell,i})^2 \right]$$

$$s.t. \;\; x_0^{\ell,i} = \omega_0^{\ell,i} - q^\ell \cdot \theta^{\ell,i}$$
$$x_s^{\ell,i} = \omega_s^{\ell,i} + (R^\ell \theta^{\ell,i})_s, \qquad \forall s.$$

2. *Arbitrageur optimization: $y^{\ell,n}$ solves*

$$\max_{y^{k,n} \in \mathbb{R}^{J^{k_2}}, y^{c,n} \in \mathbb{R}^S} \sum_k q^{k_2} \cdot y^{k,n} + q^c \cdot y^{c,n}$$

$$s.t. \;\; \sum_k R^{k_2} y^{k,n} + y^{c,n} \le 0.$$

3. *Market clearing:*

$$\sum_i \theta^{\ell,i} = \begin{pmatrix} 0 \\ \sum_n y^{\ell,n} \end{pmatrix}, \qquad \forall \ell.$$

It is implicit here that investors take asset prices as given, while arbitrageurs compete Cournot-style. Arbitrageurs maximize time zero consumption, i.e. profits from their arbitrage trades, but subject to the restriction that they are not allowed to default in any state at date 1. Equivalently, arbitrageurs need to be completely collateralized.

The first order condition for the investor's optimization problem gives us his asset demand function:

$$\theta^{\ell,i} = (\beta^{\ell,i} R^{\ell\top} \Pi R^\ell)^{-1}[R^{\ell\top} \Pi(\mathbf{1} - \beta^{\ell,i}\omega^{\ell,i}) - q^\ell],$$

where $\Pi := \text{diag } (\pi_1, \ldots, \pi_S)$ and $\mathbf{1} := (1, \ldots, 1)^\top$. We can now use the market clearing condition to deduce the price vector on exchange ℓ that

sets aggregate demand, $\sum_i \theta^{\ell,i}$, to zero for local assets and to the aggregate arbitrage supply, $y^\ell := \sum_n y^{\ell,n}$, for arbitraged assets:

$$q^k = R^{k\top}\Pi[\mathbf{1} - \beta^k\omega^k - \beta^k R^{k_2}y^k], \qquad k = 1, \ldots, K \tag{1}$$

$$q^c = \Pi[\mathbf{1} - \beta^c\omega^c - \beta^c y^c] \tag{2}$$

where $\beta^\ell := [\sum_i(\beta^{\ell,i})^{-1}]^{-1}$, and $\omega^\ell := \sum_i \omega^{\ell,i}$. Note that $(\mathbf{1} - \beta^\ell\omega^\ell)$ is exchange ℓ's autarky state price deflator. The parameter β^ℓ, which is equal to the harmonic mean of $\{\beta^{\ell,i}\}_{i=1}^{I^\ell}$, influences the "depth" of exchange ℓ, i.e. the price impact of a unit of arbitrageur trading. For instance, if $\beta^{\ell,i}$ is the same for all agents i, the larger is I^ℓ, the lower is β^ℓ.Consequently, the market impact of a trade is smaller—it can be absorbed by more investors.

It is also worth pointing out that our assumptions on preferences, in conjunction with the absence of nonnegativity constraints on consumption, guarantee that the equilibrium pricing function on an exchange does not depend on the initial distribution of endowments, but merely on the aggregate endowment of the local investors. Furthermore, prices of arbitraged assets are not affected by the payoffs of local assets. Thus the characteristics of locally traded assets have no bearing on arbitrage trades or on security design by arbitrageurs. Henceforth we will assume that all assets are arbitraged.

We now solve the Cournot game among arbitrageurs, given the demand functions (1) and (2). We focus on symmetric equilibria, i.e. equilibria in which $y^{k,n}$ is the same for all n. It turns out that the complete markets assumption on exchange c guarantees that arbitrageur n is able to choose asset supplies that generate exactly zero payoff tomorrow in all states. Let us denote the Lagrange multipliers attached to the no-default constraint in state s by ν_s, and the vector of multipliers by $\nu \in \mathbb{R}_+^S$. The first-order conditions for $y^{k,n}$ give us

$$y^{k,n} = \frac{1}{(1+N)\beta^k}(R^{k\top}\Pi R^k)^{-1}R^{k\top}\Pi\left[\mathbf{1} - \beta^k\omega^k - \Pi^{-1}\nu\right] \tag{3}$$

and

$$y^{c,n} = \frac{1}{(1+N)\beta^c}[\mathbf{1} - \beta^c\omega^c - \Pi^{-1}\nu] \tag{4}$$

which, together with

$$\sum_{k=1}^K R^k y^{k,n} + y^{c,n} = 0 \tag{5}$$

we can solve for ν. In order to characterize equilibrium arbitrage supplies, it is useful to introduce some more notation. Let

$$P^k := R^k (R^{k\top} \Pi R^k)^{-1} R^{k\top} \Pi.$$

Since P^k is idempotent, it is a projection from $\mathbb{R}^S$ onto the asset span $\langle R^k \rangle$. The angle of the projection is determined by the probabilities Π. It is convenient to state arbitrageur supplies in terms of the supply of state-contingent consumption:

Lemma 1 *Equilibrium arbitrageur supplies, for asset structure $\{R^k\}$, are given by*

$$R^k y^{k,n} = \frac{1}{(1+N)\beta^k} \cdot P^k \left(\xi^k - \sum_{j=1}^{K} \Lambda^j \xi^j \right) \tag{6}$$

$$y^{c,n} = -\sum_k R^k y^{k,n} \tag{7}$$

where

$$\xi^k := \beta^c \omega^c - \beta^k \omega^k$$

$$\Lambda^j := \left[\frac{1}{\beta^c} I + \sum_k \frac{1}{\beta^k} P^k \right]^{-1} \frac{1}{\beta^j} P^j.$$

If $R^k = R$, for all k,

$$R y^{k,n} = \frac{1}{(1+N)\beta^k} \cdot P(\xi^k - \xi^\lambda) \tag{8}$$

where

$$\xi^\lambda := \sum_{j=1}^{K} \lambda^j \xi^j$$

$$\lambda^j := \frac{\frac{1}{\beta^j}}{\frac{1}{\beta^c} + \sum_k \frac{1}{\beta^k}}.$$

If the arbitrageur trades on only two exchanges k and c,

$$R^k y^{k,n} = \frac{1}{(1+N)(\beta^k + \beta^c)} \cdot P^k \xi^k. \tag{9}$$

For intuition, focus first on the two exchange case. The supply of state-contingent consumption is proportional (with factor of proportionality $\frac{1}{(1+N)(\beta^k+\beta^c)}$) to the projection under P^k of ξ^k on the span of R^k. The vector ξ^k is the difference between the autarky state price deflators of exchanges k and c. In other words, it is the excess willingness to pay on exchange k for state-contingent consumption. We can think of it as a measure of the potential gains from trade between the two exchanges. The more useful the securities are in exploiting these gains from trade (higher $P^k\xi^k$), the greater is the demand for assets by investors, and the bigger correspondingly is the supply from arbitrageurs. The factor of proportionality is determined by two considerations. First, the deeper markets are (i.e. the lower $\beta^k + \beta^c$), the more arbitrageur n trades, since he can afford to augment his supply without affecting margins as much. And second, the supply vector is scaled to zero as competition intensifies, because the whole pie shrinks and there are more players to share the smaller pie with.

Let us now turn to the case of multiple exchanges. In determining the supply of consumption to exchange k, arbitrageurs not only need to consider the gains from trade between c and k, but also the gains from trade between c and other exchanges $k' \neq k$ on the one hand, and between k and $k' \neq k$ on the other. The most transparent case arises when all the "small" exchanges have the same payoff matrix R. The term γ^j can be interpreted as the coefficient of relative depth of exchange j vis-á-vis the total depth. We can clearly see from equation (8) that arbitrageurs supply state-contingent consumption to an exchange k when the price that agents on exchange k are willing to pay for it in excess of the price on c is higher than the average excess willingness to pay, where the average weighs the willingness to pay on each excahnge by its relative depth. This statement should be qualified, since while these are the "optimal" supplies in some sense (to be confirmed subsequently), they may not be in the span of the existing assets. Therefore, arbitrageurs will supply consumption if the excess willingness to pay, when projected onto the span of the permissible assets, is positive. Alternatively, the direction of flows between k and c depends on the magnitude of the gains from trade between k and c relative to the gains from trade between other exchanges k' and c.

It is interesting to note that even though we did not attribute any special role to exchange c other than to assume that it has complete markets (it could even be a very shallow exchange), the algebra suggests that we view it

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as a "hub" or "central exchange" with respect to which gains from trade are defined. For instance, a feasible reallocation of resources between exchanges k and k' can be viewed as two separate reallocations, each of which is in the span of c due to its complete markets : one between k and c and the opposite one between c and k'. These two reallocations are neutral on exchange c since they do not affect prices or allocations on that exchange.

We can give a less centralized interpretation as well. Noting that $\xi^c = 0$, we rewrite equation (8) as follows:

$$Ry^{k,n} = \frac{1}{(1+N)\beta^k} \cdot P \left[\sum_{j \in \{c,1,...,K\}} \lambda^j (\xi^k - \xi^j) \right].$$

Since ξ^k represents the gains from trade between exchanges k and c, $\xi^k - \xi^j$ represents the gains from trade between exchanges k and j. Indeed, $\xi^k - \xi^j$ is the difference between the state price deflators of exchanges k and j, and therefore reflects the overpricing of assets on exchange k compared to j. The expression in square brackets then represents the total gains from trade of k with all other exchanges, and looks a bit less centered on c. Again we can address the issue as to when the arbitrageur supplies state-contingent consumption to k. The condition $\xi^k - \xi^{k'} > 0$ (for some k') is clearly not sufficient, since there could be a third exchange j whose ξ^j is large enough to warrant that arbitrageurs take consumption away from both k and k' in order to transfer it to j. Above we saw that it depends on the magnitude of the gains from trade between k and c relative to the gains from trade of all other exchanges with c. Alternatively, it depends on the tradable component of the total contribution to the depth-adjusted gains from trade.

The general case with multiple exchanges offering distinct payoff matrices yields a similar conclusion, but there one also has to consider that distinct return matrices generate more general arbitrage opportunities that do not reduce to violations of the law of one price, as was the case in the discussion above. Each gain from trade has to be projected on its own payoff matrix. This implies that the no-default conditions of arbitrageurs are not mechanically satisfied as they are when the law of one price is violated. Given these caveats, we can provide the same interpretation of (6) as of equation (8). The relevant gains from trade with c are computed using the weighting matrices Λ^j, which perform the adjustment for depth and at the same time ensure that the no-default conditions are satisfied.

Lemma 1 gives us the equilibrium supply $y^{\ell,n}$ of arbitrageur n. The total equilibrium supply is then $y^\ell := Ny^{k,n}$. Substituting into the pricing equations (1) and (2) determines the equilibrium prices.

4   Optimal Security Design by Arbitrageurs

We have seen in the previous section that there is a unique CWE associated with any asset structure $\{R^k\}_{k=1}^K$. In this section we endogenize the security payoffs. Arbitrageurs play a security design game the outcome of which is an equilibrium asset structure. The payoffs of arbitrageurs are the profits they earn in the CWE associated with this asset structure. The asset structure $\{R^k\}$ is a Nash equilibrium of the security design game if no arbitrageur stands to gain by introducing additional assets that he may trade monopolistically (clearly this is also a Nash equilibrium in the associated game in which all arbitrageurs trade the additional securities). Thus the complete asset structure $R^k = I$, for all k, is a Nash equilibrium. We say that a Nash equilibrium is minimal if there is no asset structure with fewer assets that leads to the same profits for arbitrageurs. Finally, we say that an asset structure is optimal for arbitrageurs if it yields the highest profits for arbitrageurs in the associated CWE, among all possible asset structures.

Proposition 1 *Suppose $K < S$. The asset structure*

$$R^k = \xi^k - \xi^\lambda, \qquad k = 1, \ldots, K$$

is

1. *the unique minimal optimal asset structure for arbitrageurs; and*

2. *the unique minimal Nash equilibrium of the security design game.*

A reading of the optimal arbitrage supply (6) indeed suggests that the optimal security design for arbitrageurs should be as given in the proposition (it turns out that, for the optimal security design, the "netting" of gains from trade using the weighting matrices $\{\Lambda^j\}$, yields the same result as netting with the coefficients $\{\lambda^j\}$). Since the arbitrageurs' supply of state-contingent consumption is proportional to $\xi^k - \xi^\lambda$, projected down onto the span of R^k, i.e. they choose the supply closest to $\xi^k - \xi^\lambda$, it is clear that the optimal span should be exactly $\xi^k - \xi^\lambda$. In this sense, the optimal asset structure

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exploits all the existing gains from trade between the exchanges. It should be remarked that a single security on each exchange suffices for the arbitrageur to maximize his profits. On exchange k, for instance, an arbitrageur is only concerned with the one-dimensional net trade he mediates between k and c, which can be accomplished via a single security collinear with the desired net trade.

5 Concentration of Arbitrage Activity

So far we have assumed that all arbitrageurs trade the same set of securities on all exchanges. In this section we adopt a different viewpoint. An arbitrageur may introduce securities and trade on only one exchange k. As before, all arbitrageurs trade on the complete exchange c, and all arbitrageurs on a given exchange k trade the same set of securities on that exchange. However, arbitrageurs on k cannot trade on $k' \neq k$. To distinguish these two arbitraging scenarios, we refer to the situation described in the previous section as "universal arbitrage" and the case we are about to study as "restricted arbitrage".

Let N^k be the number of arbitrageurs on exchange k, with $\sum_k N^k = N$. The pricing functions (1) and (2) still apply. Equilibrium supplies are as follows:

Lemma 2 *Equilibrium arbitrageur supplies, for asset structure $\{R^k\}$ and distribution of arbitrageurs $\{N^k\}$, are given by*

$$R^k y^{k,n} = \frac{1}{(1+N^k)\beta^k + \beta^c} \cdot P^k \left(\xi^k - \sum_{j=1}^{K} \Gamma^j \xi^j \right)$$

$$y^c = -\frac{1}{\beta^c} \sum_{j=1}^{K} \Gamma^j \xi^j,$$

where

$$\Gamma^j := \left[\frac{1}{\beta^c} I + \sum_k \frac{N^k}{(1+N^k)\beta^k + \beta^c} P^k \right]^{-1} \frac{N^j}{(1+N^j)\beta^j + \beta^c} P^j.$$

If $R^k = R$, for all k,

$$Ry^{k,n} = \frac{1}{(1+N^k)\beta^k + \beta^c} \cdot P(\xi^k - \xi^\gamma)$$

where

$$\xi^\gamma := \sum_{j=1}^{K} \gamma^j \xi^j$$

$$\gamma^j := \frac{\frac{N^j}{(1+N^j)\beta^j+\beta^c}}{\frac{1}{\beta^c} + \sum_k \frac{N^k}{(1+N^k)\beta^k+\beta^c}}.$$

The interpretation of arbitrage supplies in this case is analogous to that of Lemma 1. The weights used for netting the gains from trade are different since trades between exchanges k and k' are not mediated by the same set of arbitrageurs. The exchanges are connected nevertheless through the complete exchange on which all arbitrageurs trade. Note that the weight γ^j is increasing in N^j, the number of arbitrageurs active on exchange j.

We are now ready to address questions relating to security design. For the moment we keep the distribution of arbitrageurs fixed. Then we have the following analogue of Proposition 1:

Proposition 2 *Given $\{N^k\}$, the asset structure*

$$R^k = \xi^k - \xi^\gamma, \qquad k = 1, \dots, K$$

is

1. *the unique minimal optimal asset structure for arbitrageurs; and*

2. *the unique minimal Nash equilibrium of the security design game.*

The next step is to endogenize the concentration of arbitrage activity $\{N^k\}$. To illuminate the tradeoffs involved in the choice of exchange by an arbitrageur, it is useful to first answer the following question: If all arbitrageurs had to choose the *same* exchange on which to trade, which exchange would they prefer?

Proposition 3 *Suppose arbitrageurs introduce securities and trade on the same exchange. Then the optimal exchange k^* (with optimal asset structure ξ^{k^*}) is given by*

$$k^* = \arg\max_k \ (\beta^c + \beta^k)^{-\frac{1}{2}} \cdot \|\beta^c \omega^c - \beta^k \omega^k\|_2,$$

where $\| \cdot \|_2$ is the L_2-norm. In particular, if $\beta^k = \beta^c$ for all k, then

$$k^* = \arg\max_k \|\omega^c - \omega^k\|_2.$$

If $\omega^k = \omega^c$ for all k, then

$$k^* = \arg\max_k \frac{(\beta^c - \beta^k)^2}{\beta^c + \beta^k}.$$

Consider for example the last situation. Exchanges $k = 1, \ldots, K$ differ only with respect to their preference parameters $\{\beta^k\}$. The function $f(\beta^k) := \frac{(\beta^c - \beta^k)^2}{\beta^c + \beta^k}$ is depicted in Figure 1.
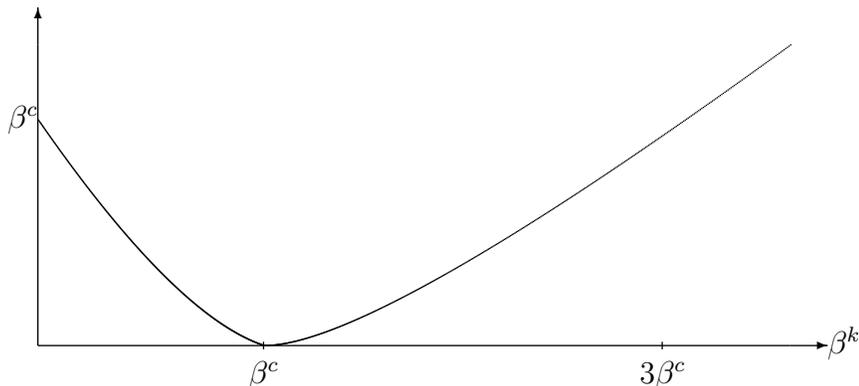

Figure 1: **The objective function of arbitrageurs**

We see that the slope of f to the left of β^c is steeper than the slope to its right, and since the domain of f is assumed to be a convex subset of $\mathbb{R}_+$, if $\beta^k > 3\beta^c$ is in the domain, the solution is always to choose exchange k. Similarly, if $3\beta^c$ is not in the domain, but a k for which β^k is close enough to zero, then k is always chosen. The reason why the optimal exchange will, depending on the domain, be either the k with the smallest or the largest β^k is as follows. $\beta^c - \beta^k$ measures the extent of the unutilized gains from trade. So gains from trade are largest the furthest from β^c the new exchange is located. But β^k also determines the shallowness of the exchange, the denominator of the expression in the proposition. The slope to the left of β^c is steeper since markets with low β^k are deeper, and therefore more attractive to arbitrageurs. Thus there is a tradeoff between gains from trade and depth.

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The best exchange may not be the deepest one, since the restriction $\beta^k \geq 0$ limits the gains from trade, in which case a very shallow exchange may be preferred to the deepest one if the gains from trade are sufficiently large to compensate for the shallowness.

From Lemma 2, we can determine the equilibrium level of profits of an arbitrageur on exchange k, for given $\{N^k\}$:

$$\Phi^k := \frac{\beta^k + \beta^c}{[(1 + N^k)\beta^k + \beta^c]^2} \cdot \left(\|\xi^k - \xi^\gamma\|_2\right)^2. \tag{10}$$

Ignoring integer constraints on the number of arbitrageurs, in equilibrium we must have $\Phi^k = \overline{\Phi}$, for all k on which there is some arbitrage activity. Since profits on an exchange go to zero as the number of arbitrageurs on that exchange tends to infinity, all exchanges attract arbitrage activity for N sufficiently large. The profit Φ^k is a product of two terms. The first term is decreasing in β^k, and therefore increasing in the depth of exchange k, while the second term captures the net gains from trade on exchange k. Consequently, there there is a tradeoff between depth and gains from trade.

Proposition 4   *1. If $N^k = N^{k'} > 0$, then*

$$\beta^k > \beta^{k'} \qquad iff \qquad \|\xi^k - \xi^\gamma\|_2 > \|\xi^{k'} - \xi^\gamma\|_2.$$

2. If $\beta^k = \beta^c$, for all k, the maximal N^k is on the exchange k^* that solves

$$k^* = \arg\max_k \|\xi^k - \xi^\gamma\|_2$$

$$= \arg\max_k \|(\omega^c - \omega^k) - \sum_{j=1}^K \frac{N^j}{2 + N^j}(\omega^k - \omega^j)\|_2.$$

6   Socially Optimal Security Design

Associated with an asset structure $\{R^k\}$ (and, in the case of restricted arbitrage, also $\{N^k\}$), there is a unique CWE with the corresponding equilibrium payoffs for each arbitrageur and investor. We say that an asset structure is socially optimal if there is no alternative asset structure that Pareto dominates it in equilibrium. Not surprisingly in view of the findings in Propositions 1 and 2, that indicated that the optimal arbitrageur-chosen securities maximize the exploitation of gains from trade, we find:

Proposition 5 *Suppose there are identical agents within each exchange. Then, whether there is universal or restricted arbitrage, the minimal optimal asset structure for arbitrageurs (which is also the minimal Nash equilibrium of the security design game) is socially optimal.*

While it is clear that K securities are sufficient to span the optimal trades between $K + 1$ agents, it is interesting to notice that the optimal security design only depends on the autarky equilibrium and does not depend on the amount supplied by arbitrageurs. For clarity we discuss the case in which there are only two exchanges k and c, so that there is a single optimal security ξ^k. (As we have remarked earlier, the case of multiple exchanges only introduces the additional complication of netting of trades.) In general, it is well-known that at an initial equilibrium the state-price deflator evaluated at that equilibrium is *locally* the most valued security for an agent, i.e. for an infinitesimal asset supply. There are two aspects to notice. First, while in general the state price deflator at the initial equilibrium is no longer the most valued security at the new equilibrium, here the autarky state prices determine the best security even at the new equilibrium. And second, this does not imply that the *difference* of state-price deflators is the optimal security, for one might think that while getting the payoffs of one's own state-price deflator may be the most valued security, it is not obvious that shorting the payoffs of the other exchange's most valued security is still part of the payoff of the optimal security. But in our setting it must be for we showed that one security is sufficient to reach the first best. Intuitively, assume that the two autarky state-price deflators are introduced on each of the two exchanges. Then the investors on k want to go long on theirs, but this requires somebody else to be the counterparty at an equilibrium, and the same applies to the investors on exchange c. Since agents are not required to short the state-price deflator of the other exchange, but are induced to do so by the prices, it follows that the difference represents the net transfers, and is exactly the right security.

Even though the securities correspond to the socially desirable ones, the equilibrium *allocation* is not socially optimal. Arbitrageurs are strategic and restrict their asset supplies in order to benefit from the markup. This implies that not all gains from trade are exhausted. It is only when the number of arbitrageurs N tends to infinity that allocations converge to Walrasian allocations, and therefore to a constrained Pareto optimum.

Proposition 5 need not hold if agents are heterogeneous. Take exchange k, for instance. Assume that J^k, the number of assets that can be introduced on k, is exogenously fixed and is at most $S-1$ (otherwise the optimal asset asset structure for investors on this exchange is the entire array of Arrow securities). We state the following proposition for the case of universal arbitrage; an analogous result holds for restricted arbitrage.

Proposition 6 *An optimal security design for agents on exchange k is the one for which the asset span is generated by the $\min(J^k, I^k)$ maximal eigenvectors of $\sum_{i=1}^{I^k} c^{k,i} [\zeta^{k,i} \zeta^{k,i \top} \Pi]$, for some positive weights $\{c^{k,i}\}$, where*

$$\zeta^{k,i} := (\beta^k \omega^k - \beta^{k,i} \omega^{k,i}) + \frac{N}{1+N} \cdot \left(\xi^k - \sum_{j=1}^{K} \Lambda^j \xi^j \right).$$

If there is no heterogeneity among agents, then $\beta^k \omega^k = \beta^{k,i} \omega^{k,i}$, and the optimal security is $\xi^k - \xi^\lambda$, which is optimal for the arbitrageurs also as seen in Proposition 5. The reason why the arbitrageurs' innovation fails to be socially optimal is the fact that arbitrageurs only cares about the aggregate valuations on the various exchanges, and they do not consider the effects of their security choice on the intra-exchange reallocations of resources that occur when investors on exchange k use the security among themselves, other than when this affects the aggregate outcome. For instance, assume that $c^{k,1} = 1$ and $c^{k,i} = 0$, $i \neq 1$. Then the socially optimal security is $\zeta^{k,1}$. Notice that the higher is N, the more closely aligned are the interests of investors and arbitrageurs.

7 Conclusion

(To be written)

A Appendix

(Incomplete)

Proof of Lemma 1 Using equations (3) and (4), we get

$$R^k y^{k,n} = \frac{1}{(1+N)\beta^k} \cdot P^k [\xi^k + (1+N)\beta^c y^{c,n}]. \tag{11}$$

Summing over k, and using (5),

$$-y^{c,n} = \frac{1}{1+N} \sum_k \frac{1}{\beta^k} P^k \xi^k + \left(\sum_k \frac{1}{\beta^k} P^k \right) \beta^c y^{c,n},$$

which can readily be solved for $y^{c,n}$:

$$y^{c,n} = -\frac{1}{(1+N)\beta^c} \cdot \sum_{j=1}^{K} \Lambda^j \xi^j.$$

Substituting this into (11) gives us the desired result (6).

For the case in which $R^k = R$ for all k, we have $P^k = P$ for all k. Using the fact that P is idempotent, it can easily be verified that $\Lambda^j = \lambda^j P$, from which (8) follows, which can then be specialized further to obtain (9). ∎

Proof of Proposition 5 (To be generalized; this proof is for the case two exchanges, k and c)

Recall that $U^k = \omega^k - q^k \cdot \theta^k + \sum_{s=1}^{S} \pi_s \left[\omega_s^k + d_s^k \cdot \theta^k - \frac{\beta^k}{2}(\omega_s^k + d_s^k \cdot \theta^k)^2 \right]$. Multiplying prices q^k from the pricing relationship (1) into asset holdings θ^k we get

$$q^k \cdot \theta^k = \mathbf{1}^\top \Pi R^k \theta^k - \beta^k (\omega^k + R^k \theta^k)^\top \Pi R^k \theta^k$$

which leads to a utility level of

$$U^k = \omega_0^k + \mathbf{1}^\top \Pi \omega^k + \beta^k (\omega^k + R^k \theta^k)^\top \Pi R^k \theta^k - \frac{\beta^k}{2}(\omega^k + R^k \theta^k)^\top \Pi R^k \theta^k - \frac{\beta^k}{2}(\omega^k + R^k \theta^k)^\top \Pi \omega^k$$

$$= \omega_0^k + \mathbf{1}^\top \Pi \omega^k - \frac{\beta^k}{2} \omega^{k\top} \Pi \omega^k + \frac{\beta^k}{2} \theta^{k\top} R^{k\top} \Pi R^k \theta^k$$

Now R^k is socially optimal for investors (on exchanges k and c) if it maximizes

$$c^c \left(\frac{y^c}{n^c} \right)^\top \Pi \left(\frac{y^c}{n^c} \right) + c^k \left(\frac{y^k}{n^k} \right)^\top R^{k\top} \Pi R^k \left(\frac{y^k}{n^k} \right)$$

Since $y^c = -R^k y^k$, this boils down to maximizing

$$y^{k\top} R^{k\top} \Pi R^k y^k$$

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Now since from equation (9) we know that $y^{k,n} = \frac{1}{1+N} \frac{1}{\beta^k + \beta^c} (R^{k\top} \Pi R^k)^{-1} R^{k\top} \Pi \xi^k$, the socially optimal R^k maximizes the expression

$$\xi^{k\top} \Pi R^k (R^{k\top} \Pi R^k)^{-1} R^{k\top} \Pi \xi^k$$

which is the same objective function as the one faced by the arbitrageurs. ∎