Bank Lending and the Term Structure[*]

Christine A. Parlour[†]   Richard Stanton[‡]   Johan Walden[§]

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Abstract

We study the implications of bank lending for the term structure. Specifically, we consider how resources are optimally allocated between a banking sector and a risky sector. To do so, we use a "two trees" framework with a risky and risk free sector, in which capital moves sluggishly between the two. We characterize equilibrium and illustrate how financial flexibility affects the term structure and social welfare. We develop a pricing equation for the equilibrium term structure and characterize the steady state distribution of growth rates in the economy. In addition, we demonstrate by example how financial innovation may increase or decrease the growth rate of the economy, how the effectiveness of monetary policy depends on the current state of the economy, and how a central bank can reduce the probability that the economy is in a low growth state.

1 Introduction

There are two goals for this paper: to develop implications of the credit channel for the term structure of interest rates; and to understand how the designers of a financial system, through their ability to allow intermediaries to redeploy capital, can affect growth rates and risk in the economy, and therefore the term structure. We take it as given that there is a credit or lending channel through which banks' lending affects the real economy. Specifically, we posit that intermediated lending and bonds are not perfect substitutes, and that banks cannot instantaneously raise new capital.[1] Thus financial frictions affect the real economy because they affect banks' propensity to lend; banks' capital being special, the growth rate of the economy is affected.[2] If through this channel the asset mix is also changed, then the aggregate risk in the economy must change.

In order to develop a model of the real term structure, we therefore take as given two distinct sectors of the economy: intermediated capital and unintermediated. Intermediated capital generates an asset with low risk and a low return, while unintermediated capital is an asset with high risk and high return. To capture these basic properties, we present a stylized, infinite horizon economy in which a central planner allocates money between an entrepreneurial sector and a banking sector. The entrepreneurial sector grows at a random rate; by contrast, the banking sector grows deterministically. This assumption captures the idea that banks add value because, through lending and monitoring, they reduce the risk associated with entrepreneurial activity. Financial and real frictions mean that capital cannot move instantaneously between the two sectors; but rather there is a maximum rate at which it can be moved from one sector to another.[3] We consider how a representative agent would value the consumption stream from each sector and therefore price assets. This is the simplest general equilibrium production economy within which we can study the effect of capital flows on the term structure and the economy.

There are two pivotal assumptions in our model: first, the banking sector has a lower risk and a lower growth rate than the unintermediated sector. Second, after a shock, the economy cannot instantaneously reallocate capital.

In terms of the risk and return of the banking sector, our framework is compatible with any model in which banks reduce the riskiness of firms' output. For example, Bolton and Freixas (2006) present a static, general equilibrium, model in which banks with profitability

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[1] A clear and precise description of how a credit channel links monetary policy actions to the real economy appears in Kashyap and Stein (1993), and also in Bernanke and Gertler (1995).

[2] Of course, banks play many roles. In addition to lending and monitoring they provide clearing and settlement services. Our model does not capture these institutional aspects of banking.

[3] These frictions can arise because of time costs in raising new bank equity and in performing due diligence before loans are issued.

"types" face an endogenous cost of issuing equity in addition to capital adequacy requirements. Bonds and bank loans are imperfect substitutes as banks, by refinancing, change the variability of projects' cashflows. Firms with high default probabilities choose costly bank financing over bonds because of these. Monetary policy affects the real economy because it affects the spread between bonds and bank loans and changes the average default probability (risk) of the undertaken projects. Specifically, a monetary contraction decreases lending to riskier firms. Further, Holmstrom and Tirole (1997) illustrates general equilibrium in which intermediaries, who are themselves subject to a moral hazard problem, exert costly effort and increase the probability of success of each entrepreneur's project.

There are many reasons why capital might not instantaneously flow between the two sectors. Capital might flow slowly from the entrepreneurial sector to the banking sector because of long term contracts and concomitant irreversible investment; for example long term market debt that is not callable. Capital might flow slowly from the banking sector to the entrepreneurial sector if banks, due to adverse selection problems, take time to raise extra equity capital. These are features of the real economy.

In addition, the speed with which capital moves into or out of the intermediated financial sector is affected by government action or regulation. Specifically, by restricting participation in synthetic banks such as hedge funds or S.I.V.'s the government restricts the flow of capital into this sector. In addition, the direct levers of monetary policy such as reserve requirements, balance sheet requirements, or access to short term financing affect the speed with which capital flows into the commercial banking sector.

We also note in passing that the existence of central banks, and other institutions that are designed to stimulate the economy counter–cyclically also imply the existence of a welfare cost to instantaneous adjustment to an optimal level. Further, specific regulations such as deposit insurance also prevent instantaneous readjustment (i.e., they preclude bank runs which are, in effect, instantaneous deleveraging). In the context of our model, these would appear as a cost associated with any change in the size of sectors in the economy and immediately translate into a restriction on the speed with which the economy adapts to a shock.

Economies are indexed by the speed with which capital moves between sectors. The central planner implements a competitive equilibrium in our economy by optimally allocating capital across sectors, given the financial frictions. Central to the intuition of our results is how funds flow given the size of each of the sectors. Suppose that the growth rate in the macro economy is lower than its historical mean. This could come about because of a shock to the unintermediated sector. In this case, the size of the banking sector is "too large" and that of the unintermediated sector is "too small." The central planner would

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therefore move capital from one sector into the other given the constraints on the capital flows in the economy. Over time, the banks would "delever" and the size of the banking sector would shrink while money would flow into the unintermediated sector which would increase.

In this framework, we characterize the optimal policy of the central planner and demonstrate the effect of sluggish capital reallocation on social welfare. We demonstrate how to price assets in this economy and, in particular, we analyze the behavior of the term structure. Because economies are indexed by the speed with which capital can be reallocated, they differ in the optimal size of the intermediated sector, growth rates and term structures. We demonstrate that the steady state growth rate of the economy can be increasing or decreasing in the rate at which capital can be reallocated. We show how small financial innovations can have a large impact on capital flows into and out of the banking sector. In addition, we characterize the equilibrium steady state distribution of growth rates and show how monetary policy reduces the likelihood of low growth states.

A vast literature develops term structures given assumptions on the evolution of the short rate. The seminal finance term structure paper is due to Cox, Ingersoll, and Ross (1985). Our aim is somewhat different, in that we are interested in the effect of the banking sector on the term structure and so consider allocation between the safe sector and the risky. Recently, a literature has developed tying financial frictions to the macro economy. For example, Jermann and Quadrini (2007) demonstrates that financial flexibility in firm financing can lead both to lower macro volatility and higher volatility at the firm level. Further, Dow, Gorton, and Krishnamurthy (2005) incorporate a conflict of interest between shareholders and managers into a CIR production economy. Auditors are essentially a proportional transaction cost levied on next period's consumption. They provide predictions on the cyclical behavior of interest rates, term spreads, aggregate investment and free cash flow.

Mechanically, our model is a "two trees" model, as presented by Cochrane, Longstaff, and Santa-Clara (2008), and further extended by Martin (2007). The fundamental difference between our approach and theirs is that the sizes of our "trees" are not exogenous, because they are the result of resource allocation decisions by the central planner. One consequence of this is that our model is characterized by a stationary solution.

Our work is also related to the literature on liquidity, and especially to Longstaff (2001), who studies portfolio choice with liquidity constraints in a model with one risky and one risk-free asset. The constraints that Longstaff (2001) imposes are similar to our sluggish capital constraints. However, there are also several differences between the two papers. Whereas Longstaff (2001) takes a partial equilibrium approach, with exogenously specified

return processes for the risky and risk-free asset, these processes are endogenously defined for us. Moreover, Longstaff (2001) allows for stochastic volatility, which we do not, but has to rely on simulation techniques for the numerical solution, since he has four state variables. This is quite nontrivial, since optimal control problems are not well suited for simulation (similarly to American option pricing problems). We need only one state variable and can therefore use dynamic programming methods to solve our model. Vergara-Alert (2007) considers an economy with two technologies, one of which is irreversible, and a duration mismatch. By contrast, we allow for some capital flows in both directions, though at a limited rate. Johnson (2007) also develops a two-sector equilibrium model, but there are no flows into or out of the risky sector.

2 Model

Consider an economy that evolves between times 0 and T, in which banks can monitor firms and in so doing reduce the variability of their output.[4] We do not explicitly model how banks do this, because our intent is to derive a term structure, but rather directly assume that money invested in a monitored firm grows deterministically. Specifically, an investment of size B_0 becomes $B = B_0 e^{rt}$, at time t. Here, T can be finite, or infinity, although we will mainly focus on the infinite horizon case. For simplicity, we focus on the case of no growth, $r = 0$, however our results are readily generalizable to nonzero deterministic returns.

In addition to the banking sector, there are unmonitored firms traded in the market. We note in passing that in any economy consistent with Modigliani and Miller (1958), the capital structure of such firms is irrelevant, and could well be decomposed into equity and risky debt.[5] The unmonitored firms grow stochastically, and pay an instantaneous dividend of $D(t)\,dt$, where $D(t) = D_0 e^{y(t)}$, $y(0) = 0$, and $dy = \mu\,dt + \sigma\,d\omega$, where μ and σ are constants. Here, ω is a standard Brownian motion, generating a standard filtration, $\mathcal{F}_t$, on $t \in [0, T)$.

As we will be considering how society allocates capital between the two sectors, we define the monitored-share, $z(s) = B(s)/(B(s) + D(s))$ Notice that, if z is constrained to be zero, then all resources are in the entrepreneurial sector and the economy collapses to that presented in Lucas (1978).[6] In what follows, we frequently describe the monitored share of

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[4]Following Leland and Pyle (1977), a plethora of corporate finance models demonstrates how a bank may monitor and increase the value of a firm. Refer, for example, to Holmstrom and Tirole (1997).

[5]The cash flows of a firm with bank debt, market debt and equity in its capital structure, for the purposes of this paper would be split between the banking sector, and the market sector. For expositional purposes, we focus on firms that are fully monitored and those that are unmonitored.

[6]The Fisherian consumption model presented in Lucas (1978) follows earlier equilibrium models such as Rubinstein (1976).

the economy as "the bank." It will sometimes be convenient to use $d = \log(1-z) - \log(z) = \log(D/B)$.

To ensure that the banking sector is never dominated by, and never dominates, the unintermediated sector, we restrict its expected return. Specifically

Condition 1 $-\frac{1}{2}\sigma^2 < \mu < \frac{1}{2}\sigma^2$.

This ensures that the risk adjusted instantaneous return is sufficiently low so that there is a role for the banking sector, and yet sufficiently high so that it is not dominated in turn.

There is a representative investor with logarithmic expected utility who consumes the output of both trees: $U(t) = E_t[\int_t^T e^{-\rho(s-t)} \log(B(s) + D(s))\, ds]$. The price of all assets are determined by her valuations, and under standard assumptions an asset that pays out $\xi(t)$ commands an initial price of

$$P_0 \;=\; (B_0 + D_0)E\left[\int_0^T \frac{e^{-\rho(s-t)}}{B(s) + D(s)}\xi(s)\, ds\right]. \tag{1}$$

Obviously, the agent's aggregate consumption determines her marginal utility and, as evinced by Equation 1 above, her valuation for all securities (including risk free ones).

The aggregate size of the economy is determined by the resources channeled through the banking sector as opposed to the unintermediated sector. Reallocation occurs if a relative fraction of total capital, $d\hat{a}$, moves from the risky to the risk-free sector. Here, $\hat{a}$ is an $\mathcal{F}_t$-adapted process. We also assume that capital can only be reallocated at finite speeds, $d\hat{a} = a\, dt$, where

$$-\lambda(z) \leq \frac{a}{(B+D)} \leq \lambda(z). \tag{2}$$

Therefore, $\lambda(z)$ represents how easy it is to move capital between the monitored and unmonitored section of the economy. The actions of the monetary authority affect the speed with which resources move between the two sectors. We consider $\lambda \equiv 0$ as a benchmark case, but in general we assume that $\lambda(z) > 0$ for all $z \in (0,1)$. The class of controls satisfying these two conditions are denoted by $\mathcal{A}_{\lambda,t,T}$, or simply by $\mathcal{A}$, when there is no confusion. Although, strictly speaking, the control is $\hat{a}$, we will represent the control by a, and write $a \in \mathcal{A}$.[7]

It is sometimes useful to decompose $\lambda(z)$ into frictions that reflect the fundamentals of the economy $\lambda_M(z)$, and those that reflect regulations imposed by the governments: $\lambda_G(z)$.

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[7]The restriction imposed by (2) leads to a qualitatively quite different situation for the central planner, compared with unconstrained optimization. As noted in Longstaff (2001), for any bounded λ, any control in $\mathcal{A}_{\lambda,t,T}$ will a.s. have bounded variation, as opposed to the optimal control in standard portfolio problems, which a.s. has unbounded variation over any time period.

If the government favors a countercyclical policy, $\lambda_G(z)$ will decrease the overall speed of adjustment.

Appendix A describes a specific set of frictions that lead to a finite λ. Our analysis of the model now proceeds by considering how a central planner would allocate funds between the two sectors given a restriction on the adjustment speed of $\lambda(z) \equiv \lambda_M(z) + \lambda_G(z)$ in Section 3, and provides a characterization of the optimal mix of intermediated and unintermediated sectors in the economy in subsection 3.1. Our solution has asset pricing implications which we develop specifically in the context of a term structure in Section 4.1. Empirical predictions and policy prescriptions follow in Section 5.

3   The Central Planner's Problem

The central planner maximizes the discounted presented value of the representative agent's utility by moving capital between the two sectors. Using the control a, the central planner hopes to achieve:

$$V(B, D, t) \equiv \sup_{a \in \mathcal{A}} E_t \left[\int_t^T e^{-\rho(s-t)} \log(B + D) \, ds \right]. \tag{3}$$

The central planner's reallocation leads to the following dynamics for the capital in the two sectors:

$$dB = a(B + D) \, dt, \tag{4}$$

$$dD = -a(B + D) \, dt + D \left(\widehat{\mu} \, dt + \sigma \, d\omega \right), \tag{5}$$

$$dz = a \, dt - z(1 - z) \left(\widehat{\mu} \, dt + \sigma \, d\omega \right) + z(1 - z)^2 \sigma^2 \, dt. \tag{6}$$

Here, $\hat{\mu} = \mu + \sigma^2/2$. Cochrane, Longstaff, and Santa-Clara (2008) characterize the "two trees" economy in terms of the relative share of each asset and also express dynamics for the share. There are two differences between the drift term for z in our formulation and in theirs that highlight the difference in our approaches. First, we allow a central planner to potentially move resources between the two sectors (our a term). Second, in our case, the difference between the drifts on the two assets is μ, that which comes from the risky tree whereas in their formulation, with two identical trees the difference between the two is zero.

We proceed by characterizing the central planner's problem for a finite T by finding a locally optimal control or reallocation (a) that will also be globally optimal. The infinite horizon case follows immediately. Given the central planner's objective, the Bellman equation for optimality is

$$\sup_{a \in \mathcal{A}} \left[V_t + \frac{1}{2} \sigma^2 D^2 V_{DD} + \left[\widehat{\mu} D - a(B + D) \right] V_D + a(B + D) V_B - \rho V + \log(B + D) \right] = 0. \tag{7}$$

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By homogeneity, we can write

$$V(B, D, t) = \frac{\log(B + D)\left(1 - e^{-\rho(T-t)}\right)}{\rho} + w(z, t), \tag{8}$$

where the normalized value function, $w(z, t) \equiv V(z, 1 - z, t)$. This equation allows us to write derivatives of V in terms of derivatives of w. Thus,

$$V_t = -e^{-\rho(T-t)} \log(B + D) + w_t; \tag{9}$$

$$V_B = \frac{1 - e^{-\rho(T-t)}}{\rho(B + D)} + w_z \frac{D}{(B + D)^2}; \tag{10}$$

$$V_D = \frac{1 - e^{-\rho(T-t)}}{\rho(B + D)} - w_z \frac{B}{(B + D)^2}; \tag{11}$$

$$V_{DD} = -\frac{1 - e^{-\rho(T-t)}}{\rho(B + D)^2} + w_z \frac{2B}{(B + D)^3} + w_{zz} \frac{B^2}{(B + D)^4}. \tag{12}$$

Substituting these into Equation (7), we obtain

$$w_t + \frac{1}{2}\sigma^2 z^2 (1 - z)^2 w_{zz} + \left[a - \widehat{\mu} z(1 - z) + \sigma^2 z(1 - z)^2\right] w_z - \rho w$$
$$+ \frac{1 - e^{-\rho(T-t)}}{\rho}\left[\widehat{\mu}(1 - z) - \frac{\sigma^2(1 - z)^2}{2}\right] = 0. \tag{13}$$

We are now in a position to characterize the optimal adjustment, a, to the banking sector. Notice that the left hand side of Equation 13 is linear in a. Therefore, a will always be either the maximum value, λ, or the minimum value, $-\lambda$; it is a bang–bang control.[8] So if $z \equiv \frac{B}{B+D}$ is "too low," the central planner will allocate resources to the banking sector at the fastest possible rate, while if z is "too high," resources will flow out of the banking sector and into the unintermediated sector. Of course "too high" and "too low" depend on how an infinitesimal change in the allocation between the sectors affects the central planner's continuation value: or w_z in our notation.

Lemma 1 *The optimal reallocation between the two sectors is*

$$a = \lambda(z) sign(w_z), \tag{14}$$

where w_z is the normalized marginal social benefit of moving capital to the banking sector.

To fully characterize a solution, we also need to solve for the central planner's optimal value function. We state this as a partial differential equation (p.d.e.).

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[8]At points where $w_z = 0$, any $a \in [-\lambda, \lambda]$ is optimal, so $a = \lambda$ is an optimal strategy at such points. However, we adopt the convention that $a = 0$ when $\lambda = 0$.

Proposition 1 *If condition 1 is satisfied, then the value function for a central planner who optimally reallocates capital between the banking and unintermediated sectors, is*

$$V(B, D, t) = \frac{\log(B + D)\left(1 - e^{-\rho(T-t)}\right)}{\rho} + w\left(\frac{B}{B + D}, t\right),$$

where $w : [0, 1] \times [0, T] \to \mathbb{R}$ *is the solution to*

$$\rho w - w_t = z(1-z)(-\hat{\mu} + (1-z)\sigma^2)w_z + \lambda(z)|w_z| + \frac{\sigma^2}{2}z^2(1-z)^2 w_{zz} + q(t, z), \quad (15)$$

$$0 = w(z, T), \quad (16)$$

and where $q(t, z) \equiv \frac{1 - e^{\rho(t-T)}}{\rho}\left((1-z)\hat{\mu} - (1-z)\frac{\sigma^2}{2}\right).$

We note that no boundary conditions are needed at $z = 0$ and $z = 1$ to obtain the solution. The reason, which we elaborate on in the proof in Appendix C, is that the p.d.e. is degenerate at the boundaries. It is hyperbolic, and the characteristic lines imply outflow at both boundaries, so no boundary conditions are needed. Indeed, it follows from the proof of proposition 1 that

Lemma 2 *a is positive close to* $z = 0$ *and a is negative close to* $z = 1$.

Armed with Proposition 1, we can characterize how the optimal mix between the banking and non banking sectors depends on the speed with which the economy can adjust, and therefore implicitly evaluate the effectiveness of monetary policy tools. In addition, we can consider the effect on social welfare of these reallocations. As a first step, in order to interpret our results, we present two benchmark cases: First, we assume that capital is infinitely flexible; second, that it is perfectly inflexible.

3.1   Benchmark Cases

To present these benchmarks more succinctly, we focus on the infinite horizon case, $T = \infty$ and we fix $B(0) + D(0) = 1$; this is without loss of generality. If capital can be moved instantaneously then, formally, $\lambda(z) = \infty$ for any z. Specifically, this means that the central planner can arbitrarily quickly move from $z = B(0)/((B(0) + D(0))$ to any z_* at $t = 0^+$. Moreover, he can choose capital reallocation strategies with unbounded variation, and specifically choose $d\hat{a} = a\,dt + b\,d\omega$ for arbitrary bounded functions, a and b. For any fixed z, the central planner can, for example, choose

$$d\hat{a} = \frac{DB}{D + B}\left(\mu\,dt + \sigma\,d\omega\right), \quad (17)$$

which implies that $dz = 0$ or, in other words, he can maintain a constant banking share in the economy.[9]

In such an economy, it suffices to directly consider the share between the banking sector and monitored sector (z) that maximizes the representative agent's expected utility. If z is constant then the total drift of the economy is also constant, and

$$\frac{d(D+B)}{D+B} = (1-z)((\mu + \sigma^2/2)\,dt + \sigma\,d\omega).$$

In this case, we have

Lemma 3 *Suppose that capital is fully flexible, $\lambda \equiv \infty$, and that the central planner chooses a constant banking share, z. Then the expected utility of the representative agent is*

$$U^\infty(z) = ((1-z)(\mu + \sigma^2/2) - (1-z)^2\sigma^2/2)/\rho^2,$$

which has a maximal value at $z_* = 1/2 - \mu/\sigma^2$.

We let the superscript ∞ denote the adjustment speed. This solution exactly mirrors the Merton (1969) solution for the portfolio choice problem of an investor allocating wealth between a risky and a risk free asset, in which the portfolio share of the risky asset is $\frac{\mu + \frac{\sigma^2}{2}}{\sigma^2}$. In this case, the risk-free asset (our banking sector), should have a portfolio weight of z_*.

Given a constant $z = z_*$ for all t, the economy "collapses" to a one tree economy, with drift term $\hat{\mu}(1 - z_*)$ and volatility $\sigma(1 - z_*)$. Thus, when $\lambda = \infty$, the economy is effectively a one tree endowment economy, and so shares all its properties. In particular, the term structure will be flat when $\lambda = \infty$.

If capital is perfectly inflexible, then $\lambda(z) = 0$ for all z. In this case, a central planner would choose the economy's initial allocation of capital (which we normalized to \$1) to maximize the expected utility of the representative investor.

Lemma 4 *Suppose that capital is inflexible, $\lambda \equiv 0$, and that the initial monitoring share is z. Then, the expected utility of the representative agent is $U^0(z) = G(z; \mu, \sigma^2, \rho)$, where*

$$
\begin{aligned}
G(z; \mu, \sigma^2, \rho) &= \frac{1}{2\rho}\Bigg(\left(2\mu^2 + \sigma^2(2\rho + q) + \mu(\sigma^2 + 2q)\right) {}_2F_1\left(1, \frac{q-\mu}{\sigma^2}, \frac{q-\mu}{\sigma^2} + 1, \frac{z}{z-1}\right) \\
&+ \; 2\frac{1-z}{z}\left(\mu^2 + \rho\sigma^2 - \mu q\right) {}_2F_1\left(1, \frac{q+\mu}{\sigma^2} + 1, \frac{q+\mu}{\sigma^2} + 2, \frac{z-1}{z}\right) \Bigg) \\
&/ \; \left(\mu^2 - \mu q + 2\rho(\sigma^2 + q)\right),
\end{aligned}
$$

$q = \sqrt{\mu^2 + 2\rho\sigma^2}$ *and* ${}_2F_1$ *is the hypergeometric function.*

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[9]Notice, that this is expression is *not* obtained from Equation 6, which is based on $d\hat{a}$ having zero quadratic variation, and therefore misses Itô terms in the general case. When $d\hat{a} = a\,dt + b\,d\omega$, the extra terms $\frac{1}{2}\frac{\partial^2}{\partial B^2}\left[\frac{B}{B+D}\right] \times (dB)^2 + \frac{\partial^2}{\partial D\partial B}\left[\frac{B}{B+D}\right] \times (dB)(dD)$ are added to (6), which leads to (17) being the condition for $dz \equiv 0$.

Here, $U^0(0) = \frac{\mu}{\rho^2}$ and $U^0(1) = 0$. The case of $U^0(0)$ corresponds to the standard Lucas Tree model, with one tree, in which bonds are in zero net supply and all resources are placed in the hands of the entrepreneurs, while the case of $U^0(1)$ is an economy with no risk, which is also a special case of the Lucas Tree model with one tree. The case of $U^0(z)$, for $z \in (0,1)$ corresponds to the two-tree model of Cochrane, Longstaff, and Santa-Clara (2008). Thus, when $\lambda = 0$, the economy is effectively a two tree endowment economy. The case when $\lambda = 0$ can also be viewed as a limit case of an economy with small but positive λ and allows us to analytically derive several differences between our model and the standard Lucas Tree model with one tree. When we wish to highlight this interpretation, we write $\lambda = 0^+$.

Although $\lambda = 0$ is formally the same model as Cochrane, Longstaff, and Santa-Clara (2008), their focus is on the risky asset. In the Subsection 4.1, we derive several novel implications for the term structure when $\lambda = 0$. This analysis complements that in Cochrane, Longstaff, and Santa-Clara (2008).

3.2 The stationary distribution of the share

In contrast to the two-trees model with inflexible capital, (the $\lambda = 0$ case), when capital is flexible, the long term share distribution is stationary. This is an appealing property of the model with capital flexibility, since it avoids the transitory interpretation that always must be associated with a nonstationary solution.

We derive the stationary probability distribution of z from the optimal control, $a \in \mathcal{A}$ and the Kolmogorov forward equation (see Björk (2004)). Using (6), we define A^*, the adjoint to the infinitesimal operator,

$$(A^*p)(z,t) \stackrel{\text{def}}{=} -\frac{\partial}{\partial z}\Big[(a - z(1-z)\hat{\mu} + z(1-z)^2\sigma^2)p\Big] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial z^2}\Big[z^2(1-z)^2p\Big].$$

We have

Proposition 2 *Given the optimal control, $a \in \mathcal{A}$, to the central planner's problem that satisfies 1. Let $\pi(t,z)$ denote the probability distribution of monitoring share, z, at time t, with initial distribution is $\pi_0(z)$ at $t = 0$. Then π is the solution to the p.d.e.*

$$
\begin{aligned}
\pi_t &= A^*\pi, \\
\pi(z,0) &= \pi_0(z), \\
\pi(0,t) &= 0, \\
\pi(1,t) &= 0, \\
t \geq 0, &\quad 0 \leq z \leq 1.
\end{aligned}
$$

We will use this method to derive the stationary solution in our discussion on empirical implications, in Section 5.

4   Asset Pricing

The actions of the central planner determine the agent's consumption and hence marginal utility. We first provide a general characterization of the price of any asset in this economy and then explicitly consider the effects on the term structure.

The price at date t of an asset that pays a terminal payoff $G_T \equiv G(B_T, D_T, t)$, and interim dividends at rate $\delta_\tau \equiv \delta(B_\tau, D_\tau, \tau)$, where $t \leq \tau \leq T$, is given by

$$P = (B_t + D_t)E_t\left[\int_t^T e^{-\rho(s-t)}\frac{\delta_s}{B_s + D_s}\,ds + e^{-\rho(T-t)}\left(\frac{G_T}{B_T + D_T}\right)\right]. \tag{18}$$

Define

$$Q(B, D, t) \equiv E_t\left[\frac{G_T}{B_T + D_T} \mid B_t = B, D_t = D\right]. \tag{19}$$

From Equation (18), we have:

$$Q(B, D, t) = \frac{e^{\rho(T-t)}P(B, D, t)}{B + D} - E_t\left[\int_t^T e^{\rho(T-s)}\frac{\delta_s}{B_s + D_s}\,ds\right]. \tag{20}$$

By iterated expectations,

$$E(dQ) = 0. \tag{21}$$

Substituting for Q and simplifying, we obtain the following p.d.e. that must be satisfied by P, subject to the terminal boundary condition $P(B, D, T) = G(B, D, T)$:

$$P_t + \frac{1}{2}\sigma^2 D^2 P_{DD} + \left[\widehat{\mu}D - a(B + D) - \frac{\sigma^2 D^2}{B + D}\right]P_D + a(B + D)P_B$$
$$- \left(\rho + \widehat{\mu}\frac{D}{B + D} - \sigma^2\frac{D^2}{(B + D)^2}\right)P + \delta(B, D, t) = 0. \tag{22}$$

For completeness, we also develop the asset pricing implications for general constant relative risk aversion utility. These are presented in Appendix B.

4.1   The Term Structure

To characterize the term structure, we consider zero coupon bonds. In this case, adapting Equation (22) yields the pricing equation.

Proposition 3 *The price, at t_0, of a τ maturity zero coupon bond, where $t_0 + \tau \leq T$, is $p(t_0, z)$, where p is the solution to the following p.d.e.*

$$p_t + \frac{1}{2}\sigma^2 z^2(1 - z)^2 p_{zz} + \left[a - \widehat{\mu}z(1 - z) + 2\sigma^2 z(1 - z)^2\right]p_z$$
$$- \left[\rho + \widehat{\mu}(1 - z) - \sigma^2(1 - z)^2\right]p = 0. \tag{23}$$

$$p(z, t_0 + \tau) \equiv 1, \tag{24}$$

$$t_0 \leq t \leq \tau,$$
$$0 \leq z \leq 1.$$

Observe that, as in the case of the value function, no boundary conditions beyond the terminal payoff are needed to ensure uniqueness. Further, if $a(z)$ is the stationary control that does not depend on T (obtained by letting $T \to \infty$) then the whole term structure, for all z and times to maturity, can be obtained by solving the p.d.e. once. Specifically, the price of a τ period zero-coupon bond is $P^\tau(z) = p(z, -\tau)$, where p solves (23) with terminal condition $p(z, 0) \equiv 1$.

Contrary to the flat term structure in the one-tree model, the yield curve in our economy is not flat. In fact, it is often upward sloping. That is, real rates display a "liquidity" or "risk" premium for longer horizons. This is due to changes in the representative agent's marginal utility and is inherent in the two trees structure and not a consequence of the central planner's reallocation of capital (although reallocation heightens the effects). To see this, we explicitly characterize the term structure in an economy in which the initial size of the two sectors is exogenous and there is no reallocation of capital between the two sectors. This corresponds to the second benchmark case in Section 3.1.

Proposition 4 *Consider an infinite horizon economy in which the relative size of the sectors is exogenous, $d = \log((1-z)/z) = \log(D/B)$. Then, the price of a τ period bond is given by*

$$
\begin{aligned}
P^\tau &= (1 + e^d) e^{-\rho\tau} \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} \frac{e^{-(y-d-\mu\tau)^2/(2\sigma^2\tau)}}{1 + e^y} dy \\
&= \frac{(1 + e^d) e^{-\rho\tau - (d+\mu\tau)^2/(2\sigma^2\tau)}}{2} \times \\
&\quad \left(\sum_{n=0}^{\infty} (-1)^n F\left(\frac{d + \mu\tau + n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right) + \sum_{n=1}^{\infty} (-1)^{n+1} F\left(\frac{-d - \mu\tau + n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right) \right),
\end{aligned}
$$

where Erfc is the error function, $Erfc(x) = (\sqrt{\pi})^{-1} \int_z^{\infty} e^{-t^2} dt$.
Further, if $\frac{d+\mu\tau}{\sigma^2\tau} = m \in \mathbb{N}$, then

$$P^\tau = \frac{(1 + e^d) e^{-\rho\tau - m^2\sigma^2\tau/2}}{2} \left(1 + 2 \sum_{n=1}^{m-1} (-1)^n e^{n^2\sigma^2\tau/2} \right).$$

Martin (2007) independently developed a somewhat similar solution method (albeit more general in that it allows for general Levy processes and multiple trees) as the one in Proposition 4. However, following Cochrane, Longstaff, and Santa-Clara (2008), he focuses on pricing of the risky asset, i.e., on price-to-dividend ratios.
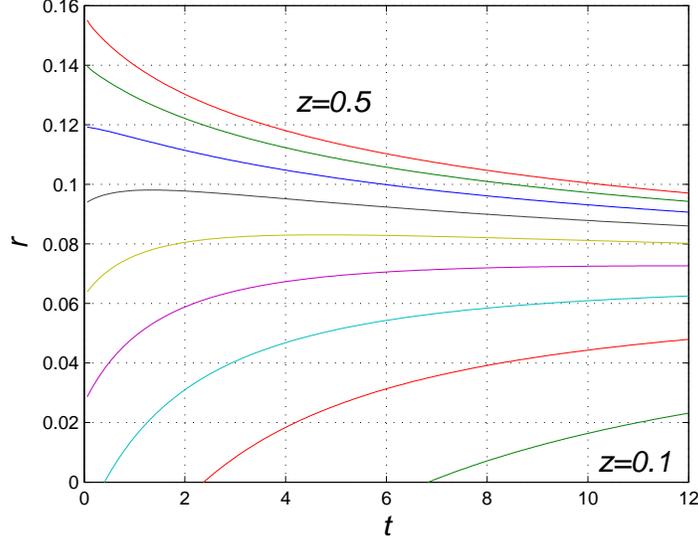
Figure 1: **Zero-coupon yield curve 0-12 years, for different choices of** z. $\mu = 1/3$, $\sigma^2 = 1$, $\rho = 0$, $\lambda(z) \equiv 0^+$, $z_* = 0.08$.

Figure 1 displays various term structures that might arise from different initial sector sizes when there is no capital reallocation. The term structure is upward sloping at short maturities for most values of z. Suppose that the economy is away from the long term socially optimal z. In this economy, the instantaneous short rate is just the representative agent's expected relative change in marginal utility. Market clearing that is imposed on the economy as it evolves means that the agent must be indifferent between consuming today and transferring consumption to tomorrow. Suppose that he expects a lower marginal utility next period. He therefore values consumption next period less (a decrease in demand for consumption tomorrow). Market clearing imposes that interest rates rise. Therefore, the term structure slopes up.

In the case of inflexible capital, we can also characterize the long end of the yield curve. To see this, define the short term rate $r_s = \lim_{\tau \searrow 0} -\frac{\log(P^\tau)}{\tau}$ and, for the case when $T = \infty$, the long term rate $r_l = \lim_{\tau \to \infty} -\frac{\log(P^\tau)}{\tau}$, where P^τ is the price of a zero-coupon bond with time τ to maturity.

Proposition 5 *For all $\lambda \geq 0$, for all $z \in [0,1]$, the short-term rate is $r_s = \rho + \hat{\mu}(1-z) - \sigma^2(1-z)^2$. Further, if $\lambda \equiv 0$, i.e., capital is inflexible, then*
(i) For $z = 0$, so that all the wealth is in the entrepreneurial sector, the long-term rate is $r_l = \rho + \hat{\mu} - \sigma^2$.
(ii) For $z = 1$, so that all the wealth is in the banking sector, the long-term rate is $r_l = \rho$.

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(iii) For $z \in (0, 1)$, the long-term rate is $r_l = \rho + \frac{(\hat{\mu} - \sigma^2/2)^2}{2\sigma^2}$.

The short rate in this economy is $\rho + \hat{\mu}(1 - z) - \sigma^2(1 - z)^2$. This can be reexpressed as the time preference of the representative agent, plus the local growth rate of the economy $(\hat{\mu}(1 - z))$ less a risk adjustment term that depends on the variability of the whole economy. (Recall, the variance of the intermediated sector is zero.) By inspection, if there is only one tree: i.e., if $z = 0$ or $z = 1$, then the term structure is flat: both the long and short rates are equal.

If there is capital in both sectors, then the long rate is a constant equal to $\rho + \frac{\mu^2}{2\sigma^2}$. We show in the appendix that this result in fact holds for general CRRA utility functions, regardless of the agent's risk aversion. This result stands in stark contrast to the one-tree model, in which the term structure is flat and the discount is very sensitive to risk aversion. If we view the case of $\lambda = 0$ as a limiting case of our model for small but positive λ, it is clear that the long rate is much less sensitive to aggregate risk aversion than a standard Lucas Tree model. Specifically, the long rate is always greater than the personal discount rate, $r_l > \rho$, regardless of the aggregate risk aversion in the economy.

Our observation is related to the risk-free rate puzzle: In the one tree model the risk free rate (both short and long) is a quadratic function of the risk aversion parameter, γ, and is therefore very sensitive to assumptions about γ. In our model (for the special case of $\lambda = 0$), the long rate is fixed, regardless of γ. As an example, if the expected growth of consumption is 1%, the volatility of consumption growth is 4% and the personal discount rate is 1%, this leads to a long rate of $r_l = 3.6\%$, regardless of γ.

The short rate depends on the size of the banking sector. In the special case in which $\lambda = 0$ and there is capital in both sectors, we can precisely characterize the conditions under which the term structure will slope up.

Corollary 1 *If $\lambda = 0$, and $q \overset{\text{def}}{=} \frac{\sqrt{-2\hat{\mu}^2\sigma^4 + 8\hat{\mu}\sigma^6 - 2\sigma^8}}{\sigma^4}$, for $z \in (0, 1)$,*
(i) The term structure slopes up or $r_l > r_s$ if $z < 1 - \frac{\hat{\mu}}{2\sigma^2} - q$ or $z > 1 - \frac{\hat{\mu}}{2\sigma^2} + q$,
(ii) $r_l < r_s$ if $1 - \frac{\hat{\mu}}{2\sigma^2} - q < z < 1 - \frac{\hat{\mu}}{2\sigma^2} + q$.
(iii) $r_l - r_s$ is decreasing in z if $z < 1 - \frac{\hat{\mu}}{2\sigma^2}$.
(iv) $r_l - r_s$ is increasing in z if $z > 1 - \frac{\hat{\mu}}{2\sigma^2}$.

We can also provide a partial characterization of the unconditional slope of the term structure. That is, we can characterize $r_l - r_s$ at the modal value z_*. Specifically,

Proposition 6 *In any economy in which $\mu > 0$ then for any λ, the term spread $r_l - r_s > 0$ at z_*.*
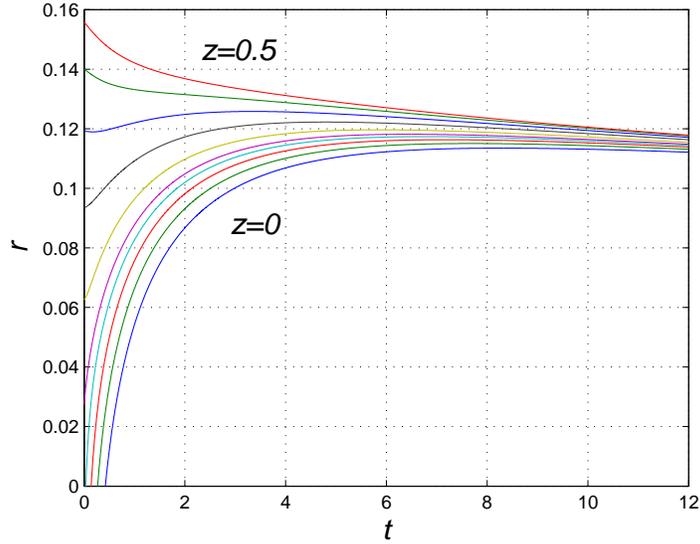
Figure 2: **Zero-coupon yield curve 0-12 years, for different choices of** z**.** $\mu = 1/3$, $\sigma^2 = 1$, $\rho = 0$, $\lambda(z) \equiv 1$, $z_* = 0.2$.

Intuitively, this arises because z_* is endogenous. Ang, Bekaert, and Wei (2007) analyze ten years of TIPS data to characterize the real term structure in the United States. They find that, unconditionally, the term structure of real rates is somewhat flat, with a peak at the one year maturity; and sometimes slopes downwards. They find that the short end is quite variable and on average there is little or no term spread. In our model the state (measured by z) is vitally important in determining the shape of the term structure and the flexibility of capital affects the stationary distribution of states.

The presence of a hump-shaped term structure for some values of z is interesting, since it is one of the stylized properties of the real world term structure, see Nelson and Siegel (1987). The curvature, however, is quite small and is even smaller for lower values of σ^2.

When $\lambda > 0$, however, a stronger hump occurs.[10] For example, compare Figure 1, in which $\lambda(z) \equiv 0^+$, with Figure 2, in which $\lambda(z) \equiv 1$. In the latter case, the yield curve is steeper. Intuitively, with flexible capital, the economy will move back to the optimal relative size quickly, and so marginal utilities will rise rapidly to the steady state value and so the term structure will be steep at short maturities and then relatively flat. In the extreme case of $\lambda = \infty$ then from the socially optimal level of z, the term structure will be flat and the

---

[10]For all numerical solutions, we have used the centralized second order finite difference stencil in space, and the first-order Euler method for the time marching. All figures can be constructed in a matter of seconds, using nonoptimized Matlab code. Codes are available from the authors upon request. In Hart and Weiss (2005), a slightly different finite difference scheme is proposed to handle the nonlinearity in the $|w_z|$ term. We have calculated the solutions with these schemes, with similar results.

pure expectations hypothesis holds. However, for $\lambda = \infty$, there are two different effects: First, a higher λ will lead to a more steeply sloped yield curve (upward or downward) when z is far from z_*. Second, the higher flexibility also implies that, on average, λ will be closer to z_*, so such events are rarer in a flexible economy. Or,

Observation 1 *Economies with high flexibility have larger slopes than economies with low flexibility, but extreme slopes are rarer in economies with high flexibility than in those with low flexibility.*

5   Empirical Predictions and Policy Prescriptions

Our model generates two types of predictions: those related to the term structure and the evolution of the economy, and those related to the effect of financial regulation on the economy. To understand both, it is useful to consider the optimal size of the banking sector and how it is affected by the speed at which capital can be reallocated. We illustrate this numerically; to do so we assume that $\lambda(z) = \lambda z(1 - z)$. This is for expositional reasons: it makes the effects easier to distinguish in the figures. The results are qualitatively the same when $\lambda(z) \equiv \lambda$ is chosen.

First, consider the effect on welfare of different rates of capital reallocation. As one expects, social welfare is highest when there are no frictions to capital flows. Figure 3 is a plot of the normalized value function as a function of the size of the banking sector (z). If capital can be instantaneously reallocated then shocks, such as a catastrophic loss in the real economy that change the relative size of the two sectors, have no effect on social welfare. For this reason, the line labeled $\lambda = \infty$ is flat. After any untoward change in the relative sizes of the two sectors, the central planner can instantaneously move the economy back to the optimal sector mix and there is no loss in welfare. Such is not the case when the reallocation rate is bounded.

There is no cost to flexibility in this economy and so the two lines labeled $\lambda = 8$ and $\lambda = 0.1$ are strictly below the welfare when there is complete flexibility. The difference between the social welfare between the fully flexible case and the inflexible case represents the social loss incurred because of sluggish reallocation of capital. Not surprisingly, the welfare loss is most severe the farther the sectors are from the optimum allocation.

Although welfare is strictly ranked, the optimal size of the intermediated sector does not change monotonically with differences in the speed with which capital can be reallocated.

Observation 2 *The optimal size of the banking sector, z^* increases in λ if the growth rate in the entrepreneurial sector is sufficiently high and decreases in the λ if the growth rate is sufficiently low.*
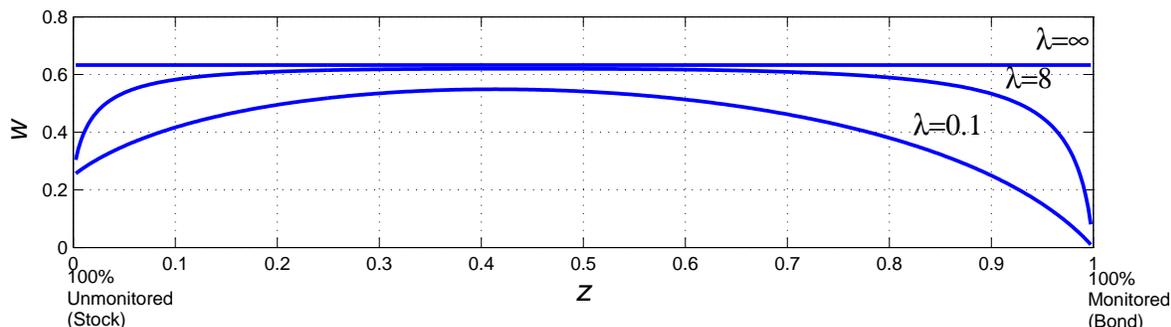
Figure 3: **Value function as a function of λ. Limiting cases are $\lambda = 0$ when there is no flexibility for capital reallocation and $\lambda \longrightarrow \infty$, which converges to full flexibility case. Parameters:** $\mu = 1/4$, $\sigma^2 = 4$, $\rho = 1$.

To see this, consider Figure 4, which illustrates the relationship between the optimal banking share, z^* and λ. Consider the case where $\hat{\mu} = 0.5$. If this is the growth rate of the entrepreneurial sector then the central planner optimally keeps half of the economy in the intermediated sector, and half in the entrepreneurial sector, irrespective of the speed at which capital moves between the sectors. However, if the growth rate in the entrepreneurial sector is lower (say $\hat{\mu} = 0.3$), then the optimal size of the banking sector is larger than a half, for all possible transfer rates while if the growth rate in the entrepreneurial sector is high (say $\hat{\mu} = 0.7$) then more of the assets are placed in the entrepreneurial sector (so z^* is lower).

If the growth rate in the entrepreneurial sector is high, then increasing the rate at which capital moves increases the optimal size of the banking sector. In this case the social cost of having an inordinately large banking sector (and therefore forgone growth) is very high. Therefore, as insurance against this state, the central planner decreases the size of the banking sector to maintain a "buffer." Because of this, for very low λ, the size of the banking sector is smaller. As λ increases, the central planner is willing to increase the size of the banking sector (alternatively decrease the size of the buffer) because the chance of the economy spending a long time in the state in which there is no growth is small. Thus, when the growth rate in the entrepreneurial sector is high, the optimal size of the banking sector is increasing in the flexibility of capital (λ).

The situation is reversed when the growth rate of capital is quite low. In this case, the cost to the central planner of ending up with too much capital in the entrepreneurial sector is high because the return is low relative to the risk. Therefore, he hedges against this possibility by maintaining a somewhat larger banking sector. As the flexibility of capital increases, he is willing to reduce the size of the banking sector as he no longer needs a buffer
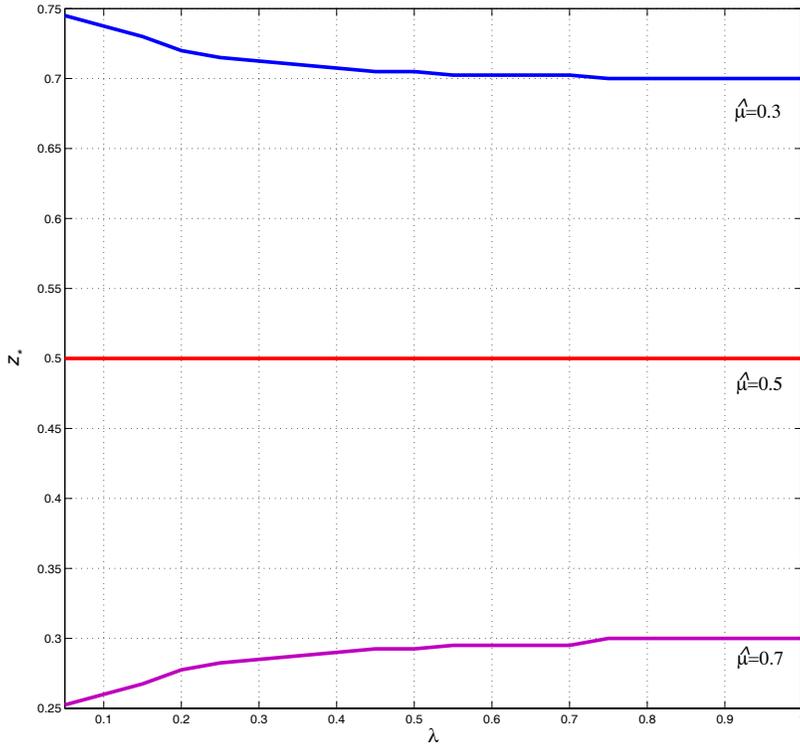
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Figure 4: **The optimal size of the banking sector, z^*, as a function of λ for different values of $\hat{\mu}$.**

against the possibility that the entrepreneurial sector will become too large.

The previous discussion suggests that the relationship between the size of the banking sector and the flexibility of capital is not monotonic. Specifically, financial innovation or government policy that increases the speed with which funds can be reallocated between sectors may, in equilibrium, decrease the size of the banking sector, or increase it. In particular, increasing the financial flexibility may decrease the growth rate of the economy.

Observation 3 *Compare two otherwise identical economies; one with $\hat{\lambda}$, and the other with $\widetilde{\lambda} > \hat{\lambda}$. Then, the economy with more capital flexibility, $\widetilde{\lambda}$ may have a lower growth rate.*

This observation follows directly from the previous one. In this economy, the larger the unintermediated sector, the higher the growth rate. Indeed, the growth rate of the economy is just $(1 - z)\hat{\mu}$. Each dollar invested in the risky sector grows at an expected rate of $\hat{\mu}$ and the banking sector (by assumption) has a growth rate of zero. Therefore increasing the flexibility of capital may decrease the growth rate of the economy. This suggests that

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cross–country regressions of economic performance (including growth rates) on proxies for financial innovation or variables that measure the speed with which capital flows between the banking and entrepreneurial sectors are complex to interpret. For example, the work of Levine (1998), drawing on that of La Porta, de Silanes, Shleifer, and Vishny (1998) considers the effect of legal protections on the development of banks and subsequent growth rates. Our analysis suggest that unambiguous causal links are difficult to find because increasing the efficiency of the banking sector may lead to an overall larger or smaller sector depending on the fundamentals of the economy.

More broadly, this observation fits into the long–running debate about the relationship between economic growth rates and financial innovation. Rather than view the view financial flexibility as a cause (Schumpeter (1911)) or a consequence (Robinson (1952)) of economic growth, we focus on economic growth as the natural consequence of the equilibrium risk appetite of a representative consumer. Specifically, the existence of high financial flexibility may induce the central planner to maintain a large banking sector and, consequently, a low stationary growth rate.

It is a stylized fact that bank lending is procyclical. Indeed, during upswings there is an increase in bank lending, whereas during downturns there is typically a decrease. The increases and decreases are not necessarily just proportional to changes in the aggregate size of the economy. Rather bank loans typically increase more than proportionally when the economy is increasing and contract more than the economy during a downturn. Indeed, a "credit crunch" during a recession illustrates the case where credit–worthy borrowers cannot find loans and the banking sector is shrinking. The countercylical pattern of bank lending appears to be an international phenomenon.[11]

In addition, a growing empirical literature finds support for the existence of a bank lending channel for monetary policy. In particular, this implies that changing banks' propensity to lend has real effects. Bernanke and Blinder (1992) demonstrates that changes in aggregate bank lending follow changes in monetary policy. In the cross–section, the literature typically examines how the loans extended by banks change in response to changes in the monetary policy. For example, Kishan and Opiela (2000) find that small, low capitalization banks are unable to increase time deposits in response to contractionary policy; necessarily decreasing their loan supply. In their 1980–1995 sample, 15.6% of the mean banking system assets are affected in this way. Kashyap and Stein (2000) also find a shift in lower capitalized banks' loan supply schedules with changes in financial flexibility and conclude that the lending channel exists.

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[11]For both US and international patterns see Bikker and Hu (2001),Borio and Lowe (2002), and Horvath (2002).

So far, we have indexed the economies by the speed with which capital reallocates. Specifically, if there is a shock to one sector and the economy is no longer at the optimal size, the central planner will increase or decrease the relative sizes of the two sectors to ensure that the economy reverts to its long term equilibrium values. Recall, from Lemma 1, the sign of the reallocation depends on the marginal normalized social benefit of changing capital between the two sectors (w_z). Figure 5 illustrates the sign of control a as a function of the size of the banking sector.
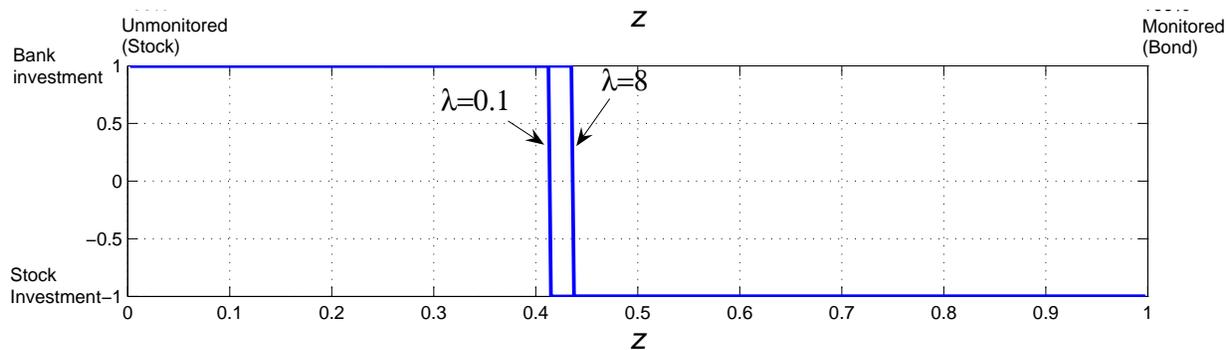

Figure 5: **Sign of control, a. For high λ, larger domain in which bank investments occur. Parameters:** $\mu = 1/4$, $\sigma^2 = 4$, $\rho = 1$.

Recall, that maximal resources are devoted to the sector that has shrunk, therefore the lines are either along the top of the box, the bottom, or switch (the vertical lines) in the middle. Clearly, this depends on the speed (λ). Therefore, the triggers (in terms of size of the banking sector) for capital reallocation differ depending on the speed with which this occurs. Specifically, consider small bank sectors so that $z \in (0, z*)$. If reallocation is rapid so that λ is large, then if a shock drives the economy into this region, the central planner will divert resources to the banking sector. By contrast, if reallocation is slow, so that λ is small, then capital optimally flows to the banking sector for $z \in (0, z')$, where $z' < z_*$. Thus, for z between z' and z_* capital is reallocated to the banking sector in the high λ case, but not in the low λ case. This asymmetry arises from the fact that utility is lower when all resources are in the banking sector than when all resources are in the entrepreneurial sector.

It also follows that, comparing two economies with slightly different λ's, the direction of capital flow is ambiguous. On one hand, a higher λ allows capital to move toward z_* faster, with increases current capital flows. On the other hand, a higher λ may result in a different optimal z_*, which can lead to a full reversal of capital flows.

Observation 4 *A financing innovation can either increase the speed at which capital flows into a sector, or reverse it.*

Specifically, in the latter case, small financing innovations and other (unanticipated) changes in flexibility can have a large impact on capital flows into and out of the banking sector.

Indeed, if the economy is in the region, $z \in (z', z_*)$, then, relaxing the constraint slightly (a marginal increase in λ) may lead to a huge shift toward bank investments (from $a = -\lambda$ to $a = \lambda$).

The states of the world in which a change in λ lead to large changes in capital flows are in regions where the value function is quite flat. Intuitively, the central planner only chooses a "buffer" if the welfare cost of such a strategy is low. Therefore, the effects of a change in financial flexibility is the largest in terms of capital flows in states of the world when the welfare gains are quite small. On the other hand, if the economy is at one of the extremes, then changing financial flexibility has no effect on the reallocation policy and capital flows, except for marginally increasing them. It will however have a large effect on social welfare.

Observation 5 *The effects on capital flows of monetary policy are the largest in states of the world in which the welfare effects are small.*

The welfare effects of monetary policy are largest in states of the world in which the effects on capital flows are small.

We use proposition 2 to develop an example that shown these effects, and also the stationary distribution of the economy. Suppose that $\mu = 1/2$, $\sigma^2 = 3$. Under full capital flexibility $\lambda = \infty$, the social optimum is to have $z_* = 1/2 - (1/2)/3 = 1/3$, iso that unmonitored stock make up two thirds of the economy. However, for small λ, the optimal banking sector size $\hat{z} \approx 0.18$. Thus, if the rate at which capital can be reallocated is sufficiently small, the central planner moves resources into the banking sector for $0.18 < z < 0.33$, even though it would be optimal to move in the other direction under capital flexibility. The reason that $\hat{z} < z_*$ is that the disutility cost of z close to 1 is very high, so the central planner chooses an extra buffer zone to ensure that this rarely happens.

To examine the role of government intervention, assume that the capital market has high flexibility for reallocation in some regions, but not in others. Specifically, suppose that

$$\lambda_M(z) = \begin{cases} 2 & z < \frac{1}{3}, \\ 0 & z \geq \frac{1}{3}. \end{cases}$$

Thus, as long as $z < 1/3$, the market is quite flexible in reallocating capital. When $z > 1/3$, however, the market has no flexibility. Instead, assume that the government, through open market and other operations, also has an impact on λ, in that

$$\lambda = \lambda_M(z) + \lambda_G,$$

where λ_G is a constant.

In the example, when $\lambda_G = 0$, we have $\hat{z} = 0.18$, whereas when $\lambda_G = 0.4$, $\hat{z}$ is 0.23 and for $\lambda_G = 1$, $\hat{z} = 0.28$. Thus, in the region $z \in (0.18, 0.23)$, which is in a higher risk state of the world than optimal, if the government through active involvement changes λ_G from 0 to 0.4, bank investments change from -2 to 2.4, i.e., a change of 4.4. The government's action is thus multiplied by 11. This is similar to the point made in observation 4.

Does this government intervention work? We can compare the stationary probability distributions for the two cases just studied. In Figure 6, the stationary distribution for the case when $\lambda_G = 0$ is shown.
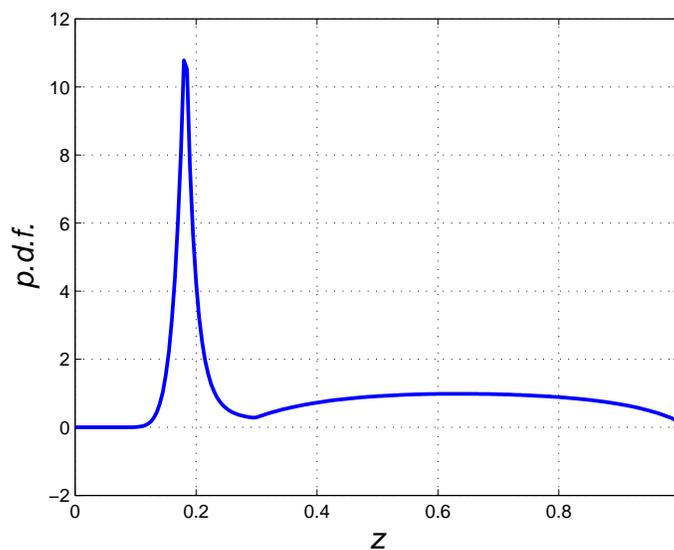

Figure 6: **Stationary distribution of z, when $\lambda_G = 0$. Parameters: $\mu = 1/2$, $\sigma^2 = 3$, $\rho = 1/4$, $\hat{z} = 0.18$.**

Compare this to the case when $\lambda_G = 0.4$ in Figure 7. The peaks of the two distributions are at slightly different points, corresponding to the different optimal controls. The major difference, however, from the perspective of the central planner is that the probability of being close to $z = 1$ is much lower in the second figure than in the first. Thus, a policy change that increases λ_G from 0 to 0.4 which is implemented when $z = 0.2$, allows the
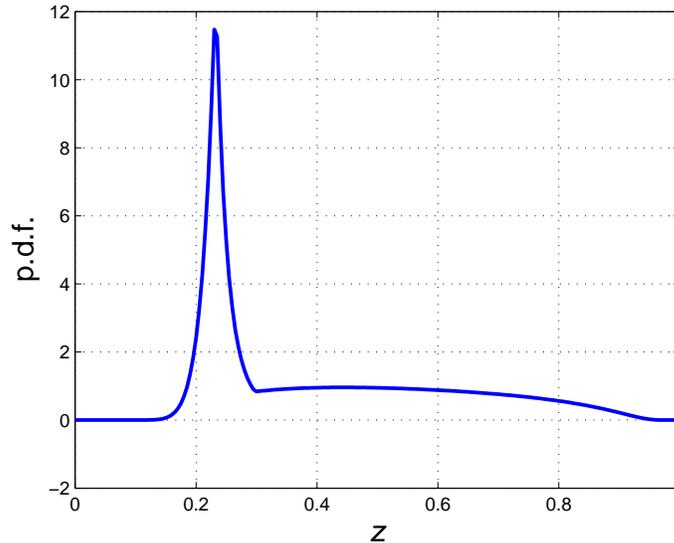
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Figure 7: **Stationary distribution of z, when $\lambda_G = 0.4$. Parameters: $\mu = 1/2$, $\sigma^2 = 3$, $\rho = 1/4$, $\hat{z} = 0.23$.**

government to signal that the risks of reaching a low-growth state of the world, in which z is close to one and the risky sector has basically shut down, are negligible. This makes the central planner less averse to the risks of high z. The central planner therefore changes control policy, a, moving $\hat{z}$ from 0.18 to 0.23, closer to the unconstrained optimum, which we know from lemma 3 is at $z_* = 0.33$. This leads to a huge shift toward bank investments.

How often will the economy be in "extreme states"? Of particular interest is the likelihood that the economy will enter a period of low growth (or negative growth relative to the long term stationary average). In our model, this occurs as the result of a negative shock in the entrepreneurial sector so that the banking sector becomes disproportionately large. Without instantaneous capital adjustment the economy will undergo a period of readjustment until it again reaches the optimal level. Thus, the probability of an is higher for the economies in which λ is lower. However, in line with the previous buffer zone argument of observation 2, the economy may be in lower growth states more often for higher λ. Thus if the recession is defined as a state of the world in which growth is low,[12] depending on how extreme the definition of what constitutes a recession, a higher λ may increase or decrease the likelihood of observing a recession.

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[12]According to the NBER's web site (http://www.nber.org/cycles.html): "The NBER does not define a recession in terms of two consecutive quarters of decline in real GDP. Rather, a recession is a significant decline in economic activity spread across the economy, lasting more than a few months, normally visible in real GDP, real income, employment, industrial production, and wholesale-retail sales."

Observation 6 *Compare two otherwise identical economies; one with $\hat{\lambda}$, and the other with $\widetilde{\lambda} > \hat{\lambda}$. Then, the economy with more capital flexibility, $\widetilde{\lambda}$ may have a higher likelihood of low-growth states, but always has a lower likelihood of extreme low-growth states.*

Finally, we can connect the term structure analysis in the previous section with the presence of recessions. We have

Observation 7 *Suppose that $\mu > 0$. Then, in a boom period ($z < z_*$), $r_l - r_s$ is decreasing in z. Also, for low enough z, $r_l < r_s$. Moreover, the expected change in future growth rate, $E[d[\hat{\mu}(1-z)]]$, is negative and decreasing in $1 - z$.*

In a recession ($z > z_*$), $r_l - r_s$ is increasing in z.

Thus, a downward sloping term structure predicts a lower growth rate of the economy, and the severity of a recession can be measured by how (upward) sloped the term structure.

6 Concluding remarks

We have developed a simple framework that incorporates monetary policy, the term structure and growth rates. The weakness of our model is that, while we can characterize the solutions, all term structures must be generated numerically. However, our framework does allow a simple, economic integration of asset pricing the intermediated finance.

The overall implication of our model is that the share of intermediated capital in the economy should be closely related to asset prices and, specifically, to the term structure of interest rates. Further, fundamental characteristics of the macro economy such as growth rates are related to the size of banking sector.

A   Some Microfoundations for our assumptions

In any model in which banks invest funds and monitor, and banks' actions are not perfectly observable; one should observe, in equilibrium, a restriction on the speed with which capital flows into the banking sector. Specifically, under a plausible set of frictions, rational investors will only commit funds slowly to the banking sector.

To see this, consider the following infinite horizon discrete time framework in which an investor in each time period invests capital in a risk-free technology. The stock of this technology at t is B_t, and αB_t matures ("dies") at B_{t+1}. The owner of the technology receives dividends in each period, rB_t. The maximum capital that can be invested is $\overline{q}B_{-1}$, and so the investor chooses an investment $0 \leq q \leq \overline{q}$. The capital at time $t + 1$ is therefore

$$B_{t+1} = B_t(1 + q_t - \alpha).$$

In the special case in which $\alpha = \overline{q}$, we can think of this as if all the maturing capital can be reinvested, but we do not have to restrict ourselves to this case.

Suppose that the investor has a personal discount factor per time period of $\rho < 1$ and the initial stock was $B_{-1} = 1$. Also suppose that the investor's utility is $U = \sum_{t=0}^{\infty} \rho^t rB_t$. Since the capital does not provide any utility other than indirectly through dividends, it is optimal for the investor to invest as much as possible in each time period, since this yields the largest possible B_t for all t. The investor will therefore choose $q_t = \overline{q}$, for all t. Hence, $B_t = B_{-1}(1 + \overline{q} - \alpha)^{t+1}$ and utility will be $U = \sum_{t=0}^{\infty} \rho^t rB_t = \sum_{t=0}^{\infty} \rho^t r(1 + \overline{q} - \alpha)^{t+1} = \frac{rB_{-1}(1+\overline{q}-\alpha)}{1-\rho(1+\overline{q}-\alpha)}$.

Consider, now the strategy of an investor who transfers resources through an intermediary. The intermediary collects a fee $\epsilon \times B_{t+1}$ at time t and has the same form of utility function as the investor. It is costly to monitor new projects, and the bank pays an up-front cost ϵc per unit of new capital being monitored. Thus, if qB_t is being invested and the bank monitors, the net benefits for the bank in period t are $\epsilon B_t(1 + q_t - \alpha - qc)$.

If the bank shirks, it no longer incurs the cost ϵcqB_t, but the technology terminates, so that $B_{t+2} = 0$, and no dividends are paid out in the future. For most reasonable parameter values, it will be socially optimal for the bank to monitor the technology. The net private benefits to the bank if it shirks are therefore $\epsilon B_t(1 + q_t - \alpha)$. Define $V(B_{t+1})$ to be the value at time $t - 1$ of the future payoffs accruing to the bank following any history, given that it optimizes its shirking strategy.

It is incentive compatible for the bank to monitor in period t if $\epsilon B_t(1+q_t-\alpha-q_tc)+\rho V(B_{t+1}) \geq \epsilon B_t(1 + q_t - \alpha)$. Suppose that q_t is constant. Then, since the problem is homogeneous, in time and in B_t (because, q_t is constant and because of the time invariant form of the utility) the bank, if it monitors in the first period, does so in all subsequent periods. Therefore, the infinite horizon payoff to the bank can be expressed as $\epsilon B_{-1}\frac{1+q-\alpha-qc}{1-\rho(1+q-\alpha)}$. [13] Therefore, the bank will monitor if

$$\epsilon B_{-1}(1 + q - \alpha) \leq \epsilon B_{-1}\frac{1 + q - \alpha - qc}{1 - \rho(1 + q - \alpha)}.$$

Some algebra leads to the observation that under some parameter restrictions, there is a $\widetilde{q} < \overline{q}$ that is the maximum amount that the investor would be willing to transfer through the bank in any period. [14]

In this simple framework, if there are up-front costs of monitoring, and the monitoring action is not observable, then it will always be optimal to cap the investment per unit time. For example, making the asset risky, having nonlinear utility functions, different personal discount rates for the bank and investor respectively will not qualitatively change the results. Obviously, if there are multiple technologies (trees), in general the cap will be different for different trees.

In principle, we could solve the agent's optimization problem in our two-tree model (with λ replacing $\widetilde{q}$). The homogeneity of the problem immediately implies that $\underline{\lambda}$ and $\overline{\lambda}$ will not depend

---

[13]This follows from $\sum_{t=0}^{\infty} \rho^t \epsilon(B_t - qcB_{t-1}) = \sum_{t=0}^{\infty} \rho^t \epsilon B_{-1}(1 + q - \alpha)^t(1 + q - \alpha - qc)$.

[14]Let $b = 1 - \alpha$. We assume that $c > 4\rho b$, and we define $\widetilde{q} = \frac{c}{2\rho} - b - \sqrt{\left(\frac{c}{2\rho} - b\right)^2 - b^2}$ and $q_2 = \frac{c}{2\rho} - b + \sqrt{\left(\frac{c}{2\rho} - b\right)^2 - b^2}$. It is easy to check that $(b + q)^2 - \frac{qc}{\rho} \geq 0$, for $q \leq \widetilde{q}$ and for $q \geq q_2$, and that $(b + q)^2 - \frac{qc}{\rho} < 0$ for $q \in (\widetilde{q}, q_2)$. Thus, if $\overline{q} < q_2$, then the investor will transfer $\widetilde{q}B_t$ per period, since any higher compensation will cause the intermediary to shirk.

on any other variables than z. Numerically, however, this is a difficult problem to solve, for two reasons. First, in solving the problem, we would have to take the strategic dynamics between the bank and investor into account. These were simple in our example, but would be more complex with an additional state variable (i.e., the share, z): The bank's value function and monitoring strategy would depend on the investor's allocation rules as a function of z, and the investor's allocation rule would, similarly, depend on the bank's monitoring strategy as a function of z. Second, in continuous time we would need a time period over which the up-front costs of monitoring new investments occur. Such a "discretization" would make the problem numerically complex, e.g., leading to a partial integro differential equation (p.i.d.e.) instead of a partial differential equation (p.d.e.). We therefore proceed by taking the existence of the financial frictions $\lambda(z)$ as given.

B Extension to general constant relative risk averse utility

The analysis above can be extended to an investor with general constant relative risk aversion with risk aversion parameter γ and utility, $U = \frac{1}{1-\gamma} E[\int_0^T e^{-\rho t} c_t^{1-\gamma}] dt$, rather than just log utility. Writing the central planner's problem as

$$V(B, D, t) \equiv \sup_{a \in \mathcal{A}} E_t \left[\int_t^T e^{-\rho(s-t)} \frac{(B+D)^{1-\gamma}}{1-\gamma} \, ds \right], \tag{25}$$

proceeding as above shows that the Bellman equation for optimality is

$$\sup_{a \in \mathcal{A}} \left[V_t + \frac{1}{2}\sigma^2 D^2 V_{DD} + [\widehat{\mu} D - a(B+D)] V_D + a(B+D)V_B - \rho V + \frac{(B+D)^{1-\gamma}}{1-\gamma} \right] = 0, \tag{26}$$

and that we can write

$$V(B, D, t) = (B+D)^{1-\gamma} \, w(z, t),$$

where w solves the partial differential equation

$$w_t + \frac{1}{2}\sigma^2 z^2 (1-z)^2 w_{zz} + \left[a - \widehat{\mu} z(1-z) + \sigma^2 \gamma z(1-z)^2 \right] w_z$$

$$- \left[\rho - \widehat{\mu}(1-\gamma)(1-z) + \frac{1}{2}\sigma^2 \gamma(1-\gamma)(1-z)^2 \right] w + \frac{1}{1-\gamma} = 0. \tag{27}$$

Proceeding as in the derivation of Equation (22), we obtain the following pricing equation for any asset P:

$$P_t + \frac{1}{2}\sigma^2 D^2 P_{DD} + \left[\widehat{\mu} D - a(B+D) - \frac{\sigma^2 \gamma D^2}{B+D} \right] P_D + a(B+D)P_B$$

$$- \left(\rho + \widehat{\mu}\gamma \frac{D}{B+D} - \frac{1}{2}\sigma^2 \gamma(\gamma+1) \frac{D^2}{(B+D)^2} \right) P + \delta(B, D, t) = 0. \tag{28}$$

For bonds, this equation can be simplified to

$$p_t + \frac{1}{2}\sigma^2 z^2 (1-z)^2 p_{zz} + \left[a - \widehat{\mu} z(1-z) + \sigma^2 (1+\gamma)z(1-z)^2 \right] p_z$$

$$- \left[\rho + \widehat{\mu}\gamma(1-z) - \frac{1}{2}\sigma^2 \gamma(1+\gamma)(1-z)^2 \right] p = 0. \tag{29}$$

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C   Proofs

Proof of Proposition 5
We here prove a more general result, corresponding to general CRRA utility. The result for r_s is standard. Since

$$P_t^\tau + \frac{1}{2}\sigma^2 z^2(1-z)^2 P_{zz}^\tau + \left[-\widehat{\mu}z(1-z) + 2\sigma^2 z(1-z)^2\right]P_z^\tau - \left[\rho + \gamma\widehat{\mu}(1-z) - \frac{1}{2}\gamma(\gamma+1)\sigma^2(1-z)^2\right]P^\tau = 0,$$

and $P^\tau(\tau, z) = 1$, it is clear that $P(0, z) = 1 - \left[\rho + \gamma\widehat{\mu}(1-z) - \frac{1}{2}\gamma(\gamma+1)\sigma^2(1-z)^2\right]\tau + o(\tau)$, for small τ, and since $-\log(1-s) = s + O(s^2)$ for small s, it is clear that $r_s = \lim_{\tau\searrow 0} -\frac{\log(P^\tau)}{\tau} = \rho + \gamma\widehat{\mu}(1-z) - \frac{1}{2}\gamma(\gamma+1)\sigma^2(1-z)^2$.

For $z = 0$ or $z = 1$, the model is a one tree model. In a one tree model, the long rate equals the short rate, $r_l = r_s$, so the results are immediately implied from the form of r_s.

For r_l, when $z \in (0, 1)$, we proceed as follows: We have

$$P^\tau = (1 + e^d)^\gamma e^{-\rho\tau}\frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty}\frac{e^{-(y-\mu\tau)^2/(2\sigma^2\tau)}}{(1 + e^{d+y})^\gamma}dy = (1 + e^d)^\gamma e^{-\rho\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2/2}}{(1 + e^d e^{x\sigma\sqrt{\tau}+\mu\tau})^\gamma}dx.$$

We study the behavior of $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2/2}}{(1+e^d e^{x\sigma\sqrt{\tau}+\mu\tau})^\gamma}dx$ for large τ. We decompose:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2/2}}{(1 + e^d e^{x\sigma\sqrt{\tau}+\mu\tau})^\gamma}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}\frac{e^{-x^2/2}}{(1 + e^{x\sigma\sqrt{\tau}+\mu\tau+d})^\gamma}dx + \frac{1}{\sqrt{2\pi}}\int_{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}^{\infty}\frac{e^{-x^2/2}}{(1 + e^{x\sigma\sqrt{\tau}+\mu\tau+d})^\gamma}dx.$$

Since $0 < e^{x\sigma\sqrt{\tau}+\mu\tau+d} \leq 1$ for $x \leq -\frac{\mu\tau+d}{\sigma\sqrt{\tau}}$, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}\frac{e^{-x^2/2}}{(1 + e^{x\sigma\sqrt{\tau}+\mu\tau+d})^\gamma}dx = C\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}e^{-x^2/2}dx = CN\left(-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}\right),$$

for some $C \in [1/2^\gamma, 1]$, where $N(\cdot)$ is the cumulative normal distribution function, $N(x) = \frac{1}{2\pi}\int_{-\infty}^{x}e^{-y^2/2}dy$. Now, we use the bound

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x}e^{-y^2/2}dy = C_2\frac{e^{-x^2/2}}{x}, \qquad C_2 \in \left[\frac{x^2}{1+x^2}, 1\right],$$

which is valid for large x, to get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}\frac{e^{-x^2/2}}{(1 + e^{x\sigma\sqrt{\tau}+\mu\tau+d})^\gamma}dx = C_2\frac{e^{-(\frac{\mu\tau+d}{\sigma\sqrt{\tau}})^2/2}}{q^\gamma} = C_2\frac{e^{-\frac{\mu^2}{2\sigma^2}\tau - \frac{\mu d}{\sigma^2} - \frac{d^2}{2\sigma^2\tau}}}{q^\gamma}, \; C_2 \in [\frac{1}{2^{\gamma+1}}, 1], \; q = \frac{\mu\tau+d}{\sigma\sqrt{\tau}}.$$

We next study the second term. First, we note that $\mu < \left(\gamma - \frac{1}{2}\right)\sigma^2$ implies that $\gamma\sigma - \frac{\mu}{\sigma} > 0$.

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Obviously, $\frac{1}{(1+e^{x\sigma\sqrt{\tau}+\mu\tau+d})^{\gamma}} \le e^{-\gamma(x\sigma\sqrt{\tau}+\mu\tau+d)}$, so

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}^{\infty} \frac{e^{-x^2/2}}{(1+e^{x\sigma\sqrt{\tau}+\mu\tau+d})^{\gamma}} dx \quad &\le \quad \frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}^{\infty} e^{-(x^2+2x\gamma\sigma\sqrt{\tau})/2-\gamma\mu\tau-\gamma d} dx \\
&= \quad \frac{1}{\sqrt{2\pi}} \int_{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}}^{\infty} e^{-(x+\gamma\sigma\sqrt{\tau})^2/2+\frac{\gamma^2\sigma^2\tau}{2}-\gamma\mu\tau-d\gamma} dx \\
&= \quad \frac{e^{-\gamma d}e^{-\tau(\gamma\mu-\gamma^2\sigma^2/2)}}{\sqrt{2\pi}} \int_{-\frac{\mu\tau+d}{\sigma\sqrt{\tau}}+\gamma\sigma\sqrt{\tau}}^{\infty} e^{-x^2/2} dx \\
&= \quad e^{-\gamma d}e^{-\tau(\gamma\mu-\gamma^2\sigma^2/2)} N\left(-\left(\gamma\sigma-\frac{\mu}{\sigma}\right)\sqrt{\tau}+\frac{d}{\sigma\sqrt{\tau}}\right) \\
&\le \quad e^{-\gamma d}e^{-\tau(\gamma\mu-\gamma^2\sigma^2/2)} \frac{e^{-\frac{d^2}{2\sigma^2\tau}-\gamma d+\frac{d\mu}{\sigma^2}-(\gamma^2\frac{\sigma^2}{2}-\gamma\mu+\frac{\mu^2}{2\sigma^2})\tau}}{q^{\gamma}} \\
&= \quad e^{-2\gamma d-\frac{d^2}{2\sigma^2\tau}+\frac{d\mu}{\sigma^2}} \times \frac{e^{\frac{-\mu^2}{2\sigma^2}\tau}}{q^{\gamma}}.
\end{aligned}
$$

Putting it all together, we get that for large τ,

$$
P^{\tau} = (1+e^d)^{\gamma}\left(\frac{\sigma\sqrt{\tau}}{\mu\tau+d}\right)^{\gamma} e^{-\frac{d^2}{2\sigma^2\tau}}\left(C_2 e^{-\frac{d\mu}{\sigma^2}}+e^{-2\gamma d+\frac{d\mu}{\sigma^2}}\right) \times e^{-\left(\frac{\mu^2}{2\sigma^2}+\rho\right)\tau}.
$$

Therefore,

$$
-\frac{\log(P^{\tau})}{\tau} = \frac{\mu^2}{2\sigma^2} + \rho + \frac{\log(Q(\tau))}{\tau}, \qquad Q(\tau) = \log\left((1+e^d)^{\gamma}\left(\frac{\sigma\sqrt{\tau}}{\mu\tau+d}\right)^{\gamma} e^{-\frac{d^2}{2\sigma^2\tau}}\left(C_2 e^{-\frac{d\mu}{\sigma^2}}+e^{-2\gamma d+\frac{d\mu}{\sigma^2}}\right)\right),
$$

and since $Q(\tau) = o(\tau)$, it is clear that $\lim_{\tau\to\infty} -\frac{\log(P^{\tau})}{\tau} = \frac{\mu^2}{2\sigma^2}+\rho$. We are done.

Proof of Proposition 1

We first note that $aw_z = \lambda(z)\,\text{sign}(w_z)w_z = \lambda(z)|w_z|$, so (15) is the same as (13). We define a solution to the central planner's optimization to be interior if $a(t,0) > 0$ and $a(t,1) < 0$ in a neighborhood of the boundaries for all $t < T$. where the radius of the neighborhoods do not depend on t. A solution is thus interior if it is always optimal for the central planner to stay away from the boundaries, $z = 0$ and $z = 1$. From our previous argument, we know that any smooth interior solution must satisfy (15). What remains to be shown is that the solution to the central planner's problem is indeed interior, and that, given that the solution is interior, equations (15) and (16) have a unique, smooth, solution, i.e., that (15) and (16) provide a *well posed* p.d.e. (Egorov and Shubin (1992)).[15]

We begin with the second part, i.e., the well posedness of the equation, given that the solution is interior. As is usual, we first study the Cauchy problem, i.e., the problem without boundaries, on the entire real line $z \in \mathbb{R}$ (or, equivalently, with periodic boundary conditions). We then extend the analysis to the bounded case, $z \in [0,1]$. Equation (15) has the structure of a generalized KPZ equation, which has been extensively studied in recent years, see Kardar, Parisi, and Zhang (1986), Gilding, Guedda, and Kersner (2003), Ben-Artzi, Goodman, and Levy (1999), Hart and Weiss (2005), Laurencot and Souplet (2005) and references therein. The Cauchy problem is well-posed, i.e., given bounded, regular, initial conditions, there exists a unique, smooth, solution. Specifically, given continuous, bounded, initial conditions, there is a unique solution that is bounded, twice continuously

---

[15]The concept of well-posedness additionally requires the solution to depend continuously on initial and boundary conditions. This requirement is natural, since we can not hope to numerically approximate the solution if it fails.

differentiable in space and once continuously differentiable in time, i.e., $w \in C^{2,1}[0,T] \times \mathbb{R}$ (see, e.g., Ben-Artzi, Goodman, and Levy (1999)).

Given that the Cauchy problem is well-posed and that the solution is smooth, it is clear that $a = \lambda(z)\,\text{sign}(w_z)$ will have a finite number of discontinuities on any bounded interval at any point in time. Moreover, given that the solution is interior, a is continuous in a neighborhood of $z = 0$ and also in a neighborhood of $z = 1$. The p.d.e.

$$0 \;=\; w_t - \rho w + (a - z(1-z)\hat{\mu} + z(1-z)^2\sigma^2)w_z + \frac{\sigma^2}{2}z^2(1-z)^2 w_{zz} + q(t,z),$$

is parabolic in the interior, but hyperbolic at the boundaries, since the $\frac{\sigma^2}{2}z^2(1-z)^2 w_{zz}$-term vanishes at boundaries. For example, at the boundary, $z = 1$, using the transformation $\tau = T - t$, the equation reduces to

$$w_\tau \;=\; -\rho w - \lambda(1)w_z.$$

Similarly, at $z = 0$, the equation reduces to

$$w_\tau \;=\; -\rho w + \lambda(0)w_z + q(t,0).$$

Both these equations are hyperbolic and, moreover, they both correspond to outflow boundaries. Specifically, the characteristic lines at $z = 0$ are $\tau + z/\lambda(0) = const$, and at $z = 1$ they are $\tau - z/\lambda(1) = const$. For outflow boundaries to hyperbolic equations, no boundary conditions are needed, i.e., if the Cauchy problem is well posed, then the initial-boundary value with an outflow boundary is well-posed without a boundary condition (Kreiss and Lorenz (1989)). This suggests that no boundary conditions are needed.

To show that this is indeed the case, we use the energy method to show that the operator $Pw \overset{\text{def}}{=} \rho w + (a - z(1-z)\hat{\mu} + z(1-z)^2\sigma^2)w_z + \frac{\sigma^2}{2}z^2(1-z)^2 w_{zz}$ is maximally semi-bounded, i.e., we use the L_2 inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$, and the norm $\|w\|^2 = \langle w, w \rangle$, and show that for any smooth function, w, $\langle w, Pw \rangle \leq \alpha\|w\|^2$, for some $\alpha > 0$.[16] This allows us to bound the growth of $\frac{d}{d\tau}\|w(t,\cdot)\|^2$ by $\frac{d}{d\tau}\|w(t,\cdot)\|^2 \leq \alpha\|w\|^2$, since $\frac{1}{2} \times \frac{d}{d\tau}\|w(t,\cdot)\|^2 = \langle w, Pw \rangle$. Such a growth bound, in turn, ensures well-posedness (see Kreiss and Lorenz (1989) and Gustafsson, Kreiss, and Oliger (1995)).

We define $I = [\epsilon, 1-\epsilon]$. Here, $\epsilon > 0$ is chosen such that w_z is nonzero outside of I for all $\tau > 0$. By integration by parts, we have

$$
\begin{aligned}
\langle w, Pw \rangle \;=\;& -\rho\|w\|^2 + \langle w, cw_z \rangle + \langle w, dw_{zz} \rangle \\
\;=\;& -\rho\|w\|^2 + \frac{1}{2}\left(\langle w, cw_z \rangle - \langle w_z, cw \rangle - \langle w, c_z w \rangle + [w^2 c]_0^1 \right) - \langle w_z, dw_z \rangle - \langle w, d_z w_z \rangle + [wdw_z]_0^1 \\
\;=\;& -\rho\|w\|^2 - \langle w, c_z w \rangle - \lambda(1)w(t,1)^2 - \lambda(0)w(0,t)^2 - \langle w_z, dw_z \rangle - \langle w, d_z w_z \rangle \\
\;\leq\;& (r - \rho)\|w\|^2 + \gamma\max_{z\in I} w(z)^2 - \langle w_z, dw_z \rangle - \langle w, d_z w_z \rangle \\
\;\leq\;& \left(r + \sigma^2 - \rho\right)\|w\|^2 + \gamma\max_{z\in I} w(z)^2 - \frac{\sigma^2}{2}\int_0^1 z^2(1-z)^2 w_z^2 dz,
\end{aligned}
$$

where $c(t,z) = a - \hat{\mu}z(1-z) + \sigma^2 z(1-z)^2$ and $d(z) = \sigma^2 z^2(1-z)^2/2$. Also, $\gamma = 2\max_{z\in I}\lambda(z)$, and

---

[16] Since we impose no boundary conditions, it immediately follows that P is maximally semi-bounded if it is semi-bounded.

$r = \max_{0 \le z \le 1} |\hat{\mu} z(1-z) - \sigma^2 z(1-z)^2|$. Here, the last inequality follows from

$$
\begin{aligned}
-\langle w_z, dw_z \rangle - \langle w, d_z w_z \rangle &= \frac{\sigma^2}{2} \int_0^1 z(1-z) \left(-z(1-z)w_z^2 - (2-4z)ww_z \right) dz \\
&\le \frac{\sigma^2}{2} \int_0^1 z(1-z) \left(-z(1-z)w_z^2 + 2|w||w_z| \right) dz \\
&\le \frac{\sigma^2}{2} \int_0^1 z(1-z) \left(-z(1-z)w_z^2 + \frac{z(1-z)}{2} w_z^2 + \frac{2}{z(1-z)} w^2 \right) dz \\
&= \sigma^2 \|w\|^2 - \frac{\sigma^2}{2} \int_0^1 z^2(1-z)^2 w_z^2 dz,
\end{aligned}
$$

where we used the relation $|u||v| \le \frac{1}{2}(\delta|u| + |v|/\delta)$ for all u, v for all $\delta > 0$. Finally, a standard Sobolev inequality implies that

$$
\gamma \max_{z \in I} w(z)^2 \le \gamma \left(\xi \int_I w_z(z)^2 dz + \left(\frac{1}{\xi} + 1 \right) \int_I w(z)^2 dz \right),
$$

for arbitrary $\xi > 0$. Specifically, we can choose $\xi = \epsilon^2(1-\epsilon)^2/(2\gamma)$ to bound

$$
\gamma \max_{z \in I} w(z)^2 - \frac{\sigma^2}{2} \int_0^1 z^2(1-z)^2 w_z^2 dz \le \gamma \left(\frac{1}{\xi} + 1 \right) \|w\|^2,
$$

and the final estimate is then

$$
\frac{d}{d\tau} \|w\|^2 \le \left(r + \sigma^2 - \rho + \frac{\gamma}{\xi} + \gamma \right) \|w\|^2.
$$

We have thus derived an energy estimate, for the growth of $\|w\|^2$, and well-posedness follows from the theory in Kreiss and Lorenz (1989) and Gustafsson, Kreiss, and Oliger (1995). Notice that we also used that $a(0, \cdot) > 0$ and $a(1, \cdot) < 0$ in the first equation, to ensure the negative sign in front of the $\lambda(0)$ and $\lambda(1)$ terms.

What remains is to show that if condition 1 is satisfied, then indeed the solution is an interior one. We first note that an identical argument as the one behind Proposition 1 in Longstaff (2001) implies that the central planner will never choose to be in the region $z < 0$ or $z > 1$, since the non-zero probability of ruin in these regions always make such strategies inferior. Since any solution will be smooth, the only way in which the solution can fail to be interior is thus if $a = 0$ for some t, either at $z = 0$, or at $z = 1$.

We note that close to time T, the solution to (13) will always be an interior one, since $\hat{\mu}(1 - z) - \frac{\sigma^2}{2}(1-z)^2$ is strictly concave, with an optimum in the interior of $[0, 1]$ and

$$
w_z(T - \tau, z) = \int_0^\tau q_z(T - s, z)ds + O(\tau^3) = \frac{\tau^2}{2} \left(-\hat{\mu} + \sigma^2(1-z) \right) + O(\tau^3),
$$

so the solution to $w_z = 0$ lies at $z_* = 1 - \frac{\hat{\mu}}{\sigma^2} + O(\tau)$, which from Condition 1 lies strictly inside the unit interval for small τ. Thus, if a solution degenerates into a noninterior one, it must happen after some time.

We next note that for the benchmark case in which $\lambda(z) \equiv 0$, i.e., for the case with no flexibility, the solution is increasing in z at $z = 0$ and decreasing in z at $z = 1$ for all t. For example, at $z = 0$, by differentiating (15) with respect to z, and once again using the transformation $\tau = T - t$, it is clear that w_z satisfies the o.d.e.

$$
(w_z)_\tau = -(\rho + \hat{\mu} - \sigma^2)w_z + q_z(T - \tau, 0), \tag{30}
$$

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and since $q_z(T - \tau, 0) > 0$ and $(w_z)(0, 0) = 0$, it is clear that $(w_z) > 0$ for all $\tau > 0$. In fact, the solution to (30) is

$$\frac{e^{-(\hat{\mu}+\rho)\tau}\left(-e^{-\tau\sigma^2}\rho + e^{\tau\hat{\mu}}(\hat{\mu} + \rho - \sigma^2) + e^{\tau(\hat{\mu}+\rho)}(-\hat{\mu} + \sigma^2)\right)}{\rho(\hat{\mu} + \rho - \sigma^2)}$$

which is strictly increasing in τ, as long as Condition 1 is satisfied. An identical argument can be made at the boundary $z = 1$, showing that $w_z(\tau, 1) < 0$, for all $\tau > 0$. Now, standard theory of p.d.e.s implies that, for any finite τ, w depends continuously on parameters, for the lower order terms, so $w_z \neq 0$ at boundaries for small, but positive, $\lambda(z)$.

For large τ, we know that w converges to the steady-state benchmark case, which has $w_z \neq 0$ in a neighborhood of the boundaries. Moreover, for small τ it is clear that $w_z \neq 0$ in a neighborhood of the boundaries according to the previous argument. Since the solution is smooth in $[0, T] \times [0, 1]$, and $w_z \neq 0$ at the boundaries for all $\tau > 0$, it is therefore clear that there is an $\epsilon > 0$, such that $w_z(t, z) > 0$ for all $\tau > 0$, for all $z < \epsilon$, and $w_z(t, z) < 0$ for all $z > 1 - \epsilon$. Thus, for $\lambda \equiv 0$, and for λ close to 0 by argument of continuity, the solution is interior.

Next, it is easy to show that for any λ, the central planner will not choose to stay at the boundary for a very long time. To show this, we will use the obvious ranking of value functions implied by their control functions: $\lambda^1(z) \leq \lambda^2(z)$ for all $z \in [0, 1] \Rightarrow w^1(\tau, z) \leq w^2(\tau, z)$ for all $\tau \geq 0$, $z \in [0, 1]$, where w^1 is the solution to the central planner's problem with control constraint λ^1, and similarly for w^2.

Specifically, let's assume that $\lambda^1 \equiv 0$, and $\lambda^2 > 0$. Now, let's assume that for all $\tau > \tau_0$, the optimal strategy in the case with some flexibility (λ^2) is for the central planner to stay at the boundary, $z = 1$, for some $\tau_0 > 0$. From (15), it is clear that $w^2(\tau, 0) = e^{-\rho(\tau-\tau_0)}w^2(\tau_0, 0)$, which will become arbitrarily small over time. Specifically, it will become smaller than $w^1(1 - \epsilon, \tau)$, for arbitrarily small $\epsilon > 0$, in line with the previous argument, since $w^1(\tau, 0) \equiv 0$ for all τ and $w_z^1(\tau, 0) < -\nu$, for large τ, for some $\nu > 0$. It can therefore not be optimal to stay at the boundary for arbitrarily large τ, since $w^2(\tau, 1 - \epsilon) \geq w^1(\tau, 1 - \epsilon) > w^2(\tau, 0)$. A similar argument can be made for the boundary $z = 0$.

In fact, a similar argument shows that the condition $w_z = 0$ can *never* occur at boundaries. For example, focusing on the boundary $z = 0$, assume that $w_z = 0$ at $z = 0$ for some τ and define $\tau_* = \inf_{\tau > 0} w_z(\tau, 0) = 0$. Similarly to the argument leading to (30), the space derivative of (13) at the boundary $z = 0$ is

$$(w_z)_\tau = -(\hat{\mu} + \rho - \sigma^2)w_z + q_z + aw_{zz}, \tag{31}$$

where $q_z = (-\hat{\mu} + \sigma^2)\frac{1 - e^{-\rho\tau}}{\rho}$ is strictly positive for all $\tau > 0$. Since, per definition, $w_z(\tau_*^-, 0) > 0$ and $w_z(\tau_*, 0) = 0$, it must therefore be that $q_z + aw_{zz} \leq 0$, which, since $a(\tau, 0) > 0$, for $\tau < \tau_*$, implies that w is strictly concave in a neighborhood of τ_* and $z = 0$. Moreover, just before τ_*, say at $\tau_* - \Delta\tau$, w_z is zero at an interior point, close to $z = 0$, because of the strict convexity of w, i.e., $w_z(\tau_* - \Delta\tau, \Delta z) = 0$. However, at Δz, w_z satisfies the following p.d.e., which follows directly from (13):

$$(w_z)_\tau = -(\hat{\mu} + \rho - \sigma^2 + O(\Delta z))w_z + (1 + O(\Delta z))q_z + O((\Delta z)^2), \tag{32}$$

and, since $w_z = 0$, this implies that

$$(w_z)_\tau = q_z + O((\Delta z)^2) > 0, \tag{33}$$

so at time τ_*, $w_z(\tau_*, \Delta z) = q_z(\tau_* - \Delta\tau, \Delta z)\Delta\tau + O((\Delta z)^2\Delta\tau) + O((\Delta\tau)^2) > 0$. However, since w_{zz} is strictly concave on $z \in [0, \Delta z]$, it can not be that $w_z(\tau_*, 0) = 0$ and $w_z(\tau_*, \Delta z) > 0$, so we have a contradiction. A similar argument can be made at the boundary at $z = 1$.

We have thus shown that the solution to (13) must be an interior one and that, given that the solution is interior, the formulation as an initial value problem with no boundary conditions (15,16) is well-posed. We are done. ∎

Proof of Lemma 4

$$
\begin{aligned}
U^0 &= E\left[\int_0^\infty e^{-\rho s} \log\left(\frac{B_0}{B_0+D_0}\left(B_0+D_0\right)\left(1+\frac{D_0}{B_0}e^{y(s)}\right)\right)ds\right] \\
&= \frac{\log(z)}{\rho} + \frac{\log(C)}{\rho} + E\left(\int_0^\infty e^{-\rho s}\log\left(1+(1/z-1)e^{y(s)}\right)ds\right] \\
&= \frac{\log(z)}{\rho} + \frac{\log(C)}{\rho} + \int_0^\infty\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}}e^{-\rho s}\log\left(1+(1/z-1)e^y\right)e^{-(y-\mu s)^2/(2\sigma^2 s)}dy\,ds \\
&= \frac{\log(z)}{\rho} + \frac{\log(C)}{\rho} + \int_{-\infty}^\infty e^{\frac{y\left(\mu-\sqrt{\mu^2+2\rho\sigma^2}\,\mathrm{sign}(y)\right)}{\sigma^2}}\frac{\log\left(1+(1/z-1)e^y\right)}{\sqrt{\mu^2+2\rho\sigma^2}}dy \\
&= \frac{\log(C)}{\rho} + G(z;\mu,\sigma^2,\rho),
\end{aligned}
$$

where

$$
\begin{aligned}
G(z;\mu,\sigma^2,\rho) &= \Bigg(\left(2\mu^2+\sigma^2(2\rho+q)+\mu(\sigma^2+2q)\right)\,{}_2F_1\left(1,\frac{q-\mu}{\sigma^2},\frac{q-\mu}{\sigma^2}+1,\frac{z}{z-1}\right) \\
&\quad + 2\frac{1-z}{z}\left(\mu^2+\rho\sigma^2-\mu q\right)\,{}_2F_1\left(1,\frac{q+\mu}{\sigma^2}+1,\frac{q+\mu}{\sigma^2}+2,\frac{z-1}{z}\right) \\
&\quad - 2\left(\mu^2-\mu q+2\rho(\sigma^2+q)\right)\log(z)\Bigg)\Big/\left(2\rho(\mu^2-\mu q+2\rho(\sigma^2+q))\right),
\end{aligned}
$$

$q=\sqrt{\mu^2+2\rho\sigma^2}$ and ${}_2F_1$ is the hypergeometric function.

Without loss of generality, we assume that $C=1$. We then immediately have $U^0(z)=\log(z)/\rho+G(z;\mu,\sigma^2,\rho)$ and it is easy to check that $U^0(0)=\mu/\rho^2$, corresponding to the classical Lucas case with log utility and a risk-free sector in zero net supply. We also have $U^0(1)=0$, corresponding to the case when the whole economy is risk-free, and $U^0=\int_0^\infty e^{-\rho s}\log(1)ds=0$. ∎

Proof of Lemma 3

For a bond maturing at date T, $G(B,D,T)=1$, and by homogeneity we can write

$$
P(B,D,t) = P\left(\frac{z}{1-z},1,t\right) \tag{34}
$$

$$
\equiv p(z,t); \tag{35}
$$

$$
P_t = p_t; \tag{36}
$$

$$
P_B = p_z\frac{\partial z}{\partial B} \tag{37}
$$

$$
= p_z\frac{D}{(B+D)^2}; \tag{38}
$$

$$
P_D = p_z\frac{\partial z}{\partial D} \tag{39}
$$

$$
= p_z\frac{-B}{(B+D)^2}; \tag{40}
$$

$$
P_{DD} = p_{zz}\left(\frac{\partial z}{\partial D}\right)^2 + p_z\frac{\partial^2 z}{\partial D^2} \tag{41}
$$

$$
= p_{zz}\frac{B^2}{(B+D)^4} + p_z\frac{2B}{(B+D)^3}. \tag{42}
$$

Substituting these into Equation (22), and simplifying, we obtain

$$p_t + \frac{1}{2}\sigma^2 z^2(1-z)^2 p_{zz} + \left[a - \widehat{\mu}z(1-z) + 2\sigma^2 z(1-z)^2 \right] p_z$$

$$- \left[\rho + \widehat{\mu}(1-z) - \sigma^2(1-z)^2 \right] p = 0. \quad (43)$$

∎

Proof of Proposition 4

For $y < 0$, we use the expansion $1/(1+e^y) = 1 - e^y + e^{2y} - \cdots$, and for $y > 0$, we use the similar expansion $1/(1+e^y) = e^{-y} - e^{-2y} + e^{3y} - \cdots$, to get an expansion[17]

$$\frac{P^\tau}{(1+e^d)e^{-\rho\tau}} = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \times$$

$$\left(\sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{0} e^{-(y-d-\mu\tau)^2/(2\sigma^2\tau)+ny} dy + \sum_{n=1}^{\infty} \int_{0}^{\infty} (-1)^{n+1} e^{-(y-d-\mu\tau)^2/(2\sigma^2\tau)-ny} dy \right).$$

Define $F(x) = e^{x^2} \text{Erfc}(x)$, where Erfc is the error function $\text{Erfc}(x) = (\sqrt{\pi})^{-1} \int_z^{\infty} e^{-t^2} dt$ (see Abramowitz and Stegun (1964)). Then, since

$$\frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{0} e^{-(y-d-\mu\tau)^2/(2\sigma^2\tau)+ny} dy = \frac{1}{2} e^{n(d+\mu\tau)+n^2\tau\sigma^2/2} \text{Erfc}\left(\frac{d+\mu\tau+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right)$$

$$= \frac{e^{-(d+\mu\tau)^2/(2\sigma^2\tau)}}{2} F\left(\frac{d+\mu\tau+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right),$$

and similarly

$$\frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{0}^{\infty} e^{-(y-d-\mu\tau)^2/(2\sigma^2\tau)-ny} dy = \frac{1}{2} e^{-n(d+\mu\tau)+n^2\tau\sigma^2/2} \text{Erfc}\left(\frac{-d-\mu\tau+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right)$$

$$= \frac{e^{-(d+\mu\tau)^2/(2\sigma^2\tau)}}{2} F\left(\frac{-d-\mu\tau+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right).$$

Therefore, we get

$$P^\tau = \frac{(1+e^d)e^{-\rho\tau-(d+\mu\tau)^2/(2\sigma^2\tau)}}{2} \left(\sum_{n=0}^{\infty} (-1)^n F\left(\frac{q+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right) + \sum_{n=1}^{\infty} (-1)^{n+1} F\left(\frac{-q+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right) \right). \quad (44)$$

Since $P^\tau = e^{-r(\tau)\tau}$, where $r(\tau)$ is the time-τ spot rate, we have

$$r(\tau) = \rho + \frac{\mu^2}{2\sigma^2}$$

$$+ \frac{1}{\tau}\left(\log\left(-\frac{1+e^d}{2} \right) + \frac{d^2}{2\sigma^2\tau} + \frac{d\mu}{\sigma^2} \right.$$

$$+ \log\left(\sum_{n=0}^{\infty} (-1)^n F\left(\frac{d+\mu\tau+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right) + \sum_{n=1}^{\infty} (-1)^{n+1} F\left(\frac{-d-\mu\tau+n\tau\sigma^2}{\sqrt{2\sigma^2\tau}} \right) \right) \right).$$

---

[17]The appropriateness of integrating to and from 0, although the two series do not converge at $y = 0$, and of moving the summation outside of the integral both follow, e.g., from dominated convergence, since the series $1 - 1 + 1 - 1 \cdots$ is bounded below by 0 and above by 1.

The formula is straightforward to use, since $F(x) \sim 1/x$ for large x.

For
$$\frac{d + \mu\tau}{\sigma^2\tau} = m \in \mathbb{N},$$

(44) reduces to a case for which closed form expressions exist, so

$$P^\tau = \frac{(1 + e^d)e^{-\rho\tau - m^2\sigma^2\tau/2}}{2}\left(1 + 2\sum_{n=1}^{m-1}(-1)^n e^{n^2\sigma^2\tau/2}\right).$$

∎

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