Advising the Management: A Theory of Shareholder Engagement*

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Abstract

We provide a model to study the effectiveness of shareholder engagement, i.e., shareholders communicating their views to management. Differences in preferences and beliefs between management and shareholders prevent effective engagement, and these frictions can be exacerbated by the firm's ownership structure if many investors choose not to become shareholders. The growth in passive ownership can counteract these effects and improve shareholder engagement. Engagement decisions of shareholders are complements under differences in beliefs, but are substitutes if the manager's and shareholders' preferences are strongly misaligned. If differences in beliefs are substantial, some degree of preference misalignment can improve shareholder engagement.

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JEL classifications: D71, D74, D82, D83, G34

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"Shareholder engagement has become one of the most talked-about issues in corporate governance, and with good reason" (Equilar, March 30, 2016).[1]

1 Introduction

Shareholder engagement, i.e., shareholders communicating to management their views on corporate policies and strategy, has become a central component of corporate governance. According to former SEC chairman Mary Schapiro, it is vital that shareholders and companies "move beyond the minimum required communications and become truly engaged" because management can "benefit from access to the ideas and the concerns investors may have." While communication between managers and shareholders was taking place in the past (e.g., Carleton, Nelson, and Weisbach, 1998; Becht et al., 2009), it has become particularly important and widespread in recent years.[2] In a survey of institutional investors, McCahery, Sautner, and Starks (2016) found that 63% of the respondents had engaged in direct discussions with top management over the previous five years. Not only actively managed investors are advising the management, but passive funds are as well, with the Big Three (BlackRock, Vanguard, and State Street) being particularly involved. For example, BlackRock's Investment Stewardship Annual Report states that in 2020, BlackRock "had over 3,000 in-depth conversations with corporate leadership," including "more than 1,000 engagements on corporate strategy and 400 engagements on the impact of COVID-19."

In addition to the growth in direct engagements, shareholders' communication with management has become more prevalent due to the increase in the number and breadth of issues that are brought up for nonbinding, i.e., advisory, shareholder votes. The Dodd-Frank requirement of a regular advisory vote on executive compensation (say-on-pay), as well as nonbinding proposals submitted by shareholders via Rule 14a-8, allow the entire shareholder

[1] Source: Equilar Blog. Equilar is the leading provider of governance tools and executive compensation data for corporations, institutional investors, and the media.

[2] The fraction of S&P 100 companies that discuss their communication with shareholders in proxy statements has increased from 2% in 2011 to 55% in 2015 (Equilar Blog, March 30, 2016). According to the management consulting firm Rivel Research Group, "effective shareholder engagement has become one of the most important, and oft heard, mantras among corporate management and investors. Institutional investors are demanding – and expect – regular, proactive communication with company management."

base to express their views and advise management on multiple issues concerning the firm's governance and strategy.

In light of the increasing prevalence and attention given to shareholder communication with management, it is important to understand when this communication is effective and what factors can enhance it. How does managerial learning from the shareholders interact with the firm's ownership structure? How does the growing ownership by passively managed funds affect shareholder engagement? And how can firms improve shareholder communication, e.g., by putting more issues up for advisory votes, adding shareholders to their boards, or changing managerial incentives? This paper provides a theory of equilibrium ownership structure and shareholder engagement that studies these questions.

In our model, the firm needs to make a decision, whose value depends on the unknown state. Investors first decide which stakes to acquire, and their decisions determine the firm's ownership structure. We view this trading stage as shaping the firm's long-term shareholder base, for example, at the time of the IPO. Then, shareholders of the firm observe private signals about the state and communicate them to the manager by sending non-verifiable messages ("cheap talk"), and the manager decides which action to take. Thus, information about the state is dispersed among investors, which creates value from shareholder engagement.

There are two sources of inefficiencies in shareholder-manager communication, which can prevent the manager from learning the most from the investors. The first source of inefficiency are communication frictions: if a shareholder has different preferences or prior beliefs from those of the manager, he may have incentives to misrepresent his information. For example, consider a firm deciding on the scale of production in a new market. The manager may have misaligned preferences and prefer a larger scale due to private benefits, giving the shareholder incentives to report more negative information than he privately has. Furthermore, the shareholder may have incentives to report more negative information if the manager has more optimistic prior beliefs about the growth of the new market. Indeed, there is growing evidence suggesting that heterogeneous beliefs are important to explain corporate finance decisions and the dynamics of asset prices and volume,[3] and Li, Schwartz-

[3] E.g., Kandel and Pearson (1995), Diether et al. (2002), Malmendier and Tate (2005), Dittmar and

Ziv, and Maug (2021) show that heterogeneity in beliefs affects how shareholders vote and trade around shareholder meetings.[4] Both misaligned preferences and differences in beliefs prevent the manager from learning shareholders' information.

In addition to communication frictions, the other impediment to managerial learning is that many potentially informed investors may choose not to become shareholders in the first place. Such investors have no incentives or ability to communicate their views to the manager, so their information does not affect corporate decision-making.

We show that these two sources of inefficiencies – communication frictions and a limited shareholder base – interact and can exacerbate each other. First, the firm's overall ownership structure affects each individual shareholder's communication with management, and in particular, a more limited shareholder base can lead to less effective advice by those investors who own the firm. In turn, the firm's ownership structure depends on how informed managerial decisions are going to be, and anticipated inefficiencies in shareholder-manager communication can dissuade many investors from holding the firm. These results have implications for the design of governance policies and suggest an important role of passively managed institutional investors for shareholder engagement.

To understand how the ownership structure affects each individual shareholder's communication with management, note that a shareholder's incentives to convey his views truthfully depend on whether the manager gets advice from other shareholders as well. In particular, we show that in the presence of heterogeneous beliefs about the optimal strategy, and if the manager's preferences are not too misaligned, shareholders' communication decisions are complements: a shareholder is more likely to communicate truthfully if he expects more other shareholders to do so. Intuitively, heterogeneous prior beliefs are a smaller impediment to communication if the manager is expected to become more informed. If the shareholder

Thakor (2007), and Thakor and Whited (2011), among others. See Hong and Stein (2007) for a survey.

[4] The survey evidence by Edmans, Gosling, and Jenter (2021) suggests that boards see themselves as maximizing shareholder value, but having different beliefs from investors about what CEO compensation contracts should look like. Likewise, there is often substantial disagreement about the effect of other governance policies, even among parties with similar interests, such as shareholders with similar portfolios. See, e.g., "A Lack of Consensus on Corporate Governance", *The New York Times* (September 29, 2015), discussing shareholder disagreements on the issue of CEO-chairman separation, and "The Proxy Access Debate", *The New York Times* (October 9, 2009), discussing disagreements about the optimal terms of proxy access.

expects the manager to get advice from many other investors, he anticipates the manager's posterior beliefs to be more congruent with his own, which improves communication between them. In contrast, if the shareholder base is limited and the manager gets advice from a small selected set of investors, then the shareholder expects their differences in beliefs to persist and has little incentive to convey his views truthfully.[5]

Not only the ownership structure affects shareholder engagement, but expected shareholder engagement affects the ownership structure as well. If the manager is expected to learn little from the shareholders, then investors who are particularly misaligned with the manager expect their belief disagreements to remain strong. Such investors expect the manager to make incorrect decisions (according to their beliefs), so they have low valuations of the stock and choose not to become shareholders. Instead, the firm is held by a subset of investors whose views about the optimal strategy are relatively more aligned with those of the manager. Moreover, because of this two-way interaction between the ownership structure and the effectiveness of shareholder engagement, multiple equilibria can arise. An equilibrium where all informed investors become shareholders and communicate their information can co-exist with an equilibrium where only a subset of investors become shareholders and managerial learning is highly limited. Intuitively, if an investor expects the firm to have few shareholders and the manager's decisions to be therefore primarily based on his prior beliefs, he expects to disagree with the manager's decisions ex-post, and hence does not invest in the firm in the first place, making this equilibrium self-fulfilling.

These results suggest an important role of passively managed funds. The unique feature of passive funds is that they are required to hold most public stocks regardless of whether their fund managers agree or disagree with the firms' policies. We show that as a result, the growth in passive funds can make managerial learning from the shareholders more effective, increase the informativeness of corporate decisions, and raise the share price. First, passive funds become shareholders and thus can engage with management even when active funds

[5] Of course, if information acquisition is costly, a limited shareholder base can also have a positive effect, as it can alleviate the free-rider problem. Since the free-rider effect is well-understood, we do not study costly information acquisition in the basic model and analyze it in an extension. See the discussion in Section 4.1.

in their position (with the same preferences and beliefs) would have not taken a stake in the firm. Moreover, because shareholders' communication decisions are complements, the growth in passive funds has an additional, positive spillover effect on the engagement of other, actively managed, funds: a larger number of active funds communicate their views to management when more passive funds are present. Finally, in the presence of equilibrium multiplicity, the growth in passive funds breaks the feedback loop between the ownership structure and managerial learning, and can eliminate the inefficient equilibrium.

When disagreements in beliefs are accompanied by a strong misalignment in preferences between the manager and shareholders (i.e., a conflict of interest), these two frictions interact and additional effects arise. First, we show that if the conflict of interest is sufficiently strong, shareholders' communication decisions become substitutes: as more other shareholders share their views with the manager, a shareholder's incentives to communicate truthfully decline. Intuitively, when the shareholder misrepresents his information to push the manager closer to his preferred decision, the shareholder is afraid to make too big of an impact and move the manager's decision too much, away even from his own preferred decision. This concern constrains misreporting if the manager reacts strongly to the shareholder's advice. However, if many other shareholders provide advice, the manager is expected to react less to the individual shareholder's advice, so this concern no longer constrains misreporting. While this substitution effect is counteracted by the complementarity effect due to managerial learning and belief convergence, the complementarity effect is dominated if the conflict of interest is sufficiently strong. Intuitively, even if the manager learns a lot from the shareholders and their posterior beliefs converge, strong preference misalignments introduce a wedge between the shareholders' and manager's preferred decisions and limit how congruent the manager and shareholders can become due to learning.

This contrast between the complementarity and substitution effects has implications for the effectiveness of governance policies that aim to promote more shareholder-manager communication. One such policy is the use of advisory votes as a way to collect shareholders' views about the firm's decisions. Advisory votes may create more problems than they solve if these votes do not provide useful information: for example, both the say-on-pay provision of the Dodd-Frank Act and Rule 14a-8 have been highly debated because of their potential downsides, such as distractions, time, and resources they may require from management.[6] Our results suggest that introducing an advisory vote on a certain decision could have a particularly positive effect on managerial learning when there is substantial heterogeneity in beliefs about this decision. In this case, the advisory vote allows a cost-effective way to get the views of a large number of shareholders, and in particular, those who would not be able to engage with management individually. This, in turn, through the complementarities in communication, may encourage truthful communication by other shareholders, who would otherwise not share their views because of strong belief disagreements with the manager. In contrast, for decisions involving a strong conflict of interest, the substitution effect dominates. We show that in this case, the information that the manager can learn from the shareholders is limited and is not improved by adding more shareholders who can communicate their views. Hence, introducing an advisory vote may not enhance managerial learning at all, and the potential downsides of such a vote become of first-order importance. For a similar reason, increasing board size by adding shareholders (e.g., venture capitalists, activist investors, or other blockholders) to the board is likely to enhance managerial learning and increase value if there are strong differences in beliefs about the strategy but is likely to decrease value if the manager's and shareholders' interests are strongly misaligned.

Given these implications, we next ask whether shareholder engagement is always enhanced by more aligned managerial preferences (e.g., through performance-sensitive compensation contracts or a more independent board). We show that the answer depends on the extent of belief disagreements about the decisions. If belief disagreements are not too strong, then more aligned managerial preferences both improve shareholder-manager communication and decrease the bias in decision-making, thereby unambiguously increasing firm

[6] For example, if a certain advisory vote is not informative, then its only effect is that it "requires companies to devote significant time and resources... and distracts management and shareholders" (see the February 3, 2020 letter to the SEC from the Corporate Governance Coalition for Investor Value). See "Debate continues over 14a-8 reform plans" at *Corporate Secretary*, Aug 4, 2020; and Thomas, Palmiter, and Cotter (2012), Section II.C "Debate over Mandatory Say on Pay" for discussions of these debates.

value. However, if the degree of belief heterogeneity is particularly high, then even if all shareholders conveyed their views, disagreements in beliefs would remain substantial, so shareholders' incentives to communicate truthfully are highly limited. In this case, some bias in the manager's preferences can improve his communication with at least some of the shareholders, by counteracting the disagreements in beliefs between them. Moreover, the positive effect of improved managerial learning can even dominate the negative effect of more biased decision-making.

Overall, our paper highlights a new informational channel through which financial markets affect the quality of managerial decisions, and thus contributes to the feedback literature (e.g., Dow and Gorton, 1997; Subrahmanyam and Titman, 1999; Goldstein and Guembel, 2008; see Bond, Edmans, and Goldstein, 2012, for a survey). We emphasize that financial markets not only influence decision-making by the information contained in the prices, but also by determining the firm's ownership structure and thus affecting which investors provide advice to management via direct engagements, advisory votes, or joining the board.

Our paper is also related to the literature that studies communication from shareholders to management (e.g., Levit, 2019; Levit, 2020), as well as from the board of directors to management (e.g., Adams and Ferreira, 2007; Harris and Raviv 2008; Baldenius, Melumad, and Meng, 2014; Chakraborty and Yilmaz, 2017). These papers analyze communication by a single agent, whereas our focus is on how communication decisions of multiple agents interact.[7] Thus, our paper contributes to the literature on cheap talk communication (Crawford and Sobel, 1982) by multiple imperfectly informed senders (Austen-Smith, 1993; Battaglini, 2004; Morgan and Stocken, 2008; Levit and Malenko, 2011; Galeotti et al., 2013). The substitution effect that arises in our model when the misalignment of preferences is substantial is related to the result in Morgan and Stocken (2008) that full information revelation is an equilibrium in a poll with a small sample, but not with a large one. While this literature only studies heterogeneous preferences, we also introduce heterogeneous beliefs and

[7] Malenko (2014), Khanna and Schroder (2015), and Chemmanur and Fedaseyeu (2018) study communication between multiple members of the board or committee, but do not analyze their communication to the manager.

show that when disagreements in beliefs are substantial, the results are the opposite of those under heterogeneous preferences, and that the two communication frictions interact in interesting ways. In addition, we highlight how in a corporate setting, the set of agents who communicate with management (i.e., the shareholder base) is itself endogenously determined by agents' preferences and beliefs, and how this, in turn, affects communication.

Finally, our paper is related to the literature on heterogeneous priors. Morris (1995) provides an overview of the heterogeneous prior assumption and discusses why it is consistent with rationality. Our model also features rational agents: although they have different priors, they are not dogmatic and rationally update their beliefs in a Bayesian way after receiving new information. There is a large theoretical literature studying the implications of heterogeneous beliefs for trading in financial markets.[8] The contribution of our paper is to examine how heterogeneous beliefs affect not only shareholders' trading decisions, but also their subsequent communication with management and the interaction between these two decisions. Boot, Gopalan, and Thakor (2006, 2008) also study differences in beliefs between shareholders and the manager, but from a very different perspective: these papers analyze the firm's choice between public and private ownership and do not feature asymmetric information and communication, which are the focus of our paper. Che and Kartik (2009), Van den Steen (2010), and Alonso and Camara (2016) study communication under heterogeneous beliefs but with only one sender and not via cheap talk, and thus do not consider the forces highlighted in our paper.[9]

2 Setup

In this section, we present a simple model, which captures a conflict of interest between the manager and shareholders, heterogeneous beliefs, and dispersed private information, and has tractable and intuitive solutions.

[8] E.g., Harris and Raviv (1993), Kandel and Pearson (1995), Banerjee, Kaniel, and Kremer (2009), Banerjee and Kremer (2010), and Kyle, Obizhaeva, and Wang (2018), among many others.

[9] Garlappi, Giammarino, and Lazrak (2017, 2021) examine group decision-making under heterogeneous beliefs but without private information and communication.

The environment consists of a firm, which is run by the manager, and a set of N investors (potential shareholders) indexed by i, $i \in \{1, ..., N\}$. The firm needs to make a decision, denoted by a continuous action $a \in \mathbb{R}$, whose value depends on the unknown state Z. If the manager takes action a in state Z, the firm delivers per-share value of

$$U(a, Z) = u_0 - (a - Z)^2, \tag{1}$$

where $u_0 > 0$ is sufficiently high, so that the equilibrium share price is always positive. The manager's interests may not be fully aligned with shareholders: his utility is

$$U_m(a, Z) = u_0 - (a - Z - b)^2, \tag{2}$$

where $b \geq 0$ measures the extent of conflicts of interest. Thus, from the shareholders' point of view, the optimal action is $a = Z$, whereas the manager's preferred action is $a = Z + b$. For example, if a refers to the firm's investment decision or how much to bid for a potential target, then b can capture the extent of empire-building preferences of the manager.

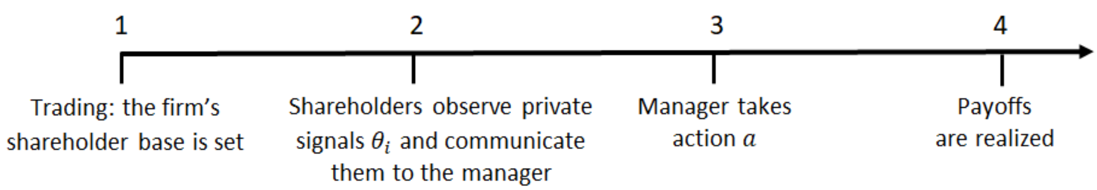

Figure 1. **Timeline of the model**.

The timing, illustrated by Figure 1, is as follows. At the initial stage, all N investors participate in the market for the firm's shares, during which the total stock of the firm is sold by the original owner (seller) in a competitive market. The stock is in unit supply, so holding α_i shares is equivalent to holding fraction α_i of the firm. Each investor submits a demand schedule that specifies the quantity he wants to buy for various prices, $\{\alpha_i(p)\}$, and the equilibrium price p^* is set to clear the market. Suppose (e.g., as in Vives, 1993) that investor i's utility from buying stake α_i is given by

$$\alpha_i \left(\mathbb{E}_i[U\left(a, Z\right)] - p\right) - \frac{\lambda}{2}\alpha_i^2, \tag{3}$$

where $U\left(a, Z\right)$ is his utility from each share and is given by (1), p is the share price, and $\lambda > 0$ captures either the holding cost due to limited diversification and risk aversion or the transaction cost due to limited liquidity. For example, λ is likely to increase with firm size and volatility because holding a given fraction of the firm is costlier when the firm is larger and more risky. Subscript i in the expectation operator captures the fact that investors could have heterogeneous beliefs, as described below.

Trading determines the firm's shareholder base, $S \subseteq \{1, ..., N\}$, which consists of all investors who hold a positive number of shares after the trading stage: $S = \{i : \alpha_i > 0\}$. After trading, each shareholder $i \in S$ learns a private signal θ_i about the state and sends a non-verifiable cheap-talk message to the manager. Investors who do not become shareholders do not communicate with the manager. After the communication stage, the manager chooses action $a \in \mathbb{R}$, and the payoffs are realized. To describe these stages in more detail, we next define the information structure of the model.

The state of the world is equal to the sum of $K \geq N$ signals:

$$Z = \sum_{i=1}^{K} \theta_i, \tag{4}$$

where $\theta_i \in \{0, 1\}$ are identically distributed binary signals: θ_i equals one with probability φ and zero with probability $1 - \varphi$. These signals are independent conditional on φ, but unconditionally correlated since φ is unknown, as described below. Signals θ_i can be thought of as different factors relevant to the decision. Information about these factors is dispersed among investors: if investor i becomes a shareholder, he privately observes θ_i and is uncertain about other signals. Such information structure is common in the literature (e.g., Harris and Raviv, 2008; Chakraborty and Yilmaz, 2017) and captures the idea that investors may have different areas of expertise and thus be informed about different aspects of the decision. For example, in the context of M&A decisions, a could be the choice of how much to bid for a potential target, and signals θ_i could capture the synergies from the merger, the intrinsic value of the target, the number of potential competing bidders and their bids, the costs of integrating the two companies, and other relevant factors. Since $K \geq N$, the model allows for residual uncertainty: all investors collectively can observe N signals at most, so $K-N$ of payoff-relevant signals always remain unknown at the decision-making stage. For simplicity, we assume that the manager is uninformed and that all investors' signals are equally important for the decision. In Section 4.3 and Section 7.1 of the Online Appendix, respectively, we show that the model can be easily extended to incorporate private information of the manager and heterogenous importance of investors' signals, without changing the results.

Investors and the manager have heterogeneous beliefs about the state: some agents are ex-ante more optimistic, while others are more pessimistic. In particular, the agents disagree about φ, the probability that each signal θ_i is equal to one: optimists have a higher expectation of φ than pessimists. The manager's prior is that φ is drawn from the Beta distribution with parameters $(\rho_m, \tau - \rho_m)$, while investor i's prior is that φ is drawn from the Beta distribution $(\rho_i, \tau - \rho_i)$.[10] Since the expected value of this Beta distribution is $\frac{\rho_i}{\tau}$, investors with a higher ρ_i are more optimistic.[11] Note that optimism in our model does not mean more positive beliefs about the value of the shares (if the action fully informed, all agents agree that firm value is u_0), but rather beliefs that a higher action should be taken. While agents may have different prior beliefs, they update their beliefs rationally (according to Bayes' rule) when they receive new information.

We look for equilibria in pure strategies at the communication stage (see Section 5 for a discussion of mixed strategy equilibria). Because signals are binary, it is without loss of generality to consider a binary message space: the communication strategy of shareholder i is a mapping from his signal $\theta_i \in \{0,1\}$ into a binary non-verifiable message $\mu_i \in \{0,1\}$. Thus, in equilibrium, each shareholder either communicates his information truthfully (i.e., $\mu_i(\theta_i) = \theta_i$ up to relabeling) or sends an uninformative (babbling) message (i.e., $\mu_i(0) = \mu_i(1)$). If there are multiple equilibria that can be Pareto-ranked in the communication

[10] That is, agent i believes that the density of φ is $f_i(\varphi) = \varphi^{\rho_i - 1}(1-\varphi)^{\tau - \rho_i - 1}\frac{\Gamma(\tau)}{\Gamma(\rho_i)\Gamma(\tau - \rho_i)}$, where $\Gamma(\cdot)$ is the gamma function.

[11] See Auxiliary Lemma A.1 in the Appendix. Notice that ρ_i also affects other moments, not only the mean. In Section 7.2 of the Online Appendix, we consider a more flexible specification in which agent i's prior of φ is characterized by the Beta distribution $(\rho_i, \tau_i - \rho_i)$. The main results extend to this setting.

subgame, we assume that the more efficient equilibrium is played.

Discussion of the model. We assume that investors trade based on their prior beliefs, but do not trade again ex-post, after learning their private signals. This simplifying assumption greatly enhances tractability: a model in which investors both trade on private information and decide how to communicate it is very difficult to analyze. There are two arguments for this assumption. First, as we discuss in the literature review, prior research has extensively studied how trading incorporates investors' private information into real decisions through its impact on prices (Bond, Edmans, and Goldstein, 2012). In contrast, our contribution is to examine how trading incorporates investors' information into real decisions through a different channel, communication: trading determines the firm's shareholder base and thus, determines which investors communicate their information to the manager via engagement, advisory voting, or being on the board. Assuming that investors do not trade based on private information allows us to abstract from the price channel and focus on the more novel communication channel. Second, we view our trading stage as determining the firm's long-term shareholder base (for example, at the time of the IPO) and ρ_i as capturing investors' beliefs at that point, e.g., how congruent they are with the overall strategic direction the management is pursuing. It is then reasonable to assume that such long-term shareholders' ownership stakes are not affected by more transitory private information that arrives later.

To make the analysis tractable, we also assume a specific communication protocol and make several assumptions about the information structure. We discuss the robustness of our results to these assumptions in Section 5.

3 Analysis of the model

3.1 Communication stage

We first characterize the action taken by the manager for a given outcome of the communication stage. Suppose that after communicating with the shareholders, the manager knows subset $R \subseteq \{1, ..., K\}$ of signals ("revealed" signals) and does not know all the other signals,

$-R \equiv \{1,...,K\}\backslash R$. We use R and $-R$ to denote signal indices and $\theta_R \equiv \{\theta_i, i \in R\}$ and $\theta_{-R} \equiv \{\theta_i, i \in -R\}$ to denote the corresponding subsets of signal realizations.

Given the quadratic payoff function, the optimal action of the manager is the sum of his bias b and his expectation of the state given his prior ρ_m and the signals he learned θ_R:

$$a_m(\theta_R) = b + \mathbb{E}_m\left(Z \mid \theta_R\right) = b + \sum_{i \in R} \theta_i + \sum_{j \in -R} \mathbb{E}_m\left[\theta_j | \theta_R\right]. \tag{5}$$

The subscript m in the expectation operator $\mathbb{E}_m$ highlights that the manager uses his own prior ρ_m to update his beliefs. In the appendix, using the properties of the Beta distribution, we show that the manager's posterior belief is that $\mathbb{E}_m(\varphi|\theta_R) = \frac{\rho_m + \sum_{i \in R} \theta_i}{\tau + |R|}$, where $|R|$ is the number of signals in R. This gives the following result:

Lemma 1 (Optimal action of the manager). *Suppose that after the communication stage, the manager knows subset R of signals. Then his optimal action is*

$$a_m(\theta_R) = b + \sum_{i \in R} \theta_i + \frac{\rho_m + \sum_{i \in R} \theta_i}{\tau + |R|}\left(K - |R|\right). \tag{6}$$

For any given information set θ_R, a higher bias b and a higher prior belief ρ_m both induce the manager to take a higher action. However, while the effect of b does not depend on the manager's information, the prior ρ_m becomes less important as the manager becomes more informed and updates his beliefs. In particular, as the set R expands, the term $\frac{K-|R|}{2\rho+|R|}$, and hence the effect of ρ_m decreases. The manager's action coincides with the optimal action from the perspective of shareholder i if $b = 0$, $K = N$, and $R = \{1, ..., N\}$.

Using Lemma 1, we next examine when shareholders will truthfully communicate their information to the manager. Consider any shareholder i and suppose that the manager knows subset $R_i \subset \{1, ..., K\}$ of signals, where R_i does not include shareholder i's signal θ_i. The manager does not know all the other signals, i.e., θ_i and all signals in the subset $-R_i \backslash \{i\}$, where as before, $-R_i \equiv \{1, ..., K\}\backslash R_i$. Suppose the manager believes the shareholder's message and uses it to update his belief about the state according to (6). If shareholder i reveals his signal truthfully, the manager's action is

$$a_m\left(\theta_{R_i},\theta_i\right) \equiv b+\theta_i+\sum_{j\in R_i}\theta_j \ +\frac{\rho_m+\theta_i+\sum_{j\in R_i}\theta_j}{\tau+|R_i|+1}\left(K-|R_i|-1\right). \tag{7}$$

If shareholder i misreports and claims that his signal is $1-\theta_i$, the manager's action is

$$a_m\left(\theta_{R_i},1-\theta_i\right) \equiv b+(1-\theta_i)+\sum_{j\in R_i}\theta_j \ +\frac{\rho_m+(1-\theta_i)+\sum_{j\in R_i}\theta_j}{\tau+|R_i|+1}\left(K-|R_i|-1\right). \tag{8}$$

Shareholder i only knows his signal θ_i and does not know the set of all the other $K-1$ signals, which we denote θ_{-i}. Thus, he compares his expected payoff from actions $a_m\left(\theta_{R_i},\theta_i\right)$ and $a_m\left(\theta_{R_i},1-\theta_i\right)$ given θ_i and his own prior belief about the distribution of those signals, and reports his signal truthfully if and only if:

$$\sum_{\theta_{-i}\in\{0,1\}^{K-1}}\left[\left(a_m\left(\theta_{R_i},\theta_i\right)-Z\right)^2-\left(a_m\left(\theta_{R_i},1-\theta_i\right)-Z\right)^2\right]P_i(\theta_{-i}|\theta_i)\leq 0, \tag{9}$$

where $P_i\left(\theta_{-i}|\theta_i\right)$ is shareholder i's belief about θ_{-i} given θ_i and his prior ρ_i. The next result characterizes the necessary and sufficient conditions for (9) to hold.

Proposition 1 (IC constraint for truthful reporting). *Suppose that the manager learns subset R_i of signals (which does not include θ_i) and does not learn all the other signals, $-R_i$. Then shareholder i reports his signal truthfully if and only if*

$$\left|\left(\tau+|R_i|+1\right)b+\left(K-|R_i|-1\right)\left(\rho_m-\rho_i\right)\right|\leq\frac{\tau+K}{2}. \tag{10}$$

As is standard in cheap talk games, communication is ineffective if the manager's preferences are sufficiently different from those of the shareholder: (10) is violated if b is large. The misalignment of preferences creates incentives to misreport, as the shareholder wants to tilt the manager's action in the direction away from the manager's bias. Similarly, communication is ineffective if the manager and shareholders have very different prior beliefs: (10) is violated if $|\rho_m-\rho_i|$ is large. For example, if the shareholder thinks that the manager is too optimistic, he wants to correct this "bias in beliefs" by reporting a more negative signal.

Thus, with a single shareholder, disagreements due to differences in preferences and differences in beliefs have similar effects. However, this is no longer true with multiple shareholders. In this case, there are communication externalities – a shareholder's incentives to communicate truthfully depend on how much the manager is expected to learn from other shareholders (i.e., $|R_i|$) – and these externalities are very different depending on the source of disagreements. To explain the intuition, we rewrite (10) in the following form:

$$2\left|b+\frac{K-|R_i|-1}{\tau+|R_i|+1}(\rho_m-\rho_i)\right| \leq 1+\frac{K-|R_i|-1}{\tau+|R_i|+1}. \tag{11}$$

The left-hand side of (11) captures the incongruence between the manager and the shareholder. For example, if the shareholder is more pessimistic than the manager ($\rho_m > \rho_i$), then the manager's preferred action is higher than that of the shareholder, both due to the manager's bias in preferences ($b > 0$) and due to his too optimistic beliefs. The right-hand side of (11) measures the manager's reaction to the shareholder's advice, i.e., by how much the manager's action changes if the shareholder misreports his signal θ_i.[12] Intuitively, the shareholder faces a trade-off: while he wants to tilt the manager in the direction of his own preferred action (the benefit of misreporting, captured by the left-hand side of (11)), he is also afraid to tilt it too much, away even from his own optimal action, i.e., to "overshoot" (the cost of misreporting). This concern makes the shareholder reluctant to misreport if the manager reacts strongly to the shareholder's advice (the right-hand side of (11) is large enough), but not otherwise. It is now easy to see that there are two opposite forces through which $|R_i|$ affects the shareholder's IC constraint:

1. Complementarity in shareholders' communication decisions. The first force is that the heterogeneity in prior beliefs becomes less important as the manager becomes more informed. This leads to shareholders' communication decisions being *complements*: the more information the manager is expected to learn from others (i.e., the higher is $|R_i|$), the more likely it is that shareholder i will also truthfully communicate his signal. Intuitively, the shareholder expects the manager to become more congruent with him as the manager learns more: the term $\frac{K-|R_i|-1}{\tau+|R_i|+1}(\rho_m-\rho_i)$ in (11) decreases in $|R_i|$. This happens due to

[12] Both of these statements follow from (7)-(8). From (7), the term under the absolute value sign on the left-hand side is the difference between the preferred actions of the manager and the shareholder given information θ_{R_i} and θ_i. From (7)-(8), the right-hand side equals $a_m(\theta_{R_i},\theta_i)-a_m(\theta_{R_i},1-\theta_i)$.

two related effects. First, once a signal is revealed, agents update their posteriors about the distribution of the state. Hence, even if the shareholder's and manager's initial beliefs are very different, the shareholder expects them to become closer following the revelation of information by other investors. Second, heterogeneous beliefs generate disagreement only over the information that is still unknown — once a signal gets revealed, all parties agree about it.[13] To see the complementarity effect most starkly, consider the extreme case of $b = 0$. Suppose that there is no residual uncertainty ($K = N$), and the manager knows all the signals except shareholder i's signal: $R_i = \{1, ..., N\} \backslash \{i\}$. Then, truthfully reporting the last remaining signal θ_i results in the manager taking the action that is optimal from the perspective of the shareholder, and hence is always incentive compatible.

The complementarity effect only arises in the presence of heterogeneous beliefs. If agents have common priors ($\rho_i = \rho_m$ for all i) and $b = 0$, then (10) is always satisfied, i.e., each shareholder has incentives to communicate his signal truthfully regardless of how many other shareholders communicate with the manager.

2. Substitution in shareholders' communication decisions. The second force is that as the manager learns from a larger number of shareholders, he reacts less to each individual shareholder's advice: the right-hand side of (11) decreases in $|R_i|$. As a result, the shareholder is less worried that misreporting his signal will tilt the manager's action too far away from the shareholder's own optimal action, i.e., the cost of misreporting declines. Hence, the shareholder is more likely to misreport when more other shareholders communicate with management, leading shareholders' communication decisions to be *substitutes*.

Proposition 1 shows that which of these two forces dominates depends on the interaction between the two communication frictions, i.e., the relation between $|\rho_m - \rho_i|$ and b. If $b = 0$, the left-hand side of (10) always decreases in $|R_i|$: if heterogeneous beliefs are the only communication friction, shareholder's communication decisions are always complements. More

[13]The first effect is captured by the denominator, $\tau + |R_i| + 1$: the manager updates his beliefs about φ and hence signals θ_{-R_i} after learning signals θ_{R_i}. The second effect is captured by the numerator, $K - |R_i| - 1$: the manager learns signals θ_{R_i} out of $\theta_{\{1,...,K\}}$.

generally, (10) implies that the complementarity effect dominates if b is sufficiently small relative to $|\rho_m - \rho_i|$. However, as b increases, the complementarity effect is eventually dominated by the substitution effect: the left-hand side of (10) increases in $|R_i|$ once b becomes sufficiently large relative to $|\rho_m - \rho_i|$. Intuitively, the misalignment of preferences limits how congruent the manager and shareholders can become due to learning: even if the manager learns a lot and his beliefs converge to those of the shareholders, strong preference misalignments introduce a wedge between the shareholders' and manager's preferred decisions.

To simplify the exposition and derive a complete, closed-form characterization of the equilibria, we assume in the remainder of the paper that there are only two types of investors:

Assumption: *Suppose $\rho_m = \rho$, $\tau = 2\rho$, and there are two types of investors: N_o optimists with $\rho_i = \rho + \Delta$, and $N_p \equiv N - N_o$ pessimists with $\rho_i = \rho - \Delta$, where $\rho > \Delta$.*

Thus, the manager believes that φ is drawn from $Beta(\rho, \rho)$, whereas optimists (pessimists) are more (less) optimistic than the manager and believe that φ is drawn from $Beta\left(\rho + \Delta, \rho - \Delta\right)$ $\left(Beta\left(\rho - \Delta, \rho + \Delta\right)\right)$. The case $\Delta = 0$ captures common priors: for example, if $\Delta = 0$ and $\rho = 1$, all agents believe that φ is uniformly distributed on $[0, 1]$. All of the parameters are publicly known.

We next use (10) to characterize the most informative equilibrium at the communication stage given any shareholder base S (as we show in the next section, the most informative equilibrium is Pareto efficient if K is large enough). Since $b \geq 0$, then for any given $|R_i|$, if (10) holds for a pessimistic shareholder ($\rho_m - \rho_i = \Delta$), it also holds for an optimistic shareholder ($\rho_m - \rho_i = -\Delta$). Intuitively, a pessimistic shareholder is worried that the manager's action will be too high both because the manager has a preference for a higher action and because he is more optimistic than the shareholder. In contrast, from an optimistic shareholder's point of view, the manager's preference for a higher action counterbalances the manager's pessimism. Essentially, optimists are more aligned with the manager than pessimists, and thus have lower incentives to misreport. This implies that without loss of generality, we can focus on equilibria in which pessimists communicate truthfully only if all optimists communicate truthfully.[14] These equilibria have the following properties:

Proposition 2 (Equilibrium at the communication stage). *The most informative equilibrium features the largest (in terms of the number of shareholders) subset of S that satisfies (10). Without loss of generality, it takes one of the following three forms: (1) all shareholders communicate truthfully; (2) all optimists communicate truthfully and some (potentially zero) pessimists communicate truthfully; (3) some (potentially zero) optimists communicate truthfully and no pessimist communicates truthfully. In addition:*

(i) If b is sufficiently small, then either all shareholders communicate truthfully (if $|S| > K - \frac{\rho+K/2}{\Delta}$) or no shareholder does (if $|S| < K - \frac{\rho+K/2}{\Delta}$).

(ii) If Δ is sufficiently small, the number of signals communicated is either $|S|$ or the floor of $\frac{\rho+K/2}{b} - 2\rho$, whichever is lower.

Statement (i) follows from shareholders' communication decisions being complements when b is small: if there exists an equilibrium in which at least one shareholder communicates truthfully, there also exists a (more informative) equilibrium in which all shareholders communicate truthfully. Moreover, because of complementarities, an equilibrium with truthful communication does not exist unless the number of shareholders $|S|$ is large enough.

Statement (ii) follows from the substitution effect, which dominates when differences in beliefs (Δ) are small relative to b. The fact that shareholders' communication decisions are substitutes implies that truthful communication by all shareholders is not possible unless their number $|S|$ is sufficiently small. Notice that the effect of $|S|$ in this case is the opposite of its effect in (i), where $|S|$ has to be sufficiently large for truthful communication to occur.

3.2 Trading stage

To solve for the equilibrium in the trading game, we first derive each investor's ex-ante expected utility from holding one share (not accounting for his holding costs) as a function

[14] Formally, as we show in the proof of Proposition 2, if there exists an equilibrium in which n_o optimists and n_p pessimists communicate truthfully, and S_o is the number of optimistic shareholders, then there exists a payoff-equivalent (in the sense of ex-ante payoffs of all shareholders and the manager) equilibrium in which $\min\{n_o + n_p, S_o\}$ optimists and $\max\{0, n_o + n_p - S_o\}$ pessimists communicate truthfully.

of the set of signals learned by the manager at the communication stage. We refer to this utility as the investor's valuation.

Lemma 2 (Ex-ante payoffs). *Suppose that in equilibrium, the manager learns subset R of the signals and does not learn all the other signals, $-R$. Then investor i's valuation of each share is given by:*

$$\begin{aligned}\mathbb{E}_i[U|R] \;=\; & u_0 - b^2 - \frac{2b(\rho-\rho_i)}{2\rho+|R|}(K-|R|) \\ & -\frac{\rho^2-\Delta^2}{2\rho(2\rho+1)}(K-|R|)\frac{2\rho+K}{2\rho+|R|} - \left[\frac{\Delta(K-|R|)}{2\rho+|R|}\right]^2 .\end{aligned} \tag{12}$$

If $K > \bar{K}$, where $\bar{K} \equiv \frac{4b\Delta\rho(2\rho+1)}{\rho^2-\Delta^2} - 2\rho$, then $E_i[U|R]$ is increasing in $|R|$ for every agent.

Intuitively, if the decision were fully informed and unbiased from an investor's perspective, his valuation would be u_0. However, the decision is biased from the investor's perspective due to the manager's misaligned preferences ($b > 0$) and different beliefs ($\rho \neq \rho_i$), which is captured by the second and third terms in (12). In addition, even if the manager has the same preferences and beliefs as the shareholder but does not have full information ($|R| < K$), the shareholder's valuation is below u_0 because the manager's decision is not fully informed. The forth and fifth terms in (12) capture this effect.

Recall that from an optimistic shareholder's point of view, the manager's preference for a higher action counterbalances the manager's pessimism. Because of this, an optimistic shareholder could even, under some circumstances, benefit from a less informed manager, i.e., a lower $|R|$. However, focusing on a large enough K (above $\bar{K}$ given by the lemma) ensures that this effect is not too strong, so that the more direct, beneficial, effect of managerial learning dominates: if $K > \bar{K}$, then all investors benefit from a more informed manager.[15] In what follows, we assume that this condition on K is satisfied, so that the most informative communication equilibrium is Pareto efficient and hence is played.

Given (3) and (12), we can calculate the demand for shares from each investor i for any

[15] The reason the positive effect dominates for large K is that by learning the signals θ_R, the manager updates his beliefs about φ and hence the signals θ_{-R}, and the set θ_{-R} is larger when K is larger.

set of signals R that the investor expects to be communicated to the manager. Maximizing (3) with respect to α, the optimal ownership stake of investor i given share price p is

$$\alpha_i(p) = \max\left\{\frac{\mathbb{E}_i[U|R] - p}{\lambda}, 0\right\}. \tag{13}$$

A larger holding cost λ decreases the investor's demand for shares, while higher expected utility $\mathbb{E}_i[U|R]$ from each share increases his demand. Given (13) and the unit supply of shares, market clearing implies $1 = \sum_{i=1}^{N} \alpha_i(p) = \sum_{i \in S} \frac{\mathbb{E}_i[U_i] - p}{\lambda}$. Hence, the equilibrium price for a given R satisfies

$$p^* = \frac{1}{|S|}\left(\sum_{i \in S} \mathbb{E}_i[U|R] - \lambda\right). \tag{14}$$

Concentrated vs. dispersed ownership. Because the manager's bias towards a higher action amplifies his disagreements with the pessimists but weakens his disagreements with the optimists, the pessimists have a lower valuation of the firm than the optimists (see the third term in (12)). As a result, the pessimists hold smaller stakes than the optimists and, if the differences in their valuations are substantial, do not hold any shares at all, resulting in more concentrated ownership. It is the interaction between the two frictions, rather than the presence of heterogeneous beliefs per se, that leads to more concentrated ownership: if the manager is unbiased ($b = 0$), the optimists' and pessimists' valuations are identical and ownership is fully dispersed.[16] The result that the interaction between the two frictions leads to more concentrated ownership does not rely on our assumption about only two types of investors and holds more generally.[17]

[16] This is because in our model, optimism does not mean a higher valuation of shares (each investor's valuation is u_0 if $a = Z$), but rather a belief that a higher action should be taken. For example, if the manager were biased towards a lower action ($b < 0$), then the optimists' valuation would be *lower* than that of the pessimists.

[17] To see this, consider a general distribution of investor beliefs $\{\rho_i, i = 1, ..., N\}$. Condition (10) implies that if preferences are aligned ($b = 0$) and there is no residual uncertainty, there exists an equilibrium in which ownership is fully dispersed (i.e., each investor's stake is $\frac{1}{N}$) and the manager learns all the signals. Indeed, in such an equilibrium, there are no disagreements in posterior beliefs, making truthful communication incentive compatible and, in turn, making the ex-ante valuations of all investors the same. In contrast, if b is large enough, truthful communication by all shareholders is no longer incentive compatible, so ex-post disagreements among investors remain, leading them to have different valuations and acquire different stakes.

Equilibria of the game. According to (13), all investors with the same prior beliefs own the same number of shares in equilibrium. Furthermore, because pessimists' valuations are lower than those of the optimists', pessimists only become shareholders if all optimists also become shareholders. As a result, the equilibria of the game take two possible forms.

The first case is that both pessimistic and optimistic investors become shareholders. This happens if the holding cost λ is sufficiently high, so that the demand for shares from the optimists declines relatively fast. Then, the shareholder base S consists of all investors, with optimists generally holding larger ownership stakes than pessimists. The set of shareholders that communicate truthfully is the largest subset of all investors for which the IC constraint (10) is satisfied; it is characterized by Proposition 2.

The second case is that only optimistic investors become shareholders, while pessimistic investors do not. This happens if the holding cost λ is sufficiently low. Then, the demand for shares from optimists does not decline very fast, and their high demand increases the share price to the level at which pessimists do not want to become shareholders. In this case, each optimist holds stake $\frac{1}{N_o}$, and the number of shareholders that communicate truthfully is the highest number in $[0, N_o]$ for which the IC constraint (10) for optimists is satisfied.

Sources of inefficiencies. There can be two sources of inefficiencies in equilibrium. One is suboptimal quality of decision-making if the manager does not learn all the available information. We will say that an equilibrium features *more informative communication* if the manager learns more signals from investors, i.e., $|R|$ is higher. Lemma 2 and the assumption $K > \bar{K}$ guarantee that if the manager learns more signals, then the expected valuation of the shares from the perspective of each investor, as well as the manager, is higher. In this sense, a greater number of signals learned by the manager means more informed and efficient decision-making. Note that managerial learning can be limited both because the firm's shareholders do not convey their views truthfully and because some potentially informed investors (pessimists in our setting) do not become shareholders in the first place.

The second source of inefficiency is suboptimal diversification by investors: the aggregate holding costs would be minimized if each investor's stake were $\frac{1}{N}$. Both inefficiencies reduce investors' combined utility from holding the stock, as well as the share price. The following proposition provides sufficient conditions under which these inefficiencies do not arise:

Proposition 3. *If $b < \frac{\rho+K/2}{2\rho+N}$, there exists $\rho(b,\Delta) > 0$, which is decreasing in b and Δ, such that if $K - N \leq \rho(b,\Delta)$, there exists an equilibrium where all investors become shareholders and truthfully communicate their information to the manager. Moreover, if $K = N$, then for any Δ, all investors acquire the same number of shares, achieving full diversification and truthful communication. If, in addition, $b = 0$, the equilibrium achieves first-best.*

The logic is as follows. Suppose that all investors indeed become shareholders. Condition $b < \frac{\rho+K/2}{2\rho+N}$ ensures that preferences are sufficiently aligned, so that all investors communicate truthfully if beliefs are aligned as well. The condition that the residual uncertainty is low, $K - N \leq \rho(b,\Delta)$, guarantees that if the manager is expected to learn all investors' information ($|R| = N$), his remaining belief disagreements with the shareholders are small, so that it is indeed incentive compatible for all investors to communicate truthfully. Together, these two conditions imply that the manager's decision is both relatively unbiased and sufficiently informed, so both optimists' and pessimists' valuations are high and they all become shareholders. Thus, an equilibrium with $S = \{1, ..., N\}$ and full communication indeed exists. Moreover, if $K = N$, then even if the original differences in beliefs are very large, the posterior beliefs of all agents are the same. In this case, the optimists and pessimists have the same ex-ante valuations and hence acquire equal stakes, achieving optimal diversification. If $b = 0$, the equilibrium achieves first-best: a social planner who maximizes the combined utility of all players would pick the same allocation of shares ($\frac{1}{N}$ to each investor) and the same corporate action ($a = Z$) as those that arise in equilibrium.

We fully characterize the set of equilibria in the appendix, after the proof of Lemma 2. In general, either one or both of the two inefficiencies are present in equilibrium. Moreover, these inefficiencies are interrelated and amplify each other. On the one hand, the fact that the manager does not learn all the information implies that ex-post, his beliefs about the state are different from those of the shareholders. Anticipating these disagreements, shareholders who are relatively less aligned with the manager (i.e., the pessimists) acquire a lower stake than the optimists and, potentially, do not acquire any shares at all. Thus, imperfect shareholder communication leads to suboptimal diversification across investors. On the other hand, the fact that some investors do not become shareholders in the first place implies that they do not engage and communicate with the manager, which leads to less informed managerial decision-making. We explore these interactions and their implications next.

4 Implications

In this section, we derive the key implications of the model. Section 4.1 focuses on the link between the firm's ownership structure and shareholder engagement. Section 4.2 analyzes the role of the firm's corporate governance quality. Section 4.3 discusses advisory shareholder voting, the advisory role of the board, and how they are affected by the manager's expertise.

4.1 Ownership structure and shareholder engagement

The analysis in Section 3 shows that differences in beliefs and misaligned preferences may prevent effective communication between shareholders and management, resulting in less informed corporate decisions. These frictions can be exacerbated by the fact that the ownership structure is itself endogenous: investors who disagree with management may choose not to become shareholders of the firm. Such investors then have no incentives or ability to communicate their information, leading to a loss of potentially valuable information for decision-making. Instead, ownership is concentrated among investors who are relatively more aligned with the management, and only they provide their advice. This concentration of ownership is more likely to happen when holding costs are relatively small – e.g., when the firm is smaller or less risky. The following proposition formalizes this intuition.

Proposition 4. *Suppose* $(2\rho + N_o + 1)\, b + (K - N_o - 1)\, \Delta \leq \rho + \frac{K}{2}$*. Then there exists* $\hat{\lambda}$ *such that for any* $\lambda > \hat{\lambda}$*, there is an equilibrium that features a more dispersed ownership structure and more informative communication than any equilibrium for* $\lambda < \hat{\lambda}$*. The equilibrium stock* *price is non-monotone in λ: it is decreasing in λ for $\lambda < \hat{\lambda}$ and $\lambda > \hat{\lambda}$, but increases discontinuously at $\lambda = \hat{\lambda}$.*

Intuitively, if the holding costs are low, $\lambda < \hat{\lambda}$, the optimistic investors' demand is high and increases the share price to the level that exceeds the pessimistic investors' valuation of the stock. Thus, the firm is entirely held by the optimists, and pessimists do not communicate their information even if they would have incentives to do so, had they owned the firm. An increase in holding costs prevents this ownership concentration and encourages more investors to hold the firm and communicate their information (the condition on b and Δ in the statement of the proposition ensures that not only optimists, but also some pessimists, have incentives to communicate truthfully). As a result, while higher holding costs generally decrease the share price (p^* decreases in λ for a given S and R in (14)), this may no longer be the case when learning from shareholders is important. As the last statement of the proposition shows, the wider shareholder base and the resulting improvement in corporate decision-making can lead the share price to increase in λ.

An important drawback of dispersed ownership, which is frequently discussed in the literature, is the free-rider problem: dispersed ownership discourages each individual shareholder from exerting effort. A related effect would arise in our model if information acquisition were costly. In Section 7.3 of the Online Appendix, we introduce costs of information acquisition and show that more dispersed ownership decreases each shareholder's incentives to acquire information. Combined with the complementarity in communication decisions, this implies that ownership should be neither too concentrated nor too dispersed for shareholder engagement to be most effective. Because the free-rider problem is well-understood in the literature, we abstract from costly information acquisition in the basic model and focus on the more novel effects coming from complementarities in shareholders' communication decisions.

4.1.1 Role of passively managed funds

As the previous discussion shows, when investors optimally pick their holdings in the firm, the shareholder base can become too limited, and the information of investors who disagree with management can be lost. This suggests an interesting distinction between the advisory role of actively managed vs. passively managed (index) funds. In recent decades, an increasing fraction of firms' ownership is comprised by passive funds (e.g., Appel et al., 2016). Passively managed funds become shareholders even if they disagree with the firm's management: as William McNabb III, former chairman and CEO of Vanguard put it, "We're going to hold your stock if we like you. And if we don't."[18] This requirement to hold the stock regardless of the fund manager's views about the company implies that the growth in passive funds can make shareholder-manager communication more effective. Not only will passive funds hold the stock and engage with management when active funds in their position would have not, but moreover, due to complementarities in communication, engagement by passive funds can have a positive spillover effect on the engagement by other firm's shareholders as well.

To show these implications, we make a small modification of the basic model. Suppose that out of N investors, L investors are required to hold $\frac{1}{N}$ shares of the firm irrespectively of the market price of the shares or their valuations. We refer to these investors as "passive." The remaining $\frac{N-L}{N}$ shares are sold in the market to the remaining $N-L$ investors, who we call "active." Suppose that optimists and pessimists are equally represented among passive and active investors: the number of optimists among passive (active) investors is $N_o\frac{L}{N}$ ($N_o\frac{N-L}{N}$). This guarantees that we keep investors' beliefs the same as we change L, i.e., there are always N_o optimists and $N-N_o$ pessimists, regardless of the number of passive investors. The basic model corresponds to $L=0$.

The assumption that each passive investor holds $\frac{1}{N}$ shares ensures that passive investors do not have price effects by changing the residual supply of shares: as we show in the proof of Proposition 5, given the same equilibrium at the communication stage, the stock price with L passive investors is the same as in the basic model without passive investors.[19] However, shareholder communication is improved by the presence of passive investors: the manager learns more information than in the basic model. As a result, managerial decisions are more

[18] See "Getting to Know You: The Case for Significant Shareholder Engagement," by William McNabb III, *Harvard Law School Forum on Corporate Governance*, June 24, 2015.

[19] If passive investors' stakes were lower (higher) than $\frac{1}{N}$, active investors would hold larger (smaller) stakes than in the basic model, leading to a lower (higher) stock price because of their holding costs.

informed, and the share price is higher:

Proposition 5. *Suppose* $(2\rho + N_o + 1)\, b + (K - N_o - 1)\, \Delta \leq \rho + \frac{K}{2}$ *and* $\lambda < \hat{\lambda}$, *as defined in Proposition 4. Then, the equilibrium with* $L > 0$ *passive investors features more informative communication and a higher share price than the equilibrium without passive investors. Informativeness of communication and the share price are weakly increasing in* L.

Intuitively, an active investor may choose to not become a shareholder if he is pessimistic and the stock price exceeds his valuation of the shares. Such active investors do not communicate with the manager, even though they would do so if they were forced to become shareholders (the parameter restrictions in Proposition 5 guarantee that their IC constraint would be satisfied). In contrast, passive investors own the shares regardless of their beliefs, i.e., even if they are pessimistic. Thus, passive investors become shareholders and provide advice to the manager even if active investors in their position (with exactly the same beliefs and preferences) would have stayed away from the firm. Overall, passive fund growth (i.e., an increase in L) makes corporate decisions more informed and increases the stock price.[20]

Not only are passive investors more likely to communicate with the manager compared to active investors in their position, but their presence also enhances the communication between the manager and *other shareholders*. This spillover effect occurs because shareholders' communication decisions are complements:

Proposition 6. *Suppose* $\lambda < \frac{4b\Delta(K-N)N_o}{2\rho+N}$ *and* $(K - N_o)\,\Delta - (2\rho + N_o)\, b > \rho + \frac{K}{2}$. *Then, without passive investors, only* N_o *optimistic investors become shareholders and not all of these optimists communicate truthfully, i.e.,* $|R| < N_o$. *If, in addition,* $\left(2\rho + N_o + N_p \frac{L}{N}\right) b + \left(K - N_o - N_p \frac{L}{N}\right) \Delta \leq \rho + \frac{K}{2}$, *then in the model with* L *passive investors,* N_o *optimists and* $N_p \frac{L}{N}$ *pessimists become shareholders, and all* $N_o + N_p \frac{L}{N}$ *shareholders communicate truthfully.*

[20] Note, however, that the equilibrium does not necessarily become more efficient as L increases. This is because even though the manager's decision becomes more informed, there is suboptimal diversification by passive investors: unless the manager becomes fully informed ($|R| = K$), optimistic (pessimistic) passive investors are restricted to holding a strictly smaller (larger) stake than they would have chosen optimally.

Intuitively, by engaging with the manager and making him more informed, passive investors reduce belief disagreements between the manager and other investors, encouraging them to communicate their information as well. As a result, whereas only a subset of optimists communicate with the manager when $L = 0$, all of the optimists (active and passive) communicate when passive investors are present. While Proposition 6 is specific in its conditions, the general intuition is that if there are substantial disagreements in beliefs, so that the complementarity effect in communication dominates the substitution effect, the presence of passive investors could facilitate communication by all shareholders, both active and passive, and thus lead to more informative decisions of the management.

Empirical implications. The model highlights two novel forces due to which greater passive fund ownership could be associated with more effective communication between shareholders and management, both in the time-series and in the cross-section. In the time-series, the rise in passive fund ownership over the last decades has coincided with the increased impact of advisory votes on firms' decisions, as well as the rise in shareholder engagement campaigns and management responsiveness to them. For example, Ferri (2012) discusses the evolution of advisory voting and concludes that until early 2000s, it was "low-impact" and that such votes "were largely ignored" by management, but that it "has become a more powerful tool" in recent years. Importantly, votes by passive funds and engagement campaigns by large index fund managers are a significant part of this overall improvement in shareholder-manager communication.[21] While these contemporaneous trends in no way show causality, to establish a more causal link between passive fund ownership and the effectiveness of shareholder communication, one could conduct cross-sectional analysis similar to the Russell-3000 reconstitution studies (e.g., Appel et al., 2016). To measure the effectiveness of shareholder communication in this setting, one could look at managerial responsiveness to advisory votes (as, e.g., in Ertimur et al., 2010; Cuñat et al., 2012; and Ferri, 2012) and to shareholder engagement campaigns (as, e.g., in Gormley et al., 2021).

[21] See, e.g., Fichtner et al. (2017). According to Larry Fink, the CEO of BlackRock, "we are taking a more active dialogue with our companies" (see "Passive Investors are Good Corporate Stewards", *Financial Times*, January 19, 2016).

Equilibrium multiplicity. While the above results emphasize how the firm's ownership structure affects managerial learning and decision-making, there is an effect in the other direction as well: shareholders' anticipation of the firm's decisions affects their valuation of the shares and the stakes they acquire. If management is expected to take decisions that differ from most investors' perspective, the firm will only attract investors whose views are aligned with this strategy. This creates a feedback loop between the ownership structure and managerial decision-making, which may lead to multiple equilibria: one where the firm is widely held and management gets advice from a large set of investors, and another where the firm is held by a subset of shareholders and managerial learning is limited. The first equilibrium features both a higher share price and higher welfare, which we define as the combined utility of all investors and the original owner (seller of the shares).[22]

Proposition 7. *Suppose that $0 < b \leq \frac{1}{2}$, and there is no residual uncertainty, $K = N$.*

(i) There always exists an equilibrium in which all N investors become shareholders, acquire equal stakes $\alpha_i = \frac{1}{N}$ in the firm, and the most informed action $a = b+Z$ is undertaken.

(ii) Suppose, in addition, that $\frac{2\rho+N_o}{\Delta}b \leq K - N_o \leq \frac{2\rho+N_o}{\Delta}b + \frac{\rho+K/2}{\Delta}$ and $\lambda < 4b\Delta\frac{(K-N_o)N_o}{2\rho+N_o}$. If $L = 0$, there also exists an equilibrium in which only optimistic investors become shareholders, action $a \neq b+Z$ is undertaken, and both welfare and the share price are lower than in the first equilibrium. If L is sufficiently large, the equilibrium in (i) is unique.

Equilibrium multiplicity arises due to the complementarity in shareholders' communication decisions. Statement (i) is a direct consequence of Proposition 3: if preferences are sufficiently aligned ($0 < b \leq \frac{1}{2}$) and $K = N$, then regardless of how strong differences in beliefs are, there exists an equilibrium in which all investors become shareholders and the manager's action reflects all available information. This equilibrium features the highest welfare and share price, both because the firm's decision is most informed, and because investors'

[22] We could alternatively define welfare as the combined utility of all players, i.e., investors, seller, and the manager, and Proposition 7 would hold as well. The reason we exclude the manager's utility from the definition of welfare is that the analysis in Section 4.2 views b, and hence the preferences of the manager, as a policy choice.

total holding costs are minimized since the stock is evenly divided among them. Statement (ii) shows that when there are no passive investors ($L = 0$), this equilibrium can co-exist with an equilibrium in which the manager's decision is not based on all the available information and total holding costs are larger. Intuitively, if only a subset of investors (optimists in our setting) are expected to become shareholders and provide advice to the manager, there are still ex-post differences in beliefs between the manager and the shareholders. Anticipating this at the trading stage, investors who are less aligned with the manager (pessimists in our setting), do not buy shares in the first place, making this equilibrium self-fulfilling. Thus, in the presence of equilibrium multiplicity, there is yet another reason why the presence of passive investors ($L > 0$) can enhance the effectiveness of shareholder communication. Since passive funds become shareholders regardless of whether their fund managers agree with the firm's CEO, their presence breaks the feedback loop between the ownership structure and managerial decision-making, and can eliminate the less efficient equilibrium.[23]

For simplicity, in the remainder of the paper, we return to the basic model without passive investors ($L = 0$), but our subsequent results would hold for $L > 0$ as well.

4.2 Corporate governance and shareholder engagement

So far, we have taken the conflict of interest b as given and examined shareholder engagement for a given b. However, b can also be viewed as a policy choice. For example, if the trading stage is interpreted as the original owners selling the firm in an IPO, then we can think of b as capturing the quality of corporate governance adopted at the IPO: more aligned managerial incentives due to better compensation contracts and a more independent board decrease b.

Does the firm always benefit from decreasing b? It turns out that the answer depends on how strong disagreements in beliefs about the firm are. If heterogeneity in beliefs is relatively small, then stronger governance, i.e., a lower b, always improves shareholder communication and yields the highest payoffs to all investors, as well as the highest possible price to the seller. However, this is no longer the case when differences in beliefs are substantial:

[23] The result that the equilibrium is unique for large L relies on the equilibrium selection criterion described above: the most informative equilibrium at the communication stage is Pareto efficient and hence is played.

Proposition 8. *(i) Suppose $\Delta \leq \frac{\rho+K/2}{K-N}$. Then $b=0$ yields the highest welfare and the highest share price compared to any $b>0$. (ii) Suppose $\Delta > \frac{\rho+K/2}{K-N}$. Then for any $b>0$ that satisfies $|(2\rho+n)\,b-(K-n)\,\Delta| \leq \rho+\frac{K}{2}$ for some $n \in [1, N_o]$, the equilibrium features more informative communication than for $b=0$. Moreover, such equilibrium can have higher welfare and a higher share price than the equilibrium for $b=0$.*

Part (i) shows that if there is relatively little disagreement in prior beliefs, the optimal governance structure features fully aligned managerial preferences, i.e., $b=0$. This is because if Δ is low enough, then the manager's and investors' posterior beliefs become sufficiently close to each other if the manager learns all investors' information. Combined with no misalignment in preferences, this makes investors and the manager sufficiently congruent and supports truthful communication by the investors. Overall, if $b=0$, all investors find it optimal to become shareholders and communicate their information, so the firm's decision is both most informed and unbiased.

Part (ii) shows that $b=0$ is not always optimal: when belief disagreements are substantial, misaligned managerial preferences can encourage *more* shareholder communication. Intuitively, if $\Delta > \frac{\rho+K/2}{K-N}$, then even if the manager learns all investors' information, the disagreement between him and investors remains substantial. Then, both the pessimists and the optimists perceive that the manager holds "incorrect" beliefs and are reluctant to communicate their information truthfully: the pessimists think that the manager is too optimistic and will take an action that is too high, while the optimists think that the manager is too pessimistic and will take an action that is too low. As a result, the only equilibrium for $b=0$ is that no shareholder communicates truthfully. In this case, some bias in preferences can align the manager with the optimistic investors: the manager's bias towards a higher action counteracts his "too pessimistic" beliefs and encourages the optimists to communicate truthfully. If the resulting improvement in communication is sufficiently large, the valuations of all investors, the share price, and welfare are all higher under $b>0$ despite more biased decision-making. The proof of Proposition 8 shows that this can indeed occur.

4.3 Nonbinding voting and the advisory role of the board

In this section, we discuss the implications of our analysis for two channels of shareholders' communication with management that have been extensively explored in the empirical literature: nonbinding (i.e., advisory) shareholder voting and the company's board of directors.

When does nonbinding voting enhance managerial learning? Shareholders of the firm have different means of communicating with management. First, they can meet and engage with management directly. Second, they can join the company's board and express their views in board meetings. However, these channels of communication are only available to the largest shareholders, as it is not feasible and worthwhile for management to meet with all of the firm's investors, or to add all of them to the board. In this sense, advisory voting offers a low-cost way to collect the views of all of the shareholders. Thus, the Dodd-Frank requirement of regular advisory votes on executive compensation, as well as investors' ability to submit proposals for an advisory vote via Rule 14a-8, can be seen as helping expand the set of shareholders who can communicate their views to management.

Both the mandatory say-on-pay requirement and Rule 14a-8 have been hotly debated because of their potential downsides, such as the time and resources they may require from management and potential distractions they can cause. Thus, to judge the overall effects of these policies, it is important to understand the extent to which expanding the set of communicating shareholders enhances managerial learning. Our results suggest that whether managerial learning is substantially improved or not depends on the extent of disagreements in prior beliefs about the decision, as well as how much shareholders' and manager's preferences regarding the decision are aligned. If belief disagreements are large and preferences are relatively aligned (Δ is large relative to b), shareholders' communication decisions are complements. In this case, expanding the set of shareholders who can convey their views has an amplified positive effect. Not only does it allow communication by the shareholders who would not be able to convey their views otherwise, but it may also have a spillover effect and encourage truthful communication by the shareholders who could convey their views (e.g., those on the board) but would not do so truthfully because of strong belief disagreements with the manager.[24] In contrast, if belief disagreements over a decision are small or conflicts of interest are substantial (Δ is small relative to b), shareholders' communication decisions are substitutes. Proposition 2(ii) then shows that the information the manager can learn is limited. Thus, expanding the set of shareholders who can communicate with management, e.g., through mandatory advisory voting on this decision, may not improve managerial learning at all, and the downsides of such votes may become first-order.

Advisory role of the board and board size. For large shareholders, joining the board of directors can be another way to communicate with managers, as advising the management is one of the most important functions of the board. For example, it is common for venture capitalists (VCs) and activist investors to take board seats (e.g., Field, Lowry, and Mkrtchyan, 2013; Bebchuk et al., 2020). Moreover, VCs often assume a "board observer" role: they attend board meetings and offer their views, but do not have board voting rights.

Increasing board size to include more shareholders who can provide advice is not always beneficial, as it brings the problems of coordination and the costs of new directors' compensation. Our results suggest that adding more advisory directors is beneficial when differences in beliefs are substantial: in this case, the complementarity effect implies that a larger board improves managerial learning. However, if conflicts of interest are substantial, the substitution effect dominates, and a larger board is more likely to decrease value. Thus, our results have implications for the literature on board size (e.g., Yermack, 1996; Coles, Daniel, and Naveen, 2008; and Jenter, Schmid, and Urban, 2019), with the caveat that they are more first-order in situations where the board's primary role is to provide advice.

[24] Many decisions that involve large disagreements in beliefs are also the decisions that involve a large degree of uncertainty. In Section 7.4 of the Online Appendix, we show that a change in parameters that increases the variance of the state from the perspective of each investor (i.e., increases uncertainty), while keeping its expected value fixed, makes the IC condition (10) more likely to be satisfied. Intuitively, higher variance means relatively uninformative priors, inducing the manager to react more strongly to the shareholder's advice. This makes it more costly for the shareholder to misreport, inducing truthful communication.

Expertise of the manager. While our focus is on the manager learning from the shareholders, the manager is likely to have private information as well. How does the manager's expertise affect his ability to learn from investors? To understand this, suppose that while investor i knows signal θ_i, the manager privately knows signals $\theta_{N+1}, ..., \theta_{N+\mathcal{M}}$, $N + \mathcal{M} \leq K$. A higher $\mathcal{M}$ corresponds to greater managerial expertise, and $\mathcal{M} = 0$ captures the basic model. Then, the arguments behind Proposition 1 imply that shareholder i reports his signal truthfully if and only if (10) holds, where R_i is now defined as the set of signals privately known by the manager combined with the subset of investors' signals (not including θ_i) that the manager is expected to learn. It follows that an equilibrium in which all investors become shareholders and communicate their information to the manager exists if and only if λ is sufficiently large and the IC condition for pessimists is satisfied:

$$(2\rho + N + \mathcal{M})\, b + (K - N - \mathcal{M})\, \Delta \leq \rho + \frac{K}{2}. \tag{15}$$

As $\mathcal{M}$ increases, (15) is more (less) likely to be satisfied if $b < \Delta$ ($b > \Delta$). Thus, whether greater managerial expertise enhances managerial learning from the shareholders depends on the interaction between the two communication frictions. The intuition is close to the intuition behind the complementarity and substitution effects in Section 3.1. If belief disagreements are substantial, the key consequence of managerial expertise is that shareholders expect their disagreements with the manager to decrease as he learns more about the decision, which increases the congruence between them and improves communication. However, if the conflict of interest is substantial, then the effect of managerial learning on congruence is limited, and the key effect is that managerial expertise decreases shareholders' costs of misreporting because the manager is expected to react less to shareholders' advice.

Empirical implications. Combining the insights about the externalities in communication and the role of managerial expertise, our model predicts that in the presence of strong belief disagreements, a shareholder's ability to influence the manager with his views is enhanced by the expertise of other shareholders and the expertise of the manager. However, as the firm's governance deteriorates and the preferences of the manager and the shareholders become less aligned, this effect weakens and is eventually reversed.

One way to test this prediction is to analyze the advisory role of the firm's directors. The literature on the board's advisory role studies how the presence of directors with a certain type of expertise is related to corporate policies and performance. For example, Dass et al. (2014) analyze directors' expertise in related industries, Güner, Malmendier, and Tate (2008) study financial expertise, and Harford and Schonlau (2013) focus on directors' experience in mergers and acquisitions. The unique prediction of our model is that the advisory role of a director (i.e., whether his information will influence the manager's decisions) should not be viewed in isolation, but depends on the expertise of the manager and the expertise of other directors. Another way to test this prediction is by studying managerial responsiveness to the advisory vote tally (as in Ertimur et al., 2010; Cuñat et al., 2012; and Ferri, 2012), and analyzing how it varies with managerial expertise and the ownership structure (number and sophistication of the firm's shareholders). To measure the extent of heterogeneity of beliefs, one could rely on several measures of belief heterogeneity proposed by the literature (e.g., Thakor and Whited, 2011; Diether et al., 2002; Malmendier and Tate, 2005).

In addition, our model predicts that greater managerial expertise will generally be associated with more dispersed ownership. To see this, note that any investor's valuation is given by (12), where R now stands for the $\mathcal{M}$ signals privately learned by the manager combined with the signals he learns from the investors. A higher $\mathcal{M}$ results in a higher $|R|$ for two reasons: the manager's own expertise and, as long as the manager's bias b is not too large, due to better learning from the shareholders. Expression (12) then implies that the optimists' and pessimists' valuations are closer to each other, leading them to acquire similar stakes and making ownership more dispersed. However, if the manager's bias b is substantial, this effect is weaker since greater expertise makes learning from the shareholders less effective.

5 Discussion and robustness

In this section, we discuss the key assumptions of the model and their role for the results.

5.1 Information structure

To make the analysis tractable and derive simple, closed-form solutions, we make specific assumptions about the information structure. As we discuss next, the complementarity and substitution effects in shareholders' communication decisions arise under many other information structures, although not all of them.

In particular, the property that drives the complementarity effect is that communication by other shareholders brings the manager's and shareholder's posterior beliefs closer to each other. Below we discuss the robustness of this property.

Heterogeneous interpretation of information. While in our model agents interpret information (i.e., signals θ_i) the same way, it is also natural to expect that they might interpret information differently. To explore this, in Section 7.5 of the Online Appendix, we follow models of differences of opinion in which agents disagree about the precision of signals (e.g., Banerjee et al., 2009; Kyle et al., 2018) and assume that each shareholder overestimates the importance of his own signal. As we show, communication decisions are complements even though agents now interpret information differently. This is because for any given realization of the shareholder's own signal, communication by other shareholders still moves the manager's posterior belief closer to that of the shareholder's. This property holds in a large class of models of different beliefs, although not in all of them.

Mixed strategy equilibria. Our focus is on pure strategy equilibria, which simplifies the analysis by making communication of each shareholder either truthful or uninformative. Under mixed strategy equilibria, communication of each shareholder can be partially informative, making the model less tractable. In unreported results, we analyze a symmetric mixed strategy equilibrium in a setting with $b = 0$ and two investors, one optimist and one pessimist. We show that the IC constraint of a shareholder is more likely to be satisfied if the probability with which the other shareholder communicates truthfully increases, suggesting that shareholders' decisions are again complements. Intuitively, even if communication of other shareholders is only partially informative, it still brings the manager's and shareholder's beliefs closer to each other.

The property that drives the substitution effect is that each subsequent signal has a smaller effect on the action of the manager, and thus communication by other shareholders decreases the manager's reaction to the shareholder's message. This property holds in a large class of models, but not in all of them. In particular:

Complementarity vs. substitutability of signals. Borgers et al. (2013) introduce the notion of substitutability vs. complementarity of signals and show that it may affect strategic interactions between agents. They call signals substitutes (complements) if the marginal impact of an additional signal on the agent's utility decreases (increases) in the number of signals.[25] In Section 7.6 of the Online Appendix, we show that signals θ_i are substitutes under this definition. Intuitively, this is because learning each additional signal leads to increasingly smaller updating of beliefs about φ. As Borgers et al. (2013) highlight, this property is not without loss of generality. However, the substitutability of signals is a very common feature in the literature and is natural in many applications, so we believe that our conclusions are applicable in many settings.

To see why the substitutability of signals may play a role, suppose that φ is a commonly known parameter (i.e., there is no learning about φ, unlike in the basic model). We consider this scenario in Section 7.6 of the Online Appendix and show that signals θ_i are then neither substitutes nor complements under the definition of Borgers et al. (2013): the extra benefit from an additional signal does not depend on the total number of signals received. As we also show, the substitution effect in communication decisions does not arise in this case because the manager's reaction to a shareholder's advice does not depend on how many other signals he learns. Thus, the substitution effect in communication is somewhat tied to

[25] Formally, Borgers et al. (2013) define signals as substitutes (complements) if the decision-maker's added utility from having two signals relative to having one signal (assuming he takes the optimal action given these signals) is smaller (larger) than his added utility from having one signal relative to no signal at all. Despite the similarity in terminology, the two notions of complementarity/substitutability are very different: Borgers et al. (2013) focus on complementarity/substitutability of *signals* in a *single* decision-maker's problem, whereas our paper studies complementarity/substitutability of *actions* in a communication game with *multiple* players.

the substitutability between agents' signals. Another implication of this result is that there is an amplified beneficial effect of diversity of expertise on corporate decision-making. If shareholders' expertise is diverse, in that they have information about different aspects of the decision (i.e., have unconditionally independent signals), then asking more shareholders for advice is useful for two reasons. First, the added value from an additional signal does not decline with the number of shareholders, and second, asking more shareholders for advice does not inhibit the communication of other shareholders.

Multiple dimensions of expertise. The previous discussion suggests, more generally, that with multiple dimensions of expertise, the substitution effect is likely to be weaker than with a single dimension of expertise. Intuitively, when a shareholder's signal not only provides noisy information about some common underlying state, but also provides information about a different, independent aspect of the decision, the manager is likely to react relatively strongly to the shareholder's advice, even if he receives advice from many other shareholders. As a result, the costs of misreporting do not decrease as much, weaking the substitution effect. We confirm this intuition in Section 7.7 of the Online Appendix, where we analyze a variation of our model with only one dimension of expertise: each shareholder receives a noisy signal about the state and has no independent expertise beyond that.[26] We show that both the complementarity effect and the substitution effect arise as well, as in the basic model, but the substitution effect is stronger relative to the basic model, so that when $b = 0$, it offsets the complementarity effect. While the offsetting result is special to the Beta distribution, the intuition that the presence of multiple dimensions of expertise weakens the substituiton effect is more general.

5.2 Communication protocols

Communication among shareholders. Our model assumes that shareholders do not communicate privately among themselves prior to communicating with management. In

[26] In contrast, in our basic model, each shareholder both has a noisy signal about the state (because θ_i provides noisy information about φ), but also has independent expertise beyond that (because conditional on φ, θ_i is independent of other shareholders' signals and is informative about the optimal decision).

practice, there are indeed limitations to such communication. First, communication with other shareholders can be viewed as "forming a group," which could require the shareholders to file form 13D or could trigger a poison pill. According to the 2011 report by Dechert LLP, "shareholder concern about unintentionally forming a group has chilled communications among large holders of shares in U.S. public companies." Second, shareholders often avoid such communication as it could be considered by management as running an activist campaign and lead to managerial retaliation. In Section 7.8 of the Online Appendix, we partly relax this assumption by considering the following change in the communication stage: first, all shareholders of the same type (i.e., with the same prior beliefs) share their signals among themselves, and then, one representative of each group communicates with the manager via cheap talk. We show that the necessary and sufficient conditions for the existence of an equilibrium where all shareholders communicate truthfully are the same as in the basic model. Thus, the results of Proposition 2 continue to hold: an equilibrium with all shareholders communicating truthfully exists if and only if the shareholder base $|S|$ is large enough when b is small, and if and only if $|S|$ is small enough when Δ is small.

Sequential communication. Our model assumes simultaneous (or, equivalently, private sequential) communication by shareholders to the manager. Instead, some shareholders could publicly announce their views, so that other shareholders can update their beliefs before communicating with the manager themselves. In Section 7.9 of the Online Appendix, we consider a variation of the model in which shareholders send public messages in a known sequence to all other shareholders and the manager. We show that the IC conditions for truthful communication are the same as in the basic model, and hence for any sequence, the equilibrium at the communication stage is the same as in the model with simultaneous communication. Intuitively, this is because what matters for the shareholder's incentives is the combined set of signals that the manager learns before taking his action, as this combined set of signals determines both the manager's reaction to the shareholder's advice and the congruence between the manager and the shareholder at the decision-making stage.

6 Conclusion

Shareholder engagement, i.e., shareholders communicating their views and information about the firm's policies to the management, has become increasingly important in recent years. This paper provides a theory of shareholder engagement and ownership structure to study how managerial learning from the shareholders can be enhanced. Differences in beliefs between shareholders and the manager, as well as misaligned preferences between them, can inhibit effective shareholder engagement by giving shareholders incentives to misrepresent their information. The ownership structure can further limit managerial learning if many informed investors choose not to become shareholders in the first place. We show that these inefficiencies – communication frictions and limited shareholder base – can amplify each other. In this case, the presence of passively managed institutional investors can enhance managerial learning, both by passive funds' own engagement, and because their engagement encourages more effective communication by active funds. We also show that communication decisions of shareholders are complements when disagreements in beliefs are substantial, but become substitutes if the manager's preferences are strongly misaligned with those of the shareholders. As a result, introducing advisory voting and adding shareholders to the firm's board can significantly improve managerial learning for decisions involving differences in beliefs, but these actions can be detrimental for decisions involving large conflicts of interest. However, some misalignment in the manager's preferences can enhance communication if differences in beliefs are especially strong.

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Appendix

Proof of Lemma 1

Since θ_i is a binary signal equal to 1 with probability φ and 0 with probability $1-\varphi$, the manager's optimal action (6) can be written as:

$$a_m(\theta_R) = b + \sum_{i\in R}\theta_i + \mathbb{E}_m\left[\varphi|\theta_i, i\in R\right](K-|R|).$$

Let $\mathbf{1}_R \equiv \sum_{i\in R}\theta_i$ be the number of signals in R equal to 1. The conditional probability that $\mathbf{1}_R$ signals out of $|R|$ are equal to one given φ is $P(\mathbf{1}_R|\varphi) = \binom{|R|}{\mathbf{1}_R}\varphi^{\mathbf{1}_R}(1-\varphi)^{|R|-\mathbf{1}_R}$. Since the prior distribution is Beta and the likelihood function is Binomial, the posterior distribution is also Beta but with different parameters (this is a known property of the Beta distribution). Formally, let $P_i(\mathbf{1}_R)$ be agent i's assessed probability that $\mathbf{1}_R$ signals out of $|R|$ are equal to 1 (over all possible values of φ). Using Bayes rule, agent i's posterior belief of φ, $P_i(\varphi|\mathbf{1}_R)$, is

$$\begin{aligned}
P_i(\varphi|\mathbf{1}_R) &= \frac{f_i(\varphi)P(\mathbf{1}_R|\varphi)}{P_i(\mathbf{1}_R)} = \frac{\varphi^{\rho_i-1}(1-\varphi)^{\tau-\rho_i-1}}{Beta(\rho_i,\tau-\rho_i)}\frac{1}{P_i(\mathbf{1}_R)}\binom{|R|}{\mathbf{1}_R}\varphi^{\mathbf{1}_R}(1-\varphi)^{|R|-\mathbf{1}_R} \\
&= \frac{1}{Beta(\rho_i,\tau-\rho_i)P_i(\mathbf{1}_R)}\binom{|R|}{\mathbf{1}_R}\times\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1},
\end{aligned}$$

which is some constant that does not depend on φ times $\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1}$. Since the posterior beliefs must integrate to one over possible values of φ, this automatically implies that the posterior belief also follows a Beta distribution with parameters $(\rho_i+\mathbf{1}_R, \tau-\rho_i+|R|-\mathbf{1}_R)$ and density

$$P_i(\varphi|\mathbf{1}_R) = \frac{1}{Beta(\rho_i+\mathbf{1}_R,\tau-\rho_i+|R|-\mathbf{1}_R)}\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1}.$$

It is known that the mean of a Beta distribution with parameters (α,β) is $\frac{\alpha}{\alpha+\beta}$. Therefore, using these expressions and the above posterior distribution, agent i's expected value of φ is $\mathbb{E}_i(\varphi|\mathbf{1}_R) = \frac{\rho_i+\mathbf{1}_R}{\tau+|R|}$, which proves the lemma.

Auxiliary Lemma A.1

Suppose $\varphi \sim Beta(\rho,\tau-\rho)$ and $X = \{x_1, x_2, \cdots, x_n\}$, where $x_i \in \{0,1\}$ are independent draws with $x_i = 1$ with probability φ. Let $\mathbf{1}_X \equiv \sum_{i=1}^n x_i$. Then

$$\mathbb{E}_X[\mathbf{1}_X] = n\frac{\rho}{\tau} \text{ and } \mathbb{E}_X[\mathbf{1}_X^2] = n\rho\,\frac{\tau-\rho+n(\rho+1)}{\tau(\tau+1)}.$$

Proof. It is known that the first two moments of a random variable X distributed according to a Beta distribution with parameters α and β are $\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$ and $\mathbb{E}[X^2] = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$. Hence, $\mathbb{E}[\varphi] = \frac{\rho}{\tau}$ and $\mathbb{E}[\varphi^2] = \frac{\rho(\rho+1)}{\tau(\tau+1)}$. Using this, we get

$$\begin{aligned}
\mathbb{E}\left[1_X\right] &= \mathbb{E}\left[\sum_{i=1}^{n} x_i\right] = n\mathbb{E}\left[x_i\right] = n\mathbb{E}\left[\varphi\right] = n\frac{\rho}{\tau};\\
\mathbb{E}\left[1_X^2\right] &= \mathbb{E}\left[\sum_{i=1}^{n} x_i^2 + \sum_{i\neq j} x_i x_j\right] = \mathbb{E}\left(n\mathbb{E}\left[x_i^2|\varphi\right] + n\left(n-1\right)\mathbb{E}\left[x_i|\varphi\right]^2\right)\\
&= n\mathbb{E}\left[\varphi\right] + n\left(n-1\right)\mathbb{E}\left[\varphi^2\right] = \frac{n\rho}{\tau\left(\tau+1\right)}\left(\tau-\rho+n\left(\rho+1\right)\right).
\end{aligned}$$

Proof of Proposition 1

Plugging (7) and (8) into (9) gives

$$\begin{aligned}
0 \geq &\sum_{\theta_{-i}\in\{0,1\}^{K-1}} \left[2\theta_i - 1 + (K-|R_i|-1)\cdot\frac{2\theta_i-1}{\tau+|R_i|+1}\right]\\
&\times\left[2b + (1-2\theta_i) - 2\sum_{j\in -R_i\setminus\{i\}}\theta_j + \frac{2(\rho_m+\mathbf{1}_{R_i})+1}{\tau+|R_i|+1}\left(K-|R_i|-1\right)\right]P_i(\theta_{-i}|\theta_i).
\end{aligned}$$

Since the first multiple in each term equals $(2\theta_i-1)\frac{\tau+K}{\tau+|R_i|+1}$, this is equivalent to

$$0 \geq (2\theta_i - 1)\sum_{\theta_{-i}} P_i(\theta_{-i}|\theta_i)\left(2b + (1-2\theta_i) - 2\sum_{j\in -R_i\setminus\{i\}}\theta_j + \frac{2(\rho_m+\mathbf{1}_{R_i})+1}{\tau+|R_i|+1}\left(K-|R_i|-1\right)\right).$$

Since $\sum_{\theta_{-R_i\setminus\{i\}}}\left(\sum_{j\in -R_i\setminus\{i\}}\theta_j\right)P_i(\theta_{-R_i\setminus\{i\}}|\theta_i,\theta_{R_i}) = \frac{\rho_i+\mathbf{1}_{R_i}+\theta_i}{\tau+|R_i|+1}\left(K-|R_i|-1\right)$, we can further simplify it to

$$(2\theta_i-1)\left[2b+(1-2\theta_i)+\frac{2(\rho_m-\rho_i)+1-2\theta_i}{\tau+|R_i|+1}\left(K-|R_i|-1\right)\right]\leq 0.$$

We consider two separate cases. If $\theta_i = 0$, the above inequality becomes:

$$2b+1+\frac{2(\rho_m-\rho_i)+1}{\tau+|R_i|+1}\left(K-|R_i|-1\right)\geq 0, \tag{16}$$

and if $\theta_i = 1$, it becomes

$$2b-1+\frac{2(\rho_m-\rho_i)-1}{\tau+|R_i|+1}\left(K-|R_i|-1\right)\leq 0. \tag{17}$$

Together we get (10), which completes the proof.

Proof of Proposition 2

Notice that the IC constraint is more lax for optimists than for pessimists. This is because

$$\begin{aligned} |(2\rho + |R_i| + 1)\, b - (K - |R_i| - 1)\, \Delta| &\leq (2\rho + |R_i| + 1)\, b + (K - |R_i| - 1)\, \Delta \\ &= |(2\rho + |R_i| + 1)\, b + (K - |R_i| - 1)\, \Delta| \end{aligned}$$

for any $b \geq 0$ and $\Delta \geq 0$. Given this, we next show that without loss of generality, the equilibrium in the communication subgame is as described in the statement of the proposition.

Consider a firm owned by S_o optimistic shareholders and S_p pessimistic shareholders. We show that if there is an equilibrium $\mathcal{E}$ in which n_o optimists and n_p pessimists communicate truthfully, then there must be a payoff-equivalent equilibrium $\mathcal{E}'$ in which $\min\{n_o + n_p, S_o\}$ optimistic shareholders and $\max\{0, n_o + n_p - S_o\}$ pessimistic shareholders communicate truthfully. Notice that the statement holds trivially if either $n_o = S_o$ or $n_p = 0$. Therefore, we consider the case $n_o < S_o$ and $n_p > 0$. The existence of equilibrium $\mathcal{E}$ implies that the pessimists' IC constraint (10) is satisfied for $|R_i| = n_o + n_p - 1$:

$$(2\rho + n_o + n_p)\, b + (K - n_o - n_p)\, \Delta \leq \rho + \frac{K}{2}. \tag{18}$$

Consider equilibrium $\mathcal{E}'$ and show that both optimists and pessimists have incentives to communicate truthfully if the manager learns $n_o + n_p - 1$ other signals. Since (10) only depends on R_i through $|R_i|$, then for any pessimist, his IC constraint in $\mathcal{E}'$ is the same as his IC constraint in $\mathcal{E}$ (i.e., (18)), and thus holds. For any optimist, his IC constraint for $|R_i| = n_o + n_p - 1$ is satisfied as well because it is more lax than for pessimists, and the pessimists' IC constraint is satisfied for $|R_i| = n_o + n_p - 1$ given (18). Thus, if $\mathcal{E}$ is an equilibrium, then $\mathcal{E}'$ is also an equilibrium. Note that the reverse is generally not true: for example, if $n_o + n_p < S_o$, then equilibrium E' requires only the IC constraint for the optimists to hold, while E requires the IC constraints for both optimists and pessimists to hold, and the latter may be violated. Finally, note that equilibria $\mathcal{E}$ and $\mathcal{E}'$ are payoff-equivalent, in the sense that the ex-ante payoffs of all players (before they learn their signals) are the same in the two equilibria. This is because as shown in Lemma 2, the valuation of shares by each investor only depends on the set R of signals that were communicated through $|R|$.

Next, we prove the other statements of the proposition. Consider statement (i). If $b = 0$, the IC constraint (10) reduces to

$$(K - |R_i| - 1)\, \Delta \leq \rho + \frac{K}{2}. \tag{19}$$

This constraint becomes more lax as the set of shareholders that communicate truthfully expands. Thus, in the most informative equilibrium, either all shareholders communicate truthfully (which happens if $K - |S| \leq \frac{\rho + K/2}{\Delta}$) or no shareholder does (if $K - |S| > \frac{\rho + K/2}{\Delta}$). By continuity, the same is true for small enough $b > 0$ if the corresponding inequalities are satisfied strongly.

Next, consider statement (ii). If $\Delta = 0$, the IC constraint (10) reduces to

$$(2\rho + |R_i| + 1)\, b \leq \rho + \frac{K}{2}. \tag{20}$$

If (20) holds for $|R_i| = |S| - 1$, then the most informative equilibrium has all shareholders communicating truthfully. In particular, if it holds strongly, i.e., $|S| < \frac{\rho+K/2}{b} - 2\rho$, then by continuity in Δ, all $|S|$ shareholders communicate truthfully for small enough $\Delta > 0$. If $|S| > \frac{\rho+K/2}{b} - 2\rho$, then given that (20) becomes tighter as $|R_i|$ increases, the number of investors that communicate in the most informative equilibrium is one plus the highest $|R_i|$ for which (20) is satisfied, i.e., the floor of $\frac{\rho+K/2}{b} - 2\rho$, and by continuity, the same is true for small enough $\Delta > 0$. Taken together, this proves statement (ii).

Proof of Lemma 2

Let $\mathbf{1}_R = \sum_{i\in R}\theta_i$ denote the number of signals equal to one in set R. Using Lemma 1, we obtain agent i's ex-ante payoff, $\mathbb{E}_i(a_m(\theta_R) - Z)^2$, as follows:

$$\mathbb{E}_i\left[U_i|R\right] = u_0 - b^2 - U_1 - U_2, \tag{21}$$

where

$$U_1 \equiv 2b\mathbb{E}_i\left[\left(\tfrac{\rho+\mathbf{1}_R}{\tau+|R|}\left(K-|R|\right) - \textstyle\sum_{j\in -R}\theta_j\right)|R\right],$$
$$U_2 \equiv \mathbb{E}_i\left[\left(\tfrac{\rho+\mathbf{1}_R}{\tau+|R|}\left(K-|R|\right) - \textstyle\sum_{j\in -R}\theta_j\right)^2|R\right].$$

Using independence of θ_j conditional on φ, and Auxiliary Lemma A.1, U_1 simplifies to

$$U_1 = 2b\frac{\rho-\rho_i}{\tau+|R|}\left(K-|R|\right). \tag{22}$$

To simplify U_2, we use the law of iterated expectations:

$$\begin{aligned} U_2 = \mathbb{E}_i&\left[\left(\tfrac{(\rho+\mathbf{1}_R)(K-|R|)}{\tau+|R|}\right)^2 - 2\tfrac{(\rho+\mathbf{1}_R)(\rho_i+1_R)(K-|R|)^2}{(\tau+|R|)^2}|R\right] \\ &+\mathbb{E}_i\left[\mathbb{E}_i\left[\left(\textstyle\sum_{j\in -R}\theta_j\right)^2|\theta_R,R\right]|R\right], \end{aligned} \tag{23}$$

where we used $\mathbb{E}_i\left[\sum_{j\in -R}\theta_j|\theta_R,R\right] = \left(K-|R|\right)\mathbb{E}_i\left[\varphi|\theta_R,R\right] = \left(\left(K-|R|\right)\right)\frac{\rho_i+1_R}{\tau+|R|}$. Consider the last term under the expectation sign:

$$\begin{aligned} \mathbb{E}_i\left[\left(\textstyle\sum_{j\in -R}\theta_j\right)^2|\theta_R,R\right] &= \mathbb{E}_i\left[\textstyle\sum_{j\in -R}Var_i\left[\theta_j|\varphi,R\right] + \varphi^2\left(K-|R|\right)^2|\theta_R,R\right] \\ &= \mathbb{E}_i\left[\textstyle\sum_{j\in -R}\varphi\left(1-\varphi\right) + \varphi^2\left(K-|R|\right)^2|\theta_R,R\right] \\ &= \tfrac{\rho_i+1_R}{\tau+|R|}\left(\left(K-|R|\right) + \left(\left(K-|R|\right)^2 - \left(K-|R|\right)\right)\tfrac{\rho_i+1_R+1}{\tau+|R|+1}\right) \\ &= \tfrac{\rho_i+1_R}{\tau+|R|}\left(K-|R|\right)\left(1 + \left(K-|R|-1\right)\tfrac{\rho_i+1_R+1}{\tau+|R|+1}\right), \end{aligned}$$

where the second equality is due to $Var_i\left[\theta_j|\varphi,R\right] = \varphi\left(1-\varphi\right)$ and the third equality is due to the fact that the agent i's posterior distribution of φ conditional on θ_R is Beta with parameters $\rho_i + 1_R$ and $\tau + |R| - \rho_i - 1_R$, whose first and second moments are, respectively, $\frac{\rho_i+1_R}{\tau+|R|}$ and $\frac{(\rho_i+1_R)(\rho_i+1_R+1)}{(\tau+|R|)(\tau+|R|+1)}$. Plugging this expression into (23) and simplifying using Auxiliary

Lemma A.1, we get

$$\begin{aligned} U_2 - \left[\tfrac{\Delta(K-|R|)}{\tau+|R|}\right]^2 &= \mathbb{E}_i\left[\tfrac{(K-|R|)(\rho_i+1_R)}{\tau+|R|} - \left(\tfrac{(K-|R|)(\rho_i+1_R)}{\tau+|R|}\right)^2 |R\right] \\ &+ \left((K-|R|)^2 - (K-|R|)\right)\mathbb{E}_i\left[\tfrac{(\rho_i+1_R+1)(\rho_i+1_R)}{(\tau+|R|+1)(\tau+|R|)}|R\right] \\ &= \left(\tfrac{(K-|R|)^2}{\tau+|R|} + (K-|R|)\right)\mathbb{E}_i\left[\tfrac{(\rho_i+1_R)(\tau+|R|-\rho_i-1_R)}{(\tau+|R|)(\tau+|R|+1)}\right] \\ &= \left(\tfrac{(K-|R|)^2}{\tau+|R|} + (K-|R|)\right)\tfrac{\rho_i(\tau-\rho_i)}{\tau(\tau+1)} = (K-|R|)\,\tfrac{\tau+K}{\tau+|R|}\,\tfrac{\frac{\tau^2}{4}-\Delta^2}{\tau(\tau+1)}. \end{aligned}$$

Combining with (21) and (22) gives (12).

Next, we study how the ex-ante payoff $\mathbb{E}_i[U|R]$ of each agent depends on $|R|$. Denote $z = \frac{K-|R|}{\tau+|R|}$ and note that $\mathbb{E}_i[U|R] = u_0 - b^2 + u(z, \rho_i)$, where

$$u(z, \rho_i) = -2b(\frac{\tau}{2} - \rho_i)z - \frac{\left(\frac{\tau^2}{4} - \Delta^2\right)(\tau + K)}{\tau(\tau+1)}z - \Delta^2 z^2.$$

Note that $\mathbb{E}_i[U|R]$ is increasing in $|R|$ if $u(z, \rho_i)$ is decreasing in $z \in \left[\frac{K-N}{\tau+N}, \frac{K}{\tau}\right]$. Differentiating with respect to z yields

$$u'(z, \rho_i) = -2b(\frac{\tau}{2} - \rho_i) - \frac{\left(\frac{\tau^2}{4} - \Delta^2\right)(\tau + K)}{\tau(\tau+1)} - 2z\Delta^2.$$

For pessimists, $\rho_i = \frac{\tau}{2} - \Delta$, and $u'\left(z, \frac{\tau}{2} - \Delta\right) < 0$. Similarly, for the manager, $\rho_i = \frac{\tau}{2}$, and $u'\left(z, \frac{\tau}{2}\right) < 0$. For optimists, $\rho_i = \frac{\tau}{2} + \Delta$. Therefore,

$$u'\left(z, \frac{\tau}{2} + \Delta\right) = 2(b - z\Delta)\Delta - \frac{\left(\frac{\tau^2}{4} - \Delta^2\right)(\tau + K)}{\tau(\tau+1)} < 2b\Delta - \frac{\left(\frac{\tau^2}{4} - \Delta^2\right)(\tau + K)}{\tau(\tau+1)}.$$

Thus, a sufficient condition for $u'(z, \rho_i) < 0$ for optimists is that $K \geq \bar{K}$, where $\bar{K} \equiv \frac{2b\Delta\tau(\tau+1)}{\frac{\tau^2}{4}-\Delta^2} - \tau$. Thus, if $K \geq \bar{K}$, then the ex-ante payoff of any agent (any investor and the manager) is increasing in $|R|$.

Characterization of equilibria in the trading game

Here, we characterize all possible equilibria in the trading game. Given that the demand function (13) is the same for all shareholders with the same belief (optimistic or pessimistic) and that it is strictly higher for optimists than pessimists (unless $K = N$ and $b = 0$, in which the two are equal), the equilibria take two possible forms:

1. **Both pessimistic and optimistic investors become shareholders**. Let R denote the subset of signals learned by the manager in the most informative equilibrium of the communication subgame as characterized by Proposition 2. Then (14) implies that the equilibrium share price is

$$p^* = \frac{N_o}{N}\mathbb{E}_o[U|R] + \frac{N_p}{N}\mathbb{E}_p[U|R] - \frac{\lambda}{N}, \tag{24}$$

where $\mathbb{E}_o[U|R]$ and $\mathbb{E}_p[U|R]$ denote the valuations of the shares (12) for the optimists and pessimists, respectively. The existence condition for this equilibrium is that the price (24) is weakly below the valuation of the shares by the pessimists. Using (12) and (24), we get:

$$\frac{\lambda}{N_o} \geq \mathbb{E}_o[U|R] - \mathbb{E}_p[U|R] = \frac{4b\Delta\,(K-|R|)}{2\rho+|R|}. \tag{25}$$

2. **Only optimistic investors become shareholders, while pessimistic investors do not**. Let R denote the subset of signals learned by the manager in the most informative equilibrium of the communication subgame as characterized by Proposition 2; $|R|$ is the highest number in $[0, N_o]$ at which the IC constraint (10) for optimists is satisfied. Given that only optimists become shareholders, (14) implies that the equilibrium share price in this case is

$$p^* = \mathbb{E}_o[U|R] - \frac{\lambda}{N_o}. \tag{26}$$

The existence condition for this equilibrium is that the price (26) strictly exceeds the valuation of the shares by the pessimists. Using (12) and (26), we get:

$$\frac{\lambda}{N_o} < \mathbb{E}_o[U|R] - \mathbb{E}_p[U|R] = \frac{4b\Delta\,(K-|R|)}{2\rho+|R|}. \tag{27}$$

Within each of these two types of equilibria, the most informative equilibrium of the communication subgame could feature communication by either a strict subset of the shareholders or by all shareholders. We next characterize all possible cases.

a. $|S| = |R| = N$: **all investors become shareholders; all shareholders communicate truthfully.** This equilibrium exists if and only if 1) the IC constraint (10) is satisfied for a pessimistic shareholder if he expects all other shareholders to communicate truthfully ($|R_i| = N-1$), and 2) each investor prefers to become a shareholder given that he expects all shareholders to communicate truthfully, i.e., (25) holds for $|R| = N$:

$$(2\rho+N)\,b + (K-N)\,\Delta \;\leq\; \rho + \frac{K}{2}, \tag{28}$$

$$\frac{\lambda}{N_o} \;\geq\; \frac{4b\Delta\,(K-N)}{2\rho+N}. \tag{29}$$

b. $|S| = N$, $|R| \in [N_o+1, N-1]$: **all investors become shareholders; all optimists and some but not all pessimists communicate truthfully.** This equilibrium exists if and only if 1) the IC constraint (10) for a pessimist is violated for $|R_i| = N-1$, i.e., (31) 2) is satisfied for some $|R_i| \in [N_o, N-2]$, and 3) each investor prefers to become a shareholder given that he expects the manager to learn $|R_i|$ signals, i.e., (25) holds for such $|R_i|$. Note that for a pessimistic shareholder (10) simplifies to

$$(2\rho + |R_i| + 1)\, b + (K - |R_i| - 1)\, \Delta \leq \rho + \frac{K}{2}. \tag{30}$$

If $b < \Delta$, the left-hand side is decreasing in $|R_i|$. Thus, if this inequality is violated for $|R_i| = N - 1$ (i.e., equilibrium with all investors communicating truthfully does not exist), it is also violated for any lower $|R_i|$ due to the complementarity effect in communication, so equilibrium with $|R| \in [N_o + 1, N - 1]$ does not exist. If $b > \Delta$, the left-hand side is increasing in $|R_i|$. Hence, in this case, there exists $|R_i| \in [N_o, N - 2]$ such that (30) is satisfied for this $|R_i|$ if and only if (30) is satisfied for $|R_i| = N_o$, i.e., (32). Finally, (25) is the least restrictive when $|R|$ is the highest possible within the set of $|R| \in [N_o + 1, N - 1]$ for which it is incentive compatible for $|R|$ investors to communicate. Thus, the conditions for this type of equilibrium are:

$$\begin{aligned}
(2\rho + N)\, b + (K - N)\, \Delta &> \rho + \frac{K}{2}, & (31)\\
(2\rho + N_o + 1)\, b + (K - N_o - 1)\, \Delta &\leq \rho + \frac{K}{2}, & (32)\\
\frac{\lambda}{N_o} &\geq \frac{4b\Delta\,(K - |R|)}{2\rho + |R|}, & (33)
\end{aligned}$$

where $|R|$ is the highest integer in $[N_o + 1, N - 1]$ for which it is incentive compatible for $|R|$ investors to communicate, or equivalently, the lowest integer $|R_i|$ in $[N_o + 1, N - 1]$ for which the IC condition for the pessimist (30) stops holding.

c. $|S| = N$, $|R| = N_o$: **all investors become shareholders; all optimists but no pessimists communicate truthfully.** This equilibrium exists if and only if 1) the IC constraint (10) is satisfied for an optimistic shareholder if he expects all other optimistic shareholders and no pessimistic shareholder to communicate truthfully ($|R_i| = N_o - 1$), i.e., (34), 2) (10) for a pessimistic shareholder is violated for all $|R_i| \in [N_o, N - 1]$, and 3) each investor prefers to become a shareholder given that he expects the manager to learn N_o signals, i.e., (25) holds for $|R| = N_o$, giving (36). Since the left-hand side of (30) increases (decreases) in $|R_i|$ if $b > \Delta$ ($b < \Delta$), the second condition holds if and only if (30) is violated for $|R_i| = N_o$ if $b \geq \Delta$, and for $|R_i| = N - 1$ if $b < \Delta$. Thus, the conditions for this equilibrium are:

$$\begin{aligned}
|(2\rho + N_o)\, b - (K - N_o)\, \Delta| &\leq \rho + \frac{K}{2}, & (34)\\
2\rho b + K\Delta + (b - \Delta)\left((N_o + 1)\, \mathbf{1}\{b \geq \Delta\} + N\mathbf{1}\{b < \Delta\}\right) &> \rho + \frac{K}{2}, & (35)\\
\frac{\lambda}{N_o} &\geq \frac{4b\Delta\,(K - N_o)}{2\rho + N_o}. & (36)
\end{aligned}$$

d. $|S| = N$, $|R| \in [0, N_o - 1]$: **all investors become shareholders; not all optimists communicate truthfully.** This equilibrium exists if and only if the IC constraint (10) is violated for an optimistic shareholder if she expects all other optimistic shareholders to communicate truthfully ($|R_i| = N_o - 1$) and if each investor prefers to become a shareholder given that she expects the manager to learn $|R|$ signals:

$$
\begin{aligned}
\left|\left(2\rho+N_o\right)b-\left(K-N_o\right)\Delta\right| &> \rho+\frac{K}{2}, & (37)\\
\frac{\lambda}{N_o} &\geq \frac{4b\Delta\left(K-|R|\right)}{2\rho+|R|}, & (38)
\end{aligned}
$$

where $|R|$ is one plus the highest integer $|R_i|$ in $[0, N_o-2]$ for which (10) for an optimist is satisfied.

e. $|S| = |R| = N_o$: **only optimists become shareholders; all shareholders communicate.** This equilibrium exists if and only if the IC constraint (10) is satisfied for an optimistic shareholder if she expects all other optimistic shareholders to communicate truthfully ($|R_i| = N_o - 1$) and if a pessimistic investor prefers to not become a shareholder under the equilibrium stock price (i.e., (27) is satisfied for $|R| = N_o$):

$$
\begin{aligned}
\left|\left(2\rho+N_o\right)b-\left(K-N_o\right)\Delta\right| &\leq \rho+\frac{K}{2}, & (39)\\
\frac{\lambda}{N_o} &< \frac{4b\Delta\left(K-N_o\right)}{2\rho+N_o}. & (40)
\end{aligned}
$$

f. $|S| = N$, $|R| \in [0, N_o-1]$: **only optimists become shareholders; not all shareholders communicate.** This equilibrium exists if and only if the IC constraint (10) is violated for an optimistic shareholder if she expects all other optimistic shareholders to communicate truthfully ($|R_i| = N_o - 1$) and if a pessimistic investor prefers to not become a shareholder under the equilibrium stock price:

$$
\begin{aligned}
\left|\left(2\rho+N_o\right)b-\left(K-N_o\right)\Delta\right| &> \rho+\frac{K}{2}, & (41)\\
\frac{\lambda}{N_o} &< \frac{4b\Delta\left(K-|R|\right)}{2\rho+|R|}, & (42)
\end{aligned}
$$

where $|R|$ is one plus the highest integer $|R_i|$ in $[0, N_o-2]$ for which (10) for an optimist is satisfied.

Proof of Proposition 3

Applying (25) for $|R| = N$, if all shareholders are expected to communicate information to the manager truthfully, then all investors choose to become shareholders if and only if $K - N \leq \frac{\lambda}{N_o}\frac{\rho+N/2}{2b\Delta}$. Using the fact that the IC condition for pessimists is harder to satisfy than for optimists, and applying (10) for $|R_i| = N-1$ and $\rho_i = \rho - \Delta$, the equilibrium in which all shareholders communicate truthfully exists if and only if

$$
K - N \leq \frac{\rho + K/2}{\Delta} - \left(2\rho + N\right)\frac{b}{\Delta}. \tag{43}
$$

Thus, for any $b < \frac{\rho+K/2}{2\rho+N}$, if $K - N \leq \min\left\{\frac{\lambda}{N_o}\frac{\rho+N/2}{2b\Delta}, \frac{\rho+K/2}{\Delta} - \left(2\rho+N\right)\frac{b}{\Delta}\right\}$, there exists an equilibrium in which all investors become shareholders and communicate information to the manager truthfully. Note that if $K = N$, this equilibrium exists if $b < \frac{1}{2}$. In this case, the manager's action is $a = b + Z$, and as follows from (12), both optimistic and pessimistic investors have the same valuation of shares. Hence, in equilibrium they acquire the same number of shares, $\frac{1}{N}$. Finally, when $b = 0$, the equilibrium achieves first-best: it features the same allocation of shares and corporate action as would be chosen by the social planner who maximizes the combined expected utility of all players.

Proof of Proposition 4

Notice that condition $(2\rho + N_o + 1)\, b + (K - N_o - 1)\, \Delta \leq \rho + \frac{K}{2}$ implies that if the shareholder base includes all investors, then the communication stage has an equilibrium in which all optimists and at least one pessimist communicate truthfully to the manager. This follows directly from (10) by plugging in $|R_i| = N_o$. Let $\hat{r}$ denote the number of signals communicated to the manager if all investors become shareholders (i.e., one plus the highest $|R_i|$ at which the IC constraint (10) for pessimistic investors is satisfied):

$$\hat{r} = \max_{|R| \in [N_o+1, N], |R| \in \mathbb{N}} \left\{ |R| : (2\rho + |R|)\, b + (K - |R|)\, \Delta \leq \rho + \frac{K}{2} \right\}. \tag{44}$$

Then, using (25), if

$$\lambda > \hat{\lambda} \equiv \frac{4b\Delta\,(K - \hat{r})\, N_o}{2\rho + \hat{r}}, \tag{45}$$

then there exists an equilibrium in which all investors become shareholders and all optimistic shareholders and either some or all pessimistic shareholders communicate truthfully. Using (24), the equilibrium stock price is:

$$p^* = \frac{N_o}{N}\mathbb{E}_o[U|\,|R| = \hat{r}] + \frac{N_p}{N}\mathbb{E}_p[U|\,|R| = \hat{r}] - \frac{\lambda}{N}, \tag{46}$$

where $\mathbb{E}_i[U|\,|R| = r]$ is the valuation of investor i if the manager learns r signals in equilibrium (by (12)), investors' valuation only depends on R through $|R|$).

Next, consider $\lambda < \hat{\lambda}$. Since $\frac{4b\Delta(K-|R|)}{2\rho+|R|}$ is strictly decreasing in $|R|$, the fact that $\lambda < \hat{\lambda}$ implies that (25) is violated for any $|R| \leq \hat{r}$. Therefore, no equilibrium in which all investors become shareholders exists. Hence, in equilibrium, only optimistic investors become shareholders and thus at most N_o shareholders communicate truthfully. Therefore, the ownership structure is less dispersed than for $\lambda > \hat{\lambda}$: if $\lambda < \hat{\lambda}$, each optimist holds $\frac{1}{N_o}$ shares and each pessimist holds zero shares, whereas if $\lambda > \hat{\lambda}$, each optimist holds fewer than $\frac{1}{N_o}$ shares and each pessimist holds a positive number of shares. Since $N_o < \hat{r}$, the manager's decision is less informed compared to $\lambda > \hat{\lambda}$, which is manifested in lower utility (12) from each investor's point of view. Using (26), the equilibrium stock price is:

$$p^* = \mathbb{E}_o[U|R] - \frac{\lambda}{N_o}, \tag{47}$$

where R is the set of shareholders that communicate truthfully when only optimistic investors become shareholders.

Finally, we examine how the stock price depends on λ, as we increase it from zero. Note that λ does not enter the IC constraints of shareholders at the communication stage, so it affects the stock price only via the holding cost and via the ownership structure. Holding the ownership structure fixed, the stock price is decreasing in λ: both (46) and (47) are decreasing in λ. However, when λ crosses $\hat{\lambda}$ from below, the ownership structure changes from only optimistic investors becoming shareholders to all investors becoming shareholders. For a given price p, the demand for shares (13) of each investor i increases discontinuously due to a jump in $E_i[U|R]$ due to an increase in the number of signals that the manager learns. Hence, the market clearing price jumps up discontinuously at $\lambda = \hat{\lambda}$.

Proof of Proposition 5

Consider $\lambda < \hat{\lambda}$, where $\hat{\lambda}$ is defined by (45). By the argument in the second paragraph of the proof of Proposition 4, in equilibrium, only optimistic investors become shareholders and thus at most N_o shareholders communicate truthfully. Let $\hat{r}_1 \in [0, N_o]$ the number of shareholders that communicate their signals truthfully in this case. The stock price is given by (26):

$$p^* = \mathbb{E}_o[U|\,|R| = \hat{r}_1] - \frac{\lambda}{N_o}, \tag{48}$$

where $\mathbb{E}_i[U|\,|R| = r]$ is investor i's valuation if the manager learns r signals in equilibrium.

Consider a model with $L > 0$ passive investors. There are two potential cases: (1) only optimistic active investors become shareholders; (2) all $N - L$ active investors become shareholders. We will show that only the first case can arise in equilibrium given that $\lambda < \hat{\lambda}$.

Consider the first case. Then, the firm has $N_o + \frac{N_p}{N}L$ shareholders. Among them, N_o are optimistic and $\frac{N_p}{N}L$ are pessimistic. By assumption in the statement of the proposition,

$$(2\rho + N_o + 1)\, b + (K - N_o - 1)\, \Delta \leq \rho + \frac{K}{2}, \tag{49}$$

and hence (10) implies that all optimistic shareholders and at least one pessimistic shareholder communicate truthfully. Therefore, the equilibrium number of signals communicated to the manager, $\hat{r}_2(L)$, is at least $N_o + 1$. Hence, the equilibrium features more informative communication (in the sense of a higher number of signals learned by the manager) and more informed corporate decision-making (in the sense of a higher expected utility (12) for each shareholder). Consider the share price. The demand from each of the $\frac{N-L}{N}N_o$ optimistic active investors is given by (13). The demand from each of the L passive investors is given by $\frac{1}{N}$. Hence, the market clearing condition is:

$$N_o \frac{N - L}{N} \left(\frac{\mathbb{E}_o[U|\,|R| = \hat{r}_2(L)] - p^*}{\lambda} \right) = 1 - \frac{L}{N}, \tag{50}$$

which yields

$$p^* = \mathbb{E}_o[U|\,|R| = \hat{r}_2(L)] - \frac{\lambda}{N_o}. \tag{51}$$

Since $\mathbb{E}_o[U|\,|R| = r]$ is strictly increasing in r, the equilibrium stock price with L passive investors, (51), exceeds the equilibrium stock price without passive investors, (51). Notice also that the presence and number of passive investors only affects the price by affecting how many signals the manager learns in equilibrium, but not by changing the residual supply of shares (due to the assumption that each passive investor demands $\frac{1}{N}$ shares): the price (51) only depends on L through $\hat{r}_2(L)$ and coincides with the price (26) without passive investors if $|R|$ is the same.

Consider the second case. Then, the firm has N shareholders, among them, N_o are optimistic and N_p are pessimistic. Given (49), all optimists and at least some pessimists communicate truthfully, and the total number of signals communicated to the manager is given by $\hat{r}$, defined by (44). Then, the market-clearing condition is:

$$\frac{N-L}{N}\left(N_o\frac{\mathbb{E}_o[U|\,|R|=\hat{r}]-p}{\lambda}+N_p\frac{\mathbb{E}_p[U|\,|R|=\hat{r}]-p}{\lambda}\right)=1-\frac{L}{N},$$

which yields

$$p^*=\frac{N_o}{N}\mathbb{E}_o[U|\,|R|=\hat{r}]+\frac{N_p}{N}\mathbb{E}_p[U|\,|R|=\hat{r}]-\frac{\lambda}{N}. \tag{52}$$

Notice again that for a given number of signals learned by the manager, the price is not affected by L and is the same as in the model without passive investors. The existence condition for this equilibrium is that the price (52) is weakly below the valuation of the shares by the pessimists. Using (12) and (52), we get:

$$\lambda\geq\frac{4b\Delta\left(K-\hat{r}\right)N_o}{2\rho+\hat{r}}=\hat{\lambda},$$

which contradicts the assumption $\lambda<\hat{\lambda}$. Hence, the second case is indeed not possible.

Finally, we examine comparative statics in L. Given L, the ownership structure has N_o optimistic shareholders and $\frac{N_p}{N}L$ pessimistic shareholders. The equilibrium number of signals communicated to the manager is given by

$$\hat{r}_2\left(L\right)=\max_{|R|\in\left[N_o+1,N_o+\frac{N_p}{N}L\right],|R|\in\mathbb{N}}\left\{|R|:\left(2\rho+|R|\right)b+\left(K-|R|\right)\Delta\leq\rho+\frac{K}{2}\right\},$$

i.e., it is determined by the highest number of signals for which the IC constraint for a pessimistic shareholder is still satisfied. Given (49), $\hat{r}_2\left(L\right)\geq N_o+1$. Notice that $\hat{r}_2\left(L\right)$ is weakly increasing in the number of passive investors L, and once it reaches $\hat{r}$, it stays constant at this level as L further increases. Since $\hat{r}_2\left(L\right)$ is weakly increasing in L and $\mathbb{E}_i[U|\,|R|=r]$ is increasing in r, the informativeness of decision-making (evaluated from either the optimist's, or the pessimist's, or the manager's point of view) is weakly increasing in L. As a consequence, the equilibrium stock price (51) is also weakly increasing in L.

Proof of Proposition 6

First, consider the case without passive investors. Notice that conditions $\lambda<\frac{4b\Delta(K-N)N_o}{\tau+N}$ and $\left(K-N_o\right)\Delta-\left(2\rho+N_o\right)b>\rho+\frac{K}{2}$ imply (41)-(42), and thus the equilibrium without passive investors is such that only optimistic investors become shareholders and not all of them communicate truthfully.

Next, consider the case with $L>0$ passive investors. Since $\lambda<\frac{4b\Delta(K-N)N_o}{\tau+N}$, we have

$$\lambda<\frac{4b\Delta\left(K-|R|\right)N_o}{\tau+|R|}\;\forall\,|R|\in\left[0,N\right]. \tag{53}$$

As shown in the proof of Proposition 5, the price for a given $|R|$ is the same as in the model without passive investors, and since (53) coincides with (27), the price strictly exceeds the valuation of the shares by the pessimists. Hence, only optimistic investors among active investors become shareholders. Thus, the ownership structure consists of N_o optimistic shareholders ($N_o \frac{L}{N}$ passive and the rest active) and $N_p \frac{L}{N}$ pessimistic shareholders (all passive). Given (10), the condition that all $N_o + N_p \frac{L}{N}$ investors communicate truthfully is:

$$\left(2\rho + N_o + N_p \frac{L}{N}\right) b + \left(K - N_o - N_p \frac{L}{N}\right) \Delta \leq \rho + \frac{K}{2}. \tag{54}$$

Finally, we show that the inequalities in the statement of the proposition define a non-empty set of parameters. This is the case when

$$\begin{aligned} \left(2\rho + N_o + N_p \frac{L}{N}\right) b + \left(K - N_o - N_p \frac{L}{N}\right) \Delta \quad &< \quad (K - N_o)\,\Delta - (2\rho + N_o)\, b \\ &\Leftrightarrow \quad \Delta > \left(\frac{2\,(2\rho + N_o)\,N}{N_p L} + 1\right) b, \end{aligned}$$

which holds for a large enough Δ.

Proof of Proposition 7

We first note that the conditions in the statement of the proposition describe a non-empty set of parameters. For example, these conditions, as well as the assumption $K > \bar{K}$, are satisfied by choosing a sufficiently large K, N_o that is close to K, and sufficiently small b and λ. The existence of the first equilibrium follows from Proposition 3: $b < \frac{\rho + K/2}{2\rho + N}$ is equivalent to $b < \frac{1}{2}$ for $K = N$. We next prove that under the additional conditions in the statement of the proposition, and if the number of passive investors $L = 0$, there exists an equilibrium where only optimists become shareholders and acquire a stake $\frac{1}{N_o}$ each, and they all truthfully communicate to the manager. First, note that if $|R| = N_0$ (i.e., the manager learns all optimists' signals), then (14) implies $p^* = \mathbb{E}_o[U|R] - \frac{\lambda}{N_o}$. The existence condition for the equilibrium in which only optimists become shareholders is that this equilibrium price strictly exceeds the value of the share by the pessimistic investor, i.e., (27). This condition is satisfied for $|R| = N_o$ by the assumption on λ in the statement of the proposition. Finally, we prove that the IC constraint (10) holds for all optimists, i.e.,

$$|(2\rho + N_o)b - (K - N_o)\Delta| \leq \rho + \frac{K}{2}.$$

Since $\frac{2\rho + N_o}{\Delta} b \leq K - N_o$ by the assumption in the statement of the proposition, this can be rewritten as

$$(K - N_o)\Delta - (2\rho + N_o)b \leq \rho + \frac{K}{2} \Leftrightarrow K - N_o \leq \frac{\rho + K/2}{\Delta} + \frac{2\rho + N_o}{\Delta} b,$$

which is satisfied by the other assumption in the statement of the proposition, completing the proof. In contrast, if the number of passive investors L is large, this equilibrium does not exist. In particular, if $L = N$, all N investors are restricted to holding $\frac{1}{N}$ shares, so only the first equilibrium remains.

Proof of Proposition 8

(i) Condition $(K-N)\,\Delta \leq \rho + \frac{K}{2}$ guarantees that if $b=0$, there is an equilibrium in which all investors acquire stakes $\frac{1}{N}$ and communicate their signals to the management truthfully. Indeed, if $b=0$, then optimists' and pessimists' valuations of the firm are the same, and hence they all become shareholders and hold stakes $\frac{1}{N}$. In addition, $(K-N)\,\Delta \leq \rho + \frac{K}{2}$ implies that (10) holds for $|R_i| = N-1$, i.e., it is incentive compatible for all N shareholders to communicate truthfully. Overall, this equilibrium features the lowest possible holding costs, the most informative communication, and (since $b=0$) unbiased decision-making. Hence, it features higher welfare (in the sense of the combined utility of all investors and the seller of the shares) and a higher share price compared to any $b>0$.

(ii) Suppose that $(K-N)\,\Delta > \rho + \frac{K}{2}$. Then (10) with $|R_i| = N-1$ implies that there is no equilibrium with fully informative communication when $b=0$. Notice that this inequality also implies that $(K-n)\,\Delta > \rho + \frac{K}{2}$ for any $n \leq N$, and thus the only equilibrium in this case is that no shareholder communicates information truthfully. As a consequence, if $b=0$, the manager takes an uninformative action, which is determined by his prior: $a_m = \frac{K}{2}$. Expression (12) implies that each investor's valuation of the share is

$$U_{b=0} = u_0 - \frac{\rho^2 - \Delta^2}{2\rho(2\rho+1)} K \frac{2\rho + K}{2\rho} - \left[\frac{K\Delta}{2\rho}\right]^2. \tag{55}$$

Hence, all investors hold the same stake $\frac{1}{N}$ in the firm, so the share price (using (24)) is

$$p^*_{b=0} = U_{b=0} - \frac{\lambda}{N}, \tag{56}$$

and the combined utility of all investors and the seller is $W_{b=0} = U_{b=0} - \frac{\lambda}{2N}$.

Next, consider $b>0$ such that there exists $n \in [1, N_o]$ for which

$$|(2\rho+n)\,b - (K-n)\,\Delta| \leq \rho + \frac{K}{2}. \tag{57}$$

The IC condition (10) implies that for any such b, there is an equilibrium in which n optimists communicate their information truthfully, and hence, it features more informative communication than the equilibrium for $b=0$.

To prove that this equilibrium can also have higher welfare and a higher share price than the one with $b=0$, we construct an example. Consider b such that (57) holds for $n = N_o$, and hence there is an equilibrium in which all optimists communicate truthfully. Take the lowest such b, i.e.,

$$\underline{b} = \frac{(K-N_o)\,\Delta - (\rho + \frac{K}{2})}{2\rho + N_o}.$$

Then the most informative equilibrium for $\bar{b}$ is one where all optimists but no pessimist communicates. This is because the IC condition for pessimists is violated, i.e., for any $n \in [N_o+1, N]$ we have $(2\rho+n)\,\underline{b} + (K-n)\,\Delta > \rho + \frac{K}{2}$. To see this, plug in $\bar{b}$ and then use the condition $(K-N)\,\Delta > \rho + \frac{K}{2}$:

$$(2\rho+n)\,\underline{b} + (K-n)\,\Delta = (2\rho+n)\,\tfrac{(K-N_o)\Delta - (\rho+\frac{K}{2})}{2\rho+N_o} + (K-n)\,\Delta >$$
$$(K-N_o)\,\Delta - (\rho + \tfrac{K}{2}) + (K-n)\,\Delta > (2K - N_o - n)\,\tfrac{\rho + K/2}{K-N} - (\rho + \tfrac{K}{2}) \geq \rho + \tfrac{K}{2},$$

as required, where the first inequality follows from $n > N_o$ and $(K - N_o)\,\Delta - (\rho + \frac{K}{2}) \geq (K - N)\,\Delta - (\rho + \frac{K}{2}) > 0$, the second inequality follows from plugging in $\Delta > \frac{\rho + K/2}{K-N}$, and the last inequality follows from $2K - N_o - n \geq 2\,(K - N)$.

Hence, in equilibrium, the manager learns N_o signals, and (12) then implies that investor i's utility of each share is:

$$u_0 - \underline{b}^2 - \frac{2\underline{b}(\rho - \rho_i)}{2\rho + N_o}\,(K - N_o) - \frac{\rho^2 - \Delta^2}{2\rho(2\rho + 1)}\,(K - N_o)\,\frac{2\rho + K}{2\rho + N_o} - \left[\frac{\Delta\,(K - N_o)}{2\rho + N_o}\right]^2. \quad (58)$$

We pick parameters such that all investors become shareholders (we verify this condition for our example). Then (24) implies that the share price is

$$p^*_{b=\underline{b}} = \frac{N_o}{N}\mathbb{E}_o[U|\,|R| = N_o] + \frac{N_p}{N}\mathbb{E}_p[U|\,|R| = N_o] - \frac{\lambda}{N}. \quad (59)$$

Consider the following parameters: $u_0 = 50$; $\rho = 7.5$; $\Delta = 2$; $K = 20$; $N = 7$; $N_o = 4$; $N_p = 3$; $\lambda = 100$. Then $K > \bar{K}$ is satisfied. Using (55) and (58), we get $U_{b=0} = 32.7$; $U_{p,b=\underline{b}} = 37.6 > U_{b=0}$; and $U_{o,b=\underline{b}} = 42.7 > U_{b=0}$, i.e., all investors' valuations are higher than their valuations for $b = 0$. Then (56) and (59) imply that $p^*_{b=\underline{b}} > p_{b=0}$, i.e., the price for $b = \bar{b}$ is higher than for $b = 0$. Next, since $p^*_{b=\underline{b}} = 26.2$, both optimists and pessimists hold shares. Using (13) to find α_o and α_p (the optimists' and pessimists' stakes, respectively), we get that welfare in this equilibrium is:

$$W_{b=\underline{b}} = N_o\left[\alpha_o U_{o,b=\underline{b}} - \frac{\lambda}{2}\alpha_o^2\right] + N_p\left[\alpha_p U_{p,b=\underline{b}} - \frac{\lambda}{2}\alpha_p^2\right],$$

which equals $W_{b=\underline{b}} = 33.6 > W_{b=0} = 18.4$. This completes the proof.

Online appendix for "Advising the Management: A Theory of Shareholder Engagement"

The online appendix presents the analysis of the extensions of the model.

7.1 General model

In this section, we consider a more general version of the model, in which different shareholders get signals of different quality, and the manager is privately informed as well. Specifically, there is a set of investors indexed by i (who observe signals $\theta_1, ..., \theta_N$), and the manager, indexed by m, who observes signal θ_{N+1}. The payoffs of each investor and the manager are as in the basic model. The state is:

$$Z = \sum_{i=1}^{N+1} c_i \theta_i, \tag{60}$$

where $c_i > 0$. Coefficients c_i can take any positive values and do not need to sum up to one. An agent with a higher c_i can be interpreted as being more informed. Thus, for simplicity, we focus on the case of no residual uncertainty, i.e., the state is perfectly known to all investors and the manager as a whole, but the model can be easily generalized further, to capture residual uncertainty.

As in the basic model, θ_i is a binary signal equal to 1 with probability φ and 0 with probability $1 - \varphi$, and agents may potentially disagree about φ: agent i's prior of φ is characterized by the Beta distribution with parameters $(\rho_i, \tau - \rho_i)$. We allow for any general set of investor beliefs, ρ_i, as well as belief ρ_m of the manager. The rest of the assumptions (e.g., the timing and the trading stage) are exactly as in the basic model. In what follows, we present the analogs of the core results in the main model for this more general setup.

Lemma OA.1 (Optimal action of the manager). *Suppose that after the communication stage, the manager knows subset R of signals. Then his optimal action is*

$$a_m(\theta_R) = b + \sum_{i \in R} c_i \theta_i + \frac{\rho_m + \sum_{i \in R} \theta_i}{\tau + |R|} \sum_{j \in -R} c_j, \tag{61}$$

where $|R|$ is the number of signals in R.

Proposition OA.1 (IC constraint for truthful reporting). *Suppose that the manager learns subset R_i of signals (which includes his own signal θ_m but not θ_i) and does not know all the other signals, $-R_i$. Then shareholder i reports his signal truthfully if and only if*

$$2 \left| b + \frac{\sum_{j \in -R_i \setminus \{i\}} c_j}{\tau + |R_i| + 1} (\rho_m - \rho_i) \right| \leq c_i + \frac{\sum_{j \in -R_i \setminus \{i\}} c_j}{\tau + |R_i| + 1}. \tag{62}$$

Condition (62) is the analog of (11) in the main model. Note that regardless of the source of communication frictions, shareholder i is more likely to report his signal truthfully if his information is more important: the IC constraint (62) is relaxed when c_i increases. Intuitively, the shareholder faces the same trade-off as described in the paper: while he wants to tilt the manager in the direction of his optimal action (the benefit of misreporting), he is also afraid to tilt it too much, away even from his own optimal action, i.e., to "overshoot" (the cost of misreporting). As the agent's information becomes more important and hence the manager is expected to react more strongly to the agent's message (as captured by the term c_i on the right-hand side), this fear makes the agent more reluctant to misreport.

To show that the existence of the complementarity and substitution effects in communication extends to this more general model, consider two extreme cases:

Case 1. $b = 0$ **but heterogeneous beliefs**

In this case, shareholder i reports his signal truthfully if and only if[27]

$$|\rho_m - \rho_i| \leq \frac{1}{2}\left[1 + c_i \frac{\tau + |R_i| + 1}{\sum_{j \in -R_i \setminus \{i\}} c_j}\right]. \tag{63}$$

Hence, as in the basic model, shareholders' communication decisions are complements: the more information the manager gets from other shareholders (i.e., the higher is $|R_i|$ and the lower is $\sum_{j \in -R_i \setminus \{i\}} c_j$), the more likely it is that shareholder i will also truthfully communicate his information.

Case 2. $b > 0$ **and homogenous beliefs**

In this case, $\rho_i = \rho_m = \rho$, so shareholder i reports his signal truthfully if and only if

$$b \leq \frac{1}{2}\left[c_i + \frac{\sum_{j \in -R_i \setminus \{i\}} c_j}{\tau + |R_i| + 1}\right]. \tag{64}$$

Hence, as in the basic model, shareholders' communication decisions are substitutes: the more information the manager gets from other shareholders (i.e., the higher is $|R_i|$ and the lower is $\sum_{j \in -R_i \setminus \{i\}} c_j$), the less likely it is that shareholder i will truthfully communicate his information.

Next, we derive each investor's valuation of the shares as a function of the information $|R|$ that the manager is expected to have at the decision-making stage.

Lemma OA.2 (Ex-ante payoffs). *Suppose that in equilibrium, the manager learns subset R of the signals and does not learn all the other signals, $-R$. Then agent i's valuation of*

[27] Here if $-R \setminus \{i\}$ is an empty set, the right-hand side of (19) is equal to infinity, i.e., (19) is always satisfied.

each share is given by:

$$\mathbb{E}_i[U_i|R] = u_0 - b^2 - \mathsf{A}_{im}(R) - \mathsf{B}_i(R) - \mathsf{C}_{im}(R), \tag{65}$$

where

$$\begin{aligned} \mathsf{A}_{im}(R) &= \tfrac{2b(\rho_m - \rho_i)}{\tau + |R|} \textstyle\sum_{j \in -R} c_j, \\ \mathsf{B}_i(R) &= \tfrac{\rho_i(\tau - \rho_i)}{\tau(\tau+1)} \left(\textstyle\sum_{j \in -R} c_j^2 + \tfrac{\left[\sum_{j \in -R} c_j\right]^2}{\tau + |R|} \right), \\ \mathsf{C}_{im}(R) &= \left[\tfrac{\rho_m - \rho_i}{\tau + |R|} \textstyle\sum_{j \in -R} c_j \right]^2. \end{aligned} \tag{66}$$

where $B_i(R), C_{im}(R)$ are decreasing in $|R|$ and increasing in any $c_j,\ j \in -R$.

It follows that as long as b is not too large, each agent's valuation is increasing in $|R|$, so the most informative communication equilibrium is Pareto efficient. Given that the complementarity and substitution properties continue to hold, it is easy to generalize the results of the paper to this setting.

7.2 A more general specification of differences in beliefs

In this section, we generalize the setting of Section 7.1 of the Online Appendix even further: we assume that agent i has a prior belief that φ is distributed according to the Beta distribution with parameters $(\rho_i, \tau_i - \rho_i)$, i.e., we allow for heterogeneous τ_i across agents. We derive the incentive compatibility constraint for this specification and show that the complementarity in communication decisions extends to this specification.

Proposition OA.2. *Suppose that the manager learns subset R_i of signals (which includes his own signal θ_m but not θ_i) and does not know all the other signals, $-R_i$. Then shareholder i reports his signal truthfully if and only if*

$$\begin{aligned} &\left| b \frac{(\tau_i + 1)(\tau_m + |R_i| + 1)}{\sum_{j \in -R_i \setminus \{i\}} c_j} + \frac{\tau_i - \tau_m}{2} + \rho_m (\tau_i + 1) - \rho_i (\tau_m + 1) \right| \\ \leq\ & \frac{(\tau_i + 1) c_i (\tau_m + |R_i| + 1)}{2 \sum_{j \in -R_i \setminus \{i\}} c_j} + \frac{\tau_i + 1}{2} - \frac{\tau_i - \tau_m}{2}. \end{aligned} \tag{67}$$

In particular, when $b = 0$, it reduces to

$$\left| \frac{\tau_i - \tau_m}{2} + \rho_m (\tau_i + 1) - \rho_i (\tau_m + 1) \right| \leq \frac{(\tau_i + 1) c_i (\tau_m + |R_i| + 1)}{2 \sum_{j \in -R_i \setminus \{i\}} c_j} + \frac{\tau_i + 1}{2} - \frac{\tau_i - \tau_m}{2}. \tag{68}$$

This inequality is relaxed as R_i expands and is always satisfied if R_i includes all signals other than θ_i.

Since the right-hand side of (68) increases as R_i expands, shareholders' decisions are complements as in the basic model: more information revealed to the manager by some shareholders encourages other shareholders to report their information truthfully.

7.3 Costly information acquisition

Suppose that shareholders are not endowed with information: instead, shareholder i can incur cost $\kappa \geq 0$ to privately observe signal θ_i, and is uncertain about other signals. The timeline is as follows. After the trading stage, all shareholders of the firm simultaneously decide whether to incur a private cost to acquire their private signals. We assume that shareholders' information acquisition decisions are observed, and this happens after the communication stage.[28] Next, all shareholders simultaneously communicate their information to the manager, and the manager takes the action that maximizes his payoff. We look for equilibria in pure strategies at the information acquisition and communication stages. For simplicity, we focus on the case $b = 0$.

Proposition OA.3 (Number of shareholders and information acquisition). *Suppose $b = 0$, so that all N investors become shareholders. Then all shareholders find it optimal to acquire information if and only if $N \leq \hat{N}(\kappa)$, where $\hat{N}(\kappa)$ decreases in κ.*

Intuitively, each shareholder's incentives to acquire information decrease in N for two complementary reasons. First, the larger is the number of shareholders, the lower is each individual shareholder's stake, and hence the stronger is the free-riding effect: the shareholder bears the full cost κ but only captures a small fraction of the benefit. Second, the larger is the number of shareholders, the larger is the aggregate information that the shareholders possess, and hence, the lower is the marginal value of any additional signal.

Thus, the requirement that shareholders must pay the information acquisition cost imposes an upper bound on the number of shareholders who can communicate their views to the manager: $N \leq \hat{N}(\kappa)$. In particular, if the shareholder base is too dispersed, an equilibrium with information acquisition and communication by all shareholders does not exist. On the other hand, the fact that shareholders' communication decisions are complements imposes a lower bound on the number of shareholders who need to communicate with the manager in order for it to be incentive compatible for them to tell the truth ($N \geq K - \frac{\rho+K/2}{\Delta}$ from Proposition 2), i.e., ownership cannot be too concentrated either.

[28] Without this assumption, a shareholder who deviates from his equilibrium strategy and does not invest in information, may want to mislead the manager and try to send a signal that he did not in fact acquire. Making the above assumption makes such deviations impossible and hence simplifies the analysis. In addition, assuming that information acquisition is observed after the communication stage rather than before simplifies the incentive compatibility constraint on information acquisition, because it implies that other shareholders do not change their behavior when one shareholder deviates to not acquiring information. However, most of the analysis would remain unchanged if information acquisition decisions were unobserved: the only difference would be an additional incentive compatibility constraint on information acquisition, which would not change the results qualitatively.

7.4 Comparative statics

Consider investor i, who believes that φ is distributed according to the Beta distibution with parameters $(\rho_i, \tau - \rho_i)$. The proof of Auxiliary Lemma A.1 in the appendix shows that from the perspective of investor i,

$$\mathbb{E}_i[\varphi] = \frac{\rho_i}{\tau}$$

and $\mathbb{E}[\varphi^2] = \frac{\rho_i(\rho_i+1)}{\tau(\tau+1)}$, and hence

$$Var_i[\varphi] = \frac{\rho_i}{\tau}\left(\frac{\rho_i+1}{\tau+1} - \frac{\rho_i}{\tau}\right) = \frac{\rho_i}{\tau}\frac{\tau - \rho_i}{\tau(\tau+1)}.$$

We are interested how the IC constraints for truthful communication are affected by the uncertainty about φ. Note that parameters τ and ρ_i affect both the mean and the variance of the distribution. We therefore would like to perform comparative statics in $Var_i[\varphi]$, while keeping the mean of the distribution fixed. To do this, suppose that we fix each investor's prior expectation of φ: denote $E_i \equiv \mathbb{E}_i[\varphi] = \frac{\rho_i}{\tau}$. Then

$$Var_i[\varphi] = E_i\frac{\tau - \tau E_i}{\tau(\tau+1)} = E_i\frac{1-E_i}{\tau+1}, \tag{69}$$

and the IC constraint (10) becomes

$$|(\tau + |R_i| + 1)\, b + (K - |R_i| - 1)\,\tau(E_m - E_i)| \leq \frac{\tau+K}{2}. \tag{70}$$

Hence, suppose we decrease τ but also simultaneously decrease all ρ_i and ρ_m proportionally to τ, in order to keep fixed the expectations of φ across investors and the manager: $E_i = \frac{\rho_i}{\tau}$ and $E_m = \frac{\rho_m}{\tau}$. Then (69) implies that this change in parameters corresponds to an increase in the variance of φ from each agent's perspective. How does the IC constraint (70) change as we make this parameter change? We consider two cases:

If $b = 0$, (70) is equivalent to

$$|(K - |R_i| - 1)\,(E_m - E_i)| \leq \frac{1}{2} + \frac{K}{2\tau}, \tag{71}$$

and it becomes more lax as τ decreases. Hence, under heterogeneous beliefs, truthful communication is more likely when there is more uncertainty.

If $\rho_i = \rho_m$ for all i, (70) is equivalent to

$$b \leq \frac{1}{2}\frac{\tau+K}{\tau + |R_i| + 1}, \tag{72}$$

which also becomes more lax as τ decreases. Hence, under heterogeneous preferences, truthful communication is also more likely when there is more uncertainty.

To understand the intuition, note that the manager's reaction to the shareholder's advice, i.e., by how much the manager's action changes if the shareholder misreports his signal θ_i, is given by $1+\frac{K-|R_i|-1}{\tau+|R_i|+1}$. Hence, as τ decreases, and thus the variance of φ increases, the manager reacts more strongly to the shareholder's advice, because high variance means relatively uninformative priors. This makes misreporting more costly and truthful communication more likely.

7.5 Heterogeneous interpretation of information

In this section, we extend the model to capture different interpretations of signals by the shareholders and the manager. In particular, we follow models of differences of opinion in which agents disagree about the precision of signals, and each agent's belief about the precision of his own signal is higher than other agents' beliefs about it (e.g., Banerjee et al., 2009; Kyle et al., 2018).

Specifically, consider the setting of Section 4.3 with $\mathcal{M} = 1$: there is a set of investors indexed by i (who observe signals $\theta_1, ..., \theta_N$), and the manager, indexed by m, who observes signal θ_{N+1}. As in the basic model, agent i's prior is that φ is drawn from the Beta distribution with parameters $(\rho_i, \tau - \rho_i)$. The key difference from the main model is that each agent overestimates the importance of his own signal: agent i believes that the state of the world is equal to

$$Z_i = \gamma\theta_i + \sum_{j\neq i} \theta_j, \tag{73}$$

where $\gamma \geq 1$ captures the extent of heterogeneous interpretation of signals. In this setting, even if the agents knew all the signals that comprise the state (assume, for simplicity, that this is indeed the case, $K = N+1$), they would still disagree about the state and thus the optimal action. Because our goal is to show that the complementarity effect in communication arises in this case as well, we assume $b = 0$ for ease of exposition.

The next result presents the constraint for truthful communication by shareholder i:

Proposition OA.4. *Suppose that the manager learns subset R_i of signals (which does not include shareholder i'signal θ_i) and does not know all the other signals, $-R_i$. Then, shareholder i reports his signal truthfully if and only if*

$$\begin{array}{ll} \textit{for } \rho_i \leq \rho_m: & \rho_m - \rho_i \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma-1)\,\frac{\tau-\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|}, \\ \textit{for } \rho_i > \rho_m: & \rho_i - \rho_m \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma-1)\,\frac{\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|}. \end{array} \tag{74}$$

This result shows that the overconfidence of each shareholder in the importance of his signal increases his incentives to report truthfully: the right-hand side of (74) increases in γ. Intuitively, if the shareholder perceives his signal to be more important than it actually is, he perceives misreporting to be costlier.

In addition, importantly, the communication decisions of the shareholders are complements: as in the basic model: the right-hand side of (74) increases as R_i expands. Overall, the key requirement for the complementarity effect is that communication of other shareholders to the manager moves the manager's and the shareholders's beliefs about the state closer to each other, which is consistent with heterogeneous interpretations of signals and does not require complete convergence of beliefs under full information. This property holds in a large class of models of different beliefs, although not in all of them.

7.6 Substitutability of shareholders' signals

In this section, we explore whether shareholders' signals in our setting are substitutes or complements under the definition of Borgers et al. (2013) and ask whether it matters for the externalities in communication.

7.6.1 Are signals complements or substitutes?

Under the definition of Borgers et al. (2013), signals are substitutes (complements) if the added effect of an additional signal on the agent's utility decreases (increases) with the number of signal, assuming the agent takes the optimal action given this information. Consider any investor j in our model with a prior belief that $\varphi \sim Beta(\rho_j, \tau - \rho_j)$. Suppose the investor knows the set R of signals. Using (6), the agent's optimal action is

$$a_j(\theta_R) = \sum_{i \in R} \theta_i + \frac{\rho_j + \sum_{i \in R} \theta_i}{\tau + |R|} \left(K - |R|\right).$$

To calculate the agent's utility from knowing $|R|$ signals, $V_j\left(|R|\right)$, we rely on Lemma OA.2 in Section 7.1 of the Online Appendix. Adapting the derivations behind Lemma 2, we get

$$V_j\left(|R|\right) = u_0 - \frac{\rho_j(\tau - \rho_j)}{\tau(\tau+1)} \left(K - |R| + \frac{\left(K - |R|\right)^2}{\tau + |R|}\right) = u_0 - \frac{\rho_j(\tau - \rho_j)}{\tau(\tau+1)} \frac{K - |R|}{\tau + |R|} \left(\tau + K\right). \tag{75}$$

Consider $\mathcal{G}(r) \equiv \frac{K-r}{\tau+r}$. Since $\mathcal{G}'(r) < 0$ and $\mathcal{G}''(r) > 0$, we have $V_j\left(|R|\right)$ increasing and concave in $|R|$. Hence, $V_j\left(|R|\right) - V_j\left(|R| - 1\right)$ decreases in $|R|$, i.e., the signals are substitutes under the definition of Borgers et al. (2013).

7.6.2 What if signals are unconditionally independent?

To explore the role of the property that signals are substitutes, we change the assumption that φ is unknown and agents form beliefs about it. Instead, we assume that φ is a commonly known parameter. In particular, $Z = \sum_{i=1}^{K} \theta_i$,where investor i observes θ_i, and $\theta_1, ..., \theta_K$ are independent binary signals equal to 1 with probability φ and 0 otherwise, where φ is known.

In this case, if investor j know the set R of signals, his optimal action is

$$a_j^{ind}(\theta_R) = \mathbb{E}\left(Z \mid \theta_R\right) = \sum_{i \in R} \theta_i + \left(K - |R|\right)\varphi.$$

Hence, the agent's utility from knowing $|R|$ signals, $V_j^{ind}(|R|)$, is now given by

$$\begin{aligned}
V_j^{ind}(|R|) &= u_0 - \mathbb{E}\left[\left(a_j^{ind}(\theta_R) - Z\right)^2 \mid \theta_R\right] = u_0 - \mathbb{E}\left[\left(\sum_{i\in R}\theta_i + (K - |R|)\,\varphi - \sum_{i=1}^{K}\theta_i\right)^2 \mid \theta_R\right] \\
&= u_0 - \mathbb{E}\left[\left(\sum_{i\in -R}(\theta_i - \varphi)\right)^2 \mid \theta_R\right] = u_0 - (K - |R|)\,Var\,(\theta_i)\,,
\end{aligned}$$

where the last equality used the fact that θ_i are unconditionally independent and have mean φ. Hence, the utility from an additional signal is $Var\,(\theta_i)$ and does not depend on the number of signals the agent has, i.e., signals are neither complements nor substitutes under the definition of Borgers et al. (2013).

In this setting, we derive each shareholder's IC constraint for truthful communication. Suppose the manager knows signals in set R_i. If shareholder i reports his signal truthfully, the manager' action is

$$a_m^{ind}(\theta_{R_i}, \theta_i) \equiv b + \theta_i + \sum_{j\in R_i}\theta_j \; + \varphi\,(K - |R_i| - 1)\,. \tag{76}$$

If shareholder i misreports, the manager's action is

$$a_m^{ind}(\theta_{R_i}, 1-\theta_i) \equiv b + (1-\theta_i) + \sum_{j\in R_i}\theta_j \; + \varphi\,(K - |R_i| - 1)\,. \tag{77}$$

The shareholder compares his expected payoff from actions $a_m^{ind}(\theta_{R_i}, \theta_i)$ and $a_m^{ind}(\theta_{R_i}, 1-\theta_i)$ and reports his signal truthfully if and only if:

$$\sum_{\theta_{-i}\in\{0,1\}^{K-1}} \left[(a_m^{ind}(\theta_{R_i}, \theta_i) - Z)^2 - (a_m^{ind}(\theta_{R_i}, 1-\theta_i) - Z)^2\right] P(\theta_{-i}) \le 0, \tag{78}$$

where $P(\theta_{-i})$ is his belief about θ_{-i}. Note that

$$(a_m^{ind}(\theta_{R_i}, \theta_i) - Z) - (a_m^{ind}(\theta_{R_i}, 1-\theta_i) - Z) = 2\theta_i - 1,$$

and

$$\begin{aligned}
&(a_m^{ind}(\theta_{R_i}, \theta_i) - Z) + (a_m^{ind}(\theta_{R_i}, 1-\theta_i) - Z) = \\
&= 2b + 1 + 2\textstyle\sum_{j\in R_i}\theta_j \; + 2\varphi\,(K - |R_i| - 1) - 2\theta_i - 2\sum_{j\neq i}\theta_j \;.
\end{aligned}$$

Hence, the shareholder reports truthfully if and only if

$$\begin{aligned}(2\theta_i-1)\sum_{\theta_{-i}\in\{0,1\}^{K-1}}\left(2b+1+2\sum_{j\in R_i}\theta_j\ +2\varphi\left(K-|R_i|-1\right)-2\theta_i-2\sum_{j\neq i}\theta_j\right)P(\theta_{-i}) &\le 0\Leftrightarrow\\ (2\theta_i-1)\left(2b+1+2\left|R_i\right|\varphi+2\varphi\left(K-|R_i|-1\right)\ -2\theta_i-2\left(K-1\right)\varphi\right) &\le 0\Leftrightarrow\\ (2\theta_i-1)\left(2b+1-2\theta_i\ \right) &\le 0.\end{aligned}$$

If $\theta_i=0$, the above inequality becomes $2b+1\geq 0$, which is always satisfied, and if $\theta_i=1$, it becomes $b\leq\frac{1}{2}$. Hence, the IC condition no longer depends on $|R_i|$.

7.7 Single vs. multiple dimensions of expertise

Our basic model features multiple dimensions of expertise: while signals θ_i all provide noisy information about φ, each of them also provides incremental information about the state Z (since θ_i are independent conditional on φ). In this section, we explore the role of this assumption by comparing our results to those in a setting with one dimension of expertise.

Specifically, suppose that agent i's belief is that state Z is a draw from the Beta distribution with parameters $(\rho_i,\tau-\rho_i)$. Each shareholder i obtains a signal $\theta_i\in\{0,1\}$ about Z, where $\theta_i=1$ with probability Z and $\theta_i=0$ with probability $1-Z$.

Suppose that after the communication stage, the manager knows subset R of signals. Then his optimal action is

$$a_m(\theta_R)=b+\mathbb{E}_m\left(Z\,|\,\theta_R\right)=b+\frac{\rho_m+\sum_{i\in R}\theta_i}{\tau+|R|}. \tag{79}$$

Suppose the manager believes the shareholder's message and uses it to update his belief about the state according to (79). If shareholder i reveals his signal truthfully, the manager's action is

$$a_m\left(\theta_{R_i},\theta_i\right)\equiv b+\frac{\rho_m+\theta_i+\sum_{j\in R_i}\theta_j}{\tau+|R_i|+1}. \tag{80}$$

If shareholder i misreports and claims that his signal is $1-\theta_i$, the manager's action is

$$a_m\left(\theta_{R_i},1-\theta_i\right)\equiv b+\frac{\rho_m+(1-\theta_i)+\sum_{j\in R_i}\theta_j}{\tau+|R_i|+1}. \tag{81}$$

Note that

$$\begin{aligned}a_m\left(\theta_{R_i},\theta_i\right)-a_m\left(\theta_{R_i},1-\theta_i\right) &= \frac{2\theta_i-1}{\tau+|R_i|+1},\\ a_m\left(\theta_{R_i},\theta_i\right)+a_m\left(\theta_{R_i},1-\theta_i\right)-2Z &= 2b+\frac{2\rho_m+1+2\left(\sum_{j\in R_i}\theta_j\right)}{\tau+|R_i|+1}-2Z.\end{aligned}$$

Hence, the IC constraint for shareholder i to communicate truthfully is:

$$-\int_0^1 \sum_{\theta_{R_i}} \frac{2\theta_i - 1}{\tau + |R_i| + 1} \left(2b + \frac{2\rho_m + 1 + 2\left(\sum_{j \in R_i} \theta_j\right)}{\tau + |R_i| + 1} - 2Z \right) f_i(Z, \theta_{R_i}|\theta_i) dZ \geq 0, \tag{82}$$

where $f_i(Z, \theta_{R_i}|\theta_i)$ is the shareholder's belief given his signal θ_i and prior beliefs. Simplifying:

$$\begin{aligned} -(2\theta_i - 1) \left(2b + \int_0^1 \sum_{\theta_{R_i}} \left(\frac{2\rho_m + 1 + 2\left(\sum_{j \in R_i} \theta_j\right)}{\tau + |R_i| + 1} - 2Z \right) f_i(Z, \theta_{R_i}|\theta_i) dZ \right) &\geq 0 \Leftrightarrow \\ -(2\theta_i - 1) \left(2b + \frac{2\rho_m + 1 + 2|R_i| \mathbb{E}_i \left[\theta_j|\theta_i\right]}{\tau + |R_i| + 1} - 2\mathbb{E}_i \left[Z|\theta_i\right] \right) &\geq 0. \end{aligned}$$

Note that $\mathbb{E}_i \left[Z|\theta_i\right] = \frac{\rho_i + \theta_i}{\tau + 1}$ and $\mathbb{E}_i \left[\theta_j|\theta_i\right] = \mathbb{E}_i \left[Z|\theta_i\right] = \frac{\rho_i + \theta_i}{\tau + 1}$. Therefore, the above expression simplifies to:

$$\begin{aligned} -(2\theta_i - 1) \left(2b + \frac{2\rho_m + 1}{\tau + |R_i| + 1} + \frac{2\mathbb{E}_i \left[Z|\theta_i\right]}{\tau + |R_i| + 1} \left(|R_i| - \tau - |R_i| - 1\right) \right) &\geq 0 \Leftrightarrow \\ -(2\theta_i - 1) \left(2b + \frac{2\rho_m + 1}{\tau + |R_i| + 1} - \frac{2\left(\rho_i + \theta_i\right)}{\tau + |R_i| + 1} \right) &\geq 0 \Leftrightarrow \\ -(2\theta_i - 1) \left(2b + \frac{2\left(\rho_m - \rho_i\right)}{\tau + |R_i| + 1} + \frac{1 - 2\theta_i}{\tau + |R_i| + 1} \right) &\geq 0. \end{aligned}$$

There are two cases to consider, $\theta_i = 1$ and $\theta_i = 0$. When $\theta_i = 1$, the IC constraint is:

$$2b + \frac{2\left(\rho_m - \rho_i\right)}{\tau + |R_i| + 1} - \frac{1}{\tau + |R_i| + 1} \leq 0.$$

When $\theta_i = 0$, the IC constraint is:

$$2b + \frac{2\left(\rho_m - \rho_i\right)}{\tau + |R_i| + 1} + \frac{1}{\tau + |R_i| + 1} \geq 0$$

Together, we get

$$2\left| b + \frac{\rho_m - \rho_i}{\tau + |R_i| + 1} \right| \leq \frac{1}{\tau + |R_i| + 1}. \tag{83}$$

As in the basic model, the left-hand side of (83) captures the incongruence between the manager and the shareholder, whereas the right-hand side of (83) measures the manager's reaction to the shareholder's advice (i.e., by how much the manager's action changes if the shareholder misreports his signal θ_i, as can be seen from (80) and (81)). Hence, both the complementarity and the substitution effects again arise in this setting. The complementarity effect is represented by the term $\frac{\rho_m - \rho_i}{\tau + |R| + 1}$: as the manager learns more signals from other shareholders, his posterior beliefs become closer to shareholder i's posterior beliefs. This increases the congruence between them (as captured by a decrease in the left-hand side of (83)) and increases the shareholder's incentives to communicate truthfully. The substitution effect is represented by the term $\frac{1}{\tau+|R_i|+1}$: as the manager learns more signals from other shareholders, he reacts less to the shareholder's advice, making misreporting more appealing. Finally, the property that a stronger misalignment of preferences limits the complementarity effect also continues to hold: as b increases, the manager's learning has a relatively smaller effect on the congruence between the manager and shareholder (as captured by the left-hand-side of (83) being increasingly limited from zero).

Note also that if $b = 0$, the effect of $|R_i|$ cancels out, i.e., the complementarity effect is fully offset by the substitution effect. While the complete offsetting of the two effects is a specific property of the Beta distribution, the more general intuition is that the substitution effect is relatively weaker in the presence of multiple dimensions of expertise (as in the basic model) than if there is a single dimension of expertise (as in the model in this section). To see why this is the case, let us compare the IC condition (83) in this setting to the corresponding IC condition in the basic model:

$$2\left|b + \frac{K - |R_i| - 1}{\tau + |R_i| + 1}(\rho_m - \rho_i)\right| \leq 1 + \frac{K - |R_i| - 1}{\tau + |R_i| + 1}. \tag{84}$$

Relative to (83), the additional additive term 1 on the right-hand side of (84) weakens the substitution effect and arises due to the multi-dimensionality of shareholders' expertise: even if the manager strongly updates his beliefs about the common state φ (as captured by a decreasing function $\frac{1}{\tau+|R_i|+1}$), he still strongly reacts to the shareholder's message if it provides information about a new dimension of uncertainty (as captured by the term 1). This encourages truthtelling and weakens the substitution effect.

7.8 Communication among shareholders

Suppose that the firm is owned by S_o optimistic and S_p pessimistic shareholders. Consider the following change in the communication stage of the game. Instead of each shareholder independently sending a binary message to the manager, suppose that all shareholders of the same type (i.e., with the same prior beliefs) share their signals among themselves, and one representative of the group then communicates with the manager via cheap talk. Since shareholders' interests within a group are fully aligned, they share their information truthfully with each other. Given this change in the setup, we effectively have a two-sender model in which one sender (representing the optimists) has signal $\Theta_o \equiv \sum_{i=1}^{S_o} \theta_i$, taking discrete values from zero to S_o, and the other sender (representing the pessimists) has signal $\Theta_p \equiv \sum_{i=S_o+1}^{S_o+S_p} \theta_i$, taking discrete values from zero to S_p. Signals θ_i for $i \in \{S_o + S_p + 1, ..., K\}$ are unknown to everyone.

Let μ_o and μ_p denote the messages of the sender representing optimistic and pessimistic shareholders, respectively. In what follows, we will derive the necessary and sufficient conditions under which there exists a fully informative equilibrium, i.e., an equilibrium in which the representatives of each group communicate all information of the group truthfully: the sender representing optimistic (pessimistic) shareholders sends message $\mu_o = \Theta_o$ ($\mu_p = \Theta_p$). The next result shows that these conditions are the same as the necessary and sufficient conditions for the existence of a fully informative equilibrium (i.e., $|R| = S_o + S_p$) without communication among shareholders.

Proposition OA.5. *The necessary and sufficient conditions for the existence of a fully informative equilibrium in the model with communication through a representative are the same as the necessary and sufficient conditions for the existence of a fully informative equilibrium in the basic model.*

The logic of this result is based on two steps. The first step is that the costs and benefits of a sender deviating from truthful communication *by one unit* (e.g., from Θ_o to $\Theta_o - 1$ or to $\Theta_o + 1$) are the same in the basic model and in the model with communication through a representative. This is because the action of the manager and the payoffs of all investors and the sender are the same in the basic model and in this extension, both if the sender tells the truth and if he deviates in his message by one unit. Thus, the IC constraints that make such deviations suboptimal are also the same. The second step concerns global deviations by the sender in the model with communication through a representative. In the model in which all shareholders communicate to the manager directly, each shareholder only has access to "small" deviations because each shareholder's signal is binary: for example, if a shareholder gets a signal of 0, he can misreport and say that his signal is 1, but he cannot misreport and say that his signal is 2. In contrast, if the representative communicates with the manager on behalf of all shareholders of the group, he can misreport by more than one unit. Nevertheless, we show that these additional deviations have no additional bite: if a deviation from truthful reporting by more than one unit is profitable to a sender, then a deviation in the same direction by one unit must also be profitable for a sender. Given these two steps, a fully informative equilibrium exists in the model with communication through a representative if and only if it exists in the basic model.[29]

7.9 Sequential communication

Suppose that the firm is owned by S shareholders. Consider the following change in the communication stage of the game. Instead of communicating simultaneously to the manager, there is sequential public communication: shareholders communicate in a known sequence $O = \{O_j, j \in S\}$ such that shareholder $j \in S$ is O_j-th to send a public message, which is

[29] Characterizing the full set of equilibria in the model with communication through a representative is more challenging than in the basic model. The reason is that in the basic model, we use the property that if the prior distribution of φ is Beta, then the posterior distribution of φ conditional on learning a number of binary signals is also Beta (with parameters that depend on the realizations of the learned signals). In contrast, in a partially informative equilibrium of the extended game, the manager will only learn that the sum of all signals of shareholders of the same type lies in some set. As a consequence, the posterior distribution of φ is no longer the Beta distribution, which makes the analysis less tractable.

observed by all the other shareholders and the manager. The next result characterizes the IC condition for truthful communication in this variation of the model.

Proposition OA.6. *Consider the sequential game described above. Suppose shareholder i expects subset $R_i \subseteq S\backslash\{i\}$ of other shareholders to communicate truthfully and other shareholders to send uninformative messages. Then, for any sequence O, shareholder i has incentives to send a truthful message if and only if (10) is satisfied.*

Hence, the IC condition for truthful communication for each shareholder is exactly the same as in the basic model. Intuitively, what matters for the shareholder's incentives is the combined set of signals that the manager learns before taking his action, as this combined set of signals determines both the manager's reaction to the shareholder's advice and the congruence between the manager and the shareholder at the decision-making stage.

It follows that for any sequence of communication, the equilibrium at the communication stage is the same as in the model with simultaneous communication. Thus, if the expectation at the trading stage is that this communication equilibrium will be played, then the solution of the trading stage will also be unchanged, leading to the same equilibrium as in the basic model. Of course, the sequential game may also have other equilibria, which we do not analyze.

Proofs for the Online Appendix

Proof of Lemma OA.1

Since θ_i is a binary signal equal to 1 with probability φ and 0 with probability $1-\varphi$, the manager's optimal action can be written as:

$$a_m(\theta_R) = b + \sum_{i\in R} c_i\theta_i + \mathbb{E}_m\left[\varphi|\theta_i, i\in R\right]\sum_{j\in -R} c_j \ .$$

Let $\mathbf{1}_R \equiv \sum_{i\in R}\theta_i$ be the number of signals in R equal to 1. The conditional probability that $\mathbf{1}_R$ signals out of $|R|$ are equal to one given φ is $P(\mathbf{1}_R|\varphi) = \binom{|R|}{\mathbf{1}_R}\varphi^{\mathbf{1}_R}(1-\varphi)^{|R|-\mathbf{1}_R}$. Since the prior distribution is Beta and the likelihood function is Binomial, the posterior distribution is also Beta but with different parameters (this is a known property of the Beta distribution). Formally, let $P_i\left(\mathbf{1}_R\right)$ be agent i's assessed probability that $\mathbf{1}_R$ signals out of $|R|$ are equal to 1 (over all possible values of φ). Using Bayes rule, agent i's posterior belief of φ, $P_i(\varphi|\mathbf{1}_R)$, is

$$\begin{aligned} P_i(\varphi|\mathbf{1}_R) &= \frac{f_i(\varphi)P(\mathbf{1}_R|\varphi)}{P_i\left(\mathbf{1}_R\right)} = \frac{\varphi^{\rho_i-1}(1-\varphi)^{\tau-\rho_i-1}}{Beta(\rho_i,\tau-\rho_i)}\frac{1}{P_i\left(\mathbf{1}_R\right)}\binom{|R|}{\mathbf{1}_R}\varphi^{\mathbf{1}_R}(1-\varphi)^{|R|-\mathbf{1}_R} \\ &= \frac{1}{Beta(\rho_i,\tau-\rho_i)P_i\left(\mathbf{1}_R\right)}\binom{|R|}{\mathbf{1}_R}\times\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1}, \end{aligned}$$

which is some constant that does not depend on φ times $\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1}$. Since the posterior beliefs must integrate to one over possible values of φ, this automatically implies that the posterior belief also follows a Beta distribution with parameters $(\rho_i+\mathbf{1}_R, \tau-\rho_i+|R|-\mathbf{1}_R)$ and density

$$P_i(\varphi|\mathbf{1}_R) = \frac{1}{Beta(\rho_i+\mathbf{1}_R, \tau-\rho_i+|R|-\mathbf{1}_R)}\varphi^{\rho_i+\mathbf{1}_R-1}(1-\varphi)^{\tau-\rho_i+|R|-\mathbf{1}_R-1}.$$

It is known that the mean of a Beta distribution with parameters (α,β) is $\frac{\alpha}{\alpha+\beta}$. Therefore, using these expressions and the above posterior distribution, agent i's expected value of φ is $\mathbb{E}_i(\varphi|\mathbf{1}_R) = \frac{\rho_i+\mathbf{1}_R}{\tau+|R|}$, which proves the lemma.

7.9.1 Proof of Lemma OA.2

Let $\mathbf{1}_R = \sum_{i\in R}\theta_i$ denote the number of signals 1 in R. Using Lemma OA.1, we obtain investor i's ex-ante payoff, $\mathbb{E}_i(a_m(\theta_R)-Z)^2$, as follows:

$$\mathbb{E}_i\left[U_i|R\right] = u_0 - b^2 - U_1 - U_2, \tag{85}$$

where

$$\begin{aligned}
U_1 &\equiv 2b\mathbb{E}_i\left[\left(\frac{\rho_m+\mathbf{1}_R}{\tau+|R|}\sum_{j\in -R}c_j - \sum_{j\in -R}c_j\theta_j\right)|R\right], \\
U_2 &\equiv \mathbb{E}_i\left[\left(\frac{\rho_m+\mathbf{1}_R}{\tau+|R|}\sum_{j\in -R}c_j - \sum_{j\in -R}c_j\theta_j\right)^2|R\right].
\end{aligned}$$

Using independence of θ_j conditional on φ, and Auxiliary Lemma A.1, U_1 simplifies to

$$U_1 = 2b\frac{\rho_m-\rho_i}{\tau+|R|}\left(\sum_{j\in -R}c_j\right) = \mathsf{A}_{im}(R). \tag{86}$$

To simplify U_2, we use the law of iterated expectations:

$$\begin{aligned}
U_2 &= \mathbb{E}_i\left[\left(\frac{(\rho_m+\mathbf{1}_R)\sum_{j\in -R}c_j}{\tau+|R|}\right)^2 - 2\frac{(\rho_m+\mathbf{1}_R)(\rho_i+1_R)\left(\sum_{j\in -R}c_j\right)^2}{(\tau+|R|)^2}|R\right] \\
&\quad +\mathbb{E}_i\left[\mathbb{E}_i\left[\left(\sum_{j\in -R}c_j\theta_j\right)^2|\theta_R, R\right]|R\right],
\end{aligned} \tag{87}$$

where we used $\mathbb{E}_i\left[\sum_{j\in -R} c_j\theta_j|\theta_R, R\right] = \left(\sum_{j\in -R} c_j\right)\mathbb{E}_i\left[\varphi|\theta_R, R\right] = \left(\sum_{j\in -R} c_j\right)\frac{\rho_i+1_R}{\tau+|R|}$. Consider the last term under the expectation sign:

$$\begin{aligned}
\mathbb{E}_i\left[\left(\sum_{j\in -R} c_j\theta_j\right)^2|\theta_R, R\right] &= \mathbb{E}_i\left[\sum_{j\in -R} c_j^2 Var_i\left[\theta_j|\varphi, R\right] + \varphi^2\left(\sum_{j\in -R} c_j\right)^2|\theta_R, R\right] \\
&= \mathbb{E}_i\left[\sum_{j\in -R} c_j^2\varphi\left(1-\varphi\right) + \varphi^2\left(\sum_{j\in -R} c_j\right)^2|\theta_R, R\right] \\
&= \frac{\rho_i+1_R}{\tau+|R|}\left(\sum_{j\in -R} c_j^2 + \left(\left(\sum_{j\in -R} c_j\right)^2 - \sum_{j\in -R} c_j^2\right)\frac{\rho_i+1_R+1}{\tau+|R|+1}\right),
\end{aligned}$$

where the second equality is due to $Var_i\left[\theta_j|\varphi, R\right] = \varphi\left(1-\varphi\right)$ and the last equality is due to the fact that the agent i's posterior distribution of φ conditional on θ_R is Beta with parameters $\rho_i + 1_R$ and $\tau + |R| - \rho_i - 1_R$, whose first and second moments are, respectively, $\frac{\rho_i+1_R}{\tau+|R|}$ and $\frac{(\rho_i+1_R)(\rho_i+1_R+1)}{(\tau+|R|)(\tau+|R|+1)}$. Plugging this expression into (87) and simplifying using Auxiliary Lemma A.1,

$$\begin{aligned}
U_2 - \mathsf{C}_{im}(R) &= \mathbb{E}_i\left[\frac{\left(\sum_{j\in -R} c_j^2\right)\left(\rho_i+1_R\right)}{\tau+|R|} - \left(\frac{\left(\sum_{j\in -R} c_j\right)\left(\rho_i+1_R\right)}{\tau+|R|}\right)^2|R\right] \\
&\quad + \left(\left(\sum_{j\in -R} c_j\right)^2 - \sum_{j\in -R} c_j^2\right)\mathbb{E}_i\left[\frac{\left(\rho_i+1_R+1\right)\left(\rho_i+1_R\right)}{\left(\tau+|R|+1\right)\left(\tau+|R|\right)}|R\right] \\
&= \left(\frac{\left(\sum_{j\in -R} c_j\right)^2}{\tau+|R|} + \sum_{j\in -R} c_j^2\right)\mathbb{E}_i\left[\frac{\left(\rho_i+1_R\right)\left(\tau+|R|-\rho_i-1_R\right)}{\left(\tau+|R|\right)\left(\tau+|R|+1\right)}\right] \\
&= \left(\frac{\left(\sum_{j\in -R} c_j\right)^2}{\tau+|R|} + \sum_{j\in -R} c_j^2\right)\frac{\rho_i\left(\tau-\rho_i\right)}{\tau\left(\tau+1\right)} = \mathsf{B}_i(R).
\end{aligned}$$

Combining with (85) and (86) completes the proof.

7.9.2 Proof of Proposition OA.1

Suppose the manager believes the shareholder's message and uses it to update his belief about the state. If shareholder i reveals his signal truthfully, the manager's action is

$$a_m(\theta_R, \theta_i) \equiv b + c_i\theta_i + \sum_{j\in R} c_j\theta_j + \frac{\rho_m + \theta_i + \sum_{j\in R}\theta_j}{\tau + 1 + |R|} \sum_{j\in -R\backslash\{i\}} c_j. \tag{88}$$

In contrast, if shareholder i misreports and claims that his signal is $1-\theta_i$, the manager's action is

$$a_m(\theta_R, 1-\theta_i) \equiv b + c_i(1-\theta_i) + \sum_{j\in R} c_j\theta_j + \frac{\rho_m + (1-\theta_i) + \sum_{j\in R}\theta_j}{\tau + 1 + |R|} \sum_{j\in -R\backslash\{i\}} c_j. \tag{89}$$

Because shareholder i does not know the realization of all other agents' ($N-1$ shareholders' and the manager's) signals, he compares his expected payoff from actions $a_m(\theta_R, \theta_i)$ and $a_m(\theta_R, 1-\theta_i)$ given his signal θ_i and his own prior belief about the distribution of those other N signals, and reports his signal truthfully if and only if:

$$\sum_{\theta_{-i}\in\{0,1\}^N} \left[(a_m(\theta_R,\theta_i) - Z)^2 - (a_m(\theta_R, 1-\theta_i) - Z)^2\right] P_i(\theta_{-i}|\theta_i) \le 0, \tag{90}$$

where θ_{-i} is the set of all signals except θ_i and $P_i(\theta_{-i}|\theta_i)$ is shareholder i's belief given his signal θ_i and his own prior.

Plugging (88) and (89) into (90) gives

$$\begin{aligned} 0 \ge \sum_{\theta_{-i}\in\{0,1\}^N} & \left[c_i(2\theta_i - 1) + (\sum_{j\in -R_i\backslash\{i\}} c_j)\cdot\frac{2\theta_i - 1}{\tau + |R_i| + 1}\right] \\ & \times \left[2b + c_i(1-2\theta_i) - 2\sum_{j\in -R_i\backslash\{i\}} c_j\theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1}\sum_{j\in -R_i\backslash\{i\}} c_j\right] P_i(\theta_{-i}|\theta_i). \end{aligned}$$

Note that the first multiple in each term equals $(2\theta_i - 1)[c_i + \frac{\sum_{j\in -R_i\backslash\{i\}} c_j}{\tau + |R_i| + 1}]$, where $c_i + \frac{\sum_{j\in -R_i\backslash\{i\}} c_j}{\tau + |R_i| + 1}$ is positive and is constant across all terms in the sum. Thus, the above inequality is equivalent to

$$0 \ge (2\theta_i - 1)\sum_{\theta_{-i}} P_i(\theta_{-i}|\theta_i)\left(2b + c_i(1-2\theta_i) - 2\sum_{j\in -R_i\backslash\{i\}} c_j\theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1}\sum_{j\in -R_i\backslash\{i\}} c_j\right).$$

Since $\sum_{\theta_{-R_i\backslash\{i\}}}\left(\sum_{j\in -R_i\backslash\{i\}} c_j\theta_j\right) P_i(\theta_{-R_i\backslash\{i\}}|\theta_i, \theta_{R_i}) = \frac{\rho_i + \mathbf{1}_{R_i} + \theta_i}{\tau + |R_i| + 1}\sum_{j\in -R_i\backslash\{i\}} c_j$, we can further simplify it to

$$(2\theta_i - 1)\left[2b + c_i(1-2\theta_i) + \frac{2(\rho_m - \rho_i) + 1 - 2\theta_i}{\tau + |R_i| + 1}\sum_{j\in -R_i\setminus\{i\}} c_j\right] \leq 0.$$

We consider two separate cases. If $\theta_i = 0$, the above inequality becomes:

$$2b + c_i + \frac{2(\rho_m - \rho_i) + 1}{\tau + |R_i| + 1}\sum_{j\in -R_i\setminus\{i\}} c_j \geq 0,$$

and if $\theta_i = 1$, it becomes

$$2b - c_i + \frac{2(\rho_m - \rho_i) - 1}{\tau + |R_i| + 1}\sum_{j\in -R_i\setminus\{i\}} c_j \leq 0,$$

Together we get (62), which completes the proof.

7.9.3 Proof of Proposition OA.2

The proof largely repeats the proof of Proposition 1 in the paper. Using $a_m(\theta_{R_i}, \theta_i)$ and $a_m(\theta_{R_i}, 1-\theta_i)$, the IC becomes

$$\begin{gathered}\textstyle\sum_{\theta_{R_i},\theta_{-R_i}} P_i(\theta_{R_i}, \theta_{-R_i}|\theta_i)\left[c_i(2\theta_i - 1) + \left(\sum_{j\in -R_i\setminus\{i\}} c_j\right)\frac{2\theta_i - 1}{\tau_m + |R_i| + 1}\right] \times \\ \left[2b + c_i(1-2\theta_i) - 2\textstyle\sum_{j\in -R_i\setminus\{i\}} c_j\theta_j + (\sum_{j\in -R_i\setminus\{i\}} c_j)\frac{2(\rho_m + \sum_{j\in R_i}\theta_j) + 1}{\tau_m + |R_i| + 1}\right] \geq 0.\end{gathered}$$

Note that $P_i(\theta_{R_i}, \theta_{-R_i}|\theta_i) = P_i(\theta_{-R_i}|\theta_{R_i}, \theta_i)P_i(\theta_{R_i}|\theta_i)$. Since $\left[c_i + \frac{\sum_{j\in -R_i\setminus\{i\}} c_j}{\tau_m + |R_i| + 1}\right] > 0$, this is equivalent to

$$-(2\theta_i - 1)\times\left[2b + c_i(1-2\theta_i) - 2\frac{\rho_i + \theta_i}{\tau_i + 1}\sum_{j\in -R_i\setminus\{i\}} c_j + (\sum_{j\in -R_i\setminus\{i\}} c_j)\frac{2\rho_m + 2\,|R_i|\,\frac{\rho_i + \theta_i}{\tau_i + 1} + 1}{\tau_m + |R_i| + 1}\right] \geq 0$$

or equivalently,

$$-(2\theta_i - 1)\times\left[2b + c_i(1-2\theta_i) + (\sum_{j\in -R_i\setminus\{i\}} c_j)\left[\frac{2\rho_m(\tau_i + 1) + \tau_i + 1 - 2(\rho_i + \theta_i)(\tau_m + 1)}{(\tau_i + 1)(\tau_m + |R_i| + 1)}\right]\right] \geq 0$$

Considering two cases ($\theta_i = 1$ and $\theta_i = 0$) and simplifying the expressions (similar to the proof of Proposition 1), we obtain (67). It is easy to see that (67) is equivalent to the condition (62), which was derived for the same setting but with $\tau_i = \tau$ for all i.

7.9.4 Proof of Proposition OA.3

Let $V_i(r)$ denote investor i's payoff before acquiring and learning his private signal, which is given by (12) with $|R| = r$. Let S be the firm's shareholder base. Suppose there is an equilibrium in which all shareholders in S acquire information, which, in turn, requires that they communicate it truthfully. Consider the shareholder's decision to acquire information in this equilibrium. If shareholder i acquires his signal, his expected utility is $\alpha_i V_i(|S|) - \kappa$. If the shareholder deviates and does not acquire his signal, his expected utility is $\alpha_i V_i(|S| - 1)$: because information acquisition decisions are observed after the communication stage, the shareholder's deviation does not change other shareholder' incentives to communicate truthfully, but at the decision-making stage, the manager will make his decision knowing that the shareholder is uninformed. Hence, the incentive compatibility condition on information acquisition by all shareholders is that:

$$V_i(|S|) - V_i(|S| - 1) \geq \frac{\kappa}{\alpha_i} \tag{91}$$

for all i. To analyze these conditions, consider the function $V_i(r) - V_i(r-1)$. Using (12) and denoting $\mathcal{G}(r) \equiv \frac{K-r}{2\rho+r}$, we get

$$V_i(r) = u_0 - b^2 - 2b(\rho - \rho_i)\mathcal{G}(r) - \frac{\rho^2 - \Delta^2}{2\rho(2\rho+1)}\mathcal{G}(r)(2\rho + K) - [\Delta\mathcal{G}(r)]^2. \tag{92}$$

Note that $\mathcal{G}(r) > 0$, and that $\mathcal{G}(r)$ and hence $\mathcal{G}^2(r)$ decrease in r. In addition, $\mathcal{G}''(r) > 0$, and hence $(\mathcal{G}^2)''(r) > 0$ as well. It follows that the function $V_i(r)$ is increasing and concave in r, and hence $V_i(r) - V_i(r-1)$ is decreasing in r. If $b = 0$, then optimists and pessimists have the same valuations of the stock and hence hold the same number of shares, i.e., $S = \{1, ..., N\}$, $V_i(r) = V(r)$ for all i, and $\alpha_i = \frac{1}{N}$. Hence, (91) is equivalent to $\frac{V_i(N)-V_i(N-1)}{N} \geq \kappa$. Since $V_i(r) - V_i(r-1)$ is decreasing in r, $\frac{V_i(r)-V_i(r-1)}{r}$ is decreasing in r as well. Let $\hat{N}(\kappa)$ be the highest value of r for which $\frac{V_i(r)-V_i(r-1)}{r} \geq \kappa$, and note that $\hat{N}(\kappa)$ is weakly decreasing in κ. Then, the incentive compatibility condition on information acquisition is satisfied for all shareholders if and only if $N \leq \hat{N}(\kappa)$.

7.9.5 Proof of Proposition OA.4

Suppose that the manager expects shareholder i to report his signal truthfully, and consider shareholder i's decision whether to do so. If shareholder i reveals his signal truthfully, the manager's action is

$$a_m(\theta_{R_i}, \theta_i) \equiv \theta_i + \gamma\theta_m + \sum_{j \in R_i \setminus \{m\}} \theta_j + \frac{\rho_m + \theta_i + \sum_{j \in R_i} \theta_j}{\tau + 1 + |R_i|}(N - |R_i|). \tag{93}$$

In contrast, if shareholder i misreports, the manager's action is

$$a_m\left(\theta_{R_i}, 1-\theta_i\right) \equiv (1-\theta_i) + \gamma\theta_m + \sum_{j\in R_i\setminus\{m\}} \theta_j \;\; + \frac{\rho_m + (1-\theta_i) + \sum_{j\in R_i}\theta_j}{\tau+1+|R_i|}\left(N-|R_i|\right). \quad (94)$$

Truthful reporting is optimal if and only if

$$\sum_{\theta_{-i}\in\{0,1\}^N} \left[\left(a_m\left(\theta_{R_i},\theta_i\right)-Z_i\right)^2 - \left(a_m\left(\theta_{R_i},1-\theta_i\right)-Z_i\right)^2\right] P_i(\theta_{-i}|\theta_i) \leq 0 \quad (95)$$

for each $\theta_i \in \{0,1\}$, where $Z_i = \gamma\theta_i + \sum_{j\neq i}\theta_j$. Simplifying, (95) reduces to

$$(2\theta_i - 1)\left[(1-2\gamma\theta_i) + 2(\gamma-1)\frac{\rho_i+\theta_i}{\tau+1} + \frac{2(\rho_m-\rho_i)+1-2\theta_i}{\tau+1+|R_i|}\left(N-|R_i|\right)\right] \leq 0.$$

If $\theta_i = 1$, we have:

$$\rho_m - \rho_i \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma-1)\frac{\tau-\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|},$$

which is trivially satisfied for $\rho_m \leq \rho_i$, but may be violated if $\rho_m > \rho_i$. If $\theta_i = 0$, we have:

$$\rho_i - \rho_m \leq \frac{1}{2}\frac{N+1+\tau}{N-|R_i|} + (\gamma-1)\frac{\rho_i}{\tau+1}\frac{\tau+1+|R_i|}{N-|R_i|},$$

which is trivially satisfied for $\rho_m \geq \rho_i$, but may be violated if $\rho_m < \rho_i$. Combining the two cases proves the proposition.

7.9.6 Proof of Proposition OA.5

Consider the equilibrium in which $\mu_o = \Theta_o$ and $\mu_p = \Theta_p$. Given these strategies, the manager's action as a function of messages μ_o, μ_p is

$$a_m\left(\mu_o,\mu_p\right) = b + \mu_o + \mu_p + \frac{\rho+\mu_o+\mu_p}{2\rho+S_o+S_p}\left(K-S_o-S_p\right). \quad (96)$$

First, consider the sender representing optimists. If he sends a truthful message, the manager's action will be $a_m(\Theta_o,\Theta_p)$, while if she sends a message $\mu \neq \Theta_o$, the manager's action will be $a_m(\mu,\Theta_p)$, where $a_m(\cdot,\cdot)$ is given by (96). The sender finds it optimal to communicate truthfully if and only if

$$\sum_{\theta_{-i}\in\{0,1\}^{K-S_o}} \left[\begin{array}{c} (b + \frac{\rho+\Theta_o+\Theta_p}{2\rho+S_o+S_p}\left(K-S_o-S_p\right) - \sum_{i=S_o+S_p+1}^{K}\theta_i)^2 \\ -(b+\mu+\frac{\rho+\mu+\Theta_p}{2\rho+S_o+S_p}\left(K-S_o-S_p\right) - \Theta_o - \sum_{i=S_o+S_p+1}^{K}\theta_i)^2 \end{array}\right] P_i(\theta_{-i}|\sum_{i=1}^{N_o}\theta_i = \Theta_o) \leq 0.$$

Simplifying this expression, we obtain

$$(\Theta_o - \mu)\,\frac{K+2\rho}{2\rho+S_o+S_p}\left[\begin{array}{c} 2b+\frac{2\rho+\Theta_o+2S_p+\mu}{2\rho+S_o+S_p}\left(K-S_o-S_p\right) \\ +\mu-\Theta_o-2\left(K-S_o-S_p\right)\mathbb{E}_o\left[\theta_i\middle|\sum_{i=1}^{N_o}\theta_i=\Theta_o\right]_i \end{array}\right]\leq 0$$

for all $\mu\in\{0,1,...,S_o\}$, which is equivalent to

$$(\Theta_o-\mu)\left[(2\rho+S_o+S_p)\,b-(K-S_o-S_p)\,\Delta-(\Theta_o-\mu)\left(\frac{K}{2}+\rho\right)\right]\leq 0.$$

Consider deviations by one signal (i.e., from a truthful message Θ_o to $\mu=\Theta_o-1$ and $\mu=\Theta_o+1$). A deviation to $\mu=\Theta_o-1$ is not profitable if and only if

$$(2\rho+S_o+S_p)\,b-(K-S_o-S_p)\,\Delta\leq\rho+\frac{K}{2}.$$

A deviation to $\mu=\Theta_o+1$ is not profitable if and only if

$$(K-S_o-S_p)\,\Delta-(2\rho+S_o+S_p)\,b\leq\frac{K}{2}+\rho.$$

Taken together, we have

$$\left|(2\rho+S_o+S_p)\,b-(K-S_o-S_p)\,\Delta\right|\leq\rho+\frac{K}{2}, \tag{97}$$

which coincides with the IC constraint in the basic model without communication among shareholders. We next show that if truthful reporting dominates sending $\mu=\Theta_o-1$, then it dominates sending any $\mu<\Theta_o-1$. To see this, note that

$$\begin{aligned} & (2\rho+S_o+S_p)\,b-(K-S_o-S_p)\,\Delta-(\Theta_o-\mu)\left(\frac{K}{2}+\rho\right) \\ < \;& (2\rho+S_o+S_p)\,b-(K-S_o-S_p)\,\Delta-\left(\frac{K}{2}+\rho\right)\leq 0, \end{aligned}$$

where the first inequality follows from the strict monotonicity of the expression in μ, and the second inequality is the condition that truthful reporting dominates sending $\mu=\Theta_o-1$. Next, we show that if truthful reporting dominates sending $\mu=\Theta_o+1$, then it dominates sending any $\mu>\Theta_o+1$:

$$\begin{aligned} & (K-S_o-S_p)\,\Delta+(\Theta_o-\mu)\left(\frac{K}{2}+\rho\right)-(2\rho+S_o+S_p)\,b \\ < \;& (K-S_o-S_p)\,\Delta-\left(\frac{K}{2}+\rho\right)-(2\rho+S_o+S_p)\,b\leq 0, \end{aligned}$$

where the first inequality follows from the strict monotonicity of the expression in μ, and the second inequality is the condition that truthful reporting dominates $\mu = \Theta_o + 1$. Hence, (97) is both the necessary and the sufficient condition for truth-telling of the sender representing optimists (under the assumption that the sender representing pessimists reports Θ_p truthfully).

Second, consider the sender representing pessimists. By the argument identical to the argument for the sender representing optimists, this sender finds it optimal to report Θ_p truthfully if and only if

$$(\Theta_p - \mu)\left[(2\rho + S_o + S_p)\, b + (K - S_o - S_p)\,\Delta - (\Theta_o - \mu)\left(\frac{K}{2} + \rho\right)\right] \leq 0$$

for all $\mu \in \{0, 1, ..., S_p\}$. Notice that this inequality holds automatically for all $\mu > \Theta_p$, since the first multiple of the expression is negative and the second multiple is positive. Thus, it is sufficient to consider deviations to $\mu \leq \Theta_p - 1$. In this case, $\Theta_p - \mu > 0$, so the above inequality is equivalent to

$$(2\rho + S_o + S_p)\, b + (K - S_o - S_p)\,\Delta \leq (\Theta_o - \mu)\left(\frac{K}{2} + \rho\right).$$

Since the left-hand side does not depend on μ and the right-hand side is strictly decreasing in μ, it is necessary and sufficient to verify that the inequality holds for $\mu = \Theta_p - 1$, in which case:

$$(2\rho + S_o + S_p)\, b + (K - S_o - S_p)\,\Delta \leq \frac{K}{2} + \rho.$$

Notice that both inequalities are identical to the conditions for existence of the truth-telling equilibrium in the basic model.

7.9.7 Proof of Proposition OA.6

Consider shareholder i who is O_ith to send the message. Let $B_i \equiv R_i \cap \{j \in S, O_j < O_i\}$ be the set of shareholders that are expected to communicate truthfully who communicate before shareholder i. Similarly, let $A_i \equiv R_i \cap \{j \in S, O_j > O_i\}$ be the set of shareholders that are expected to communicate truthfully who communicate after shareholder i. By definition, $|R_i| = |B_i| + |A_i|$.

Consider the incentive constraint of shareholder i. Given the message of shareholder i and the belief that shareholders in set R_i communicate truthfully, the manager's action is given by (7) and (8) for thruthful and non-truthful messages of shareholder i. Given that shareholder i already observes messages of shareholders in B_i, he has incentives to communicate truthfully if and only if

$$\sum_{\theta_{-i} \in \{0,1\}^{K-|B_i|-1}} \left[(a_m(\theta_{R_i}, \theta_i) - Z)^2 - (a_m(\theta_{R_i}, 1 - \theta_i) - Z)^2\right] P_i(\theta_{-(i,B_i)} | \theta_i, \theta_{B_i}) \leq 0, \quad (98)$$

where $\theta_{-(i,B_i)}$ refers to the set of all signals that exclude the signal of shareholder i and the signals of shareholders in set B_i, and θ_{B_i} refers to the set of all signals of shareholders in set B_i. Plugging (7) and (8) into (98) gives

$$0 \geq \sum_{\theta_{-(i,B_i)}} \left[2\theta_i - 1 + (K - |R_i| - 1) \cdot \frac{2\theta_i - 1}{\tau + |R_i| + 1}\right]$$
$$\times \left[2b + (1 - 2\theta_i) - 2 \sum_{j \in -R_i \setminus \{i\}} \theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1} (K - |R_i| - 1)\right] P_i(\theta_{-(i,B_i)}|\theta_i, \theta_{B_i}).$$

Note that the first multiple in each term equals $(2\theta_i - 1)\frac{\tau+K}{\tau+|R_i|+1}$. Thus, the above inequality is equivalent to

$$0 \geq (2\theta_i - 1) \sum_{\theta_{-(i,B_i)}} P_i(\theta_{-(i,B_i)}|\theta_i, \theta_{B_i}) \left(2b + (1 - 2\theta_i) - 2 \sum_{j \in -R_i \setminus \{i\}} \theta_j + \frac{2(\rho_m + \mathbf{1}_{R_i}) + 1}{\tau + |R_i| + 1} (K - |R_i| - 1)\right).$$

Since $\sum_{\theta_{-(i,B_i)}} \left(\sum_{j \in -R_i \setminus \{i\}} \theta_j\right) P_i(\theta_{-(i,B_i)}|\theta_i, \theta_{R_i}) = \frac{\rho_i + \mathbf{1}_{R_i} + \theta_i}{\tau + |R_i| + 1} (K - |R_i| - 1)$, we can further simplify it to

$$(2\theta_i - 1) \left[2b + (1 - 2\theta_i) + \frac{2(\rho_m - \rho_i) + 1 - 2\theta_i}{\tau + |R_i| + 1} (K - |R_i| - 1)\right] \leq 0.$$

Considering $\theta_i = 0$ and $\theta_i = 1$, we get the same IC constraint as in the base model, (10).