

# Financial Fragility <sup>1</sup>

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## **Abstract**

A financial system is fragile if a small event has a large effect. Sunspot equilibria, where the endogenous variables depend on extrinsic uncertainty, provide an extreme illustration. However, fundamental equilibria, where outcomes depend only on intrinsic uncertainty, can also be fragile. We study the relationship between sunspot equilibria and fundamental equilibria in a model of financial crises. The model has multiple equilibria, but only some of these are the limit of fundamental equilibria when the fundamental uncertainty becomes vanishingly small. We show that under certain conditions the only robust equilibria are those in which extrinsic uncertainty gives rise to asset price volatility and financial crises.

JEL Classification: D5, D8, G2

# 1 Excess sensitivity and sunspots

There are numerous historical examples of *financial fragility*, where events that are small in relation to the economy as a whole have a significant impact on the financial system. For example, Kindleberger (1978, pp. 107-108) argues that the immediate cause of a financial crisis

“... may be trivial, a bankruptcy, a suicide, a flight, a revelation, a refusal of credit to some borrower, some change of view which leads a significant actor to unload. Prices fall. Expectations are reversed. The movement picks up speed. To the extent that speculators are leveraged with borrowed money, the decline in prices leads to further calls on them for margin or cash, and to further liquidation. As prices fall further, bank loans turn sour, and one or more mercantile houses, banks, discount houses, or brokerages fail. The credit system itself appears shaky and the race for liquidity is on.”

More recently, Kaminsky and Schmukler (1999) have documented the difficulty of identifying the changes in fundamentals that triggered changes in asset prices during the Asian crises of 1997. This paper uses a simple version of the general equilibrium model introduced in Allen and Gale (2000a), henceforth AG, to investigate how small events can have large consequences that result in a financial crisis.

We model liquidity preference in the standard way by assuming that consumers have stochastic time preferences. There are three dates,  $t = 0, 1, 2$ , and all consumption occurs at dates 1 and 2. There are two types of consumers, *early consumers*, who only value consumption at date 1 and *late consumers*, who only value consumption at date 2. Consumers are identical at date 0 and learn their true type, “early” or “late”, at the beginning of date 1.

There are two types of investments in this economy, short-term investments, which yield a return after one period and long-term investments, which take two periods to mature. There is a trade-off between liquidity and returns: long-term investments have a higher yield but take longer to mature.

Banks are modeled as risk-sharing institutions that pool consumers’ endowments and invest them in a portfolio of long- and short-term investments.

In exchange for their endowments, banks give consumers a deposit contract that allows a consumer the option of receiving either a fixed amount of consumption at date 1 or a fixed amount of consumption at date 2. This provides individual consumers with insurance against liquidity shocks by intertemporally smoothing the returns paid to depositors. There is free entry into the banking sector: in a competitive equilibrium banks must maximize the expected utility of depositors in order to attract customers.

As a benchmark, we first consider an economy in which there is no aggregate uncertainty. More precisely, we assume that both asset returns and the total number of early consumers in the economy are non-stochastic. Despite the absence of aggregate uncertainty, individual banks can be uncertain about the number of early consumers among their depositors and, hence, about their demand for liquidity at date 1. The asset market plays a crucial role in providing liquidity at date 1. Banks that have an above average demand for liquidity can sell long-term assets to banks that have a lower than average demand. Because the number of early consumers is constant across the entire economy, the aggregate demand for and supply of liquidity are in balance and the asset market serves to reallocate liquidity among banks as necessary.

The asset market is important because it integrates the financial system and allows banks to share liquidity. However, it is also a source of financial fragility. To see this, we have to understand the relationship between asset prices and the bank's decision to sell assets. If the asset price is high, the bank can meet its liquidity needs by selling a small proportion of its long-term assets. As the asset price falls, the bank has to sell a larger proportion of its long-term assets. In extreme cases, the asset price is so low that the bank cannot meet its commitments and defaults. At that point it is forced to liquidate all its long-term assets. This "backward bending" supply curve for long-term assets explains why the asset market can clear at more than one price. The asset price and the quantity of long-term assets supplied move in opposite directions, so that the *value* of assets supplied does not change very much.

The existence of multiple, market-clearing, asset prices is the key to our explanation of endogenous financial crises. By a crisis we mean a profound drop in the value of asset prices which affects the solvency of a large number of banks and their ability to meet their commitments to their depositors. If the price movement is large enough some banks are forced into liquidation, but there may also be crises in which banks avoid default, although their

balance sheet is under extreme pressure.

We distinguish equilibria in several ways. In the first place, we distinguish *fundamental equilibria*, in which the endogenous variables are functions of the exogenous primitives or fundamentals of the model (endowments, preferences, technologies) from *sunspot equilibria*, in which endogenous variables may be influenced by extraneous variables that have no direct impact on fundamentals. In a fundamental equilibrium, a crisis is driven by exogenous shocks to fundamentals, such as asset returns or liquidity demands. In the absence of aggregate real shocks, asset prices are non-stochastic and a crisis cannot arise. In a sunspot equilibrium, by contrast, asset prices fluctuate in the absence of aggregate real shocks and crises appear to occur spontaneously.

So far, we have suggested there might be multiple equilibria, only some of which are characterized by crises. We want to go further and suggest that some equilibria are more *robust* than others. To test for robustness we perturb the benchmark economy by introducing a small amount of aggregate uncertainty. Specifically, we assume there is a small amount of uncertainty about the total number of early consumers in the economy. However small the amount of aggregate uncertainty, the equilibria of the perturbed economy always exhibit crises with positive probability. Furthermore, the probability of a crisis is bounded away from zero as the aggregate uncertainty becomes vanishingly small. Thus, in a robust equilibrium of the limit economy there must be extrinsic uncertainty. The fundamental equilibrium of the limit economy is not robust.<sup>1</sup>

These results help us understand the relationship between two traditional views of financial crises. One is that they are spontaneous events, unrelated to changes in the real economy. Historically, banking panics were attributed to “mob psychology” or “mass hysteria” (see, e.g., Kindleberger (1978)). The modern theory explains banking panics as equilibrium coordination failures (Bryant (1980), Diamond and Dybvig (1983)). An alternative view is that financial crises are a natural outgrowth of the business cycle (Gorton (1988), Calomiris and Gorton (1991), Calomiris and Mason (2000), Allen and Gale (1998, 2000a-d)). The formal difference between these two views is whether a crisis is generated by intrinsic or extrinsic uncertainty. *Intrinsic uncertainty* is caused by stochastic fluctuations in the primitives or fundamentals of the

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<sup>1</sup>The limit economy has two types of equilibria with extrinsic uncertainty. In a trivial sunspot equilibrium, prices are random but the allocation is essentially the same as in the fundamental equilibrium and no banks default. In a non-trivial sunspot equilibrium, the equilibrium allocation is random as well as the prices.

economy. Examples would be exogenous shocks that effect liquidity preferences or asset returns. *Extrinsic uncertainty*, often referred to as sunspots, by definition has no effect on the fundamentals of the economy.<sup>2,3</sup>

Our model combines the most attractive features of both traditional approaches. Like the sunspot approach, it produces large effects from small shocks. Like the real business cycle approach, it makes a firm prediction about the conditions under which crises will occur.<sup>4</sup> However, it is important to distinguish the present model of systemic or economy-wide *crises* from models of individual bank runs or *panics*. Panics are the result of coordination failure (Bryant (1980), Diamond and Dybvig (1983)). If late consumers expect a run, it is optimal for them to join it; if late consumers do not expect a run, it is optimal for them to wait until date 2 to withdraw. Whether a run occurs depends entirely on the depositors' expectations, not on the value of the bank's assets. Furthermore, a run on a single bank can occur independently of what is happening to other banks.

In the crisis model, by contrast, default only occurs if it is unavoidable, that is, the value of the bank's portfolio is too low to allow the bank to meet its commitments.<sup>5</sup> Furthermore, default occurs as part of a general crisis. Banks fail because asset prices are too low and asset prices are low because banks are selling assets when liquidity is scarce. From the point of view of a single, price-taking bank, default results from an exogenous shock to asset

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<sup>2</sup>Strictly speaking, much of the banking literature exploits multiple equilibria without addressing the issue of sunspots. We adopt the sunspots framework here because it encompasses the standard notion of equilibrium and allows us to address the issue of equilibrium selection.

<sup>3</sup>The theoretical analysis of sunspot equilibria began with the seminal work of Cass and Shell (1982) and Azariadis (1981), which gave rise to two streams of literature. The Cass-Shell paper is most closely related to work in a Walrasian, general-equilibrium framework; the Azariadis paper is most closely related to the macroeconomic dynamics literature. For a useful survey of applications in macroeconomics, see Farmer (1999); for an example of the current literature in the general equilibrium framework see Gottardi and Kajii (1995, 1999).

<sup>4</sup>Models of sunspot phenomena typically have many equilibrium, including as a special case the fundamental equilibria in which extrinsic uncertainty has no effect on endogenous variables. Thus, financial fragility remains a possibility but not a necessity.

<sup>5</sup>This is a refinement of the equilibrium concept. We assume that late consumers withdraw at the last date whenever it is incentive compatible for them to do so. Bank runs occur only when it is impossible for the bank to meet its obligations in an incentive-compatible way. Such runs are called essential in AG to distinguish them from the coordination failures in Diamond and Dybvig (1983).

prices. From the point of view of the banking sector as a whole, the “shock” is endogenous. But note that only the simultaneous failure of a non-negligible number of banks can affect asset prices.

Our work is related to the wider literature on general equilibrium with incomplete markets (GEI). As is well known, sunspots do not matter when markets are complete. (For a precise statement, see Shell and Goenka, (1997)). The incompleteness in our model reveals itself in two ways. First, sunspots are assumed to be non-contractible, that is, the deposit contract is not explicitly contingent on the sunspot variable. In this respect we are simply following the incomplete contracts literature (see, for example, Hart (1995)). Secondly, there are no markets for Arrow securities contingent on the sunspot variable, so financial institutions cannot insure themselves against asset price fluctuations associated with the sunspot variable. This is the standard assumption of the GEI literature (see, for example, Geanakoplos (1990) or Magill and Quinzii (1996)).

There is a small but growing literature related to financial fragility. Financial multipliers were introduced by Bernanke and Gertler (1989). In the model of Kiyotaki and Moore (1997), the impact of illiquidity at one link in the credit chain has a large impact further down the chain. Chari and Kehoe (2000) show that herding behavior can cause a small information shock to have a large effect on capital flows.

The rest of the paper is organized as follows. Section 2 contains the basic assumptions of the model. Section 3 describes the optimal contracts offered by banks and the rules governing default and liquidation. Section 4 defines equilibrium. A few special cases of the model are considered in Section 5. Section 6 characterizes the equilibria of the limit economy, in which there is no aggregate uncertainty. The analysis of equilibrium in the perturbed economy is contained in Section 7. Here we show that, in any equilibrium of the perturbed economy, crises occur with positive probability. We also show that the limit of a sequence of equilibria corresponding to a sequence of perturbed economies is an equilibrium in the limit economy and we characterize the limit equilibria. A series of numerical examples are presented in Section 8 to illustrate the properties of the model. Some readers may find it advantageous to read this after Section 5. Section 9 contains concluding remarks. Proofs are gathered in Section 10.

## 2 Assets and preferences

The model we use is a special case of AG.

*Dates.* There are three dates  $t = 0, 1, 2$  and a single good at each date. The good can be used for consumption or investment.

*Assets.* There are two assets, a short-term asset (the *short* asset) and a long-term asset (the *long* asset).

- The short asset is represented by a storage technology. Investment in the short asset can take place at date 1 or date 2. One unit of the good invested at date  $t$  yields one unit at date  $t + 1$ , for  $t = 0, 1$ .
- The long asset takes two periods to mature and is more productive than the short asset. Investment in the long asset can only take place at date 0. One unit invested at date 0 produces  $r > 1$  units at date 2.

*Consumers.* There is a continuum of ex ante identical consumers, whose measure is normalized to unity. Each consumer has an endowment  $(1, 0, 0)$  consisting of one unit of the good at date 0 and nothing at subsequent dates. There are two (ex post) types of consumers at date 1, *early consumers*, who only value consumption at date 1, and *late consumers*, who only value consumption at date 2. If  $\eta$  denotes the probability of being an early consumer and  $c_t$  denotes consumption at date  $t = 1, 2$ , the consumer's ex ante utility is

$$u(c_1, c_2; \eta) = \eta U(c_1) + (1 - \eta)U(c_2).$$

The period utility function  $U : \mathbf{R}_+ \rightarrow \mathbf{R}$  is twice continuously differentiable and satisfies the usual neoclassical properties,  $U'(c) > 0$ ,  $U''(c) < 0$ , and  $\lim_{c \searrow 0} U'(c) = \infty$ .

*Uncertainty.* There are three sources of intrinsic uncertainty in the model. First, each individual consumer faces idiosyncratic uncertainty about her preference type (early or late consumer). Secondly, each bank faces idiosyncratic uncertainty about the number of early consumers among the bank's depositors. For example, different banks could be located in regions subject to independent liquidity shocks. Thirdly, there is aggregate uncertainty about the fraction of early consumers in the economy. Aggregate uncertainty



is represented by a state of nature  $\theta$ , a non-degenerate random variable with a finite support and a density function  $f(\theta)$ . The bank's idiosyncratic shock is represented by a random variable  $\alpha$ , with finite support and a density function  $g(\alpha)$ . The probability of being an early consumer at a bank in state  $(\alpha, \theta)$  is denoted by  $\eta(\alpha, \theta)$ , where

$$\eta(\alpha, \theta) \equiv \alpha + \varepsilon\theta$$

and  $\varepsilon \geq 0$  is a constant. We adopt the usual "law of large numbers" convention and assume that the fraction of early consumers at a bank in state  $(\alpha, \theta)$  is identically equal to the probability  $\eta(\alpha, \theta)$ . The economy-wide average of  $\alpha$  is assumed to be constant and equal to the mean  $\bar{\alpha} = \sum \alpha g(\alpha)$ . Thus, there is aggregate intrinsic uncertainty only if  $\varepsilon > 0$ .

*Information.* All uncertainty is resolved at date 1. The true value of  $\theta$  is publicly observed,<sup>6</sup> the true value of  $\alpha$  for each bank is publicly observed, and each consumer learns his type, i.e., whether he is an early consumer or a late consumer.

*Markets.* There are no markets for hedging against aggregate uncertainty at date 0, for example, there are no Arrow securities contingent on the state of nature  $\theta$ . At date 1, there is a market in which future (date-2) consumption can be exchanged for present (date-1) consumption. If  $p(\theta)$  denotes the price of one unit of future consumption at date 2 in terms of present consumption at date 1, then one unit of the long asset is worth  $p(\theta)r$  at date 1 in state  $\theta$ .

Markets are incomplete at date 0 but complete at date 1. We assume that market participation is incomplete: financial institutions such as banks can participate in the asset market at date 1 but individual consumers cannot.<sup>7</sup>

### 3 Banking

Banks are financial institutions that provide investment and liquidity services to consumers. They do this by pooling the consumers' resources, investing them in a portfolio of short- and long-term assets, and offering consumers

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<sup>6</sup>It is not strictly necessary to assume that  $\theta$  is observed. In equilibrium, all that agents need to know is the equilibrium price  $p(\theta)$ , which may or may not reveal  $\theta$ .

<sup>7</sup>As Cone (1983) and Jacklin (1986) showed, if consumers have access to the capital market, it is impossible for banks to offer risk sharing that is superior to the market.

future consumption streams with a better combination of asset returns and liquidity than individual consumers could achieve by themselves. Banks also have access to the interbank capital market, from which consumers are excluded.

Banks compete by offering deposit contracts to consumers in exchange for their endowments and consumers respond by choosing the most attractive of the contracts offered. Free entry ensures that banks earn zero profits in equilibrium. The deposit contracts offered in equilibrium must maximize consumers' welfare subject to the zero-profit constraint. Otherwise, a bank could enter and make a positive profit by offering a more attractive contract.

Anything a consumer can do, the bank can do. So there is no loss of generality in assuming that consumers deposit their entire endowment in a bank at date 0.<sup>8</sup> The bank invests  $y$  units per capita in the short asset and  $1 - y$  units per capita in the long asset and offers each consumer a deposit contract, which allows the consumer to withdraw either  $d_1$  units at date 1 or  $d_2$  units at date 2. Without loss of generality, we set  $d_2 = \infty$ . This ensures that consumers receive the residue of the bank's assets at date 2 which must be optimal given banks are maximizing the expected utility of depositors. Then the deposit contract is characterized by the middle-period payment

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<sup>8</sup>This is not simply an application of the Modigliani-Miller theorem. The consumer may do strictly better by putting all his "eggs" in the bank's "basket". Suppose that the deposit contract allows the individual to hold  $m$  units in the safe asset and deposit  $1 - m$  units in the bank. The bank invests  $y$  in the short asset and  $1 - m - y$  in the long asset. If the bank does not default at date 1, the early consumers receive  $d + m$  and the late consumers receive

$$\frac{p(\theta)r(1 - m - y) + y - \eta(\theta)d}{(1 - \eta(\theta))p(\theta)} + m.$$

If the bank defaults, early and late consumers receive

$$p(\theta)r(1 - m - y) + y + m.$$

Suppose that  $m > 0$  and consider a reduction in  $m$  and an increase in  $y$  and  $d$  of the same amount. It is clear that the early consumers' consumption is unchanged. So is the late consumers' consumption if the bank defaults. The change in the late consumers' consumption when the bank does not default is

$$\frac{\Delta y - \eta(\theta)\Delta d}{(1 - \eta(\theta))p(\theta)} + \Delta m = \frac{-\Delta m}{p(\theta)} + \Delta m \geq 0$$

because  $p(\theta) \leq 1$  and  $\Delta m < 0$ . Thus, it is optimal for the bank to choose  $m = 0$ .

$d_1 \equiv d$ .

Given  $p(\theta)$  denotes the price of date 2 consumption at date 1 in state  $\theta$ , then the value of the bank's assets at date 1 is  $y + p(\theta)r(1 - y)$ .

A consumer's type is private information. An early consumer cannot misrepresent his type because he needs to consume at date 1; but a late consumer can claim to be an early consumer, withdraw  $d$  at date 1, store it until date 2 and then consume it. The deposit contract is *incentive compatible* if and only if the residual payment to late consumers at date 2 is at least  $d$ . Since the late consumers are residual claimants at date 2, it is possible to give them at least  $d$  units of consumption if and only if

$$\eta(\alpha, \theta)d + (1 - \eta(\alpha, \theta))p(\theta)d \leq y + p(\theta)r(1 - y). \quad (1)$$

The left hand side is a lower bound for the present value of consumption when early consumers are given  $d$  and late consumers are given at least  $d$ . The right hand side is the value of the portfolio. Thus, condition (1) is necessary and sufficient for the deposit contract  $d$  to satisfy incentive compatibility and the budget constraint simultaneously. We often refer to the inequality in (1) as the *incentive constraint* for short.

In what follows, we assume that bank runs occur only if they are unavoidable. In other words, late consumers will withdraw at date 2 as long as the bank can satisfy the incentive constraint. If (1) is violated, then all consumers will want to withdraw at date 1. In the event of bankruptcy, the bank is *required* to liquidate its assets in an attempt to provide the promised amount  $d$  to the consumers who withdraw at date 1. Whatever withdrawal decisions consumers make, the consumers who withdraw at date 2 will receive less than the consumers who withdraw at date 1. Hence, in equilibrium, all consumers must withdraw at date 1. Then each consumer receives the liquidated value of the portfolio  $y + p(\theta)r(1 - y)$ .

Let  $x_t(d, y, \alpha, \theta)$  denote the consumption at date  $t$  if the bank chooses  $(d, y)$  and the bank is in state  $(\alpha, \theta)$  at date 1. Let  $x = (x_1, x_2)$ , where

$$x_1(d, y, \alpha, \theta) = \begin{cases} d & \text{if (1) is satisfied} \\ y + p(\theta)r(1 - y) & \text{otherwise,} \end{cases}$$

$$x_2(d, y, \alpha, \theta) = \begin{cases} \frac{y + p(\theta)r(1 - y) - \eta d}{(1 - \eta)p(\theta)} & \text{if (1) is satisfied} \\ y + p(\theta)r(1 - y) & \text{otherwise,} \end{cases}$$

and  $\eta = \eta(\alpha, \theta)$ . Using this notation, the bank's decision problem can be written as

$$\begin{aligned} \max \quad & E[u(x(d, y, \alpha, \theta), \eta(\alpha, \theta))] \\ \text{s.t.} \quad & 0 \leq d, 0 \leq y \leq 1. \end{aligned} \tag{DP1}$$

An ordered pair  $(d, y)$  is *optimal* for the given price function  $p(\cdot)$  if it solves (DP1).

## 4 Equilibrium

The bank's decision problem is "non-convex". To ensure the existence of equilibrium, we take advantage of the convexifying effect of large numbers and allow for the possibility that ex ante identical banks will choose different deposit contracts  $d$  and portfolios  $y$ . Each consumer is assumed to deal with a single bank and each bank offers a single contract. In equilibrium, consumers will be indifferent between banks offering different contracts. Consumers allocate themselves to different banks in proportions consistent with equilibrium.

To describe an equilibrium, we need some additional notation. A partition of consumers at date 0 is defined by an integer  $m < \infty$  and an array  $\rho = (\rho_1, \dots, \rho_m)$  of numbers  $\rho_i \geq 0$  such that  $\sum_{i=1}^m \rho_i = 1$ . Consumers are divided into  $m$  groups and each group  $i$  contains a measure  $\rho_i$  of consumers. We impose an arbitrary bound  $m$  on the number of groups to rule out pathological cases.<sup>9</sup> The banks associated with group  $i$  offer a deposit contract  $d_i$  and a portfolio  $y_i$ , both expressed in per capita terms. An *allocation* consists of a partition  $(m, \rho)$  and an array  $(d, y) = \{(d_i, y_i)\}_{i=1}^m$  such that  $d_i \geq 0$  and  $0 \leq y_i \leq 1$  for  $i = 1, \dots, m$ .

To define the market clearing conditions we need some additional notation. Let  $\bar{x}(d_i, y_i, \alpha, \theta) = (\bar{x}_1(d_i, y_i, \alpha, \theta), \bar{x}_2(d_i, y_i, \alpha, \theta))$  denote the bank's demand for goods. If the bank is solvent then a fraction  $\eta(\alpha, \theta)$  get paid  $x_1(d_i, y_i, \alpha, \theta)$  at date 1 and a fraction  $(1 - \eta(d_i, y_i, \alpha, \theta))$  get paid  $x_2(d_i, y_i, \alpha, \theta)$  at date 2. Then the bank's total demand for goods is given by

$$\bar{x}(d_i, y_i, \alpha, \theta) = (\eta(\alpha, \theta)x_1(d_i, y_i, \alpha, \theta), (1 - \eta(d_i, y_i, \alpha, \theta))x_2(d_i, y_i, \alpha, \theta)).$$

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<sup>9</sup>In general, two groups are sufficient for existence of equilibrium.

If the bank is bankrupt, on the other hand, then everyone gets paid  $x_1(d_i, y_i, \alpha, \theta) = x_2(d_i, y_i, \alpha, \theta)$  at date 1 and the bank's total demand for goods is

$$\bar{x}(d_i, y_i, \alpha, \theta) = (x_1(d_i, y_i, \alpha, \theta), 0).$$

An allocation  $(m, \rho, d, y)$  is *attainable* if it satisfies the market-clearing conditions

$$\sum_i \rho_i E [\bar{x}_1(d_i, y_i, \alpha, \theta)] \leq \sum_i \rho_i y_i, \quad (2)$$

and

$$\begin{aligned} \sum_i \rho_i \{E [\bar{x}_1(d_i, y_i, \alpha, \theta) + \bar{x}_2(d_i, y_i, \alpha, \theta)]\} \\ = \sum_i \rho_i \{y_i + r(1 - y_i)\}, \end{aligned}$$

for any state  $\theta$ . In the market-clearing conditions, we take expectations with respect to  $\alpha$  because the cross-sectional distribution of idiosyncratic shocks is assumed to be the same as the probability distribution. The first inequality says that the total demand for consumption at date 1 is less than or equal to the supply of the short asset. The inequality may be strict, because an excess supply of liquidity can be re-invested in the short asset and consumed at date 2. The second condition says that total consumption at date 2 is equal to the return from the investment in the long asset *plus* the amount invested in the short asset at date 1, which is the difference between the left and right hand sides of (2).

In equilibrium, it must be the case that  $p(\theta) \leq 1$ . Otherwise banks could make an arbitrage profit at date 1 by selling goods forward and investing the proceeds in the short asset. If  $p(\theta) < 1$  then no one is willing to invest in the short asset at date 1 and (2) must hold as an equation. A price function  $p(\cdot)$  is *admissible* (for the given allocation) if it satisfies the following complementary slackness condition:

$$\begin{aligned} \text{For any state } \theta, p(\theta) \leq 1 \text{ and } p(\theta) < 1 \text{ implies that (2)} \\ \text{holds as an equation.} \end{aligned}$$

Now we are ready to define an equilibrium.

An *equilibrium* consists of an attainable allocation  $(m, \rho, d, y)$  and an admissible price function  $p(\cdot)$  such that, for every group  $i = 1, \dots, m$ ,  $(d_i, y_i)$  is optimal given the price function  $p(\cdot)$ .

## 5 A first look at equilibrium

### 5.1 Autarkic equilibria

In order to illustrate the model and its properties, we begin with some special cases. Consider first the case in which there is no aggregate uncertainty ( $\varepsilon = 0$ ) and banks receive no idiosyncratic shocks ( $\alpha$  is a constant). In this case banks have no need to rely on the asset market to provide liquidity. Autarky is optimal. To see this, consider the problem faced by a central planner. In the absence of aggregate uncertainty, optimal risk sharing requires the consumption allocation to be non-stochastic. An early consumer receives  $c_1$  and a late consumer receives  $c_2$ , where  $c_1$  and  $c_2$  are constants. The total demand for consumption is  $\alpha c_1$  at date 1 and  $(1 - \alpha)c_2$  at date 2. The planner provides consumption at each date by holding the asset with the highest return. Thus, consumption at date 1 is provided by holding the short asset and consumption at date 2 is provided by holding the long asset. So the planner chooses  $y$  to satisfy  $\alpha c_1 = y$  and  $(1 - \alpha)c_2 = r(1 - y)$ . In the first best (i.e., ignoring the incentive constraint), the allocation  $(c_1, c_2, y)$  is chosen to solve the following problem:

$$\begin{aligned} \max \quad & \alpha U(c_1) + (1 - \alpha)U(c_2) \\ \text{s.t.} \quad & 0 \leq y \leq 1 \\ & \alpha c_1 = y \\ & (1 - \alpha)c_2 = r(1 - y). \end{aligned}$$

This problem uniquely determines  $(c_1, c_2, y)$ . Note that the optimal consumption allocation must satisfy the first-order condition

$$U'(c_1) = rU'(c_2),$$

which implies that  $c_1 < c_2$ , so the incentive constraint is strictly satisfied. This means that the incentive-efficient and Pareto-efficient allocations are identical.

Now consider the case of the bank. On the one hand, the bank cannot do better than the first best. On the other hand, it can achieve the first best by choosing the same portfolio  $y$  and setting  $d = c_1$ . To show that this is an equilibrium, we only need to find a price that makes the portfolio choice optimal at date 0 and clears the market at date 1. The first order condition for (DP1) for the choice of  $y$  given that the incentive constraint is satisfied,

there is no default and  $\varepsilon = 0$ , is

$$EU'(c_2) \left[ \frac{1 - p(\theta)r}{p(\theta)} \right] = 0. \quad (3)$$

Since there is no aggregate uncertainty, it is natural to assume that there is a non-stochastic price  $p(\theta) = \bar{p}$  at date 1. It then follows from (3) that

$$\bar{p} = \frac{1}{r}.$$

At this price, one unit invested in either asset at date 0 yields one unit at date 1. The banks will hold only the long asset between date 1 and date 2 because at this price the yield on the short asset, one, is dominated by the return on the long asset,  $r$ . Thus,  $(d, y, \bar{p})$  defines an equilibrium.

The equilibrium is special in several ways. First, there is no aggregate uncertainty, intrinsic or extrinsic. Secondly, the allocation is Pareto-efficient. Finally, the market plays no role apart from determining a market-clearing price. Banks can achieve the first best while remaining autarkic.

Autarky is not actually necessary for equilibrium: since banks are indifferent between the two assets at date 0, they may choose to hold different portfolios initially and then use the asset market at date 1 to reshuffle their liquidity holdings. Thus, an attainable allocation  $(\rho, m, d, y)$  is an equilibrium allocation for the price  $\bar{p}$  if  $d_i = c_1$  for all  $i$  and the market-clearing condition

$$\sum_i \rho_i \alpha d_i = \alpha c_1 = \sum_i \rho_i y_i$$

is satisfied. So there is a continuum of equilibria differing in the portfolios chosen but identical in terms of consumption and consumer welfare.

One further point we can make using this special case is to show how random prices can arise in equilibrium. Consider the symmetric equilibrium in which all banks choose the same portfolio  $y_i = \alpha c_1$  and suppose that in place of the constant price  $\bar{p}$  we choose a price function  $p(\theta)$ . In order to maintain equilibrium, we need to satisfy two market-clearing conditions. First, we have to make banks willing to hold the portfolio  $y$  between date 0 and date 1 and, secondly, we have to make the banks willing to hold (only) the long asset between date 1 and date 2. It follows from (3) that the first condition is satisfied if

$$E \left[ \frac{1}{p(\theta)} \right] = r. \quad (4)$$

The second condition is satisfied if  $p(\theta) \leq 1$ . There are clearly many price functions that will satisfy these conditions. The uncertainty about prices that can arise in equilibrium is an example of extrinsic uncertainty or sunspot activity. This is a trivial example because the sunspot activity has no effect on the equilibrium allocation.

Note that the construction of the trivial sunspot equilibrium requires that the banks all hold the same amount of the short asset  $y_i = \alpha d_i$ . It is only because banks are autarkic that price fluctuations have no impact on the equilibrium allocation. For this reason, although there is a continuum of trivial sunspot equilibrium, differing in the equilibrium price function, they have identical allocations, including the choice of portfolio.

## 5.2 Non-autarkic equilibria

In the preceding example, markets could be used in equilibrium, but they played no essential role in achieving the first-best outcome. To illustrate a less trivial role for markets, consider the case where there is again no aggregate uncertainty ( $\varepsilon = 0$ ) but banks face idiosyncratic risk ( $\alpha$  is a non-degenerate random variable). Once again, because there is no aggregate uncertainty, it is natural to consider an equilibrium in which the price of future consumption  $\bar{p}$  is non-stochastic. As before, equilibrium requires  $\bar{p} = 1/r$ . If a bank chooses  $(d, y)$  at date 0 the incentive constraint at date 1 is

$$\alpha d + \bar{p}(1 - \alpha)d \leq y + \bar{p}r(1 - y) = 1.$$

Since the value of  $\alpha$  is uncertain, the bank may well choose  $(d, y)$  so that there is a positive probability of default at date 1. The avoidance of default requires the bank to lower  $d$  or increase  $y$  and either change will be costly in terms of the consumers' expected utility. For example, if the proportion of early consumers is low ( $\alpha = \alpha_L$ ), with high probability and high ( $\alpha = \alpha_H$ ), with low probability, then the choice of  $d$  will be close to what it would be in an equilibrium with a constant  $\alpha = \alpha_L$ . This value of  $d$  satisfies  $\alpha_L d + \bar{p}(1 - \alpha_L)d \leq 1$  but it may be that  $\alpha_H d + \bar{p}(1 - \alpha_H)d > 1$ . If the probability of  $\alpha_H$  is not too great, it is optimal for the bank to default in that state, rather than distort its decisions in the more likely state  $\alpha = \alpha_L$ . Default in this case is a means of introducing greater flexibility into the risk sharing contract.

While uncertainty about liquidity preference, as measured by  $\alpha$ , is a plausible reason for default, it is not our main interest here. So, in what follows,



we assume the parameters of the model are such that it is never optimal for the bank to choose default in the equilibrium with the non-stochastic price  $\bar{p}$  and in the absence of aggregate uncertainty. Then the bank's choice of  $(d, y)$  will always satisfy the incentive constraint and the decision problem can, without loss of generality, be written as follows:

$$\begin{aligned} \max \quad & E \left[ \alpha U(d) + (1 - \alpha) U \left( r \frac{1 - \alpha d}{1 - \alpha} \right) \right] \\ \text{s.t.} \quad & \alpha d + \bar{p}(1 - \alpha)d \leq 1. \end{aligned}$$

In the absence of default, the early consumers receive  $d$  and the late consumers receive the remainder. Since the value of the portfolio is  $y + \bar{p}r(1 - y) = 1$  at date 1 the late consumers receive  $1 - \alpha d$  in present value terms, which is worth  $r(1 - \alpha d)/(1 - \alpha)$  per capita at date 2. This explains the objective function of the bank. We can add the incentive constraint without loss of generality because we have assumed that it is not optimal to allow default. The value of  $d$  that solves this problem must satisfy the first order condition

$$E \left[ \alpha U'(d) - \alpha r U' \left( r \frac{1 - \alpha d}{1 - \alpha} \right) \right] \geq 0$$

with equality if the incentive constraint is not binding. Once again, the deposit contract is uniquely determined by this condition, but the choice of portfolio is not. Since banks are indifferent between the two assets at date 0, they can hold any amount of the short asset as long as the market clearing condition is satisfied at date 1. Whatever amount of the short asset they hold, they need to use markets to provide liquidity for their depositors. Because the bank's demand for liquidity  $\alpha d$  is random, there is no way the bank can remain autarkic and achieve optimal risk sharing.

Precisely because all banks rely in a non-trivial way on the market for liquidity, a change in price will have real effects on the consumption allocation available to the bank's depositors. If sunspot equilibria occur, they have real effects.

### 5.3 Aggregate intrinsic uncertainty

The third case we want to look at is one in which there is idiosyncratic uncertainty at the level of the banks ( $\alpha$  random) and a small amount of aggregate (intrinsic) uncertainty, that is,  $\varepsilon > 0$ . Somewhat surprisingly, it turns out that introducing a small amount of aggregate uncertainty leads

to a big change in the properties of the equilibrium set. In particular, the “natural” equilibrium of the limit economy, in which there is no aggregate uncertainty in prices or allocation, has no counterpart here. The first thing we note is that, unlike in the limit economy, in any equilibrium there must be a positive probability of default. To see this, suppose that, contrary to what we want to show, there does exist an equilibrium in which the banks never default. Then the market clearing condition at date 1 implies

$$(\bar{\alpha} + \varepsilon\theta) \sum_i \rho_i d_i \leq \sum_i \rho_i y_i,$$

for every value of  $\theta$ , where  $\bar{\alpha}$  is the mean of  $\alpha$ . If the inequality is strict there is an excess supply of liquidity and  $p(\theta) = 1$ . The right hand side represents the supply of liquidity, which is inelastic and independent of  $\theta$ . The left hand side represents the demand for liquidity, which is inelastic and linearly dependent on  $\theta$ . Then if  $\varepsilon > 0$  and the distribution of  $\theta$  is not degenerate, the inequality must be strict at least with positive probability. This means that  $p(\theta) = 1$  with positive probability. Clearly, we cannot have  $p(\theta) = 1$  with probability one because this would violate condition (3). In fact,  $p(\theta) = 1$  with probability one implies that the return on the long asset between date 0 and date 1 is  $p(\theta)r > 1$ , so the short asset is dominated,  $y_i = 0$ , and the market clearing condition cannot be satisfied at date 1. In order to satisfy condition (3) with a positive supply of liquidity at date 1 we must have  $p(\theta) < 1/r$  with positive probability. To sum up, either default occurs with positive probability in equilibrium or there is non-negligible price asset-price volatility.

Thus, introducing a small amount of uncertainty changes the properties of equilibrium. Notice that these properties hold for every value of  $\varepsilon > 0$  however small. It is not surprising that a large amount of aggregate uncertainty leads to price volatility and default. What these examples suggest is that the introduction of a small amount of aggregate uncertainty leads to a sharp change in the nature of equilibrium. In other words, at least some of the equilibria of the limit economy with  $\varepsilon = 0$  are not robust. To see which equilibria are robust, we need to characterize the equilibrium set of the limit economy. We do this in the next section.

## 6 Equilibrium in the limit

In this section we characterize the equilibria of the limit economy in which  $\varepsilon = 0$ . There is no aggregate intrinsic uncertainty in the model, but there may still be aggregate extrinsic uncertainty (sunspots). We first classify equilibria in the limit economy according to the impact of extrinsic uncertainty. An equilibrium  $(m, \rho, d, y, p)$  in the limit economy is a *fundamental equilibrium* (FE) if  $x(d_i, y_i, \alpha, \theta)$  is almost surely constant for each  $i$  and  $\alpha$  and if  $p(\theta)$  is almost surely constant. In that case, the sunspot variable  $\theta$  has no influence on the equilibrium values. An equilibrium  $(m, \rho, d, y, p)$  of the limit economy is a *trivial sunspot equilibrium* (TSE) if  $x(d_i, y_i, \alpha, \theta)$  is constant for each  $i$  and  $\alpha$  and  $p(\theta)$  is not constant. In this case, the sunspot variable  $\theta$  has no effect on the allocation of consumption but it does affect the equilibrium price  $p(\theta)$ . An equilibrium  $(m, \rho, d, y, p)$  which is neither a FE nor a TSE is called a *non-trivial sunspot equilibrium* (NTSE), that is, a NTSE is an equilibrium in which the sunspot variable has some non-trivial impact on the allocation of consumption.

We saw examples of these different kinds of equilibria in Section 5. In the economy with no aggregate uncertainty ( $\varepsilon = 0$ ) and no idiosyncratic uncertainty for banks ( $\alpha$  constant), the FE has a non-stochastic price  $\bar{p}$  and a non-stochastic consumption allocation  $(c_1, c_2)$ . For the same economy, we also saw that there are many TSE, in which the real allocation is the same as in the FE but the equilibrium price  $p(\theta)$  varies with  $\theta$ . By contrast, in the economy with no aggregate uncertainty ( $\varepsilon = 0$ ) and idiosyncratic uncertainty for banks ( $\alpha$  is not a constant), the banks must use the market to obtain liquidity so fluctuations in asset prices do have real effects. In other words, any sunspot equilibrium is a NTSE.

We can also classify equilibria according to the variety of choices made by different groups of banks. An equilibrium  $(m, \rho, d, y, p)$  is *pure* if each group of banks makes the same choice:

$$(d_i, y_i) = (d_j, y_j), \forall i, j = 1, \dots, m.$$

An equilibrium  $(m, \rho, d, y, p)$  is *semi-pure* if the consumption allocations are the same for each group of banks:

$$x(d_i, y_i, \alpha, \theta) = x(d_j, y_j, \alpha, \theta), \forall i, j = 1, \dots, m.$$

Otherwise,  $(m, \rho, d, y, p)$  is a *mixed* equilibrium where different types of bank provide different allocations (but the same ex ante expected utility). In our

first look, we saw examples of each of these three types. For example, in the case of the economy with no aggregate uncertainty ( $\varepsilon = 0$ ) and no idiosyncratic uncertainty for banks ( $\alpha$  constant), there is a unique pure FE, namely the one in which  $\alpha d_i = y_i$  for all  $i$ . However, for the same economy there is also a large number of semi-pure equilibria FE in which the consumption allocation is the same for all banks but the portfolios are different. In a TSE, however, price fluctuations require that each bank be autarkic ( $\alpha d_i = y_i$ ) so any TSE is pure. The role of mixed equilibria is to ensure existence when default introduces non-convexities.

## 6.1 Equilibrium with non-stochastic $\alpha$

In the special case with  $\alpha$  constant the following theorem partitions the equilibrium set into two cases with distinctive properties.

**Theorem 1** *Suppose that  $\alpha$  is almost surely constant and let  $(\rho, m, x, y, p)$  be an equilibrium of the limit economy, in which  $\varepsilon = 0$ . There are two possibilities:*

- (i)  $(\rho, m, x, y, p)$  is a semi-pure, fundamental equilibrium in which the probability of default is zero;*
- (ii)  $(\rho, m, x, y, p)$  is a pure, trivial sunspot equilibrium in which the probability of default is zero.*

**Proof.** See Section 10. ■

By definition, an equilibrium must be either a FE, TSE, or NTSE. What Theorem 1 shows is that a NTSE does not occur and each of the remaining cases is associated with distinctive properties in terms of symmetry and probability of default.

Because the incentive constraint does not bind in either the FE or TSE, both achieve the first-best or Pareto-efficient allocation. No equilibrium can do better. Any bank can guarantee this level of utility by choosing  $\alpha d_i = y_i$ , where  $d_i$  is the deposit contract chosen in the FE. For this choice of  $(d_i, y_i)$  prices have no effect on the bank's budget constraint and, because we assume crises are essential, the depositors will receive the first-best consumption. In a NTSE, by contrast, agents receive noisy consumption allocations. Because they are risk averse, the noise in their consumption allocations is inefficient. Since we have seen that the bank can guarantee more to the depositors, this kind of equilibrium cannot exist.

## 6.2 Equilibrium with stochastic $\alpha$

Banks may choose  $(d_i, y_i)$  so that they are forced to default in some states because of uncertainty about the idiosyncratic shock  $\alpha$ . In order to distinguish crises caused by aggregate extrinsic uncertainty from defaults caused by idiosyncratic shocks, we will assume that the parameters are such that default is never optimal in the FE. In that case, the bank's optimal choice of  $(d_i, y_i)$  must satisfy the incentive constraint, so the bank's optimal decision problem can be written as follows. At the equilibrium price  $\bar{p} = 1/r$ , the value of the bank's assets at date 1 is  $y_i + \bar{p}r(1 - y_i) = 1$ , independently of the choice of  $y_i$ . In the absence of default, the budget constraint implies that the consumption at date 2 is given by  $r(1 - \alpha d_i)/(1 - \alpha)$ . The incentive constraint requires that  $r(1 - \alpha d_i)/(1 - \alpha) \geq d_i$ . Thus, the decision problem can be written as:

$$\begin{aligned} \max \quad & E[\alpha U(d_i) + (1 - \alpha)U(r(1 - \alpha d_i)/(1 - \alpha))] \\ \text{st} \quad & r(1 - \alpha d_i)/(1 - \alpha) \geq d_i, \forall \alpha. \end{aligned}$$

This is a convex programming problem and has a unique solution for  $d_i$ . As noted,  $y_i$  is indeterminate, but the equilibrium allocation must satisfy the market-clearing condition

$$\sum_i \rho_i \bar{\alpha} d_i = \sum_i \rho_i y_i$$

where  $\bar{\alpha} = E[\alpha]$ . Note that there is a single *pure* FE  $(\rho, m, d, y, \bar{p})$ , in which  $y_i = E[\alpha d_i]$  for every  $i = 1, \dots, m$ .

In the case of idiosyncratic shocks, the following theorem partitions the equilibrium set into two cases, FE and NTSE.

**Theorem 2** *Let  $(\rho, m, d, y, p)$  be an equilibrium of the limit economy, in which  $\varepsilon = 0$ . There are two possibilities:*

- (i)  $(\rho, m, d, y, p)$  is a semi-pure, fundamental equilibrium in which the probability of default is zero;
- (ii)  $(\rho, m, d, y, p)$  is a non-trivial sunspot equilibrium which is pure if the probability of default is zero.

**Proof.** See Section 10. ■

The fundamental equilibrium is semi-pure for the usual reasons and there is no default by assumption. Unlike the case with no idiosyncratic shocks

(non-stochastic  $\alpha$ ), there can be no trivial sunspot equilibrium. To see this, suppose that some bank group  $i$  with measure  $\rho_i$  chooses  $(d_i, y_i)$  and that default occurs with positive probability. Then consumption for both early and late consumers is equal to  $y_i + p(\theta)r(1 - y_i)$ , which is independent of  $p(\theta)$  only if  $y_i = 1$ . In that case there is no point in choosing  $d > 1$  and so no need for default. If there is no probability of default, then the consumption of the late consumers is

$$\frac{y_i + p(\theta)r(1 - y_i) - \alpha d_i}{(1 - \alpha)p(\theta)}.$$

This is independent of  $p(\theta)$  only if  $y_i = \alpha d_i$ , which cannot hold unless  $\alpha$  is a constant. Thus, there can be no TSE.

The only remaining possibility is a NTSE. If the probability of default is zero, then the bank's decision problem is a convex programming problem and the usual methods suffice to show uniqueness of the optimum choice of  $(d, y)$  under the maintained assumptions.

As long as there is no possibility of default, an equilibrium  $(m, \rho, d, x, p)$  must be either a FE or a TSE. If there is a positive probability of default, the equilibrium  $(m, \rho, d, x, p)$  must be a NTSE. Any NTSE is characterized by a finite number of prices. To see this, let  $B \subset \{1, \dots, m\}$  denote the groups that default at some state  $\theta$ . If  $p(\theta) < 1$  then complementary slackness and the market-clearing condition (2) imply that

$$\sum_{i \in B} \rho_i w_i(\theta) + \sum_{i \notin B} \alpha \rho_i d_i = \sum_i \rho_i y_i.$$

Substituting for  $w_i(\theta) = y_i + p(\theta)r(1 - y_i)$  gives

$$\sum_{i \in B} \rho_i (y_i + p(\theta)r(1 - y_i)) + \sum_{i \notin B} \alpha \rho_i d_i = \sum_i \rho_i y_i,$$

or

$$p(\theta) = \frac{\sum_i \rho_i y_i - \sum_{i \in B} \rho_i y_i - \sum_{i \notin B} \alpha \rho_i d_i}{\sum_{i \in B} \rho_i r(1 - y_i)},$$

where the denominator must be positive. If not, then  $\sum_{i \in B} \rho_i r(1 - y_i) = 0$ , which implies that  $y_i = 1$  for every  $i \in B$ . The budget constraint implies that  $x_{i1}(\theta) \leq 1$  almost surely, in which case there can be no default, contradicting

the definition of  $B$ . So the denominator must be positive. The expression on the right hand side can only take on a finite number of values because there is a finite number of sets  $B$ . Thus, there can be at most a finite number of different prices observed in equilibrium.

We have proved the following corollary.

**Corollary 3** *Let  $(\rho, m, d, y, p)$  be an equilibrium. There are three possibilities:*

- In a FE,  $p(\cdot)$  has a single value:  $p(\theta) = 1/r$  almost surely;*
- In a TSE,  $p(\cdot)$  can have a finite or infinite number of values;*
- In a NTSE,  $p(\cdot)$  has a finite number of values.*

The properties of equilibria in the limit economy are summarized in the following table.

Symmetry	Type	Default	# of $p(\cdot)$
Semi-pure	FE	No	One
Pure	TSE	No	Finite or infinite
Mixed or pure	NTSE	Possible	Finite

## 7 Limits of equilibria

We have defined a *fundamental equilibrium* as one in which the endogenous variables are functions of the exogenous primitives or fundamentals of the model (endowments, preferences, technologies). In the special case  $\varepsilon = 0$ , there is idiosyncratic uncertainty about consumers' types and the proportion of types at a given bank, but no aggregate uncertainty about the proportion of early consumers in the economy as a whole. By definition, in that case, asset prices are non-stochastic in a fundamental equilibrium and crises cannot occur. When  $\varepsilon > 0$  is small, the amount of aggregate uncertainty is small but positive. By definition, every equilibrium is fundamental. The state  $\theta$  represents an exogenous real shock to liquidity demand, so the variation in asset prices  $p(\theta)$  represents intrinsic rather than extrinsic uncertainty.

We think of extrinsic uncertainty as an extreme case of excess sensitivity to small amounts of intrinsic uncertainty. The limit economy with no aggregate intrinsic uncertainty is approximated by a perturbed economy with a small but positive amount of intrinsic uncertainty. An equilibrium of the limit

economy with no aggregate intrinsic uncertainty is *robust* if the equilibrium does not change very much when the economy is perturbed by introducing a small amount of intrinsic uncertainty. In this section, we test the robustness of an equilibrium *in the limit* by examining the *limit* of sequences of equilibria as  $\varepsilon > 0$  becomes vanishingly small.

The next theorem characterizes the properties of equilibria in perturbed economies.

**Theorem 4** *Let  $(m, \rho, d, y, p)$  denote an equilibrium for a perturbed economy in which  $\varepsilon > 0$ . Then either there is a positive probability of default or there is non-trivial price volatility, i.e.,  $p(\theta) = 1$  with positive probability and  $p(\theta) < 1/r$  with positive probability.*

**Proof.** See Section 10. ■

We note several other properties of equilibrium with  $\varepsilon > 0$ . First, there cannot be a state in which all banks are in default, for this would imply  $p(\theta) = 0$  which is inconsistent with equilibrium. Hence, any equilibrium with default must be mixed. Secondly, as we have seen, if there is no default there must be a positive probability that  $p(\theta) = 1$ . Finally, in the absence of default, for any value of  $\varepsilon > 0$ , the volatility of asset prices, as measured by the variance of  $p(\cdot)$ , is bounded away from zero. Both assets are held at date 0 in equilibrium and this requires that the high returns to the long asset, associated with  $p(\theta) = 1$ , must be balanced by low returns associated with  $p(\theta) < 1/r$ . These properties are all preserved in the limit as  $\varepsilon \rightarrow 0$ .

The next result shows that the limit of a sequence of equilibria is an equilibrium in the limit.

**Theorem 5** *Consider a sequence of perturbed economies corresponding to  $\varepsilon = 1/q$ , where  $q$  is a positive integer, and let  $(m^q, \rho^q, d^q, y^q, p^q)$  be an equilibrium for the corresponding perturbed economy. For some convergent subsequence  $q \in Q$  let*

$$(m^0, \rho^0, d^0, y^0, p^0) = \lim_{q \in Q} (m^q, \rho^q, d^q, y^q, p^q).$$

*If  $d_i^0 > 0$ , and  $0 < y_i^0 < 1$  for  $i = 1, \dots, m^0$  then  $(m^0, \rho^0, x^0, y^0, p^0)$  is an equilibrium of the limit economy.*

**Proof.** See Section 10. ■



Theorem 5 shows that, under certain conditions, the limit of a sequence of equilibria as  $\varepsilon \rightarrow 0$  is an equilibrium of the limit economy where  $\varepsilon = 0$ . We are also interested in the opposite question, namely, which equilibria of the limit economy in which  $\varepsilon = 0$  are limits of equilibria from the perturbed economy in which  $\varepsilon > 0$ ? This requirement of lower hemi-continuity is a test of *robustness*: if a small perturbation of the limit economy causes an equilibrium to disappear, we argue that the equilibrium is not robust. Since there are many equilibria of the limit economy, it is of interest to see whether any of these equilibria can be eliminated by being shown to be non-robust.

We have shown in Theorem 4 that equilibria of the perturbed economy are characterized by default or non-trivial asset price uncertainty. Furthermore, because the random variable  $\theta$  has a finite support, the probability of these events is bounded away from zero, uniformly in  $\varepsilon$ . The fundamental equilibrium of the limit economy with  $\varepsilon = 0$  has none of these properties. However, this does not by itself prove that the fundamental equilibrium is not robust. It could be the limit of a sequence of equilibria of the perturbed economy if the fraction of banks that defaults in equilibrium converges to zero as  $q \rightarrow \infty$ . However, if the fundamental equilibrium were the limit referred to in Theorem 5, then it must be the case that (a) default is optimal in the limit and (b) there is no price volatility in the limit. These two properties can be shown to be inconsistent.

**Corollary 6** *If  $(m^0, \rho^0, x^0, y^0, p^0)$  is the equilibrium of the limit economy mentioned in Theorem 5, then  $(m^0, \rho^0, x^0, y^0, p^0)$  is not a FE of the limit economy, i.e., it must be either a NTSE or a TSE.*

**Proof.** Suppose that, contrary to what we wish to prove,  $(m^0, \rho^0, x^0, y^0, p^0)$  is the fundamental equilibrium. Then  $p^0(\theta) = 1/r < 1$  with probability one and, hence,  $p^q(\theta) < 1$  for all  $\theta$  and all  $q$  sufficiently large. From Theorem 4 we know that this implies some group  $i$  defaults with positive probability for all sufficiently large  $q$ . Thus, in the limit, default must be optimal for group  $i$  and  $\rho_i^q \rightarrow \rho_i^0 = 0$ . However, we assumed in Section 6.2 that default is not optimal for stochastic  $\alpha$  and it is clear that it is not optimal with non-stochastic  $\alpha$  because

$$U(1) < \max_{\alpha c_1 + (1-\alpha)c_2/r=1} \{\alpha U(c_1) + (1-\alpha)U(c_2)\}.$$

This contradiction proves that  $(m^0, \rho^0, x^0, y^0, p^0)$  is not a FE, i.e., the FE is not robust. ■

This shows that the limit of a sequence of equilibria of the perturbed economy must be either a TSE or NTSE, but not a FE. So this approach of regarding sunspot equilibria as a limiting case of fundamental equilibria actually eliminates the FE in the limit economy or, in other words, selects the sunspot equilibria as the only robust equilibria.

## 8 Numerical examples

This section develops a series of examples to illustrate the main results of the paper.

*Example 1*

$$\begin{aligned} U(c_t) &= Ln(c_t) \\ r &= 1.5 \\ \alpha &= 0.8 \end{aligned}$$

$$\theta = \begin{cases} 0 & w. pr. 0.65 \\ 1 & w. pr. 0.35 \end{cases}$$

The equilibrium with a non-stochastic price is shown in Panel 1.1A of Table 1. This is the “natural” equilibrium. The short term asset is used to provide for the consumption needs of the early consumers and the long term asset is used to provide for the needs of the late consumers. There is no default. The price of consumption at date 2 in terms of consumption at date 1,  $p(\theta) = \bar{p} = 2/3$ , is simply equal to the marginal rate of substitution between consumption at those two dates. The equilibrium could be autarkic with each bank operating on its own. There is no need for trade. The banks could also use the markets to trade in which case they can hold different individual amounts of the safe and risky assets as long as the aggregate amounts of the safe and risky assets are the same as those in Panel 1.1A.

Next consider what happens if an arbitrarily small amount of aggregate uncertainty is introduced so that  $\varepsilon > 0$  but for the moment still assume  $\alpha = 0.8$  is a constant. The “natural” equilibrium with a non-stochastic price, no default and all banks providing the same allocation is not robust. There does not exist an equilibrium for the economy with aggregate uncertainty in which prices are non-stochastic and default occurs with probability zero. The aggregate supply of liquidity at date 1 is fixed by the portfolio decisions of

banks at date 0. However the demand for liquidity can take on two values depending on whether  $\theta = 0$  or 1. If default is not to occur then it must be the case that in aggregate  $y \geq (0.8 + \varepsilon)d$  when  $\theta = 1$  which implies  $y > 0.8d$  when  $\theta = 0$ . Given there is excess liquidity at date 1 we must have  $p(0) = 1$  in order for banks to be willing to hold this liquidity between dates 1 and 2. If price is to be non-stochastic  $p(1) = 1$  also. When  $p$  is always 1 for each 1 unit of the good invested the long asset pays off  $rp(\theta) = 1.5$  at date 1 while the short asset only pays off 1. The long asset therefore strictly dominates the short asset and nobody will invest in the short asset. This is inconsistent with  $y \geq (0.8 + \varepsilon)d$  so there cannot be an equilibrium with a non-stochastic price and no default.

Now if default occurs with positive probability it cannot be an equilibrium for all banks to provide the same allocation. In this case all banks will go bankrupt together. But this cannot be an equilibrium because then  $p(\theta) = 0$  with positive probability and a bank could do better by remaining solvent in this state and buying up all the assets.

What does equilibrium look like? Clearly price must be non-stochastic. As the argument with all banks going bankrupt suggests, there can be two different strategies that banks can follow in equilibrium. One type follows a safe strategy (denoted with a subscript  $s$ ) and stays solvent while the other follows a risky strategy (denoted with the subscript  $r$ ) and goes bankrupt when prices are low.

To see what this equilibrium looks like it is helpful to notice that when  $\varepsilon = 0$  in Example 1 the equilibrium in Panel 1.1A is not the only equilibrium. There is also an equilibrium when  $p(0) = 0.943$  and  $p(1) = 0.432$ . The full details of the equilibrium are given in Panel 1.1B. In this sunspot equilibrium the safe banks choose the same portfolio  $y_s = 0.8$  and set  $d_s = 1$ . These banks thus provide the same consumption  $c_s^1 = 1$  and  $c_s^2 = 1.5$  as the banks in Panel 1.1A. The crucial point is that with these prices there is another strategy that yields the same expected utility to their depositors as that given by the safe banks. The risky banks choose  $y_r = 0$ . When  $p(0) = 0.943$  they liquidate some of their long term asset in the market and use the proceeds to pay  $c_r^1 = d_r^1 = 1.414$  to early consumers and  $c_r^2 = 1.5$  to late consumers. However, when  $p(0) = 0.432$  the risky banks are no longer solvent. They are unable to pay  $d_r^1 = 1.414$  to early and late consumers so there is a run on these banks. They go bankrupt and are forced to liquidate all of their long term asset in the market. They obtain  $1.5 \times 0.432 = 0.648$  and pay this out to early and late consumers so  $c_r^2 = c_r^1 = 0.648$ . With  $\varepsilon = 0$  market clearing requires that

$\rho_1 = 1$  and  $\rho_2 = 0$  so that there is only entry by the safe banks. In order for there to be entry by the risky banks it is necessary that  $\varepsilon > 0$ . In contrast to the fundamental equilibrium in Panel 1.1A this sunspot equilibrium is robust and there exists an equilibrium close to it with positive  $\varepsilon$ .

Next consider an example like Example 1 but with  $\varepsilon = 0.010$ . The corresponding equilibrium is given in Panel 1.2 of Table 1. In this case  $\rho_1 = 0.996$  and  $\rho_2 = 0.004$  so that there is entry by both types of bank. This equilibrium is a fundamental equilibrium rather than a sunspot equilibrium since with  $\varepsilon > 0$  the uncertainty associated with  $\theta$  is now intrinsic rather than extrinsic. The nature of equilibrium is the same as before except there is entry by the risky banks and there is a positive volume of trade. When  $\theta = 0$ , which occurs 65 percent of the time,  $p(0) = 0.940$  and the risky banks liquidate a portion of their assets in the market at date 1 to obtain the liquidity necessary to pay their early consumers. The safe banks supply the necessary liquidity to the market. When  $\theta = 1$ , which occurs the remaining 35 percent of the time, banks are faced with extra liquidity demand. The risky banks must sell more of their long term asset to satisfy this extra demand. This small change leads to a collapse in price. As the price falls the risky banks cease to be solvent and are subject to runs. They go bankrupt and liquidate all of their assets in the market. There is “cash in the market pricing” where the price  $p(1) = 0.430$  is the ratio of the liquidity supplied by the safe banks to the long assets supplied because of the liquidation of the risky banks.

Table 1 illustrates the case where there is default and price volatility. A second possibility is that there is no default and price volatility. This can be illustrated with the following.

*Example 2*

$$\begin{aligned} U(c_t) &= Ln(c_t) \\ r &= 1.5 \\ \alpha &= 0.5 \end{aligned}$$

$$\theta = \begin{cases} 0 & \text{w. pr. } 0.7 \\ 1 & \text{w. pr. } 0.3 \end{cases}$$

The equilibria for this example are shown in Table 2. The main difference is that in these equilibria there is no entry by risky banks since they are strictly worse off. The safe banks hold enough liquidity to meet depositors’ demand when  $\theta = 1$ . This means they have surplus liquidity when  $\theta = 0$  and

this is why  $p(0) = 1$ . This is the only price at which they are willing to hold surplus liquidity.

Thus, introducing a small amount of uncertainty changes the properties of equilibrium. Either there is default with positive probability and/or there will be a non-trivial amount of price volatility. Perhaps most important is the fact that in all the examples the robust equilibria display financial fragility. A small change in aggregate liquidity preference leads to a large change in asset prices.

To see the effect of introducing randomness in  $\alpha$  in the context of a numerical example consider the following variation on Example 1.

*Example 3*

$$U(c_t) = Ln(c_t) \\ r = 1.5$$

$$\alpha = \begin{cases} 0.75 & w. pr. 0.5 \\ 0.85 & w. pr. 0.5 \end{cases}$$

$$\theta = \begin{cases} 0 & w. pr. 0.65 \\ 1 & w. pr. 0.35 \end{cases}$$

The equilibria that occur in this example are shown in Table 3. It can be seen that the fundamental equilibrium in Panel 3.1A is unchanged from Panel 1.1A. However the sunspot equilibria in Panels 1.1B and 3.1B are not the same. Trade occurs at the low price and this adversely affects the welfare of the safe banks as well as the risky banks. In Panel 3.1B,  $d_s$  is lowered to 0.995 from 1 in Panel 1.1B to reduce the effects of trading at this low price. Similarly,  $y_s$  is increased to 0.810 from 0.800. Expected utility is lowered from 0.081 in Panel 3.1A to 0.080 in Panel 3.1B. As with the autarkic equilibria introducing a small amount of aggregate uncertainty eliminates equilibria with a non-stochastic price. Only the sunspot equilibrium is robust. Panel 3.2 shows the equilibrium with intrinsic uncertainty  $\varepsilon = 0.010$  close to the sunspot equilibrium. In this case  $\rho_1 = 0.996$  and  $\rho_2 = 0.004$  so that there is entry by both types of bank.

## 9 Concluding remarks

In this paper, we have investigated the relationship between intrinsic and extrinsic uncertainty in a model of financial crises. Our general approach is to regard extrinsic uncertainty as a limiting case of intrinsic uncertainty. In our model, small shocks to the demand for liquidity are associated with large fluctuations in asset prices. These price fluctuations can cause financial crises to occur with positive probability. This is the sense in which there is financial fragility. In the limit, as the liquidity shocks become vanishingly small, the model converges to one with extrinsic uncertainty. The limit economy has three kinds of equilibria,

- fundamental equilibria, in which there is neither aggregate uncertainty nor a positive probability of crisis;
- trivial sunspot equilibria, in which prices fluctuate but the real allocation is the same as in the fundamental equilibrium;
- and non-trivial sunspot equilibria, in which prices fluctuate and financial crises can occur with positive probability.

Introducing small shocks into the limit economy destabilizes the first type of equilibrium, leaving the second and third as possible limits of equilibria of the perturbed economy. If  $\alpha$  is a constant, the limiting equilibrium as  $\varepsilon \rightarrow 0$  is a trivial sunspot equilibrium. If  $\alpha$  is random, the limiting equilibrium as  $\varepsilon \rightarrow 0$  is a non-trivial sunspot equilibrium. We argue that only the sunspot equilibria are robust, in the sense that a small perturbation of the model causes a small change in these equilibria. This selection criterion provides an argument for the relevance of extrinsic uncertainty and the necessity of financial crises.

Although crises in the limit economy arise from extrinsic uncertainty, the causation is quite different from the bank run story of Diamond and Dybvig (1983). In the Diamond-Dybvig story, bank runs are spontaneous events that depend on the decisions of late consumers to withdraw early. Given that almost all agents withdraw at date 1, early withdrawal is a best response for every agent; but if late consumers were to withdraw at date 2, then late withdrawal is a best response for every late consumer. So there are two “equilibria” of the coordination game played by agents at date 1, one with a bank run and one without. This kind of coordination failure plays no

role in the present model. In fact, coordination failure is explicitly ruled out: a bank run occurs only if the bank cannot simultaneously satisfy its budget constraint and its incentive constraint. When bankruptcy does occur, it is the result of low asset prices. Asset prices are endogenous, of course, and there is a “self-fulfilling” element in the relationship between asset prices and crises. Banks are forced to default and liquidate assets because asset prices are low and asset prices are low as a result of mass bankruptcy and the association liquidation of bank assets.

One interesting difference between the present story and Diamond and Dybvig (1983) is that here a financial crisis is a *systemic event*. A crisis occurs only if the number of defaulting banks is large enough to affect the equilibrium asset price. In the Diamond-Dybvig model, by contrast, bank runs are an idiosyncratic phenomenon. Whether a run occurs at a particular bank depends on the decisions taken by the bank’s depositors. It is only by coincidence that runs are experienced by several banks at the same time.

At the heart of our theory is a pecuniary externality: when one group of banks defaults and liquidates its assets, it forces down the price of assets and this may cause another group of banks to default. This pecuniary externality may be interpreted as a form of contagion.

Allen and Gale (2000b) describes a model of contagion in a multi-region economy. Bankruptcy is assumed to be costly: long-term projects can be liquidated prematurely but a fraction of the returns are lost. This deadweight loss from liquidation creates a spillover effect in the adjacent regions where the claims on the bankrupt banks are held. If the spillover effect is large enough, the banks in the adjacent regions will also be forced into default and liquidation. Each successive wave of bankruptcies increases the loss of value and strengthens the impact of the spillover effect on the next region. Under certain conditions, a shock to one small region can propagate throughout the economy. By contrast, in the present model, a bank’s assets are always marked to market. Given the equilibrium asset price  $p$ , bankruptcy does not change the value of the bank’s portfolio. However, if a group of banks defaults, the resulting change in the price  $p$  may cause other banks to default, which will cause further changes in  $p$ , and so on. The “contagion” in both models is instantaneous.

Several features of the model are special and deserve further consideration.

Pecuniary externalities “matter” in our model because markets are incomplete: if banks could trade Arrow securities contingent on the states  $\theta$ , they

would be able to insure themselves against changes in asset values (AG). No trade in Arrow securities would take place in equilibrium, but the existence of the markets for Arrow securities would have an effect. The equilibrium allocation would be incentive-efficient, sunspots would have no real impact, and there would be no possibility of crises.

It is important that small shocks lead to large fluctuations in asset prices (and large pecuniary externalities). We have seen, in the case of trivial sunspot equilibria, that small price fluctuations have no real effect. What makes the pecuniary externality large in this example is inelasticity of the supply and demand for liquidity. Inelasticity arises from two assumptions. First, the supply of liquidity at date 1 is fixed by the decisions made at date 0. Secondly, the assumption of Diamond-Dybvig preferences implies that demand for consumption at date 1 is interest-inelastic. This raises a question about the robustness of the results when more general preferences are allowed.

One justification for the Diamond-Dybvig preferences is that they provide a cheap way of capturing, within the standard, Walrasian, auction-market framework, some realistic features of alternative market clearing mechanisms. In an auction market, prices and quantities adjust simultaneously in a *tatonnement* process until a full equilibrium is achieved. An alternative mechanism is one in which quantities are chosen before prices are allowed to adjust. An example is the use of market orders. If depositors were required to make a withdrawal decision before the asset price was determined in the interbank market, the same inelasticity of demand would be observed even if depositors had preferences that allowed for intertemporal substitution. There may be other institutional structures that have the qualitative features of our example. An investigation of these issues goes far beyond the scope of the present paper, but it is undoubtedly one of the most important topics for future research.

## 10 Proofs

### 10.1 Proof of Theorem 1

By definition, an equilibrium  $(\rho, m, d, y, p)$  must be either a FE, TSE, or NTSE. The theorem is proved by considering each case in turn.

Case (i). If  $(\rho, m, d, y, p)$  is a FE then by definition the price  $p(\theta)$  is



almost surely constant and, for each group  $i$ , the consumption allocation  $x(d_i, y_i, \alpha, \theta)$  is almost surely constant. In particular,  $x_1(d_i, y_i, \alpha, \theta) = d_i$  with probability one so there is no default in equilibrium.

Let  $\bar{p}$  denote the constant price and  $c_i = (c_{i1}, c_{i2})$  the consumption allocation chosen by banks in group  $i$ . The decision problem of a bank in group  $i$  is

$$\begin{aligned} \max \quad & [\alpha U(c_{i1}) + (1 - \alpha)U(c_{i2})] \\ \text{s.t.} \quad & c_{i1} \leq c_{i2}, 0 \leq y_i \leq 1 \\ & \alpha c_{i1} + (1 - \alpha)\bar{p}c_{i2} \leq y_i + \bar{p}r(1 - y_i). \end{aligned}$$

Clearly,  $y_i$  will be chosen to maximize  $y_i + \bar{p}r(1 - y_i)$ . Then the strict concavity of  $U(\cdot)$  implies that  $c_i$  is uniquely determined and independent of  $i$ . Thus, the equilibrium is semi-pure:  $c_i = c_j$  for any  $i$  and  $j$ .

Case (ii). Suppose that  $(\rho, m, d, y, p)$  is a TSE. Then by definition  $p(\theta)$  is not almost surely constant and, for each group  $i$ , the consumption allocation  $x(d_i, y_i, \alpha, \theta)$  is almost surely constant. In particular,  $x_1(d_i, y_i, \alpha, \theta) = d_i$  with probability one so there is no default in equilibrium.

Let  $c_i$  denote the consumption allocation chosen by banks in group  $i$ . The budget constraint at date 1 reduces to

$$\alpha c_{i1} - y_i = -p(\theta)((1 - \alpha)\alpha c_{i2} - r(1 - y_i)), \text{ a.s.}$$

Since  $p(\theta)$  is not almost surely constant, this equation can be satisfied only if

$$\alpha c_{i1} - y_i = (1 - \alpha)c_{i2} - r(1 - y_i) = 0.$$

Then the choice of  $(c_i, y_i)$  must solve the problem

$$\begin{aligned} \max \quad & E [\alpha U(c_{i1}) + (1 - \alpha)U(c_{i2})] \\ \text{s.t.} \quad & c_{i1} \leq c_{i2}, 0 \leq y_i \leq 1 \\ & \alpha c_{i1} = y_i, (1 - \alpha)c_{i2} = r(1 - y_i). \end{aligned}$$

The strict concavity of  $U(\cdot)$  implies that this problem uniquely determines the value of  $c_i$  and hence  $y_i$ , independently of  $i$ . Consequently, the equilibrium is pure.

Case (iii). Suppose that  $(\rho, m, d, y, p)$  is a NTSE. The allocation of consumption for group  $i$  is  $x(d_i, y_i, \alpha, \theta)$ , and the expected utility of each group is the same

$$E [u(x(d_i, y_i, \alpha, \theta), \alpha)] = E [u(x(d_j, y_j, \alpha, \theta), \alpha)], \forall i, j.$$

The mean allocation  $\sum_i \rho_i x(d_i, y_i, \alpha, \theta)$  satisfies the market clearing conditions for every  $\theta$  and hence consumption bundle  $E[\sum_i \rho_i x(d_i, y_i, \alpha, \theta)]$  is feasible for the planner. Since agents are strictly risk averse,

$$u\left(E\left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta)\right], \alpha\right) > \sum_i \rho_i E[u(x(d_i, y_i, \alpha, \theta), \alpha)].$$

This contradicts the equilibrium conditions, since the individual bank could choose

$$y_0 = \alpha d_0 = \alpha E\left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta)\right]$$

and achieve a higher utility.

## 10.2 Proof of Theorem 2

Again we let  $(\rho, m, d, y, p)$  be a fixed but arbitrary equilibrium and consider each of three cases in turn.

Case (i). If  $(\rho, m, d, y, p)$  is a FE  $p(\theta)$  is almost surely constant and, for each group  $i$  and each  $\alpha$ , the consumption allocation  $x(d_i, y_i, \alpha, \theta)$  is almost surely constant. In particular,  $x_1(d_i, y_i, \alpha, \theta) = d_i$  with probability one so there is no default in equilibrium.

Let  $\bar{p}$  denote the constant price and  $c_i(\alpha) = (c_{i1}(\alpha), c_{i2}(\alpha))$  the consumption allocation chosen by banks in group  $i$ . The decision problem of a bank in group  $i$  is

$$\begin{aligned} \max \quad & E[\alpha U(c_{i1}(\alpha)) + (1 - \alpha)U(c_{i2}(\alpha))] \\ \text{s.t.} \quad & c_{i1}(\alpha) \leq c_{i2}(\alpha), 0 \leq y_i \leq 1 \\ & \alpha c_{i1}(\alpha) + (1 - \alpha)\bar{p}c_{i2}(\alpha) \leq y_i + \bar{p}r(1 - y_i). \end{aligned}$$

Clearly,  $y_i$  will be chosen to maximize  $y_i + \bar{p}r(1 - y_i)$ . Then the strict concavity of  $U(\cdot)$  implies that  $c_i(\alpha)$  is uniquely determined and independent of  $i$  (but not of  $\alpha$ ). Thus, the equilibrium is semi-pure:  $c_i = c_j$  for any  $i$  and  $j$ .

Case (ii). Suppose that  $(\rho, m, d, y, p)$  is a TSE. Then by definition  $p(\theta)$  is not almost surely constant and, for each group  $i$  and  $\alpha$ , the consumption allocation  $x(d_i, y_i, \alpha, \theta)$  is almost surely constant. In particular,  $x_1(d_i, y_i, \alpha, \theta) = d_i$  with probability one so there is no default in equilibrium.

Let  $c_i(\alpha)$  denote the consumption allocation chosen by banks in group  $i$ . The budget constraint at date 1 reduces to

$$\alpha c_{i1}(\alpha) - y_i = -p(\theta)((1 - \alpha)\alpha c_{i2}(\alpha) - r(1 - y_i)), \text{ a.s.}$$

Since  $p(\theta)$  is not almost surely constant, this equation can be satisfied only if

$$\alpha c_{i1}(\alpha) - y_i = (1 - \alpha)c_{i2}(\alpha) - r(1 - y_i) = 0.$$

Since  $\alpha$  is not constant this can only be true if  $c_{i1}(\alpha) = c_{i2}(\alpha) = 0$ , a contradiction. Thus, there cannot be a TSE when  $\alpha$  is not constant.

Case (iii). The only remaining possibility is that  $(\rho, m, d, y, p)$  is a NTSE. If there is no default in this equilibrium, then each bank in group  $i$  solves the problem

$$\begin{aligned} \max \quad & E \left[ \alpha U(d_i) + (1 - \alpha) U \left( \frac{y_i + p(\theta)r(1 - y_i) - \alpha d_i}{(1 - \alpha)p(\theta)} \right) \right] \\ \text{st} \quad & \frac{y_i + p(\theta)r(1 - y_i)}{(1 - \alpha)p(\theta)} \geq d_i. \end{aligned}$$

This is a convex programming problem and it is easy to show that the strict concavity of  $U(\cdot)$  uniquely determines  $(d_i, y_i)$ . Thus a NTSE without default is pure.

### 10.3 Proof of Theorem 4

Suppose that the probability of default in  $(\rho, m, d, y, p)$  is zero. Then for each group  $i$ ,  $x_1(d_i, y_i, \alpha, \theta) = d_i$  almost surely and the market-clearing condition (2) implies

$$\sum_i E [\rho_i \eta(\alpha, \theta)] d_i \leq \sum_i \rho_i y_i. \quad (5)$$

There are two cases to consider. In the first case,  $\sum_i \rho_i y_i = 0$ . Then  $d_i = 0$  for every  $i$  and the utility achieved in equilibrium is

$$E \left[ \eta(\alpha, \theta) U(0) + (1 - \eta(\alpha, \theta)) U \left( \frac{r}{1 - \eta(\alpha, \theta)} \right) \right].$$

By holding a small amount  $\delta > 0$  of the short asset, positive consumption could be guaranteed at the first date. Optimality requires that

$$\begin{aligned} & E \left[ \eta(\alpha, \theta) U \left( \frac{\delta}{\eta(\alpha, \theta)} \right) \right] + E \left[ (1 - \eta(\alpha, \theta)) U \left( \frac{r(1 - \delta)}{1 - \eta(\alpha, \theta)} \right) \right] \\ & \leq E \left[ \eta(\alpha, \theta) U(0) + (1 - \eta(\alpha, \theta)) U \left( \frac{r}{1 - \eta(\alpha, \theta)} \right) \right]. \end{aligned}$$

for any  $\delta > 0$ . In the limit as  $\delta \rightarrow 0$ ,

$$E[U'(0)] - E \left[ U' \left( \frac{r}{1 - \eta(\alpha, \theta)} \right) \right] \leq 0,$$

which contradicts the assumption that  $U'(0) = \infty$ .

In the second case,  $\sum_i \rho_i y_i > 0$ . Then the market-clearing condition (5) and the fact that  $\eta(\alpha, \theta) = \alpha + \varepsilon\theta$  together imply that

$$\sum_i \rho_i \eta(\alpha, \theta) d_i < \sum_i \rho_i y_i$$

with positive probability. The complementary slackness condition implies that  $p(\theta) = 1$  with positive probability and the short asset will be dominated by the long asset at date 0 unless  $p(\theta) < 1/r$  with positive probability. Thus, the price volatility is non-trivial if the probability of default is zero.

## 10.4 Proof of Theorem 5

Continuity and the convergence of  $\{(\rho^q, m, d^q, y^q, p^q)\}$  immediately implies the following properties of the limit point  $(\rho^0, m, d^0, y^0, p^0)$ :

- (i)  $\sum_i \rho_i^0 = 1$  and  $\rho_i^0 \geq 0$  for every  $i$  so  $(\rho^0, m)$  is a partition.
- (ii) For every  $i$ ,  $(d_i^0, y_i^0) \in \mathbf{R}_+ \times [0, 1]$ . The market-clearing conditions

$$\sum_i \rho_i^0 E [\alpha \bar{x}_1(d_i^0, y_i^0, \alpha, \theta) | \theta] \leq \sum_i \rho_i^0 y_i^0,$$

and

$$\begin{aligned} \sum_i \rho_i^0 E \{ \alpha \bar{x}_1(d_i^0, y_i^0, \alpha, \theta) + (1 - \alpha) \bar{x}_2(d_i^0, y_i^0, \alpha, \theta) | \theta \} \\ = \sum_i \rho_i^0 \{ y_i^0 + r(1 - y_i^0) \} \end{aligned}$$

are satisfied in the limit and the complementary slackness condition holds. Thus,  $(d^0, y^0)$  is an attainable allocation.

It remains to show that  $(d_i^0, y_i^0)$  is optimal for each  $i$ . Let  $W^q(d_i, y_i, \alpha, \theta)$  denote the utility associated with the pair  $(d_i, y_i)$  in the perturbed economy corresponding to  $\varepsilon = 1/q$  when the price function is  $p^q$  and let  $W^0(d_i, y_i, \alpha, \theta)$  denote the utility associated with the pair  $(d_i, y_i)$  in the limit economy corresponding to  $\varepsilon = 0$ , where the price function is  $p^0$ . The function  $W_i^0(\cdot)$  is discontinuous at the bankruptcy point defined implicitly by the condition

$$(\alpha + (1 - \alpha)p^0(\theta)) d_i = y_i + p^0(\theta)r(1 - y_i). \quad (6)$$

If (6) occurs with probability zero in the limit, then it is easy to see from the assumed convergence properties that

$$W^q(d_i, y_i, \alpha, \theta) \rightarrow W^0(d_i, y_i, \alpha, \theta), \text{ a.s.}$$

and hence

$$\lim_{q \rightarrow \infty} E [W^q(d_i, y_i, \alpha, \theta)] = E [W^0(d_i, y_i, \alpha, \theta)].$$

Let  $(d_i^0, y_i^0)$  denote the pair corresponding to the limiting consumption allocation  $x_i^0$  and let  $\{(d_i^q, y_i^q)\}$  denote the sequence of equilibrium choices converging to  $(d_i^0, y_i^0)$ . There may exist a set of states  $(\alpha, \theta)$  with positive measure such that  $(d_i^q, y_i^q)$  implies default in state  $(\alpha, \theta)$  for arbitrarily large  $q$  but that  $(d_i^0, y_i^0)$  does not imply default in state  $(\alpha, \theta)$ . Then at least we can say that

$$\liminf_q W^q(d_i^q, y_i^q, \alpha, \theta) \leq W^0(d_i^0, y_i^0, \alpha, \theta), \text{ a.s.}$$

and this implies that

$$\liminf_q E [W^q(d_i^q, y_i^q, \alpha, \theta)] \leq E [W^0(d_i^0, y_i^0, \alpha, \theta)].$$

Now suppose, contrary to what we want to prove, that  $(d_i^0, y_i^0)$  is not optimal. Then there exists a pair  $(d_i, y_i)$  such that  $E [W^0(d_i, y_i, \alpha, \theta)] > E [W^0(d_i^0, y_i^0, \alpha, \theta)]$ . If  $d_i = 0$  then (6) holds with probability zero and it is clear that for some sufficiently large value of  $q$ ,  $E [W^0(d_i, y_i, \alpha, \theta)] > E [W^q(d_i^q, y_i^q, \alpha, \theta)]$ , contradicting the equilibrium conditions. If  $d_i > 0$ , then either the critical condition (6) holds with probability zero or we can find a slightly lower value  $d' < d$

that does satisfy the critical condition. To see this, note first that the critical condition uniquely determines the value of  $p^0(\theta)$  as long as

$$(1 - \alpha)d \neq r(1 - y)$$

which is true for almost every value of  $d$ . Secondly, if the value of  $p^0(\theta)$  for which the critical condition is satisfied is an atom, we can always find a slightly smaller value  $d'_i < d_i$  such that the value of  $p^0(\theta)$  for which the critical condition is satisfied is not an atom. Furthermore, reducing  $d$  slightly will at most reduce the payoff by a small amount, so for  $d'_i < d_i$  and close enough to  $d_i$  we still have  $E[W^0(d'_i, y_i, \alpha, \theta)] > E[W^0(d_i, y_i^0, \alpha, \theta)]$ . Then this leads to a contradiction in the usual way.

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**Table 1**  
Equilibria for Example 1

#	$\varepsilon$	$\begin{bmatrix} E[U_s] \\ E[U_r] \end{bmatrix}$	$\begin{bmatrix} y_s \\ y_r \end{bmatrix}$	$\begin{bmatrix} (c_s^1(0), c_s^2(0)) \\ (c_s^1(1), c_s^2(1)) \\ (c_r^1(0), c_r^2(0)) \\ (c_r^1(1), c_r^2(1)) \end{bmatrix}$	$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
1.1A	0	0.081	0.800	(1.000, 1.500)	0.667
1.1B	0	$\begin{bmatrix} 0.081 \\ 0.081 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ 0 \end{bmatrix}$	$\begin{bmatrix} (1.000, 1.500) \\ (1.000, 1.500) \\ (1.414, 1.500) \\ (0.648, 0.648) \end{bmatrix}$	$\begin{bmatrix} 0.943 \\ 0.432 \end{bmatrix}$
1.2	0.010	$\begin{bmatrix} 0.078 \\ 0.078 \end{bmatrix}$	$\begin{bmatrix} 0.810 \\ 0.016 \end{bmatrix}$	$\begin{bmatrix} (0.998, 1.491) \\ (0.998, 1.520) \\ (1.405, 1.486) \\ (0.651, 0.651) \end{bmatrix}$	$\begin{bmatrix} 0.940 \\ 0.430 \end{bmatrix}$

**Table 2**  
Equilibria for Example 2

#	$\varepsilon$	$\begin{bmatrix} E[U_s] \\ E[U_r] \end{bmatrix}$	$\begin{bmatrix} y_s \\ y_r \end{bmatrix}$	$\begin{bmatrix} (c_s^1(0), c_s^2(0)) \\ (c_s^1(1), c_s^2(1)) \\ (c_r^1(0), c_r^2(0)) \\ (c_r^1(1), c_r^2(1)) \end{bmatrix}$	$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
2.1A	0	0.203	0.500	(1.000, 1.500)	0.667
2.1B	0	$\begin{bmatrix} 0.203 \\ 0.111 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0 \end{bmatrix}$	$\begin{bmatrix} (1.000, 1.500) \\ (1.000, 1.500) \\ (1.500, 1.500) \\ (0.563, 0.563) \end{bmatrix}$	$\begin{bmatrix} 1.000 \\ 0.375 \end{bmatrix}$
2.2	0.010	$\begin{bmatrix} 0.199 \\ 0.110 \end{bmatrix}$	$\begin{bmatrix} 0.508 \\ 0 \end{bmatrix}$	$\begin{bmatrix} (0.996, 1.497) \\ (0.996, 1.507) \\ (1.500, 1.500) \\ (0.561, 0.561) \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.374 \end{bmatrix}$

**Table 3**  
Equilibria for Example 3

#	$\varepsilon$	$\begin{bmatrix} E[U_s] \\ E[U_r] \end{bmatrix}$	$\begin{bmatrix} y_s \\ y_r \end{bmatrix}$	$\begin{bmatrix} (c_s^1(0), c_{sL}^2(0), c_{sH}^2(0)) \\ (c_s^1(1), c_{sL}^2(1), c_{sH}^2(1)) \\ (c_r^1(0), c_{rL}^2(0), c_{rH}^2(0)) \\ (c_r^1(1), c_{rL}^2(1), c_{rH}^2(1)) \end{bmatrix}$	$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
3.1A	0	0.081	0.800	(1.000, 1.500)	0.667
3.1B	0	$\begin{bmatrix} 0.080 \\ 0.080 \end{bmatrix}$	$\begin{bmatrix} 0.798 \\ 0 \end{bmatrix}$	$\begin{bmatrix} (0.998, 1.648, 1.283) \\ (0.998, 1.648, 1.283) \\ (1.364, 1.502, 1.503) \\ (0.682, 0.682, 0.682) \end{bmatrix}$	$\begin{bmatrix} 0.910 \\ 0.455 \end{bmatrix}$
3.2	0.010	$\begin{bmatrix} 0.077 \\ 0.077 \end{bmatrix}$	$\begin{bmatrix} 0.809 \\ 0 \end{bmatrix}$	$\begin{bmatrix} (0.995, 1.414, 1.624) \\ (0.995, 1.700, 1.366) \\ (1.360, 1.502, 1.503) \\ (0.678, 0.678, 0.678) \end{bmatrix}$	$\begin{bmatrix} 0.940 \\ 0.430 \end{bmatrix}$