The Insurance is the Lemon: Failing to Index Contracts[*]

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Abstract

We introduce a model to explain the widespread failure to index contracts to aggregate indices, despite the apparent risk-sharing benefits of indexation. Our model features these benefits, but demonstrates that asymmetric information about the ability of indices to measure underlying aggregate states can lead to risk-sharing failures and non-indexation. Suppose that a borrower receives an offer from a lender that features higher repayments in "good" states, in exchange for lower repayments in "bad" states. To make such an offer, a lender must ask for higher average repayments, because the lender is exposed to these aggregate risks. The borrower, however, is concerned that she is paying something for nothing; if the index is a poor measure of the true aggregate state, the cost of this contract might exceed its benefits. We provide conditions under which this effect is strong enough to cause the borrower to reject this contract, and choose a conventional, non-contingent contract instead. Under these conditions, many equilibria are possible, and they can be Pareto-ranked; the use of non-contingent contracts can be viewed as a coordination failure.

1   Introduction

Contracts often fail to share aggregate risk, despite the availability of indices that are related to the relevant risks. A leading example of this phenomena is standard mortgage contracts, in which a homeowner borrows from a lender, using a home as collateral. Homeowners face the risk that their home will decline in value, and could choose to hedge this risk, either through the mortgage contract or through another financial arrangement. Mortgage lenders are often more equipped to bear this risk than borrowers, and as such, it would be natural for lenders to offer borrowers repayment terms that are conditional on aggregate house prices. Our view is that there is a puzzle– homeowners could benefit from mortgage products that offered more risk sharing, mortgage lenders could feasibly offer such products by conditioning on house price indices, and yet such products are rarely available.

In this paper, we develop a model in which the failure of household financial contracts to condition on indices is an equilibrium outcome. In our model, a household seeks financing from set of financiers, whom we call lenders. There are potential risk sharing benefits between lenders and this household over some unobservable state (e.g. house prices). Since this state is unobservable, in order to realize any risk sharing benefits, contracts must condition on some imperfect measurement of the state (e.g. a house price index). Lenders know the true joint distribution of the index and the underlying state, i.e. the quality of the index, while borrowers do not.

At least two types of equilibria can arise in the model. In the first type of equilibrium, all lenders offer a contract which features the optimal amount of insurance conditional on the true quality of the index. Such an equilibrium always exists and features no loss in efficiency due to asymmetric information about the index. In the second type of equilibrium, lenders offer a contract that does not condition on the index. To see why such an equilibrium can obtain, consider the borrower's response when a single lender deviates and offers a contingent contract. To at least break even on such a contract, the lender must charge the household an insurance premium. At the same time, the household will be concerned that the index is in fact uncorrelated with the risk she is aiming to insure. As a result, the borrower will reject the indexed mortgage in favor of a

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standard non-contingent contract.

Our model in fact features many equilibria, depending on which contracts occur in equilibrium and which contracts do not. However, not every contract cannot occur in equilibrium; to be part of an equilibrium, a contract must be preferable to any "deviating" mortgage design, under the beliefs the homeowner holds after seeing the deviating contract offered. As a result, the equilibria that we construct involve contracts that are Pareto-efficient for some lender types, even though they may be inefficient for other lender types. We provide general conditions under which this risk-sharing failure can arise in equilibrium.

We also show an example in which all Pareto-efficient contracts are debt contracts, in which the level of debt is indexed. This example model features asymmetric information about idiosyncratic states, in the spirit of Townsend [1979], Gale and Hellwig [1985], and Hart and Moore [1998]. The asymmetric information problem pins down the structure of the contract conditional on the aggregate index. This asymmetric information also explains why using the aggregate index, rather than the idiosyncratic outcome, is useful– the aggregate index is not subject to the same information problems.

In this example, the "best" equilibrium contract in our model is a state-contingent debt contract, reminiscent of the one described by Innes [1993]. However, there is also an equilibrium in which the contract is a standard (non-indexed) debt contract. These equilibria are not equivalent, from a welfare perspective; for almost all lender types, ex-ante, the "best" contract can Pareto-dominate the non-contingent contract. This example is our resolution to the puzzle motivating this paper: contingent debt contracts might be optimal, but concerns about the relevance of the index can generate equilibria in which non-contingent contracts are used.

Our model builds on the literature on incomplete contracts, surveyed by Tirole [1999]. Papers focusing on incomplete contracts and asymmetric information include Spier [1992], Allen and Gale [1992], and Aghion and Hermalin [1990], among others.[1] Our model differs from most of

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[1]Papers that endogenize contractual incompleteness, but do not emphasize asymmetric information, include Anderlini and Felli [1994], Battigalli and Maggi [2002], Bernheim and Whinston [1998], Dewatripont and Maskin [1995], Kvaløy and Olsen [2009], Tirole [2009].

this literature in several respects. First, our model emphasizes competitive markets, rather than bilateral negotiation. Second, our model is focused on asymmetric information about the quality of the index, rather than the "fundamentals."

Our main intuition is similar in spirit to Spier [1992]; however, our argument does not hinge on transactions costs. Formally, the model is similar in some respects to Allen and Gale [1992], although the focus of that paper is the manipulability of the index. One can also view our model as related to models of insurance, in the vein of Rothschild and Stiglitz [1976]. The key difference between our model and those models is that our model places the information advantage and the competition on the same side of the market (with lenders), rather than on opposite sides of the market. Loosely speaking, the key intuition in our model is that the insurance itself might be a "lemon," in the sense of Akerlof [1970].

We also employ a general space of states and contracts. As a result, there is a great deal of scope for signaling, in contrast with the previous literature (in Spier [1992] and Aghion and Hermalin [1990], the contract space has one or two dimensions). As a consequence of this ability to signal, to generate our results, homeowners must be somewhat "paranoid," in the sense that they place non-zero probability on the index being irrelevant.

The failure of risk-sharing in our model can be described as a coordination failure. Individual lenders do not offer contracts with desirable risk-sharing features, because they fear losing business to their competitors. However, if all of the lenders offered such a contract, the offer would lose its signaling properties, and be accepted by the borrower, improving ex-ante welfare. As a result, there is the potential for policy to improve welfare in our model by ruling out undesirable equilibria. Our model does not feature any externalities as a result of this risk-sharing failure; the existence of such externalities would provide an additional motivation for policy interventions.

Our model is motivated by the example of homeowners and mortgages, although the model is abstract and could easily apply to other settings. In the context of home ownership, as noted by Sinai and Souleles [2005], purchasing a house hedges a homeowner against changes in future rents. Nevertheless, homeowners are exposed to both price and rent risks, and these could be hedged

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through the mortgage contract. Of course, as noted by Case et al. [1995] and Shiller [2008], homeowners could also hedge these risks through other financial markets, although this almost never actually happens. This failure to hedge might be explained by household's limited access to such markets, or by the sophistication required to hedge in this manner. However, these arguments suggest that it would be profitable for a financial intermediary to provide hedging services, and mortgage lenders appear to be ideally situated to do this as part of mortgage contracts. Proposals for mortgage reform after the recent financial crisis (Mian and Sufi [2015]) have advocated this approach. Although rare, shared appreciation mortgages are legal in the United States and used, for example, by Stanford University faculty who borrow from Stanford to purchase a house. [2]

We begin in section §2 by describing our model setup. In section §3, we discuss the zero-profit condition that arises from competition in our model, and characterize the "best" equilibria, which features contingent contracts. In section §4, we illustrate the intuition of our results using an example model. In section §5, we discuss our most general results, which describe assumptions under which risk-sharing fails and non-contingent contracts arise in equilibrium. In section §6, we provide an example security design problem that satisfies the assumptions in the preceding setting. In section §7, we describe a number of variations and extensions to our basic framework. Section 8 concludes.

2   Model Setup

In this section, we introduce our general model. In the general model, we treat the indirect utility functions of the borrower and lender as primitives, imposing minimal assumptions on their properties. In the example model we discuss in section §4, we specify particular functional forms.

At date zero, a borrower wishes to raise $K > 0$ dollars to pursue a project (e.g. purchasing a home). After raising the funds from a lender and investing them in the project, at date 1 an aggregate index $z \in Z$ and an aggregate state $a \in A$ will be determined. Additional, non-contractible

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[2]Stanford mortgages are indexed to an appraisal, rather than a local house price index, and involve renegotiation when the homeowner makes major investments.

actions might be taken by the borrower or lender at this point, and then the contract will determine the payoffs of the borrower and lender.

The aggregate index $z \in Z$ should be thought of an index based on the aggregate state $a \in A$. For simplicity, we will assume that both A and Z are finite, totally ordered sets. We will write $a \succ a'$ to denote the idea that the aggregate state $a \in A$ is "better than" the aggregate state $a' \in A$, and use the same notation for the aggregate index values. In the context of mortgages, the aggregate state $a \in A$ might influence house prices, borrower income, and the cost of capital for lenders. The aggregate index $z \in Z$ is an index that, perhaps imperfectly, measures these things, such as a local area house price index, a wage index, or an interest rate. Our notation will assume that A and Z are finite sets, but nothing relies on this.

The aggregate state influences the distribution of the borrower's idiosyncratic outcomes, $i \in I$. For a mortgage borrower, idiosyncratic outcomes could include the borrower's particular house price or income. The set I can be finite or infinite. A contract is a function $s : I \times Z \to \mathbb{R}^+$ that takes the idiosyncratic outcome i and aggregate index z and determines a payment from the borrower to the lender. We use the notation $s_z : I \to \mathbb{R}^+$ to refer to the "conditional contract," which is the contract for a particular value of the aggregate index, and the notation $s_{i,z}$ to refer to the value of security for a particular idiosyncratic state and index value.

The idiosyncratic outcomes may or may not be observable or contractible, and might be influenced by the borrower's behavior. Conditional on any particular index value $z \in Z$, the set of feasible contracts is S_I, which reflects the limits on the observability or verifiability of the idiosyncratic outcomes. We assume that S_I is compact, convex and contains the contract that pays nothing. The feasible set of contracts is S, which is the product space $Z \times S_I$, restricted to monotone contract designs. A contract s is monotone increasing if, for all $i \in I$ and $z, z' \in Z$ such that $z \succ z'$, $s_{i,z} \geq s_{i,z'}$. A contract is monotone decreasing if $s_{i,z} \leq s_{i,z'}$ for all $i \in I$, $z, z' \in Z$ with $z \succ z'$. The set S contains only monotone contracts, meaning that all $s \in S$ are either monotone increasing, monotone decreasing, or both. We discuss our motivation for this assumption below.

Given a particular state $a \in A$ and index value $z \in Z$, the borrower's indirect utility function

is $\phi(s_z, a)$. We refer to this as an indirect utility function because it summarizes the borrower's payoff, given some underlying relationship between the aggregate state a, conditional contract s_z, and the distribution of idiosyncratic outcomes. Similarly, we denote the lender's payoff as $\psi(s_z, a)$. As mentioned previously, in the general model, we treat these functions as primitives that satisfy several properties. First, we assume that both of these functions are continuous, and that $\psi(s_z, a)$ is weakly positive. The borrower's utility function also satisfies a monotonicity property: if $s'_{i,z} \geq s_{i,z}$ for all $i \in I$ and some $z \in Z$, then $\phi(s'_z, a) \leq \phi(s_z, a)$ for that $z \in Z$ and all $a \in A$. Intuitively, if the borrower must pay weakly more regardless of the outcome, he is worse off. For the lender, this property does not necessarily hold.

Second, we assume that there is a well-defined solution to the "ex-post" optimal contracting problem. That is, for any constant $\lambda \geq 0$ and any probability distribution $\pi(a) \in \mathcal{P}(A)$, the problem

$$s(\pi) \in \arg\max_{s \in S} \sum_{a \in A} \pi(a)\{\phi(s, a) + \lambda \psi(s, a)\}$$

has a convex set of solutions. As a result, there would never be any benefit to randomizing contracts.

The dependence of the borrower and lender's marginal utilities (with respect to contract payoffs) on the aggregate state is the force that causes them to want to condition their contract on the aggregate state. Of course, they cannot directly condition their contract on the aggregate state, only on the index. We next describe the relationship between the index and the aggregate state.

We define $\theta(a, z)$ as the joint distribution of the aggregate state and the aggregate index. This joint distribution is common knowledge amongst the lenders, but is not known to the borrower; it is the "type" in our adverse selection problem. The types θ are drawn from a convex set Θ. All of the elements of Θ have the same marginal distributions for $a \in A$ and $z \in A$, which we denote $p(a)$ and $q(z)$, respectively. Without loss of generality, we assume these marginal distributions have full support over A and Z, respectively. The borrower's prior belief over these types is $\mu(\theta)$. We assume that an uninformative type, $\theta_0(a, z) = p(a)q(z)$, is always in the support of μ.

In effect, the borrower is uncertain about the relationship between the aggregate index and the

aggregate state. A homeowner, for example, might not be certain how the S&P Case-Shiller index for his metro area is related to the price of his particular house. We assume that the borrower is aware of the marginal distributions, to abstract from the problems generated by that type of asymmetric information and focus on the borrower's doubt about the relevance of the index.

Having defined the indirect utility functions, contracts, and lender types, we are now in a position to describe the market for contracts. Let L be the set of lenders, with $|L| \geq 2$ lenders, each of whom can post a contract. After these lenders post contracts, the borrower can pick whichever one she prefers, or chose to forgo the investment opportunity. The outside options for both borrower and lender are zero. Note that, from the borrower's perspective, lenders are perfect substitutes.

Let $S^L = \biguplus_{l \in L} s^l$ be the multi-set containing the securities offered by each lender, given the common type θ. From lender l's perspective, the payoff of offering a contract s^l, when the other lenders offer contracts $S^{-l} = S^L \setminus \{s^l\}$ and the common type is θ, is

$$\rho(s^l, S^L)\{-K + \sum_{a \in A, z \in Z} \theta(a,z)\psi(s^l_z, a)\},$$

where $\rho(s^l, S^L)$ is the probability that the buyer accepts the contract s^l, given the contracts posted. This notation implicitly assumes that the buyer's decision does not depend on the identity of the lender, only on the contract that the lender offers. We will assume this in the equilibria we study, and note that it is consistent with the assumption that the borrower's utility does not depend on the lender she chooses, only on the design of the contract.

Assuming the borrower chooses to borrow, his expected payoff for security s is

$$\sum_{a \in A, z \in Z, \theta' \in \Theta} \mu(\theta'; S^L)\theta'(a,z)\phi(s_z, a),$$

where $\mu(\theta; S^L)$ denotes the beliefs the borrower holds about the distribution of the lender's common type θ, after observing S^L.

These beliefs are central to our theory. The borrower does not observe the lender's common type θ; initially, she has prior $\mu(\theta)$ over the set of types Θ, but might refine these beliefs based

on the menu of securities offered. It is important to note that, because the type θ is common across lenders, an optimal mechanism could allow the borrower to solicit this information and then negotiate a contract (Cremer and McLean [1988]). The market structure we impose, which we believe is realistic in many contexts, prevents the buyer from conducting this sort of auction.

Having discussed the basic structure of the model, we next describe the equilibrium concept and the refinements for off-equilibrium beliefs that we employ.

2.1 Equilibrium Definition

The basic equilibrium concept we use is perfect Bayesian. Given the strategies of the other lenders and the buyer, and the common type θ, we require that lender l posts

$$s^l \in \arg\max_{s \in \bar{S}} \rho(s, S^{-l} \uplus \{s\})\{-K + \sum_{a \in A, z \in Z} \theta(a,z) \psi(s_z, a)\},$$

if that strategy yields weakly positive profits, and otherwise does not participate. That is, each lender's choice of contract maximizes her utility, given the strategies of the other lenders and borrower.

If the borrower is offered any contracts, she must choose a strategy $\rho(s^l, S)$ such that, if $\rho(s^l, S) > 0$, then

$$s^l \in \arg\max_{s \in S} \sum_{a \in A, z \in Z, \theta \in \Theta} \mu(\theta; S) \theta(a,z) \phi(s_z, a),$$

and

$$\sum_{a \in A, z \in Z, \theta \in \Theta} \mu(\theta; S) \theta(a,z) \phi(s_z, a) \geq 0.$$

In words, the borrower must maximize his utility given the menu of contracts being offered.

The equilibrium strategies of the lenders create a correspondence $S^*(\theta)$ that describes the menu of securities that might be offered, given the common type. If the buyer observes a menu S for which there exists a type θ' such that $S = S^*(\theta')$, then she must update her beliefs according to

Bayes' rule:

$$\mu(\theta;S) = \frac{\mu(\theta)1(S = S^*(\theta))}{\sum_{\theta' \in \Theta} \mu(\theta')1(S = S^*(\theta'))}.$$

This does not, of course, pin down what the buyer believes when he observes some menu S that could not have been generated from the equilibrium strategies $S^*(\theta)$, for any $\theta \in \Theta$. For the purposes of determining if a conjectured set of strategies is an equilibrium, we only need to consider menus S that differ from a menu $S^*(\theta')$ for a single lender.

The result we are building towards is that there are many equilibria. This would, of course, be expected in the absence of refinements for off-equilibrium beliefs. Without refinements, the borrower can in effect dictate the contract by forming pessimistic beliefs when offered any other contract, justifying rejection. For this reason, we employ the standard $D1$ equilibrium refinement (Banks and Sobel [1987]). This refinement captures the intuition that, if confronted with a "deviating" contract, the borrower should belief the lender is the type that would benefit most from this deviation. We believe our results are robust to using other refinements that provide a similar intuition.

In our context, the borrower's strategies are simply the probability of accepting a particular contract. A lender of type θ offering contract s', instead of a contract s, would benefit, given that the buyer accepted the deviating contract with probability ρ, if

$$\rho\{-K + \sum_{a \in A, z \in Z} \theta(a,z)\psi(s'_z,a)\} \geq \rho(s,S^{-l} \uplus \{s\})\{-K + \sum_{a \in A, z \in Z} \theta(a,z)\psi(s_z,a)\}.$$

The types for whom the set of $\rho \in [0,1]$ satisfying this condition is maximal are the types with positive support in the buyer's beliefs, $\mu(\theta;S)$. Looking ahead, we will show that in equilibrium, lender profits are zero, due to the effects of competition. As a result, the $D1$ refinement will simply state that the buyer must place her the support of her beliefs on types that would weakly profit from offering the deviating contract, if that contract was accepted.

Our analysis will focus on a particular set of equilibria, symmetric pure-strategy equilibria. These equilibria are pure strategy equilibria and symmetric in the sense, for all types $\theta \in \Theta$, either

all of the lenders offer the same security with certainty, $s(\theta)$, or none of the lenders offer a security. They are also symmetric in the sense that the borrower, faced with a menu of identical securities, chooses each lender with probability $|L|^{-1}$.

3 Preliminary Analysis

We begin our analysis by focusing on the effects of competition. Consider a symmetric pure-strategy equilibrium, and imagine that the lender's profits from offering this contract are strictly positive. Intuitively, this could not be an equilibrium. Suppose a lender offered a deviating contract s' such that $s'_{i,z} \leq s_{i,z}$, strictly for some $i \in I, z \in Z$ that occur with positive probability. The buyer would be better off regardless of her beliefs, and therefore accept the contract with probability one. The lender, by sacrificing some profit, would capture the entire market, and be better off. Because of the monotonicity property of the buyer's indirect utility function and the continuity property of the lender's indirect utility function, standard Bertrand competition effects apply, and profits must be zero in equilibrium.

Lemma 1. *In any symmetric pure-strategy equilibrium, lender profits must be zero.*

Proof. See the appendix, section A.1. □

We next introduce an assumption to ensure that there are contracts which can satisfy both the lender and borrower's participation constraint.

Assumption 1. *There exists a contract $s \in S$ that offers weakly positive utility to the borrower, while satisfying the lender's participation constraint. That is, the problem*

$$\max_{s \in S} \sum_{a \in A, z \in Z} \theta_0(a,z)\phi(s_z,a)$$

subject to the constraint $\sum_{a \in A, z \in Z} \theta_0(a,z)\psi(s_z,a) \geq K$ is feasible and has a weakly positive solution.

Because we have assumed that the marginal distributions are the same for all types $\theta \in \Theta$, this assumption is sufficient to ensure that for any type, there is a contract that both the borrower and lender would be willing to accept under full information.

Next, we discuss the existence of a "best" equilibrium. Consider a symmetric, pure-strategy equilibrium, described by an offer of the contract $s(\theta)$. Suppose that the mapping between types θ and securities $s(\theta)$ is one-to-one. In this case, in equilibrium, the borrower knows the lenders' common type. Define a set of full-information optimal contracts as

$$\bar{s}(\theta) \in \arg\max_{s \in S} \sum_{a \in A, z \in Z} \theta(a,z)\phi(s_z,a),$$

subject to the constraint that $\sum_{a \in A, z \in Z} \theta(a,z)\psi(s_z,a) = K$. By 1, the solution to the above maximization can offer the buyer weakly positive utility for all types $\theta \in \Theta$.

A set of full-information optimal contracts is on the Pareto frontier for all θ, and offers the lender zero profit. As a result, for any deviating contract a lender might be willing to offer, if the borrower correctly inferred the lenders' true type, the borrower would weakly prefer the full-information optimal contract being offered. The D1 refinement in our model allows the borrower to make this inference, and the presence of a competing lender allows the borrower to choose the equilibrium full-information optimal contract instead of the deviating contract. The following proposition summarizes this logic:

Proposition 1. *The pure-strategy symmetric equilibrium $s(\theta) = \bar{s}(\theta)$ exists.*

Proof. See the appendix, section A.2. □

The above proposition describes a "best" pure-strategy symmetric equilibrium, in which a full-information optimal contract is offered. Our main results describe the conditions under which another type of pure-strategy symmetric equilibrium exists. This alternative equilibrium is notable because it uses a non-contingent contract, is a pooling equilibrium, and is Pareto-inferior to the "best" equilibrium, from an ex-ante perspective.

We say a contract is "non-contingent" if $s_z = s_{z'}$ for all $z, z' \in Z$; that is, the contract does not make use of the aggregate index. We will consider the existence of a non-contingent contract pooling equilibrium, in which, for all $\theta \in \Theta$,

$$s(\theta) = s^* \in \arg\max_{s \in S} \sum_{a \in A, z \in Z} p(a)q(z)\phi(s_z, a)$$

subject to the constraint that $\sum_{a \in A, z \in Z} p(a)q(z)\psi(s_z, a) = K$. By 1, this contract can offer the buyer weakly positive utility for all types $\theta \in \Theta$.

To ensure that our results are not trivial, we will assume that this equilibrium (if it exists) is ex-ante Pareto-inferior to the "best" equilibrium described previously. This assumption rules out some un-interesting cases, such as when the type space contains only uninformative indices, or when there is a full information optimal contract that is always equal to the non-contingent contract.

Assumption 2. *There exists a type $\theta \in \Theta$ for which $\mu(\theta) > 0$ and $s^* \neq \bar{s}(\theta)$ for all full-information optimal contracts.*

We will begin our discussion by analyzing our example model. The example model will provide a clear intuition about why a non-contingent equilibrium can exist, even though it is sub-optimal. We will then state a general condition on the indirect utility functions that is sufficient to guarantee the existence of a non-contingent equilibrium.

4   Risk-Sharing Failure: An Example

In this section, we introduce a simple version of our model, with functional forms for the indirect utility functions, to demonstrate the basic intuition behind our results. We start by assumption that the states $a \in A$ are ordered, from "bad" to "good." We use the notation $a \succ a'$ to say that a is "better" than a'.

We use the indirect utility functions

$$\phi(s,a) = \delta(a)(x(a) - s)$$

and

$$\psi(s,a) = \beta(a)s$$

for the borrower and lender, respectively. In this example, $x(a)$ should be interpreted as the borrower's idiosyncratic endowment. The idiosyncratic states are not contractible, and are one-to-one with the aggregate state. The space of contracts is a bounded, weakly positive payment for each realization of the index: $s_z \in [0,B]$, for some $B > K$.

The marginal utilities (with respect to security payments) $\delta(a) > 0$ and $\beta(a) > 0$ play a key role in our example. We normalize δ and β, given the marginal distribution of aggregate states, $p(a)$:

$$\sum_{a \in A} p(a)\delta(a) = \sum_{a \in A} p(a)\beta(a) = 1.$$

Because there is no room for inter-temporal trade in our model– the amount raised initially, K, is fixed– this normalization is without loss of generality.

To derive our result in this example, we impose a specific functional form on the marginal utilities.

Assumption 3. *For some $\alpha \in [0,1)$, $\beta(a) - 1 = \alpha(\delta(a) - 1)$.*

This assumption ensures that the borrower is always "more risk averse" than the lender, with respect to aggregate risks. As a result, it is never efficient for the borrower to insure the lender against idiosyncratic risk. We will not make any assumption on the set of types Θ, which explains why we require such a strong assumption on the marginal utilities.[3] In section 4.1 below, we use a weaker assumption on the structure of the marginal utilities, along with an assumption on the type space Θ, to generate the same equilibrium.

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[3]In fact, our results under 3 do not depend on the monotonicity of the security designs.

We are now in position to present our main result for this example.

Proposition 2. *In the example model, under assumptions 3, there exists a symmetric pure-strategy equilibrium in which $s(\theta) = s^*$.*

Proof. See the appendix, section A.3. □

The proof of this proposition involves considering potential "deviating" securities, and demonstrating the borrower can rationally reject those deviations. The first thing to note about this equilibrium, as opposed to the "best" equilibrium, is that the borrower can infer nothing from the securities that are offered by non-deviating lenders. As a result, there is still asymmetric information between the borrower and lenders, and the borrower updates his beliefs based only on the offer of the deviating lender.

The borrower's beliefs are restricted by the D1 refinement, which (as discussed previously) requires that the borrower place the support of her beliefs on types who could weakly profit from offering the deviating security, if the buyer accepted it. The proof holds fixed the lender's true type, θ, and considers two cases: either the deviating security "hedges" the lender, or it does not. For the purposes of this discussion, we will say that a security "hedges" the lender if the lender values it weakly more than a constant security with the same expected value.

Suppose that the deviating security hedges the lender. In this case, because the lender and borrower agree on which states are "good" and "bad" (assumption 3), providing this insurance to the lender is costly for the borrower. In fact, because the borrower's marginal utility is more sensitive to the aggregate state than the lender's (assumption 3), providing this insurance costs the borrower more than it benefits the lender. As a result, there is no "price" (reduction in payments) that the lender can offer the borrower such that both the lender and borrower are better off than in the non-contingent equilibrium. This argument does not depend on adverse selection– the borrower can place her beliefs entirely on the lender's true type in this case, and it doesn't matter. Because the insurance "goes the wrong way," the borrower can reject any deviating security that hedges the lender and that the lender would be willing to offer.

Now suppose that the deviating security does not hedge the lender, and in fact the lender values the security less than a constant security with the same expected value. In this case, if the borrower believes the lenders true type, he might accept the security, because the security provides the borrower with insurance. Of course, because the lender values the security less than its expected value, and must break-even, the lender would only offer securities whose expected value is greater than the expected value of the non-contingent security. This creates the "lemons" problem: a lender with an uninformative index would also happily offer such a security, because she would get the higher expected value without suffering any of the costs of providing insurance. The off-equilibrium beliefs of the borrower can therefore place all of their support on the uninformative type, and reject these deviations.

To summarize, the equilibrium is sustained by a conflict between the ability of lenders with informative indices to separate and the efficient direction of insurance provided by the contract. The only way for an lender with a high quality index to prove her index is good is to "purchase" insurance from the borrower, but because of the borrowers comparative risk aversion, this is never better than using a non-contingent contract.

4.1 A Second Example

In this example, we continue to use the indirect utility functions defined in the previous example, but relax our assumption of a linear structure in marginal utilities. The results in this section rely on the monotonicity of security designs with respect to the index. The intuition is similar to the idea in the previous section, in the sense that our assumptions will divided potential securities into two groups: those that insure the borrower and not the lender, and those that insure the lender but not the borrower. The role of the assumptions is to rule out the possibility of a security that would have positive insurance value to both the borrower and the lender.

We will offer two versions of the assumptions that are somewhat similar and work in essentially the same way. Both sets of assumptions rely on the idea that the aggregate states $a \in A$ and aggregate index values $z \in A$ are ordered. We will say that a function $f : A \to R$ is weakly decreasing

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if, for all $a, a' \in A$ such that $a \succ a'$, $f(a) \le f(a')$. We begin with the following set of assumptions:

Assumption 4. *The marginal utility of the lender, $\beta(a)$, is weakly decreasing in a. The difference of the marginal utilities, $\delta(a) - \beta(a)$, is also weakly decreasing in a. For all $\theta \in \Theta$, if $z \succ z'$, then the conditional distribution $\frac{\theta(\cdot, z)}{q(z)}$ weakly first-order stochastically dominates $\frac{\theta(\cdot, z')}{q(z')}$.*

This set of assumptions implies that better aggregate states are, intuitively, ones with lower marginal utility for both the borrower and lender. Moreover, the borrower is "more risk averse" than the lender in the sense that her marginal utility fluctuates more than the lender's marginal utility. Finally, we assume that "better" realizations of the index $z \in Z$ result in "better" marginal distributions of the aggregate state, in the sense of first-order stochastic dominance. In the terminology of Athey and Levin [1998], every type θ corresponds to a signal structure that is ordered with respect to the set of non-decreasing functions.

A slightly different set of assumptions allows us to achieve the same result. In this case, our results build on the literature on log-supermodular functions, in particular Athey [2002], described in Gollier [2004].

Assumption 5. *The marginal utility of the lender, $\beta(a)$, is weakly decreasing in a. The ratio of the marginal utilities, $\frac{\delta(a)}{\beta(a)}$, is also weakly decreasing in a. For all $\theta \in \Theta$, $\theta(a, z)$ is log-supermodular.*

The assumption that θ is log-supermodular (or "stochastically affiliated") is weaker than the assumption of conditional first-order stochastic dominance in the previous assumption. However, it also captures the notion that better values of the index are associated with between values of the aggregate state. Conversely, the assumption that the ratio of marginal utilities is decreasing, along with the assumption that $\beta(a)$ is weakly decreasing, implies that the difference of the marginal utilities is weakly decreasing. That is, 5 is weaker than 4 in terms of assumptions on the types $\theta \in \Theta$, but makes stronger assumptions about the behavior of the marginal utilities.

Under either of these assumptions, we can demonstrate the existence of a non-contingent equilibrium.

Proposition 3. *In our example model, if one (or both) of 4 and 5 hold, there exists a symmetric pure-strategy equilibrium in which $s(\theta) = s^*$.*

Proof. See the appendix, section A.4. Under 4, this is a special case of the general results in proposition 4. ☐

The intuition behind this proof is similar to the one behind our first example. The assumptions we use in the more general case are designed to ensure that there is some notion of "ordering" which creates a conflict between the desire of the agents to share risk and the desire of lenders with a good index to separate from lenders with a bad index. Under our assumptions, this conflict cannot be resolved, and the non-contingent equilibrium exists. Of course, the full-information optimal contract is also an equilibrium.

The purpose of these examples is to illustrate the economic intuition behind our main results, which apply to a more general class of indirect utility functions. We describe these results in the next section.

5 Risk-Sharing Failure with Security Design

In the previous section, we demonstrated an example of how a non-contingent equilibrium can arise, even though it is Pareto-inferior to the full-information optimal contract, which is also an equilibrium. In this section, we provide a sufficient condition that summarizes the same ideas, but for a particular class of indirect utility functions $\phi(s,a)$ and $\psi(s,a)$. Our assumptions about this class of indirect utility functions are motivated by security design problems that have been studied in the literature. These security design problems share a common feature, which is that, holding the aggregate state $a \in A$ fixed, the set of Pareto-optimal securities is a family of security designs indexed by a single parameter. For example, in many models of security design (e.g. Hart and Moore [1998], Innes [1990], Townsend [1979]), regardless of the parameters, the set of Pareto-optimal securities is the set of debt contracts.

This fact implies that (in those papers) any debt contract is "renegotiation-proof," in the following sense. If the agents chose a debt contract, perhaps conditional on some realization of the index $z \in Z$, and then learned the true aggregate state $a \in A$, there would be no benefit to a renegotiation of the contract. More formally, consider the "social welfare function," conditional on a particular aggregate state and security,

$$U(s,a;\lambda) = \phi(s,a) + \lambda \psi(s,a).$$

If the set of Pareto-optimal securities in each aggregate state a is the set of debt contracts, then, for a fixed debt contract, there exists a Pareto-weight $\lambda(a)$ such that the given debt contract is the maximizer of the social welfare function.

Our key assumption in this setting is that the optimal non-contingent contract, s^*, is renegotiation-proof in this sense.

Assumption 6. *Given the non-contingent contract s^*, for each state $a \in A$, there exists a Pareto-weight $\lambda(a)$ such that*

$$s^* = \arg\max_{s \in S_I} U(s;a, \lambda(a)).$$

This assumption is somewhat weaker than the assumption that the set of Pareto-optimal securities is the same for all states; it only requires that the non-contingent optimal contract be in that set for all states. Note that, because s^* offers a positive payoff to lender, the Pareto-weights $\lambda(a)$ are all strictly positive. Additionally, by the definition of s^*, there exists a Pareto-weight λ^* such that

$$s^* = \arg\max_{s \in S_I} \sum_{a \in A} p(a)U(s;a, \lambda^*).$$

Our next set of assumptions imposes concavity on the lender's indirect utility function.

Assumption 7. *For all $a \in A$ and $s \in S_I$, the lender's indirect utility function, $\psi(s,a)$, and the social welfare function under Pareto-weight λ^*, $U(s,a;\lambda^*)$, are concave in s. For all $a \in A$, the indirect utility functions $\psi(s,a)$ and $\phi(s,a)$ are locally Lipschitz continuous in s.*

Loosely speaking, 6 and 7 ensure that the logic applied in our example (proposition 3) apply

in the more general security design case. The concavity of the lender's indirect utility function ensures that only monotone decreasing security designs can be used to "screen out" the uninformative types. The concavity of the social welfare function ensures that, if we can rule out "small" security deviations, we can rule out all deviations. These assumptions make our proof easier, but are likely stronger than necessary for our results. They also allow us to define a notion of marginal utility, which we will use in our last set of assumptions.

Recall that the "subdifferential" of a convex function $f(x)$ is a set of vectors $x^* \in \partial f(x)$ such that, for any x, x',

$$f(x) + x^* \cdot (x' - x) \leq f(x').$$

We will denote Clarke's generalized subdifferential as $\bar{\partial}$. These generalized subdifferentials can be defined on a larger set of functions, including all locally Lipschitz-continuous functions.

Our next assumption introduces an ordering on the Pareto weights and lender "marginal utilities", which are defined using the subdifferentials of the lender's indirect utility function. We avoid assuming differentiability because many hidden information security design problems (such as Townsend [1979] or Hart and Moore [1998]) do not have differentiable indirect utility functions. These assumptions are a generalized version of the assumptions we used in 4.

Assumption 8. *Given the non-contingent contract s^*, the Pareto-weight of the non-contingent security, $\lambda(a)$, is weakly decreasing in a. For all $\theta \in \Theta$, if $z \succ z'$, then the conditional distribution $\frac{\theta(\cdot, z)}{q(z)}$ weakly first-order stochastically dominates $\frac{\theta(\cdot, z')}{q(z')}$. There exist sequences of subdifferentials, $w_a^* \in \bar{\partial}(-\phi(s^*, a))$ and $x_a^* \in \bar{\partial}(-\psi(s^*, a))$, defined over $a \in A$, such that $\lambda(a)x_a^* + w_a^* = \vec{0}$ and x_a^* is weakly increasing.*

This assumption captures the same ideas behind our example: better aggregate states are ones in which, at least under the non-contingent optimal contract, the lender's marginal utility is low, but the borrower's marginal utility is lower. Moreover, better values of the index $z \in Z$ are associated with better distributions of the aggregate state. The existence of these sequences of supergradients whose weighted sum is zero is guaranteed by the Pareto-optimality of s^*, 6, and by the properties of

Clarke's generalized subdifferential. The additional assumption imposed by 8 is the monotonicity of the marginal utilities (x_a^*). Assumption 8 discusses the case when bad states are associated with higher "marginal utilities" for the borrower and lender, which is intuitive. However, the proof would also work if the Pareto weights $\lambda(a)$ and marginal utilities x_a^* were weakly increasing in a. The key idea is that the lender and borrower's marginal utilities co-move together, and that the borrowers marginal utilities are more sensitive to the aggregate state than the lenders.

Under these assumption, we prove a general result.

Proposition 4. *In our general model, under assumptions 6, 7, and 8, there exists a symmetric pure-strategy equilibrium in which $s(\theta) = s^*$.*

Proof. See the appendix, section A.7. □

This proposition establishes a sufficient for the existence of a non-contingent equilibrium. Intuitively, if it is not efficient for the borrower to hedge the lender, the deviations necessary to separate from the type with the uninformative index are never welfare-improving. Our assumptions are designed to ensure that this is the case. Given a particular specification for the indirect utility functions, our assumptions can be checked, and used to demonstrate the existence of a risk-sharing failure. We speculate that similar results could be shown under assumptions analogous to 5, instead of 8.

In the next section of the paper, we discuss a standard security design problem that satisfies the assumptions of our general model.

6 A Mortgage Example

In this section, we discuss a simple example of mortgage lending that illustrates the intuition of the results of the previous section. Specifically, we consider a setting in which a mortgage borrower has hidden information about her endowment. As a result, the borrower can only raise financing through debt-like contracts, in which the house serves as collateral, along the lines of Hart and

Moore [1998]. These contracts sometimes cause inefficient liquidation in equilibrium. Moreover, the degree of inefficiency of liquidation depends on the aggregate state $a \in A$. As a result, there are benefits to writing contracts in which the face value of the debt depends on the aggregate state. We will show that under mild assumptions, this model of mortgage lending satisfies the assumptions given in the previous section. Therefore, there exists an equilibrium in which the face value of the debt does not depend on the aggregate state, despite the benefits of such contracts.

The model setup is a specialized version of the general setup described in section §2, subject to a technical caveat. There is an aggregate index $z \in Z$ and aggregate state $a \in A$, exactly as described in that section. We will assume that the set of lender types Θ satisfies the first-order stochastic dominance condition of 8. We will specialize the indirect utility functions and space of idiosyncratic states to a particular model of mortgage lending.

The borrower's has an idiosyncratic endowment at date 1, which is a random variable $e \sim F(e)$, with full support on $[0, \bar{e}]$.[4] The realization of e is hidden to lenders and non-verifiable; the borrower can make a report $\tilde{e}$. In the notation of our general model, the idiosyncratic states I are pairs $(e, \tilde{e})$. A security is a map from these idiosyncratic states and the aggregate index to payments, $Z \times I \to \mathbb{R}^+$, as in the general model. The set of admissible contracts conditional on the index value, S_I, is restricted to depend only on the report, $\tilde{e}$, and not on the actual endowment e. For mathematical convenience, we also restrict the set S_I to be weakly increasing in the report, $\tilde{e}$.

In this model, the contract payment is not necessarily the payment the lender will receive. The borrower has an inalienable option to sell the house, which results in proceeds $L(a) \in [0, K]$. The lender has priority to the proceeds from the sale of the house, meaning that $L(a)$ first goes towards paying off the contract, and anything remaining goes to the borrower. However, if $L(a)$ is insufficient to cover the payment demanded by the contract, the lender has no additional recourse. We have assumed that, in all states $a \in A$, the lender recovers less than the cost of their initial investment liquidation occurs.

If the borrower chooses to sell the house, she bears a private, monetary cost $C(a) \geq 0$. We

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[4]The distribution of e could depend on the aggregate state $a \in A$; such dependence would not change the results.

assume that the private cost of liquidation is large relative to the endowment, $C(a) \geq \bar{e}$ for all $a \in A$, and that liquidation is inefficient, $C(a) \geq L(a)$ for all $a \in A$. If the borrower is unable to pay the required repayment out of her endowment, he must liquidate.

The borrower is risk-neutral over the value of her endowment net of the contract repayments and/or liquidation costs. Given an endowment e, a contract s, a realization of the index z, and an aggregate state a, the borrower will choose an optimal report $\tilde{e}^*(e; s_z, a)$ and liquidation probability $\rho^*(e; s_z, a)$ to solve

$$(\tilde{e}^*(e; s_z, a), \rho^*(e; s_z, a)) = \arg\min_{\tilde{e}, \rho} \{(1 - \rho)s_z(\tilde{e}) + \rho(\min\{s_z(e), L(a)\} - L(a) + C(a))\}$$

subject to the constraint that, if the borrower does not liquidate, repayment must be feasible:

$$e - (1 - \rho)s_z(\tilde{e}) - \rho(\min\{s_z(e), L(a)\} - L(a)) \geq 0.$$

In other words a feasible strategy is one for which the borrower has enough cash, either via her endowment or from the proceeds from liquidation, to cover the required repayment.

Because $s(\tilde{e})$ is weakly increasing, it is without loss of generality to assume that $\tilde{e}^*(e; s_z, a) = 0$ in all states. Moreover, the borrower will default only when she is unable to make the required repayment:

$$\rho^*(e; s_z, a) = \mathbf{1}(e \geq s_z(0)).$$

In words, the borrower's payment in always weakly increasing in her report, regardless of her liquidation decision, and as such she will always report the minimum possible endowment. Moreover, since liquidation is inefficient, the borrower will only default when her endowment is insufficient to pay the lowest possible amount she could pay given s_z. Thus, any security in this model implies an allocation that is equivalent to a security that do not depend on the report $s_z(\tilde{x}) = \bar{s}_z$ for all $\tilde{x}$. Such securities are essentially defaultable mortgages as they require the payment of some face value $\bar{s}_z$. If repayment is not made, liquidation (i.e., foreclosure) occurs and the lender receives the minimum of the liquidation value and the face value. In this way, this simple model gives rise to

the common structure of mortgage lending that we observe in practice.

We denote the lender's marginal utility in aggregate state a as $\beta(a) > 0$. For the borrower, the marginal utility is endogenous to the problem. For the lender, who is presumed to be connected to the broader financial markets, is risk-neutral with respect to the borrower's idiosyncratic outcome but not with respect to aggregate risk. We normalize the marginal utility so that

$$\sum_{a \in A} p(a)\beta(a) = 1.$$

We are now in a position to see that our model of mortgage lending satisfies assumptions 6, 7, and 8 of the previous section. The indirect utility function of the lender is given by

$$\psi(s_z, a) = \beta(a)(1 - F(s_z(0))s_z(0) + \beta(a)F(s_z(0))\min\{s_z(0), L(a)\}. \tag{1}$$

while the borrower's indirect utility function is given by

$$\phi(s_z, a) = E[e] - (1 - F(s_z(0))s_z(0) - F(s_z(0))(\min\{s_z(0), L(a)\} + C(a) - L(a)) \tag{2}$$
$$= E[e] - \beta(a)^{-1}\psi(s_z, a) - F(s_z(0))(C(a) - L(a)).$$

We next discuss our assumptions on the magnitudes and orderings of the variables. We will show that these assumptions are sufficient to establish that the assumption in the general model apply to this particular example.

Assumption 9. *The liquidation value, $L(a)$, is weakly increasing in a. The lender's marginal utility, $\beta(a)$, is weakly decreasing in a, and the lender's risk-neutral valuation of the collateral, $\beta(a)L(a)$, is weakly decreasing in a. The inefficiency of liquidation, $C(a) - L(a)$, is decreasing in a.*

Intuitively, good states are ones with high liquidation values, low marginal utilities, and relative efficiency conditional on liquidation. The assumption that $\beta(a)L(a)$ is weakly decreasing ensures that the lender's marginal utility of promised payments is higher in bad states than good. That is,

the higher expected value of a promised payment (due to higher liquidation values) is more than offset by the lower marginal utility in good states of the world.

Assumption 10. *The endowment is uniformly distributed on $[0, \bar{e}]$, with $\bar{e} \geq 4K$. For all $a \in A$,*

$$\beta(a) \geq \sum_{a \in A} p(a) \frac{\bar{e}}{C(a)}.$$

The quantity

$$\frac{\bar{e} + C(a) - L(a)}{\beta(a)\bar{e}}$$

is decreasing in a.

The assumption of a uniform distribution makes the other assumptions in 10 easy to state. The assumption that $\bar{e} \geq 4K$ ensures that default will not occur too frequently under the non-contingent contract, s^*. It is necessary to ensure that the lender benefits (weakly) when the borrower promises more repayment, in the neighborhood of s^*. The lower bound on $\beta(a)$, which is of course less than one by assumption, ensures that the social welfare function is concave by ruling out states where the lender was very lower marginal utilities. The last part of the assumption says that the decreasing inefficiency of liquidation in good states dominates the lender's decrease in marginal utility. As a result, the borrower is "more risk averse" with respect to aggregate risk than the lender.

The following three lemmas verify that the assumptions of the general model are satisfied by these indirect utility functions.

Lemma 2. *The indirect utility functions defined in equation* (2) *and equation* (1)*, under assumptions 9 and 10, satisfy 6.*

Proof. See the appendix, section A.9. □

Lemma 3. *The indirect utility functions defined in equation* (2) *and equation* (1)*, under assumptions 9 and 10, satisfy 7.*

Proof. See the appendix, section A.10. □

Lemma 4. *The indirect utility functions defined in equation* (2) *and equation* (1)*, under assumptions 9 and 10, satisfy 8.*

Proof. See the appendix, section A.11. ☐

Because the indirect utility functions in this example satisfy the conditions of our general theorem, we have the following corollary:

Corollary 1. *In the mortgage model described in this section, there exists an equilibrium characterized by non-contingent debt contracts.*

Proof. This follows from lemma 2, lemma 3, lemma 4, and proposition 4. ☐

The result in corollary 1 summarizes our answer to the question, "why aren't contracts indexed?" It illustrates a concrete example in which debt contracts indexed to an aggregate index would be welfare-improving compared to debt contracts that are not indexed, but the latter arise in equilibrium due to adverse selection about the quality of index.

The intuition behind the non-contingent equilibrium in the context of this simple example of mortgage lending hinges on the incentives for lenders to offer contingent securities even when the index is a poor measurement of the true aggregate state. To see this, suppose there are just two aggregate states, g(good) and b (bad), two possible indices, an index that is perfectly correlated with the aggregate state and one that is uncorrelated, and that the borrowers endowment is uniformly distributed on [0,1]. Further suppose that $N-1$ lenders offer a mortgage with a face value $s^\star$ that is not contingent on the outcome of the index. How will the borrower respond if the Nth lender offers a contingent on the index? The answer to this question hinges on how the offer depends on the index. First, consider the case when the offered security has a higher face value in the bad index state than the good index state, i.e., $s_g < s_b$. If the index is informative, the lender would be willing to pay an insurance premium to the lender, but by our assumptions on the inefficiency of liquidation relative the lenders preferences β, this premium must be below what the borrower would require to accept the contract in favor of the non-contingent contract. Thus, the borrower will reject such an offer.

Perhaps the more interesting case is when the offered security has a higher face value in the good index state than the bad index state, i.e., $s_g > s_b$. This is a type of "shared appreciation" mortgage in that the borrower owes more in the state that house prices increase. For simplicity, let $s_g > L(g)$ and $s_b < L(b)$. Let $p(g)$ and $p(b)$ be the probabilities of the good state and bad state, respectively. If the lender knows that the index is perfectly correlated with aggregate state, then in order for the lender to break even, we must have

$$p(g)\beta(g)((1-s_g)s_g + s_g L(g)) + p(b)\beta(b)s_b \geq K.$$

Since $\beta(g) < 1 < \beta(b)$, $s_g > L(g)$, and $s_b < L(b)$, we have

$$p(g)\beta(g)((1-s_g)s_g + s_g L(g)) + p(b)\beta(b)s_b < p(g)\beta(g)((1-s_g)s_g + s_g L(g)) + p(b)\beta(b)s_b$$
$$< \beta(g)(p(g)((1-s_g)s_g + s_g L(g)) + p(b)s_b)$$
$$< p(g)((1-s_g)s_g + s_g L(g)) + p(b)s_b.$$

This in turn implies that

$$p(g)((1-s_g)s_g + s_g L(g)) + p(b)s_b > K.$$

In other words, the contract must pay an insurance premium to the lender. But that means that the lender would be just as willing to offer this contract if she knows that the index is uninformative. Thus, D1 requires that the borrower believe that the index is uninformative, and she will reject it. To sum up, no matter what alternative to $s^\star$ the Nth lender offers, either the security insure the lender, which in which the borrower will reject, or the security "insures" the borrower, in which case she must believe that the index is uninformative in response. Regardless, the Nth lender has no incentive to deviate and $s^\star$ is an equilibrium.

To see why it is surprising that the non-contingent equilibrium exists in this simple example of mortgage lending, it is informative to consider a possible equilibrium in which the security is

contingent on the index. Suppose the index is perfectly correlated with the aggregate state, an $N-1$ lenders offer the best contingent security. That is, when the index is informative, $N-1$ lenders offer the security $\{s_g^\star, s_b^\star\}$ that maximizes the borrower's indirect utility subject to lenders participation constraint given that the index is informative. Given our assumptions on $C(a) - L(a)$ and $\beta(a)$, this will call for the mortgage to have a higher face value in the good state of the index and will reduce the inefficiency from liquidation realized in equilibrium, relative to a non-contingent mortgage. It is then immediately clear that the Nth lender cannot profitable deviate, since any alternative to $\{s_g^\star, s_b^\star\}$ that satisfies the lenders participation constraint would leave the borrower with lower utility and she would reject it. When the index is independent of the aggregate state, all lenders will simply offer the optimal non-contingent contract and the argument ruling out a profitable deviation is equivalent to the case when the index is informative. Thus is possible to support an equilibrium allocation equivalent to the first-best, i.e., as-if all agents have common knowledge about the quality of the index. This intuition reinforces our main point. Although asymmetric information information about the index quality does not preclude the possibility of first best risk sharing, it can lead a "bad" equilibrium in which securities, in this case mortgages, are not contingent on the index.

7 Variations and Extensions

In this section, we discuss modifications and extensions to the model. We begin by discussing a model with positive profits for lenders, that nevertheless retains the competition between lenders. In this case, our results go through essentially unchanged. We then discuss what would happen with a single, monopoly lender. We will see that neither the "best" equilibrium or the non-contingent equilibrium arise in the presence of a monopoly.

7.1 Profitable Lending

In this extension, we describe a model in which lenders make positive profits in equilibrium, but nevertheless face competition. We introduce profits into the economy by assuming that each lender faces a convex cost in the number of loans she makes, and that there is a unit mass of borrowers.[5] Let Q_l be the number of loans made by lender l. A lender of type θ who makes Q loans using contract s earns

$$\Pi(s, Q, \theta) = Q\{ \sum_{a \in A, z \in Z} \theta(a, z) \psi(s_z, a)\} - C(Q),$$

where $C(Q)$ is a convex, twice differentiable function with $C(0) = 0$ and $C'(|L|^{-1}) = K$.

In this analysis, the $D1$ refinement can be more complicated than in our baseline model. If one considers a deviation in which the lender offers a single, marginal borrower a different contract, then the criteria the same as in our main analysis, because (in equilibrium) the marginal profit of each lender is zero. If however, the lender contemplates a deviation in which he offers a deviating contract to all borrowers, then substantial profits could be at stake, because the average profits of lenders are positive.

In this case, the D1 refinement requires that the borrower place her beliefs on the lender type who would break-even under the smallest amount of the demand for the deviating contract. This is equivalent to saying that the borrower must believe the lender is of a type for whom the difference between the marginal profit of the deviating contract and the marginal profit of the equilibrium contract is maximal.

Surprisingly, perhaps, our non-contingent equilibrium exists under the same conditions in this model. The intuition comes from the proof description in section §4. When a lender with a "good" index offers a contract that insures the borrower, the lender requires a higher expected value of repayments to be indifferent between the deviating contract and the non-contingent contract. However, a lender with an irrelevant index could offer the same deviating contract at a profit, and therefore (in the case of profitable lending) the borrower must believe that the lender is of this type,

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[5]Introducing profits in this way is an old idea, described in the textbook of Tirole [1988].

or of a type that is even worse from the perspective of the borrower.

7.2   Monopoly Lending

In this extension, we consider what type of equilibrium can obtain when the lender has monopoly power. Specifically, we assume that a single lender can make a take it or leave it offer to the borrower and that if the borrower rejects this offer, she receives an exogenous outside option $\bar{\phi}$. In this case, subject to some restrictions on the space of possible indices Θ and preferences, the lender can credibly signal the quality of the index by offering a contract that partially insures the aggregate state.

To simplify the analysis, consider the special case of our example model in which there a just two possible qualities for the index. In this case, it is without loss of generality to assume that the index is either perfectly correlated with, or independent of, the aggregate state. If the borrower knows the quality of the index, then the unique equilibrium calls for the lender to offer the borrower a contract that is non-contingent if the index is independent of the aggregate state, or only call for repayment in the good state if the index is perfectly correlated with the index. This is essentially identical to the fully efficient equilibrium that obtains under competition, save for the fact that the lender now captures all the surplus.

However, the full-information equilibrium is not obtainable under asymmetric information. The key intuition is that under this equilibrium, the lender's payoff is the sum of the surplus generated by financing the household and the surplus generated by insuring the household. The latter is always greater when the index is more correlated with the aggregate state. Thus, if the lender does not have to take any further action to receive the surplus generated by the insurance with an informative index, then the lender will always attempt to collect this payoff, even when the index is poor quality. In the language of standard signaling games, the low type will always seek to emulate the high type if she can do so without cost.

Interestingly, a pooling equilibrium can also not obtain under monopoly lending with asymmetric information about the quality of the index. Indeed, such an equilibrium would violate D1.

To see this, note that it is less costly for the lender to offer a contract that calls

8   Conclusion

We have introduced a theory to explain the widespread lack of indexation present in contracts. Intuitively, when a borrower is offered a contract that includes insurance, she is concerned that the insurance is not actually relevant for the risks she faces. Under the conditions described in our model, this effect is strong enough to allow the borrower to reject that offer, and choose instead a contract without insurance from a different lender. As a result, equilibria that feature little or no risk-sharing can arise, even though they are ex-ante Pareto-dominated by equilibria that feature full risk-sharing. This phenomena can be viewed as a coordination failure, and policies that induce the offer of contracts with desirable risk-sharing properties can improve welfare.

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A Proofs

A.1 Proof of lemma 1

First, note that, for any values of θ for the which the lenders do not offer a security, profits are zero.

Proof by contradiction: suppose that there exists a symmetric pure-strategy equilibrium such that, for some values of $\theta \in \Theta$, the security $s(\theta)$ is offered and equilibrium lender profits are strictly positive.

Let θ' and $s' = s(\theta')$ denote the equilibrium type and security for which lender profits are positive. In this equilibrium, each lender earns

$$|L|^{-1}(\sum_{a \in A, z \in Z} \theta'(a,z)\psi(s'_z,a) - K) > 0.$$

Consider a deviation by some lender to the security $s'' = \alpha s'$, for $\alpha \in (0,1)$. By assumption, $s'' \in S$. By the monotonicity property of the borrower's indirect utility function, $\phi(s_z,a)$, we have

$$\sum_{a \in A, z \in Z} \theta(a,z)\phi(s''_z,a) > \sum_{a \in A, z \in Z} \theta(a,z)\phi(s'_z,a)$$

for all $\theta \in \Theta$. It follows that, regardless of the beliefs the borrower forms off-equilibrium, she will accept security s'' if offered, for any value of $\alpha \in [0,1)$.

The change in profits for the deviating lender are

$$\sum_{a \in A, z \in Z} \theta'(a,z)(\psi(s''_z,a) - |L|^{-1}\psi(s'_z,a)).$$

By the continuity of ψ and the fact that $|L| > 1$, there exists an $\alpha \in (0,1)$ such that this quantity is positive. It follows that an equilibrium with lender profits cannot exist.

A.2 Proof of proposition 1

By 1, this equilibrium delivers weakly positive utility for the borrower. Therefore, the borrower is willing to participate, and lenders earn zero profits (by the construction of $\bar{s}(\theta)$) and therefore are also willing to participate.

Now consider a deviation by a single lender: suppose some lender of type θ offers security s' instead of $\bar{s}(\theta)$, and would weakly profit from doing so if the security was accepted. Because the lender can weakly profit from offering this deviation, the borrower is free to place the full support of her beliefs on the lender's true type. Because the security $\bar{s}(\theta)$ is on the Pareto-frontier, and offers zero profit to lenders, it follows that the borrower must be weakly worse off using security s', and therefore would prefer the security $\bar{s}(\theta)$. Because there is more than one lender ($|L| > 1$), the borrower can choose the non-deviating lender and reject the deviating security. Given that the security will be rejected, the lender does not profit from offering it, and therefore $s(\theta) = \bar{s}(\theta)$ is an equilibrium.

A.3 Proof of proposition 2

The non-contingent security s^* has payoffs that do not depend on the index. As a result, it offers zero profits for the lender, regardless of the lender's type, by the assumption that all $\theta \in \Theta$ have the same marginal distribution with respect to the aggregate state. By 1, s^* can deliver positive utility to the borrower, and therefore the participation constraints are satisfied in this equilibrium.

We now consider deviations from this equilibrium. We will divide our analysis into two cases: when the deviating security s', offered by a lender of type θ, hedges the lender,

$$\sum_{a \in A, z \in Z} \theta(a,z)\beta(a)s'_z \geq (\sum_{a \in A} p(a)\beta(a))(\sum_{z \in Z} q(z)s'_z),$$

and when that condition does not hold. We will assume that the security can be profitably offered

by the lender of the given type:

$$\sum_{a \in A, z \in Z} \theta(a,z) \beta(a) s'_z \geq K.$$

Consider the first case when the security hedges the lender in this way. Suppose that the security s' satisfies these conditions and is offered to the borrower. The borrower is able, under the D1 refinement, to place the support of her beliefs entirely on the lender's true type.

By 3, for any probability distribution $m(a) \in \mathscr{P}(A)$,

$$\sum_{a \in A} m(a) \delta(a) \geq \sum_{a \in A} m(a) \beta(a).$$

By the assumption that that $K > 0$ and $\beta(a) > 0$, the deviating security s' must contain a strictly positive payment. Therefore, we can define a probability distribution by

$$m(a) = \frac{\sum_{z \in Z} \theta(a,z) s'_z}{\sum_{a \in A, z \in Z} \theta(a,z) s'_z} = \frac{\sum_{z \in Z} \theta(a,z) s'_z}{\sum_{z \in Z} p(z) s'_z}$$

Therefore,

$$\sum_{a \in A, z \in Z} \theta(a,z) \delta(a) s'_z \geq \sum_{a \in A, z \in Z} \theta(a,z) \beta(a) s'_z.$$

By the fact that the lender is willing to offer the security,

$$\sum_{a \in A, z \in Z} \theta(a,z) \delta(a) s'_z \geq K.$$

Now consider the payoff under the security s^*. We have

$$\sum_{a \in A, z \in Z} \theta(a,z) \beta(a) s^* = K$$

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for all $\theta \in \Theta$. Using the definition of the indirect utility functions in our example,

$$\sum_{a \in A, z \in Z} \theta(a,z)(\phi(s'_z,a) - \phi(s^*,a)) = \sum_{a \in A, z \in Z} \theta(a,z)\delta(a)(s^* - s'_z)$$

$$= K - \sum_{a \in A, z \in Z} \theta(a,z)\delta(a)s'_z.$$

Using the result derived above,

$$\sum_{a \in A, z \in Z} \theta(a,z)(\phi(s'_z,a) - \phi(s^*,a)) \leq 0,$$

and therefore any deviation in this category can be rejected.

Next, consider the case of

$$\sum_{a \in A, z \in Z} \theta(a,z)\beta(a)s'_z < (\sum_{a \in A} p(a)\beta(a))(\sum_{z \in Z} q(z)s'_z).$$

This can be rewritten as

$$\sum_{a \in A, z \in Z} \theta(a,z)\beta(a)s'_z < \sum_{a \in A, z \in Z} \theta_0(a,z)\beta(a)s'_z,$$

and by the fact that the lender of type θ is willing to offer s'_z,

$$\sum_{a \in A, z \in Z} \theta_0(a,z)\beta(a)s'_z > K.$$

Suppose that the borrower believes the lender's type is θ_0, which he is allowed to do under the D1 refinement by the condition just derived. By the Pareto-efficiency of s^* and the fact that

$$\sum_{a \in A, z \in Z} \theta_0(a,z)\beta(a)s^* = K,$$

it must be the case that

$$\sum_{a \in A, z \in Z} \theta_0(a,z)(\phi(s'_z,a) - \phi(s^*,a)) < 0,$$

and therefore the security can be rejected.

It follows that every deviation that can be offered by a lender of arbitrary type θ can be rejected, and therefore $s(\theta) = s^*$ is an equilibrium.

A.4   Proof of proposition 3

By the arguments in the proposition 2, the participation constraints for the borrower and lender are satisfied. We therefore consider only deviations, in which a single lender offers security s' instead of s^*. Again, we divide our argument into two cases. Consider first securities that could be offered by the type with the uninformative index. In this case, the argument is identical to the proof of proposition 2, and these securities can be rejected.

We therefore consider securities for which, under the true type θ, the lender would be willing to offer the security,

$$\sum_{a \in A, z \in Z} \theta(a,z)\beta(a)s'_z \geq K$$

and the uninformative type would not be willing to offer the security,

$$K > (\sum_{a \in A} p(a)\beta(a))(\sum_{z \in Z} q(z)s'_z).$$

By the fact that all types share the same marginal distributions, this is equivalent to

$$\sum_{a \in A, z \in Z} \theta(a,z)\beta(a)s'_z > (\sum_{a \in A} p(a)\beta(a))(\sum_{z \in Z} q(z)s'_z).$$

Using the definition of the indirect utility functions in our example,

$$\sum_{a\in A, z\in Z} \theta(a,z)(\phi(s'_z,a) - \phi(s^*,a)) = \sum_{a\in A, z\in Z} \theta(a,z)\delta(a)(s^* - s'_z)$$

$$= K - \sum_{a\in A, z\in Z} \theta(a,z)\delta(a)s'_z.$$

It follows that if we can show that $\sum_{a\in A, z\in Z} \theta(a,z)\delta(a)s'_z \geq K$, we will have proven that the offer of security s' can be rejected.

We consider the proofs under the two assumptions separately from this point onward.

A.4.1 Proof under 4

Define

$$\hat{\beta}(z;\theta) = \sum_{a\in A} \theta(a,z)\beta(a).$$

Under 4, as discussed in Athey and Levin [1998], by the fact that $\frac{\theta(a,z)}{q(z)}$ weakly F.O.S.D. $\frac{\theta(a,z')}{q(z')}$ for $z \succ z'$, and that $\beta(a)$ is weakly decreasing, $\hat{\beta}(z;\theta)$ is weakly decreasing in z, for all $\theta \in \Theta$. We can write

$$\sum_{a\in A, z\in Z} \theta(a,z)\beta(a)s'_z = \sum_{z\in Z} q(z)\hat{\beta}(z;\theta)s'_z.$$

Note that

$$\sum_{z\in Z} q(z)\hat{\beta}(z;\theta) = \sum_{a\in A, z\in Z} \theta(a,z)\beta(a) = \sum_{a\in A} p(a)\beta(a).$$

Under the condition that type θ but not θ_0 is willing to offer the security, we have

$$\sum_{z\in Z} q(z)\hat{\beta}(z;\theta)s'_z > (\sum_{z\in Z} q(z)\hat{\beta}(z;\theta))(\sum_{z\in Z} q(z)s'_z).$$

By assumption, s'_z is either weakly monotone increasing or weakly monotone decreasing (or both). As derived above, $\hat{\beta}(z;\theta)$ is weakly decreasing. It follows by the covariance rule (Gollier [2004], section 6.4) that if s'_z were weakly monotone increasing, the above inequality could not hold. Therefore, s'_z is weakly monotone decreasing.

Define

$$\hat{\delta}(z;\theta) = \sum_{a \in A} \theta(a,z)\delta(a)$$

Under 4, by the fact that $\frac{\theta(a,z)}{q(z)}$ weakly F.O.S.D. $\frac{\theta(a,z')}{q(z')}$ for $z \succ z'$, and that $\delta(a) - \beta(a)$ is weakly decreasing, $\hat{\delta}(z;\theta) - \hat{\beta}(z;\theta)$ is weakly decreasing in z, for all $\theta \in \Theta$. It follows by the covariance rule that

$$\sum_{z \in Z} q(z)(\hat{\delta}(z;\theta) - \hat{\beta}(z;\theta))s'_z \geq (\sum_{z \in Z} q(z)(\hat{\delta}(z;\theta) - \hat{\beta}(z;\theta)))(\sum_{z \in Z} q(z)s'_z).$$

By our normalization,

$$\sum_{z \in Z} q(z)(\hat{\delta}(z;\theta) - \hat{\beta}(z;\theta)) = 0.$$

Therefore,

$$\sum_{z \in Z} q(z)\hat{\delta}(z;\theta)s'_z \geq \sum_{z \in Z} q(z)\hat{\beta}(z;\theta))s'_z \geq K.$$

It follows that

$$\sum_{a \in A, z \in Z} \theta(a,z)(\phi(s'_z,a) - \phi(s^*,a)) \leq 0,$$

and therefore this security can be rejected.

A.4.2   Proof under 5

This proof use many of the properties of log-supermodular functions described in Gollier [2004], which is based on Athey [2002].

Define the function

$$u(a,x) = \begin{cases} \delta(a) & x = 0 \\ \beta(a) & x = 1 \\ 1 & x = 2 \end{cases}.$$

Under the assumptions of 5, the function $u(a,x)$ is log-supermodular: for $a \succ a'$ and $x > x'$,

$$\frac{u(a',x')}{u(a',x)} \geq \frac{u(a,x')}{u(a,x)}.$$

This is a consequence of the fact that $\frac{\delta(a)}{\beta(a)}$ is weakly decreasing, and that $\beta(a)$ is weakly decreasing.

By the log-supermodularity of $\theta(a,z)$, the function

$$h(a,z,x;\theta) = \theta(a,z)u(a,x)$$

is log-supermodular. By proposition 20 of Gollier [2004], it follows that

$$H(z,x;\theta) = \sum_{a\in A} h(a,z,x;\theta)$$

is log-supermodular. Note that using the definitions in the previous section,

$$\hat{\delta}(z;\theta)q(z) = H(z,0;\theta)$$

$$\hat{\beta}(z;\theta)q(z) = H(z,1;\theta)$$

$$q(z) = H(z,2;\theta)$$

Using the definitions above, the functions $\hat{\beta}(z;\theta)$ and

$$\frac{\hat{\delta}(z;\theta)}{\hat{\beta}(z;\theta)}$$

are weakly decreasing, and therefore $\hat{\delta}(z;\theta) - \hat{\beta}(z;\theta)$ is weakly decreasing. The argument proceeds identically from these facts.

A.5 Proof of Positive Marginal Utility

Lemma 5. *Under the assumptions of the general model, for all $a \in A$, $\partial_i \psi(s^*,a) \geq 0$ for all $i \in I$ such that $s_i^* > 0$.*

Proof. Argument goes here. □

A.6 Additional Lemma

Lemma 6. *Under the assumptions of proposition 4, for all $a \in A$, $\partial_i \psi(s^*, a) \geq 0$ for all $i \in I$ such that $s_i^* > 0$.*

Proof. Argument goes here. □

A.7 Proof of proposition 4

The non-contingent security s^* has payoffs that do not depend on the index. As a result, it offers zero profits for the lender, regardless of the lender's type, by the assumption that all $\theta \in \Theta$ have the same marginal distribution with respect to the aggregate state. By 1, s^* can deliver positive utility to the borrower, and therefore the participation constraints are satisfied in this equilibrium. We therefore consider only deviations, in which a single lender offers security s' instead of s^*. We will assume that the security can be profitably offered by the lender of type θ':

$$\sum_{a \in A, z \in Z} \theta'(a, z) \psi(s_z', a) \geq K.$$

We divide the proofs into two cases. First, consider the case in which a lender of type θ_0 (the uninformative index type) could also offer the security. In this case,

$$\sum_{a \in A, z \in Z} \theta_0(a, z) \psi(s_z', a) \geq K.$$

Suppose that the borrower, under the D1 refinement, the borrower places full support on the belief that the lender is of type θ_0. By the fact that the participation constraint binds for the s^* security, we can write

$$\sum_{a \in A, z \in Z} \theta_0(a, z) \psi(s^*, a) = K$$

and therefore

$$\sum_{a \in A, z \in Z} \theta_0(a, z)(\psi(s_z', a) - \psi(s^*, a)) \geq 0.$$

By the Pareto-optimality of s^*, it follows that

$$\sum_{a\in A, z\in Z} \theta_0(a,z)(\phi(s'_z,a) - \phi(s^*,a)) \leq 0$$

and therefore the offered deviation can be rejected.

Now consider securities that could not be offered by the uninformative type:

$$\sum_{a\in A, z\in Z} \theta_0(a,z)\psi(s'_z,a) < K.$$

By the feasibility of the offer under type θ', we have

$$\sum_{a\in A, z\in Z} (\theta'(a,z) - \theta_0(a,z))\psi(s'_z,a) \geq 0.$$

By construction,

$$\sum_{a\in A, z\in Z} \theta'(a,z)\psi(s^*,a) = \sum_{a\in A, z\in Z} \theta_0(a,z)\psi(s^*,a) = K,$$

and therefore

$$\sum_{a\in A, z\in Z} (\theta'(a,z) - \theta_0(a,z))(\psi(s'_z,a) - \psi(s^*,a)) \geq 0.$$

Suppose that, under the D1 refinement, the borrower places full support on the lender's true type, θ', which he is allowed to do by the feasibility of θ'. By the concavity of ψ (7)

$$-\psi(s'_z,a) \geq (s'_z - s^*) \cdot x_a^* - \psi(s^*,a),$$

where x_a^* is the supergradient defined in 8. It follows that we must have

$$\sum_{a\in A, z\in Z} (\theta'(a,z) - \theta_0(a,z))(s'_z - s^*) \cdot x_a^* \leq 0.$$

By the fact that x_a^* is weakly increasing in a and the FOSD ordering in 8 (see Athey and Levin

[1998]), it must be case that

$$x_z^* = \sum_{a \in A} \frac{\theta'(a,z)}{q(z)} x_a^*$$

is weakly increasing. Moreover,

$$\sum_{z \in Z} q(z) x_z^* = \sum_{a \in A, z \in Z} \theta'(a,z) x_a^* = \sum_{a \in A, z \in Z} \theta_0(a,z) x_a^*.$$

Our expression can be written as

$$\sum_{z \in Z} q(z)(s_z' - s^*) \cdot (x_z^* - \sum_{z' \in Z} x_{z'}^*) \leq 0.$$

By assumption, s_z', and therefore $s_z' - s^*$, is either weakly monotone increasing or weakly monotone decreasing (or both). As was just shown, x_z^* is weakly increasing. It follows by the covariance rule (Gollier [2004], section 6.4) that if s_z' were weakly monotone increasing, the above inequality could not hold. Therefore, s_z' is weakly monotone decreasing.

Now consider the social welfare function, $U(s,a;\lambda)$. By 6, there exists a $\lambda(a) \geq 0$ such that

$$s^* = \arg\max_{s \in S} U(s,a;\lambda(a)),$$

and therefore,

$$\vec{0} \in \bar{\partial}(-U(s^*,a;\lambda(a))).$$

Next, consider the "social welfare" gains from security s_z', for Pareto weight λ^*:

$$\sum_{a \in A, z \in Z} \theta'(a,z)(U(s_z',a;\lambda^*) - U(s^*,a;\lambda^*)).$$

By the concavity of U, for any

$$\hat{y}_a^* \in \bar{\partial}(-U(s^*,a;\lambda^*)),$$

we have

$$\sum_{a \in A, z \in Z} \theta'(a,z)(U(s^*,a;\lambda^*) - U(s'_z,a;\lambda^*)) \geq \sum_{a \in A, z \in Z} \theta'(a,z)(s'_z - s^*) \cdot \hat{y}^*_a.$$

By the property that

$$\partial(-U(s^*,a;\lambda^*)) = \partial(-\phi(s^*,a)) + \partial(-\lambda^*\psi(s^*,a)),$$

there exists a $\hat{y}^*_a$ such that

$$\hat{y}^*_a = w^*_a + \lambda^* x^*_a,$$

and therefore by assumption

$$\hat{y}^*_a = (\lambda^* - \lambda(a))x^*_a.$$

Moreover, by the Pareto-optimality of s^* under θ_0, for all $s \in S_I$,

$$\sum_{a \in A} p(a)(s - s^*) \cdot \hat{y}^*_a = 0$$

and therefore, for all $s \in S_I$,

$$\lambda^* \sum_{a \in A} p(a)(s - s^*) \cdot x^*_a = \sum_{a \in A} \lambda(a)p(a)(s - s^*) \cdot x^*_a.$$

It follows that

$$\sum_{a \in A, z \in Z} \theta'(a,z)(U(s^*,a;\lambda^*) - U(s'_z,a;\lambda^*)) \geq - \sum_{a \in A, z \in Z} \theta'(a,z)(\lambda(a) - \lambda^*)(s'_z - s^*) \cdot x^*_a.$$

By the assumption that $\lambda(a)$ and $-x^*_a$ are weakly decreasing (7) and $x^*_a \geq 0$, $-(\lambda(a) - \lambda^*)x^*_a$ is also weakly decreasing. It follows by the conditional FOSD argument used previously that

$$\hat{x}^*_z = -\sum_{a \in A} \frac{\theta'(a,z)}{q(z)}(\lambda(a) - \lambda^*)x^*_a$$

is weakly decreasing in z. Moreover, for all (non-contingent) $s \in S_I$,

$$\sum_{z \in Z} q(z)(s - s^*) \cdot \hat{x}_z^* = -\sum_{a \in A} \theta'(a,z)(\lambda(a) - \lambda^*)(s - s^*) \cdot x_a^*$$

$$= -\sum_{a \in A} \theta_0(a,z)(\lambda(a) - \lambda^*)(s - s^*) \cdot x_a^*$$

$$= 0.$$

Therefore,

$$-\sum_{a \in A, z \in Z} \theta'(a,z)(\lambda(a) - \lambda^*)(s_z' - s^*) \cdot x_a^* = \sum_{z \in Z} q(z)(s_z' - s^*) \cdot \hat{x}_z^*$$

$$\geq 0,$$

where the latter follows from the covariance rule and the fact that $s_z' - s^*$ is monotone decreasing. It follows that

$$\sum_{a \in A, z \in Z} \theta'(a,z)(U(s^*, a; \lambda^*) - U(s_z', a; \lambda^*)) \geq 0,$$

or equivalently,

$$\sum_{a \in A, z \in Z} \theta'(a,z)(\phi(s^*, a) - \phi(s_z', a)) \geq \lambda^* \sum_{a \in A, z \in Z} \theta'(a,z)(\psi(s_z', a) - \psi(s^*, a))$$

By the fact that the security is feasible for type θ', we have

$$\sum_{a \in A, z \in Z} \theta'(a,z)(\psi(s_z', a) - \psi(s^*, a)) \geq 0,$$

and therefore

$$\sum_{a \in A, z \in Z} \theta'(a,z)(\phi(s^*, a) - \phi(s_z', a)) \geq 0.$$

It follows that the security can be rejected, and the equilibrium exists.

A.8 Additional Lemmas for Mortgage Example

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Lemma 7. *In the mortgage example, the borrower's indirect utility function $\phi(s,a)$, described in equation (2), is weakly monotone decreasing in s, for all $a \in A$.*

Proof. Consider a perturbation

$$\hat{s}(\tilde{e}) = s(\tilde{e}) + \varepsilon \omega(\tilde{e}),$$

such that there exists a $\delta > 0$ such that $\hat{s}(\tilde{e})$ is weakly increasing in $\tilde{e}$ for $\varepsilon \in [0, \delta)$ and $\omega(\tilde{e}) \geq 0$ for all $\tilde{e}$.

We have

$$\frac{\partial \phi(\hat{s},a)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{-\min\{s(0),L(a)\}+s(0)-(\bar{e}-s(0))-\mathbf{1}(s(0)<L(a))s(0)]-C(a)+L(a)}{\bar{e}}\omega(0).$$

$\square$

When $L(a) > s^*$,

$$\frac{\partial \phi(\hat{s},a)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{-\bar{e}-C(a)+L(a)}{\bar{e}}\omega(0),$$

which is negative by the assumption that $C(a) > L(a)$,

Otherwise,

$$\frac{\partial \phi(s,a)}{\partial s} = \frac{2s(0)-\bar{e}-C(a)}{\bar{e}}\omega(0),$$

which is negative by the assumption that $C(a) > \bar{e}$ and the requirement that $s(\tilde{e}) \leq \bar{e}$.

Lemma 8. *In the mortgage problem described in section §6, the non-contingent optimal contract satisfies $L(a) \leq s^*(0) \leq \frac{1}{2}\bar{e}$ for all $a \in A$.*

Proof. By definition,

$$s^* \in \arg\max_{s \in S_I} \sum_{a \in A} p(a)\phi(s,a)$$

subject to

$$\sum_{a \in A} p(a)\psi(s;a) \geq K.$$

By 7, it is without loss of generality to assume

$$s^* \in \arg\min_{s \in S_I} s(0)$$

subject to $\sum_{a \in A} p(a)\psi(s;a) \geq K$.

By definition,

$$\psi(s,a) \leq \beta(a)s(0).$$

It follows that

$$K \leq \sum_{a \in A} p(a)\psi(s^*;a) \leq s^*(0)\sum_{a \in A} p(a)\beta(a),$$

and therefore, by the normalization that $\sum_{a \in A} p(a)\beta(a) = 1$, $s^*(0) \geq K$. By the assumption that $L(a) \leq K$ for all $a \in A$, $s^*(0) \geq L(a)$.

By the assumption that $L(a) > 0$ and $s(0) \geq 0$, □

$$\psi(s,a) \geq \beta(a)(1 - \frac{s(0)}{\bar{e}})s(0).$$

It follows by the normalization that

$$\sum_{a \in A} p(a)\psi(s;a) \geq (1 - \frac{s(0)}{\bar{e}})s(0).$$

Therefore,

$$(1 - \frac{s(0)}{\bar{e}})s(0) \geq K$$

is sufficient to establish feasibility. By the assumption that $4K < \bar{e}$ (10), and the fact that the left hand side of this equation is maximized by $2s(0) = \bar{e}$, it follows that there exists an $\tilde{s} \in (0, \frac{1}{2}\bar{e})$ such that

$$(1 - \frac{\tilde{s}}{\bar{e}})\tilde{s} = K,$$

and therefore, for the constant security with value $\tilde{s}$ (call this $\hat{s}$),

$$\sum_{a \in A} p(a)\psi(\hat{s};a) \geq K.$$

Because $\hat{s}$ is a feasible solution, it follows that $s^*(0) \leq \hat{s}(0) \leq \frac{1}{2}\bar{e}$.

A.9   Proof of lemma 2

First, as noted in the text, it is without loss of generality to assume that the contract s^* does not depend on the report. Abusing notation, we will let s^* denote the scalar value of $s^*(\tilde{e})$ for all $\tilde{e}$.

To satisfy 6, we must show that there exists a $\lambda(a)$ such that

$$s^* \in \arg\max_{s \in S_I} U(s; a, \lambda(a)).$$

In our problem,

$$U(s; a, \lambda(a)) = E[e] + (\lambda(a)\beta(a) - 1)\beta(a)^{-1}\psi(s,a) - \frac{s(0)}{\bar{e}}(C(a) - L(a))$$

Consider the perturbation

$$s(\tilde{e}) = s^* + \varepsilon\omega(\tilde{e}).$$

such that $\omega(\tilde{e})$ is weakly increasing in the report. The first variation is necessary for optimality:

$$\frac{\partial}{\partial\varepsilon}U(s; a, \lambda(a))|_{\varepsilon=0} \leq 0.$$

Consider in particular the case in which $\omega(\tilde{e}) = \bar{\omega}$. In this case, we must have

$$\frac{\partial}{\partial \varepsilon} U(s; a, \lambda(a))|_{\varepsilon=0} = 0.$$

In fact, it is easy to see that if there are no constant perturbations that are welfare-improving, there are no local perturbations that are welfare-improving. For a constant perturbation,

$$\frac{\partial}{\partial \varepsilon} U(s; a, \lambda(a))|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} (\lambda(a)\beta(a) - 1)\beta(a)^{-1}\psi(s,a)|_{\varepsilon=0} - \frac{\bar{\omega}}{\bar{e}}(C(a) - L(a)),$$

and

$$\beta(a)^{-1} \frac{\partial \psi(s,a)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\min\{s^*, L(a)\} - s^*}{\bar{e}}\bar{\omega} + (1 - \frac{s^*}{\bar{e}})\bar{\omega} + \mathbf{1}(s^* < L(a))\frac{s^*}{\bar{e}}\bar{\omega}.$$

By lemma 8, $s^* \geq L(a)$ for all $a \in A$. Since these equations must hold for any $\bar{\omega}$, we must have

$$(\lambda(a)\beta(a) - 1)\frac{L(a) + \bar{e} - 2s^*}{\bar{e}} - \frac{C(a) - L(a)}{\bar{e}} = 0. \tag{3}$$

Scaling by $\bar{e}$,

$$(\lambda(a)\beta(a) - 1)(L(a) + \bar{e} - 2s^*) = C(a) - L(a).$$

By lemma 8 and the assumption that $0 < L(a) < C(a)$, there exists a $\lambda(a)$ that solves these equations, with

$$\lambda(a)\beta(a) > 1.$$

We have shown that there exists a $\lambda(a)$ such that there are no local, welfare-improving perturbations. We now show that $U(s; a, \lambda(a))$ is concave in s. We can write

$$U(s; a, \lambda(a)) = E[e] + (\lambda(a)\beta(a) - 1)\beta(a)^{-1}\psi(s,a) - \frac{s(0)}{\bar{e}}(C(a) - L(a)). \tag{4}$$

By the fact that $\lambda(a)\beta(a) > 1$, the concavity of ψ (section A.10) establishes the concavity of $U(s; a, \lambda(a))$. It follows that the first-variation condition is sufficient, and therefore s^* is indeed a

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maximizer for the value of $\lambda(a)$ derived above.

A.10   Proof of lemma 3

That the indirect utility functions are locally Lipschitz follows from continuity. The definition of ψ is

$$\psi(s,a) = \beta(a)[(1 - \frac{s(0)}{\bar{e}})s(0) + \frac{s(0)}{\bar{e}}\min\{s(0), L(a)\}].$$

We can write this as

$$\psi(s,a) = \beta(a)\min(s(0), \frac{L(a)s(0) - s(0)^2}{\bar{e}}).$$

It follows that $\psi(s,a)$ is the min of two concave functions, and therefore concave.

Next, we show that $U(s,a;\lambda^*)$ is concave. We begin by deriving the properties of λ^*. We have

$$s^* \in \arg\max_{s \in S_I} \sum_{a \in A} p(a)U(s,a;\lambda^*).$$

Using the analog of equation (3), we can write

$$\sum_{a \in A}(\lambda^*\beta(a) - 1)p(a)[\bar{e} + L(a) - 2s^*] = \sum_{a \in A} p(a)(C(a) - L(a)).$$

By lemma 8, we have $0 < \bar{e} + L(a) - 2s^* \leq \bar{e}$.

$$\lambda^* \sum_{a \in A} \beta(a)\bar{e} \geq \sum_{a \in A} p(a)(C(a) + \bar{e} - 2s^*),$$

and therefore

$$\lambda^* \geq \sum_{a \in A} p(a)\frac{C(a)}{\bar{e}} > 1.$$

By 10, it follows that

$$\lambda^*\beta(a) > 1.$$

By the analog for λ^* of equation (4), and the concavity of ψ, the concavity of $U(s,a;\lambda^*)$ follows.

A.11 Proof of lemma 4

By construction, because ψ is concave, and only the first element of s affects the payoffs in the set S_I,

$$x_a^* = \beta(a) \begin{bmatrix} \frac{(\min\{s^*, L(a)\} - s^*) + \bar{e} - s^* + \mathbf{1}(s^* < L(a))s^*}{\bar{e}} \\ 0 \\ \vdots \end{bmatrix}$$

is a Clarke subgradient of $\psi(s^*, a)$. By the optimality of s^*, and the associated FOC (equation (3))

$$w_a^* = -\beta(a)^{-1} x_a^* + \begin{bmatrix} \frac{C(a) - L(a)}{\bar{e}} \\ 0 \\ \vdots \end{bmatrix}$$

is a Clarke subgradient of $\phi(s^*, a)$. By lemma 8,

$$x_a^* = \beta(a) \begin{bmatrix} \frac{L(a) - 2s^* + \bar{e}}{\bar{e}} \\ 0 \\ \vdots \end{bmatrix}.$$

The assumption that $\beta(a)$ and $\beta(a)L(a)$ are weakly decreasing (9), along with $\bar{e} \geq 2s^*$ (lemma 8) establish that x_a^* is weakly monotone decreasing.

Using the proofs of section A.9, the Pareto weights are

$$\lambda(a) = \frac{C(a) - L(a) + \bar{e} + L(a) - 2s^*}{\beta(a)(\bar{e} + L(a) - 2s^*)}$$

We can rewrite this as

$$\lambda(a) = \frac{C(a) - L(a)}{\beta(a)\bar{e}f(a)} + \beta(a)^{-1},$$

where $f(a) = \frac{\bar{e}}{\bar{e} + L(a) - 2s^*}$.

By the assumption that $\beta(a)$ is weakly decreasing, $\beta(a)^{-1}$ is weakly increasing. By 10,

$$\lambda(a) = \frac{C(a) - L(a) + \bar{e}}{\beta(a)\bar{e}}$$

is decreasing in a, and therefore

$$\frac{C(a) - L(a)}{\beta(a)\bar{e}}$$

is decreasing in a.

By lemma 8, $f(a) > 1$. By the assumption that $L(a)$ is increasing (9), $f(a)$ is decreasing. It follows that

$$(f(a) - 1)\frac{C(a) - L(a)}{\beta(a)\bar{e}}$$

is decreasing. We have

$$\lambda(a) = (f(a) - 1)\frac{C(a) - L(a)}{\beta(a)\bar{e}} + \frac{C(a) - L(a)}{\beta(a)\bar{e}} + \beta(a)^{-1},$$

which is the sum of two decreasing functions, and therefore decreasing.