Bargaining with Arrival of New Traders*

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Abstract

We study a general model of dynamic bargaining between a seller and a privately informed buyer, with arrival of exogenous events. Events can represent arrival of competing buyers (or sellers) or release of information. We characterize the unique limit of stationary equilibria of these games as the time between offers goes to zero. The possibility of arrivals leads to new equilibrium dynamics. First, the no-delay part of the Coase conjecture no longer holds: even in the limit, there is considerable delay of trade in equilibrium and the seller slowly screens out buyers with higher valuations. Second, the inability to commit to future prices (and hence the Coasian forces) drive the seller payoffs down to his outside option. The limit of equilibria is very tractable, allowing us to establish many comparative statics and utilize the model to answer many applied questions. For example, we show applications in which when buyer valuations fall, average transaction prices drop and the time on the market gets longer. Moreover, for high enough arrival rates, the division of surplus and equilibrium dynamics are driven more by the relative competition from new traders than on the relative discount factors. Finally, even when multiple buyers can arrive, the expected time to trade is a non-monotonic function of the arrival rate.

In bargaining theory (and practice) outside options play an important role. Often, an important outside option is to wait for new developments. Maybe another agent will show up and offer better terms of trade, maybe new information will arrive reducing the information asymmetry, etc. Traders compare these potential benefits to costs of delaying trade and the risk that over time the opportunity to trade might disappear (or that unfavorable information will arrive etc.). In this paper we study a general bargaining model that captures outside options of this nature and characterize their impact on the dynamics of bargaining.

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For example, suppose you have put your house on the market. So far only one buyer has expressed interest. He informs you that your original price is too high and asks you to reduce it. What do you consider before responding? Out of many factors that you may take into account two important ones are: 1) How likely it is that other serious buyers will show up in the short run. 2) How likely it is that if you wait to reduce the price, the current buyer will find another house and "disappear". In fact, these risks in many situations are likely to be more important in evaluating the relative costs and benefits of delay than the standard discounting costs that play a crucial role in many bargaining models. This intuition is confirmed by our model.

New traders arriving over time is a common feature of many markets (housing, labor, financial markets to name a few). A key characteristic of such markets is that trade/bargaining over price take time and the bargaining dynamics are heavily influenced by the market conditions. For example, the asking price of a house takes time to drop, and how long it takes may depend on whether it is a "sellers' market" or a "buyers' market." We shed some light on how such external conditions affect the dynamics of bargaining.

We start with an abstract, general bargaining game: there is a buyer and a seller. The seller has an asset that he values at zero (normalization). The buyer has private information about his value, v, only the prior distribution of values, $F(v)$, is commonly known. The seller makes offers to the buyer and the buyer accepts or rejects. Over time, the seller can reduce his asking price and he cannot commit to future prices. Over time an event can arrive that ends the game (event arrives according to a Poisson process). This abstract event represents arrival of a new trader or information (we analyze such special cases in applications).

We characterize stationary equilibria of this game. Our first main result is that arrivals induce delay in equilibrium. That is, unlike in the classic Coase-conjecture dynamics, there is inefficient delay even if the seller makes offers frequently. The discrete time game is very difficult to analyze, but we show that as the length of periods shrinks to zero (allowing the seller to change asking price frequently) the equilibrium becomes incredibly tractable. This allows us to obtain a very clear understanding of the equilibrium dynamics. In particular, the seller's inability to commit to prices decreases his payoffs to his expected outside option (that he can achieve by simply waiting), showing that some of the Coasian forces/features are still present. Finally, prices have the “no-ex-post regret” property – every type pays the present expected payoff the seller expects from him upon arrival.

The intuition is as follows. Suppose that the buyer follows a stationary reservation price strategy $P(v)$, which can be interpreted as a demand function and says that all types above v accept prices below $P(v)$, independently of the history of the game. Moreover, suppose that the seller has a strictly increasing cost $c(v) < v$ of serving type v. The reason $c(v)$ is strictly increasing in our paper is that we assume that upon arrival the seller expects a higher revenue if v is higher. For example, if the event stands for arrival of a new buyer and is followed by an auction of the asset, the higher is the value of the first buyer, the higher is the expected revenue from the auction. $c(v)$

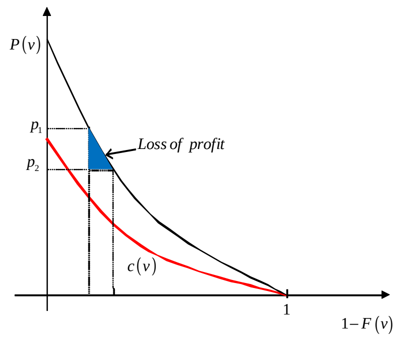

Figure 1:

represents the expected discounted profit from this auction conditional on the type of the buyer and economically is the alternative cost of selling to type v today. Alternatively, like in papers on bargaining with interdependent valuations, Robert Evans (1989), Daniel Vincent (1989) and Raymond Deneckere and Meng-Yu Liang (2006), the physical cost of serving type v may depend on v.[1]

Now, suppose that in equilibrium $P(v)$ is above $c(v)$, as shown in Figure 1. What is the seller's best response? In the discrete time it is in general very difficult to calculate. The seller wants to collect the area between $P(v)$ and $c(v)$ as quickly as possible but he is facing a complicated tradeoff: decreasing the price faster allows him to collect the profit faster but forces him to sacrifice some profit because if a positive mass of types trade in a period, the price drops discontinuously, leaving small triangles, like the one between p_1 and p_2 in Figure 1, unextracted. However, if we consider the continuous time limit, the second force disappears since the seller can smoothly decrease prices to collect the whole area between the curves. And doing it as fast as possible economizes on the delay costs. Hence, if the reservation price strategy $P(v)$ were above $c(v)$ for an interval of types prices would drop discretely in an instant. But, if this were the case, the reservation price strategy could not have been an equilibrium in the first place: the a buyer with a high value would be better off to wait for an instant and trade at a much lower price. That tells us that the equilibrium reservation price strategy cannot be like the one on Figure 1.

Next, suppose that $P(v)$ and $c(v)$ are configured like in Figure 2. The seller's best response would be then to sell to types in region A as quickly as possible and not to sell to types in region B at all, but rather wait for the arrival of the event. However, if immediate trade is efficient (as we assume), it cannot be an equilibrium in region B that trade stops: if the buyer expects no trade until arrival then his reservation prices would be higher than $c(v)$.

Applying this reasoning to all configurations of $P(v)$ and $c(v)$ we get that the only possibility is $P(v) = c(v)$: every type pays the seller's alternative cost. It also means that the seller's ex-ante

[1] For example, the negotiations are about an insurance policy and higher types have a higher probability of having a claim.

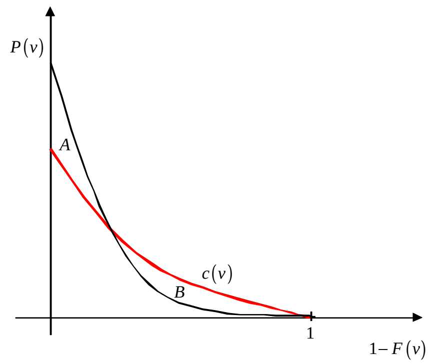

Figure 2:

expected profit is equal to the outside option (and in case $c(v)$ is a physical cost, the profit is zero). Moreover, we can pin down the speed at which the seller is screening down the demand curve as follows. When $P(v) = c(v)$ the seller is indifferent over the speed at which he screens the types.[2] However, the buyer is not: if the seller is expected to reduce prices faster, the reservation prices, $P(v)$ drop, if slower, they go up. There is a unique speed of reducing prices so that each type v is willing to pay $P(v) = c(v) > 0$ instead of either accepting a higher price earlier or waiting for a lower price later. If the seller was to reduce prices slower, then $P(v) > c(v)$ in some region and the seller would like to deviate and speed up the trade.

We formalize this intuition by looking at a sequence of games with period length $\Delta \to 0$ and show that indeed the stationary equilibria of these games converge to the described behavior: the seller slowly reduces prices, each buyer type pays the seller's alternative cost and the seller payoff is reduced to his outside option.

The benchmark to compare our results to is the stationary equilibrium of our bargaining game but without the arrivals. In that case, as $\Delta \to 0$ trade becomes immediate and prices converge to the lowest buyer's type. This is the remarkable Coase conjecture result shown by Drew Fudenberg, David K. Levine and Jean Tirole (1985) and Faruk Gul, Hugo Sonnenschein and Robert Wilson (1986) (henceforth FLT and GSW, respectively). We show that in that same environment, allowing for external events dramatically changes equilibrium dynamics. The Coase conjecture results can be illustrated as a limit case of our model: making $c(v)$ flatter and flatter (for example, by decreasing the arrival rate, which reduces the option value of waiting), the model and its equilibria converge to the Coasian dynamics: immediate trade at the cost to the seller.

Equilibria of dynamic bargaining games are generally quite intractable, reducing their appeal for use in more applied work and severely limiting the possibility of doing comparative statics analysis. Our analysis of the limit as the seller loses all commitment power (and hence can adjust offers

[2] This indifference happens only in the continuous time limit. In discrete time the seller plays a strict best response and $P(v)$ is slightly above $c(v)$.

continuously) yields relatively simple expressions for equilibrium strategies, opening the doors for many applications and empirical predictions.

For example, we show that:

- (Proposition 2) If the value of the seller's outside option is less sensitive to the current buyer's valuation then trade takes place at a faster rate. In the limit, if the outside value is independent of the value of the current bargaining party then trade either takes place immediate or only upon the arrival of the outside option.

- (Proposition 3.i) If the outside options of both the seller and the buyer are independent of the distribution of types then the path of prices through time, as well as reservation prices of different buyer types are also independent of the distribution.[3] Hence, in such environments the equilibrium structure is robust to the details of the distribution of values.

- (Proposition 3.ii) Nonetheless, the average equilibrium outcomes still depend on the distribution of values. For example, when buyer valuations fall (in a first-order-stochastic-dominance sense), average transaction prices drop and the time on the market increases.

- (Section III.A) In the setting of a thin market where the events stand for competing traders that can arrive on both sides of the market (under the pure common value assumption), we show how the thickness of the market and the relative likelihood of the seller being on the short side of the market determines the equilibrium prices and time on the market. Moreover, if the arrival rates of other traders are high enough, then the division of surplus and equilibrium dynamics are driven more by the relative chances of being on the short side of the market rather than on the relative discount factors – the impatience caused by the arrival rates of competition dominates the time discounting.

In most of the applications we consider, we assume that only one event can arrive – even though we allow for different types of events, we assume that upon arrival the game ends with some payoffs. A general analysis of markets with many opportunities to trade is complicated, and one way to interpret our general model is that the reduced-form payoffs upon arrival of the event are not really terminal payoffs in the game but rather represent expected continuation payoffs from possibly continued bargaining. In the last part of the paper, we explicitly analyze a game with infinitely many potential buyers.[4]

In terms of relationship to the existing literature, the main intuition why there is delay in equilibrium in our model is, as we described above, closely related to the bargaining models with

[3] This condition holds, for example, if the arrival stands for the buyer's type being revealed and the game continuing as full information bargaining. It also holds if arrival means that a second buyer shows up with the same valuation as the current buyer and an English auction is held upon arrival.

[4] To simplify the analysis we assume that upon the arrival of a new buyer the seller can make a last take it or leave it offer to the current party he is bargaining with. If rejected, the old trader disappears and the bargaining restarts with the new trader.

interdependent values. In these models if the lowest possible value is below the average cost, delay must occur. The reasoning is that otherwise the individual rationality constraint of at least one of the agents would be violated: if the low buyer types do not lose money, no delay implies prices have to drop close to the lowest type quickly and hence all prices have to be low, but then the seller would lose money on average. The main difference between our model and the previous work on bargaining with interdependent values is that the interdependence is created by market conditions and hence we can obtain interesting insights about thin markets. Additionally, we are the first to characterize the limit of the equilibrium strategies in this environment (Deneckere and Liang (2006) prove that in the limit there is delay and the seller's profits converge to zero, but do not characterize equilibrium prices or how the delay depends on the parameters of the model. One difficulty that they face is that they consider a model with finite number of types, which complicates the analysis). Hence, we believe our contribution lies also on the methodological side: by focusing on the continuous time limit we managed to greatly simplify the analysis and that allowed us to apply the model to many different situations and provide additional insights.

Beyond interdependent values, there is a rich literature about equilibrium delay in bargaining.[5] In terms of contributing to that literature, the novelty is that even in the simplest FLT/GSW framework adding only the possibility of arrival of a second buyer leads to delay. Such arrivals are a natural possibility of real-life transactions and hence can be a common reason for delay. Moreover, our paper can be viewed as a generalization of the Coasian dynamics: we show how the Coasian forces reduce the seller's payoff down to his outside option and allow us to easily pin down the equilibrium strategies.

Finally, there are also other bargaining papers that allow for arrival of new traders (in particular buyers) without obtaining equilibrium delay. The difference in results is caused by different assumptions about post-arrival competition, mainly that the post-arrival profits do not depend on the current buyer type. For example, Roman Inderst (2008) only allows the seller to choose whether to keep the original buyer or switch to the new one but if he does switch, then the value of the original value is irrelevant for his continuation value. As a result, in his model the Coase conjecture continues to hold.[6]

The paper is organized as follows: Section I presents the general model. Section II characterizes the equilibrium of the game in the continuous time limit. Section III presents applications of the general model. Section IV discusses an extension to allow multiple arrivals of buyers and Section V concludes. Most proofs are in the Appendix.

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[5] For example, delay occurs in a model with two sided private information about fundamentals and overlap in values (e.g. Peter Cramton 1984, Kalyan Chatterjee and Larry Samuelson 1987 or In-Koo Cho 1990), with irrational players (Dilip Abreu and Faruk Gul 2000), with higher order beliefs (Yossi Feinberg and Andrzej Skrzypacz 2005) with disagreement about continuation play (Muhamet Yildiz 2004) or with the possibility that players can commit to not responding to offers (Anat Admati and Motty Perry 1987).

[6] The same happens in Alberto Trejos and Randall Wright (1995) where the newly arrived traders simply displace the old ones.

I The Model

We start with a general bargaining game with arrival of a new event. In Section III we analyze in detail several applications and in particular a model where the event stands for the arrival of a new trader.

A General Bargaining

There is a seller and a buyer. The seller has an indivisible good (or asset) to sell. The buyer has a privately known type $v \in [0,1]$ that represents his value of the asset. v is distributed according to a *c.d.f.* $F(v)$ which is an atomless distribution with full support and density $f(v)$. The seller's value of the asset is zero.[7]

Time is discrete and periods have length Δ. The timing within periods is as follows. In the beginning of the period an event arrives with probability $1-e^{-\Delta\lambda}$ that ends the game (λ represents a Poisson arrival rate; for now, we treat the event as a reduced form of some continuation play). If the event does not arrive, the seller makes a price offer p. The buyer then decides whether to accept this price or to reject it. If he accepts, the game ends. If he rejects, the game moves to the next period.

A strategy of the seller is a mapping from the histories of rejected prices to current period price offers. A strategy of the buyer of type v is a mapping from the history of rejected prices to an acceptance strategy (which specifies the set of prices that the buyer accepts in the current period).

The payoffs are as follows. If the game ends with the buyer accepting price p at time t, then the seller's payoff is $e^{-rt}p$ and the buyer's payoff is $e^{-rt}(v-p)$, where r is a common discount rate.[8] If the game ends with the event arriving at time t, then the payoffs are:

$$e^{-rt}W(v) \text{ for the buyer,}$$
$$e^{-rt}\Pi(v) \text{ for the seller.}$$

Finally, define $V_A(k) = \int_0^k \Pi(v)\frac{f(v)}{F(k)}dv = E[\Pi(v)|v \leq k]$ as the seller's expected payoff conditional on the arrival of the event and buyer type being distributed according to a truncated $F(v)$ over $v \in [0,k]$.

To justify the reduced-form payoffs $W(v)$ and $\Pi(v)$, consider the following examples. Let the arrival represent a second buyer arriving and suppose the seller runs an English auction upon arrival. If the buyers' valuations are i.i.d. then $\Pi(v) = \int_0^1 \min\{x,v\}\,dF(x)$ and $W(v) = \int_0^v F(x)\,dx$. If their values are perfectly correlated, then $\Pi(v) = v$ and $W(v) = 0$. The arrival could also represent

[7] The only non-trivial assumption about the range of v and the seller's value is that the seller's value is no lower than the lowest buyer's value - i.e. the "no-gap case". The rest is a normalization.

[8] We focus on the case $\Delta \to 0$, i.e. no bargaining frictions, so it is more convenient to count time in absolute terms rather than in periods. Period n corresponds to real time $t = n\Delta$.

the buyer's information becoming public and the beginning of a bargaining game with complete information.[9] We provide additional examples later.

We assume:

Assumption 1

i) $\frac{e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)}{1-e^{-\Delta(r+\lambda)}}\left(\Pi\left(v\right)+W\left(v\right)\right)<v$ for all $v>0$.

ii) $W\left(v\right)$ is continuous and increasing, with $v-W\left(v\right)$ strictly increasing.

iii) $\Pi\left(v\right)$ is continuous, strictly increasing and differentiable.

iv) $\Pi\left(0\right)=W\left(0\right)=0$.

These assumptions are not too restrictive and are satisfied in many environments (including the examples above).[10]

Condition (i) is assumed so that from the point of view of the two parties involved in the negotiation delay is inefficient and if it were not for the information frictions there would be no delay in equilibrium. If it was violated delay would be a natural consequence of waiting for the total surplus to grow.[11] (ii) simply states that higher types are more eager to trade immediately. This guarantees that the *skimming property* holds (see below). The properties of $\Pi\left(v\right)$ in (iii), in particular $\Pi'\left(v\right)>0$, play an important role in the equilibrium dynamics - they are necessary for slow screening over types in equilibrium. We discuss this in more detail in Section II. (iv) is assumed to simplify the analysis since it saves us from solving for a fix point problem to find the relevant lowest type that trades. In Section IV we analyze an environment in which parts (i) and (iii) and (iv) of Assumption 1 are relaxed.

B Stationary Equilibrium

As usual (in dynamic bargaining games), in any equilibrium the buyer types remaining after any history are a truncated sample of the original distribution (even if the seller deviates from the equilibrium prices). This is due to the *skimming property* which states that in any sequential equilibrium after any history of offered prices p^{t-1} and for any current offer p_t, there exists a cutoff valuation $\kappa(p_t,p^{t-1};\Delta)$ such that buyers with valuations exceeding $\kappa(p_t,p^{t-1};\Delta)$ accept the offer p_t and buyers with valuations less than $\kappa(p_t,p^{t-1};\Delta)$ reject it. Best responses satisfy the skimming property because it is more costly for the high types to delay trade than it is for the low types (it can be easily shown using the assumption that $v-W\left(v\right)$ is strictly increasing, see FLT Lemma 1 for an analogous proof).[12]

[9] If in the complete information bargaining game the seller has bargaining power α and trade is efficient, then $\Pi\left(v\right)=\alpha v$ and $W\left(v\right)=\left(1-\alpha\right)v$.

[10] For comparison, Inderst's (2008) model violates (iii) because the outside option of the seller is not increasing in the current buyer's valuation in his environment.

[11] A sufficient condition is $\Pi\left(v\right)+W\left(v\right)\leq v$.

[12] The skimming property (implied by Assumption 1 (ii)) differentiates our model from the dynamic market for lemons / dynamic signaling models with arrival of new information in Ilan Kremer and Skrzypacz (2007) or Brendan Daley and Brett Green (2008). In these models higher types of the informed player are less eager to trade quickly than the low types which dramatically changes the equilibrium dynamics, in particular creating periods with no trade.

The current cutoff k hence describes the payoff-relevant state of the game and is a natural state variable on which the seller can condition his strategy. If in equilibrium the seller conditions his offers only on the cutoff k and the buyer has an acceptance policy that is independent of the history of the game then we call this equilibrium stationary. The classic papers in dynamic bargaining (FLT, GSW, Lawrence Ausubel and Raymond Deneckere (1989), henceforth AD) have shown existence of stationary equilibria and that these equilibria all satisfy the Coase conjecture: as $\Delta \to 0$ the expected time to trade converges to zero and the profit of the seller converges to zero (and prices converge to seller's cost). As shown by AD, there can also exist non-stationary equilibria that exhibit delay and a positive seller's payoff even as $\Delta \to 0$. Since one of our goals is to show that equilibrium delay is a consequence of the arrival of external events alone, we limit our analysis to stationary equilibria.

Formally, a stationary equilibrium is characterized by two functions (κ, P) :

1. A buyer's acceptance rule $\kappa(p; \Delta)$ that specifies the lowest type that accepts offer p.

2. A seller's pricing rule $P(k; \Delta)$ that specifies the price he offers given truncated beliefs $F(v)$ over $v \in [0, k]$.

A pure stationary equilibrium characterized by (κ, P) is a profile of strategies such the seller offers $P(1)$ in the first period and then in any future period, if $p_{\min}$ is the lowest offered price in the past, he offers $P(\kappa(p_{\min}; \Delta); \Delta)$; the buyer follows the acceptance strategy $\kappa(p; \Delta)$ on and off the equilibrium path. In other words, if the seller ever deviates, the equilibrium strategies call for a return to the equilibrium path as if the seller made the offer $p_{\min}$ in the last period. A general stationary equilibrium allows additionally for mixing by the seller over some prices. However, as shown by AD (Proposition 4.3) in any stationary equilibrium the seller's pricing rule is pure along the equilibrium path except for possibly the first price, $P(1; \Delta)$.[13] We will refer to (κ, P) as strategies with the understanding that these functions induce proper equilibrium strategies.

Note that the pair (κ, P) determines the future sequence of prices starting at any history described by k : the current equilibrium price is $p = P(k; \Delta)$, the next period price is $P(\kappa(p; \Delta); \Delta)$ and so on. They induce a decreasing step function $K(t; \Delta)$ which specifies the highest remaining type in equilibrium as a function of time (with $K(0; \Delta) = 1$) and a decreasing step function $T(v; \Delta)$ (with $T(1; \Delta) = 0$) which specifies the time at which each type v trades conditional on no arrival. For notational purposes, we let $k_+ = \kappa(P(k; \Delta); \Delta)$ denote next period cutoff given current cutoff k and the strategies (κ, P).

Definition 1 *Functions (κ, P) describe a stationary sequential equilibrium if after every history with the induced belief $v \in [0, k]$:*

a) given κ, for any k, the seller maximizes his expected discounted payoffs by choosing price $p =$

[13] Additionally, there can be randomization off the equilibrium path if the seller deviates to a price p' such that $k' = \kappa(p'; \Delta)$ and yet $p' \neq \max\{p | \kappa(p; \Delta) = k'\}$. This can happen only if $\kappa(p; \Delta)$ is constant over a range and would never be a seller best response since he could increase the price without changing the probability of trade. After such a deviation the seller randomizes between prices p_1 and p_2 to rationalize the acceptance of p' by type k'. The prices p_1 and p_2 are the maximum and minimum elements of the seller maximization problem given the cutoff k'.

$P(k;\Delta)$.

b) for any k, given the induced future sequence of prices, the acceptance strategy κ maximizes the buyer expected discounted payoff.

Remark: To fully specify the equilibrium strategies, the functions (κ, P) may need to be augmented by an appropriate mixed strategies off-the equilibrium path, as discussed above (yet, the equilibrium-path behavior is completely described by (κ, P)).

AD call stationary equilibria weak-Markov (and strong-Markov when $\kappa(p;\Delta)$ is strictly increasing, which implies that there is no randomization off-equilibrium). The existence of these equilibria is proven in FLT and in AD for the game without arrival of events; and in Deneckere and Liang (2006) in a setup with interdependent values. These proofs can be extended to the present setup. Since we are in the no-gap case these equilibria may not be unique. [14]

Let $V(k;\Delta)$ be the expected continuation payoff of the seller given a cutoff k and the strategy pair (κ, P). We can express $V(k;\Delta)$ recursively as:

$$V(k;\Delta) = \left(1 - e^{-\Delta\lambda}\right) V_A(k) + e^{-\Delta\lambda}\left[\left(\frac{F(k) - F(k_+)}{F(k)}\right) P(k;\Delta) + \frac{F(k_+)}{F(k)} e^{-\Delta r} V(k_+;\Delta)\right] \quad (1)$$

The seller's strategy is a best response to the buyer's strategy $\kappa(p;\Delta)$ if:

$$P(k;\Delta) \in \arg\max_p \left[\left(\frac{F(k) - F(\kappa(p;\Delta))}{F(k)}\right) p + \frac{F(\kappa(p;\Delta))}{F(k)} e^{-\Delta r} V(\kappa(p;\Delta);\Delta)\right] \quad (2)$$

This best response problem captures the seller's lack of commitment: in every period he chooses a price to maximize his payoff (instead of committing to a whole sequence of prices at time 0).

Instead of analyzing the seller's full best response problem, we will focus on the following necessary conditions. Given an equilibrium $, \kappa, P$ let Ω_Δ be the set of cutoff types on the equilibrium path. That is, $k \in \Omega_\Delta$ if and only if there exists a $t \geq 0$ such that in this equilibrium $k = K(t;\Delta)$ (Ω_Δ depends on Δ and the particular equilibrium, but we omit the second dependence in notation). A necessary condition for seller's optimality is that for all $k \in \Omega_\Delta$:

$$V(k;\Delta) = \max_{\substack{k_+ \in \Omega_\Delta, \\ k_+ \leq k}} \left(1 - e^{-\Delta\lambda}\right) V_A(k) + e^{-\Delta\lambda}\left[\left(\frac{F(k) - F(k_+)}{F(k)}\right) \kappa^{-1}(k_+;\Delta) + \frac{F(k_+)}{F(k)} e^{-\Delta r} V(k_+;\Delta)\right] \quad (3)$$

where $\kappa^{-1}(k_+;\Delta)$ is the price the seller asks on the equilibrium path to reach the new cutoff k_+ (and we require that $P(k;\Delta) = \kappa^{-1}(k_+;\Delta)$ for a k_+ that solves (3)).[15]

Regarding the buyer best response, we will focus on the necessary conditions that the buyer plays

[14] Finally, in case the equilibria are not unique, there also exist equilibria in which the seller randomizes in the first period over a set of prices that correspond to a set of equilibria without initial randomization.

[15] Note that since we have restricted in (3) the seller to choose k_+ only from Ω_Δ, $\kappa^{-1}(k_+;\Delta)$ is well defined: it is the price he asks on the equilibrium path in time $T(k_+;\Delta) - 1$.

a best response for all histories such that the current cutoff $k \in \Omega_\Delta$, i.e. histories in which the seller offered only equilibrium-path prices. Denote by $B(v;\Delta)$ the expected payoff of buyer with value v (at the beginning of the game). Looking at the direct-revelation representation of the buyer's strategy, he plays a best response (to the seller's equilibrium strategy) if and only if:

$$B(v;\Delta) = \max_{v'} e^{-(r+\lambda)T(v';\Delta)} \left(v - P\left(k(v');\Delta\right)\right) + \int_0^{T(v';\Delta)} \lambda W(v) e^{-(\lambda+r)s} ds \tag{4}$$

and $v' = v$ is a solution to this problem, where $k(v') = K(T(v';\Delta);\Delta)$ is the highest type that trades at time $T(v';\Delta)$. In words, the buyer can mimic another type v' to trade at a different price and time, $P(k(v');\Delta)$ and $T(v';\Delta)$. The first part on the RHS reflects the surplus from trading before the arrival of an event and the second part stands for the possibility that the arrival happens before $T(v')$.

We prove in the Appendix in Lemma 2 (*No Quiet Period*) that for every Δ, in every stationary equilibrium, there is trade with positive probability in every period. As a result, a necessary condition for the buyer's strategy, $\kappa(p;\Delta)$, to be a best response is that for every k that is reached on the equilibrium path and type $k_+ = \kappa(P(k;\Delta);\Delta)$ we have:

$$\underbrace{k_+ - P(k;\Delta)}_{\text{trade now}} = e^{-\Delta r}\left(1 - e^{-\Delta\lambda}\right)\underbrace{W(k_+)}_{\text{arrival}} + e^{-\Delta(r+\lambda)}\underbrace{(k_+ - P(k_+;\Delta))}_{\text{trade tomorrow}} \tag{5}$$

The interpretation is that the lowest type trading today (the new cutoff type k_+) has to be indifferent between accepting $P(k;\Delta)$ today and trading next period at $P(k_+;\Delta)$ (while facing the risk of arrival and getting $W(k_+)$ instead).

Instead of working with the general buyer problem, we will describe the equilibria using only the necessary optimality condition, (5).[16]

C (Continuous-time) Limit of Equilibria

The equilibrium strategies in discrete time are known to be in general analytically intractable (other than in special cases, for example for uniform distribution of values, see Nancy Stokey (1981), and in our game even in the uniform case they are not tractable). In the Appendix we analyze the stationary equilibria in discrete time and show that as $\Delta \to 0$ they all (even if they are not unique or not pure) converge to the same equilibrium path limit.[17] In contrast, this continuous-time limit of equilibria turns out to be relatively easy to characterize so that in the rest of the paper we focus on this limit. The full characterization of this limit is the focus of the next section and the main result of the paper.

[16] The RHS of (4) is supermodular in v and v' if $T(v')$ is weakly decreasing. Hence, the skimming property guarantees that the local incentive compatibility conditions - (5) - are not only necessary for optimality of the buyer's strategy but they are also sufficient.

[17] The general intuition behind the proof is related to the uniform convergence of equilibria shown in AD.

Theorem 1 *There exist strictly increasing functions* $V(k)$, $P(k)$ *and a strictly decreasing function* $K(t)$ *such that for any sequence of games indexed by the period lengths* Δ *asymptotically decreasing to* 0 *and any selection of stationary equilibria of these games* $\{\kappa(p;\Delta), P(k;\Delta)\}$ *and the corresponding sequences* $\{V(k;\Delta), K(t;\Delta)\}$ *we have that as* $\Delta \to 0$, $V(k;\Delta) \to V(k)$, $P(k;\Delta) \to P(k)$ *and* $K(t;\Delta) \to K(t)$. *That is,* $V(k)$, $P(k)$ *and* $K(t)$ *describe the unique limit of any selection of stationary equilibria for a sequence of games with* $\Delta \to 0$.

The main force behind the proof comes from taking the continuous-time limits of the optimality conditions that have to be satisfied in any equilibrium, (3) and (5). The proof of this result is in the Appendix, but we recommend the Reader to study Section II first since the proof relates to some of the expressions derived there.

As we show below, focusing the analysis on the limit as $\Delta \to 0$ is very convenient in terms of analytical tractability. The theorem above guarantees that learning about the limit we also learn about stationary equilibria of the game for small Δ since they all converge to the same (unique) limit.

II Characterization of $V(k)$, $P(k)$ and $K(t)$

We now heuristically characterize the limiting functions $V(k)$, $P(k)$ and $K(t)$ (these results are proved more formally in the Appendix). We first take the continuous-time limit of the seller's problem to derive $V(k)$ and $P(k)$. Second, we take the continuous-time limit of the Buyer's problem and derive $K(t)$. We then establish some important properties of the limit. At some points of the analysis it is convenient to use the continuous and strictly decreasing function $T(v) = K^{-1}(v)$ which specifies the (equilibrium path) time at which a buyer of type v trades (and is the unique limit of the corresponding $T(v;\Delta)$ functions).

Seller's problem. For any $\Delta > 0$ we can write (3) also as:

$$V(k;\Delta) = \max_{\substack{\frac{k_+ - k}{\Delta} \leq 0 \\ k_+ \in \Omega_\Delta}} \left(1 - e^{-\Delta\lambda}\right) V_A(k) + e^{-\Delta\lambda}\left[\left(\frac{F(k) - F(k_+)}{F(k)}\right)\kappa^{-1}(k_+;\Delta) + \frac{F(k_+)}{F(k)} e^{-\Delta r} V(k_+;\Delta)\right] \tag{6}$$

Subtracting $e^{-\Delta r} V(k;\Delta)$ from both sides, dividing by Δ and taking the limit as $\Delta \to 0$, we can show that the seller's best response problem can be thought of as finding the optimal rate at which he goes through types:[18]

[18] When taking the limit we use the following claims: a) Prices become continuous in time and in types (Lemma 3, *Prices Don't Jump*) and b) that the probability of trade in every period converges to zero ($F(k_+) \to F(k)$, Lemma 4, *No Atoms*). Since $T(v)$ is continuous and strictly decreasing, the set Ω_Δ becomes dense in the limit, covering the whole interval $[0,1]$.

$$rV(k) = \max_{\dot{K}\in(-\infty,0]} \lambda\left(V_A(k) - V(k)\right) + \left(P(k) - V(k)\right)\frac{f(k)}{F(k)}\left(-\dot{K}\right) + V'(k)\dot{K} \tag{7}$$

where $\dot{K} = K'(t)$ is the speed at which the seller screens the types in equilibrium.[19]

This condition has a direct interpretation if we know that in the limit the seller smoothly screens down the "demand curve" (as proven in the two lemmas referenced above). The left-hand side is the expected equilibrium payoff expressed in flow terms. The right hand side represents the possible sources of the flow: upon arrival of the event (which happens with a probability flow λ) the game ends with the seller earning in expectation $V_A(k)$ (and since the game ends he forgoes $V(k)$). With a flow probability $\frac{f(k)}{F(k)}\left(-\dot{K}\right)$ the buyer accepts current offer, $P(k)$, which also ends the game. Finally, if the game does not end immediately, the continuation payoff drops, as the seller becomes more pessimistic about v, as captured by $V'(k)\dot{K}$.

Note that (7) is linear in $\dot{K}$.[20] This linearity is the source of Coasian dynamics when $\lambda = 0$. In that case, for any strictly increasing $P(k)$ the seller wants to run down the demand function as fast as possible. Therefore the equilibrium $P(k)$ in the limit becomes flat at 0.The outside option in our model provides a counterbalance for the seller's temptation to run down the demand curve, leading to a strictly downward-sloping $P(k)$.

If the coefficients on $\dot{K}$ in (7) add up to something negative (over a range of k), the seller would maximize payoffs by trading as fast as possible, which would violate in the limit Lemma 4 (*No Atoms*). If the coefficients on $\dot{K}$ add up to something positive (over a range of k), then the seller would maximize payoffs by not trading over that range at all, so that after reaching this range the trade would stop. That can be shown to be inconsistent with equilibrium following the reasoning in Lemma 2 (*No Quiet Period*). Therefore, in the limit of equilibria the coefficients on $\dot{K}$ need to add up to 0 (almost everywhere) and the seller has to be indifferent over all possible $\dot{K}$ making any interior $\dot{K}$ optimal:

$$\begin{aligned} \left(P(k) - V(k)\right)\frac{f(k)}{F(k)} &= V'(k) \\ &\Updownarrow \\ P(k) &= \frac{\partial}{\partial k}\left[V(k)F(k)\right]/f(k) \end{aligned} \tag{8}$$

If (8) holds, then we can calculate $V(k)$ by simply substituting (8) in (7) :

$$V(k) = \frac{\lambda}{\lambda + r}V_A(k) \tag{9}$$

[19]More precisely, $\dot{K} = \lim_{\Delta\to 0}\frac{k_+ - k}{\Delta}$.

[20]The intuition for linearity is that the seller has the option to "speed up the clock" and smoothly screen down the demand function twice as fast. We use this reasoning in Lemma 5 (Payoffs Converge) to show that $V(k;\Delta) \to \frac{\lambda}{\lambda+r}V_A(k)$.

That implies the equilibrium prices must satisfy:

$$P(k) = \frac{\lambda}{\lambda + r}\Pi(k) \tag{10}$$

These two equations pin down the unique candidates for the limiting functions $P(k)$ and $V(k)$.

Note that, interestingly, $V(k)$ has the property that at any point in the game (for any k) the expected payoff of the seller is equal to his payoff from waiting for the arrival of the event. Hence, although the Coase conjecture does not hold in terms of the price dropping immediately to zero, the Coasian dynamics force down the seller's profit down to his outside option. Moreover, for each type k, $P(k)$ is exactly the expected present value the seller would have earned from this type if he waited for the arrival - a kind of no-ex-post regret property - upon the price being accepted the seller does not regret not slowing down the trade.

In discrete time $\frac{\lambda}{\lambda+r}V_A(k)$ bounds equilibrium payoffs from below ($V(k;\Delta) \geq \frac{\lambda}{\lambda+r}V_A(k)$) because the seller has the option not to trade until the arrival.[21] In discrete time the seller can commit not to reduce the price for Δ units of time. This allows him to earn more than his outside option, which makes the analysis of the equilibrium much more difficult. In the continuous-time limit the seller loses all commitment power and as we have shown $V(k;\Delta) \to \frac{\lambda}{\lambda+r}V_A(k)$: the value of the outside option also becomes the upper bound on equilibrium payoffs![22] This generalization of the Coase conjecture (that as the bargaining frictions disappear, the seller cannot earn more than his outside option) makes the limit very simple and intuitive.

Buyer's problem. We now turn to the buyer's best response problem. Recall that for any $\Delta > 0$ we have a necessary optimality condition:

$$\underbrace{k_+ - P(k;\Delta)}_{\text{trade now}} = e^{-\Delta r}\left(1 - e^{-\Delta\lambda}\right)\underbrace{W(k_+)}_{\text{arrival}} + e^{-\Delta(r+\lambda)}\underbrace{(k_+ - P(k_+;\Delta))}_{\text{trade tomorrow}} \tag{11}$$

Subtracting $e^{-\Delta(r+\lambda)}(k_+ - P(k;\Delta))$ from both sides, dividing by Δ and taking $\Delta \to 0$ we get the following limit of the indifference condition:[23]

$$r(k - P(k)) + \lambda(k - P(k) - W(k)) = -P'(k)\dot{K} \tag{12}$$

It also has a direct interpretation: the LHS is the cost of delaying trade (due to discounting and possibility of arrival) and the RHS is the benefit of waiting from the reduction in price. The benefits and costs are evaluated at the current cutoff type.

[21] If τ is the random Poisson arrival time, then $\frac{\lambda}{\lambda+r} = E\left[e^{-r\tau}\right]$ is the expected present value of a dollar received at the arrival time.

[22] See Lemma 5 for a formal derivation of this statement.

[23] For this result we use Lemma 6, which shows that $P(k;\Delta)$ converges to $P(k)$, and Lemma 4, which shows that there are no atoms of trade in the limit.

Using $P(k)$ we found above, the buyer's indifference condition allows us to find $K(t)$. Substituting (10) in (12) yields:

$$-\dot{K} = (\lambda + r) \frac{(r+\lambda) K(t) - \lambda (\Pi(K(t)) + W(K(t)))}{\lambda \Pi'(K(t))} \tag{13}$$

which together with the boundary condition $K(0) = 1$ pins down $K(t)$.[24] By assumption, for all $v > 0$, $\frac{\lambda}{\lambda+r}(\Pi(v) + W(v)) < v$ and $\Pi'(v) > 0$, so the numerator and denominator are strictly positive for all $K(t) > 0$. Therefore, this differential equation (with the boundary condition) uniquely defines a strictly decreasing and continuous $K(t)$.

Additionally, we can calculate the buyer's expected payoff using (4). Note that the limit $B(v) = \lim_{\Delta \to 0} B(v; \Delta)$ is simply equal to:

$$B(v) = \max_{v'} e^{-(r+\lambda)T(v')} \left(v - P\left(v'\right)\right) + \int_0^{T(v')} \lambda W(v) e^{-(\lambda+r)s} ds \tag{14}$$

We can either use $\{T(v), P(v)\}$ to calculate $B(v)$ directly, we can apply the envelope theorem:

$$B'(v) = e^{-(r+\lambda)T(v)} + \frac{\lambda}{\lambda + r}\left(1 - e^{-(r+\lambda)T(v)}\right) W'(v) \tag{15}$$

and use the boundary condition $B(0) = 0$ to pin down $B(v)$.

Summarizing this Section:

Theorem 2 *The unique limit of stationary equilibria* $\{V(k), P(k), K(t)\}$ *is characterized by* (9), (10), (13) *and the boundary condition* $K(0) = 1$.

A Properties of the Limit of Equilibria.

We now present some general properties of the unique limit of equilibria.

In the previous section we implicitly characterized $T(v) = K^{-1}(v)$, the time at which type v trades conditional on no arrival. If we interpret that the arrival of the event ends the game with an immediate trade (which is true in the applications we present in Section III and not true in the application with multiple arrivals in Section IV), we can further define the expected time at which

[24] The boundary condition is proven in Lemma 7, which shows that there is no atom of trade at time 0, but rather the seller starts smoothly screening the buyer types immediately.

type v trades, $\tau(v)$. It takes into account the possibility that arrival takes place before $T(v)$:

$$\begin{aligned}\tau(v) &= \underbrace{\left(\int_0^{T(v)} \lambda e^{-\lambda s} ds\right)}_{\text{Pr arrival before } T(v)} \times \underbrace{\left(\int_0^{T(v)} s \frac{\lambda e^{-\lambda s}}{1-e^{-\lambda T(v)}} ds\right)}_{E[\text{arrival time } |<T(v)]} \\ &\quad + \underbrace{\left(1-\int_0^{T(v)} \lambda e^{-\lambda s} ds\right)}_{\text{Pr no arrival before } T(v)} \underbrace{T(v)}_{\substack{\text{time to trade conditional} \\ \text{on no arrival}}}\end{aligned}$$

Finally, we can define the (unconditional) expected time to trade as $\int_0^1 \tau(v)\, dF(v)$.

Proposition 1

(i) (Delay): For all $0 < \lambda < \infty$ *the expected time to trade is strictly positive.*
(ii) (Coase conjecture): as $\lambda \to 0$, *the expected time to trade and transaction prices converge to* 0 *for all types (i.e.* $T(v) \to 0$ *and* $P(k) \to 0$*).*

Part (i) shows that when the bargaining is subject to external influences, delay is to be expected, which is one of our main results.

It follows directly from our characterization, but the intuition is as follows: suppose that there is no delay in equilibrium. Then the transaction prices for all types have to be close to zero, implying seller's payoff close to zero, in particular, less than $\frac{\lambda}{\lambda+r} V_A(k) > 0$. But that leads to a contradiction since the seller can guarantee himself that by just waiting for the arrival of the event. Moreover, the bargaining cannot stop at any type $k > 0$ with the buyer and seller waiting for the arrival of the event, since then for all types $v \leq k$ the reserve price would be $P(v) = v - \frac{\lambda}{\lambda+r} W(v)$. But then, by Assumption 1 that $v > \frac{\lambda}{\lambda+r}(W(v) + \Pi(v))$, the seller would be strictly better off screening through the types quickly than waiting for the arrival.

Part (ii) shows that our limit of equilibria converges to the equilibria in GSW and FLT: as we take the probability of arrivals to zero (convergence of the model) trade takes place immediately and the buyer captures all the surplus (convergence of equilibrium).

Since both results follow directly from our characterization above, we omit the proof of this proposition.

Arrival of new traders or outside options is necessary for delay but another important ingredient for slow equilibrium screening is that the seller's outside value depends on the buyer's type. In particular, we can establish the following general comparative statics:

Proposition 2

(i) Consider two environments, one with $\Pi_1(v)$ *and the other with* $\Pi_2(v)$ *and either* $W_1(v) = W_2(v)$ *or* $\Pi_1(v) + W_1(v) = \Pi_2(v) + W_2(v)$. *Then if* $\Pi_1'(v) > \Pi_2'(v)\ \forall v > 0$, *the expected time to trade is*

shorter in the environment with $\Pi_2(v)$ *.*

(ii) In the limit as $\Pi'(v) \to 0\ \forall v$*, expected time to trade converges to zero and the buyer asymptotically captures all the surplus.*

The second part of this Proposition shows that the Coase conjecture holds in the limit as $\Pi'(v) \to 0\ \forall v$. Given our assumption $\Pi(0) = 0$, $\Pi'(v) \to 0\ \forall v$ implies that $\Pi(v) \to 0\ \forall v$. To separate slope versus level effects consider the case where $\Pi'(v) = 0$ *but* $\Pi(v) = c > 0\ \forall v$ (that is, the arrival stands for somebody coming to offer the seller price c). In this case, in equilibrium the seller offers price $p = \frac{\lambda}{\lambda+r}c$ and either trade happens immediately or there is no trade until arrival. For there to be trade with delay it is necessary that $\Pi'(v) > 0$. Intuitively, the seller makes a first offer $p = \frac{\lambda}{\lambda+r}\Pi(1)$. Since this offer is accepted by the highest types, the seller's outside option decreases a bit and next period he is willing to make lower offers. In this way he slowly skims through all buyer types.

But why does it happen slowly? Why don't we get almost immediately to $p = 0$, like in the Coase conjecture? The reason is that if the seller ran 'the clock' too fast then some buyer types would have an incentive to wait for a lower price - their reservation prices would decrease. But then the seller would prefer to stop trading, since he would get a higher expected payoff from just waiting for an arrival than from trading at these low prices. On the other hand, the seller cannot run too slowly through the demand either, since then the reservation prices would be so high, that the seller would prefer to collect the whole area below the demand *before* the arrival. Therefore the speed at which price decreases has to be such that the reservation prices of the buyer keep the balance between the incentive to speed up and slow down the trade.

Following this logic, if $\Pi_1'(v) \geq \Pi_2'(v)\ \forall v$, then under Π_1 the seller's outside option drops faster as his belief of the current buyer cutoff type falls. This makes prices as a function of k decrease at a faster rate for the steeper $\Pi(v)$. Hence, if the seller ran the clock (with respect to $K(t)$) at the same speed, prices would drop faster in time under Π_1. But then the buyers would have an incentive to wait for lower prices, leading to a contradiction that the k changes through time. To keep the current cutoff types willing to trade at the current prices the seller has to go through the types slower under Π_1, as claimed in the first part of the last proposition.

This result allows us to compare our dynamics to existing literature. For example, in Inderst (2008) (and other papers that have the new buyers replace the existing buyer), $\Pi'(v) = 0$ and there is no delay.[25] Taking the limit $\Pi'(v) \to 0$ in our model leads to the same limiting outcome. It is essential for there to be delay that the outside value of the seller depends on the buyer's type. The more sensitive the outside value of the seller to the buyer's private information, the greater the delay/inefficiency. As we explained in the Introduction, the correlation of the seller's outside option with the buyer value endogenously creates a bargaining environment with interdependent values,

[25] Inderst (2003) assumes that upon the second buyer arriving, the seller can only choose to keep on bargaining with the current buyer or switch to bilateral bargaining with the new one, which implies $\Pi'(v) = 0$ in his model.

as studied by Evans (1989), Vincent (1989) and Deneckere and Liang (2006), and hence the main economic intuition behind the delay is similar to that in those papers.

The next proposition characterizes how the time on the market and the ex-ante expected payoffs depend on the distribution of values:

Proposition 3 *Suppose* $\Pi(v)$ *and* $W(v)$ *are independent of the distribution of values,* $F(v)$ *.*
(i) The limit (of stationary equilibria) $P(k)$ *and* $K(t)$ *are independent of the distribution of values,* $F(v)$*.*
(ii)(Weak markets and time on the market) Consider two distributions of buyer's values F *and* H *such that* F *first order stochastically dominates* H*. The expected time to trade is longer if the distribution of values is* H *(and average prices are lower).*
(iii) (Dispersion of values and efficiency of trade) Suppose $\Pi(v)$ *and* $W(v)$ *are weakly convex. Consider two distributions of values* F *and* H *such that* F *second order stochastically dominates* H*. Then the ex-ante expected sum of payoffs is higher under distribution of values* H*.*

To illustrate this surprising result (part (i) - that the equilibrium $P(k)$ and $K(t)$ can be independent of the distribution of values) consider the following example. Suppose that the event represents an arrival of one more buyer who has the same valuation as the original buyer. Upon arrival the seller runs an English auction.[26] As a result, $\Pi(v) = v$ and $W(v) = 0$ independently of the distribution. In that case, the proposition states that the equilibrium path of prices $P(k)$ and the times at each type trades, $T(v)$ (and hence, how prices change over time) are independent of $F(v)$!

The intuition behind this result is as follows. The equilibrium outcome can be thought of as a screening of the buyer types with a menu of prices and times to trade. Now, since the inability to commit drives the expected payoff of the seller down to his outside option for any k ($V(k) = \frac{\lambda}{\lambda+r} V_A(k)$), he collects from each type just the outside option from this type. That is, the prices are $P(k) = \frac{\lambda}{\lambda+r}\Pi(k)$, independently of the distribution. But then, like in any separating equilibrium, the whole menu is independent of the distribution as well! Mechanically, $K(t)$ is pinned down by the indifference condition of the buyers. Since the current marginal buyer's incentives do not depend on the distribution (unless $W(v)$ does) the limit $K(t)$ is independent of $F(v)$. Clearly low valuation buyers would like the seller not to spend time sorting through high types. The problem is that they have no credible way in which to signal to the seller that they have a low value.

Part (ii) of the proposition follows then immediately: since $T(v) = K^{-1}(t)$ and $P(k)$ are independent of the distribution, and T is decreasing while P is increasing, the average time to trade is higher and transaction prices lower under the weaker distribution that puts higher weight on the lower types.

Finally, part (iii) follows from payoff calculations: $V(1) = \frac{\lambda}{\lambda+r} E[\Pi(v)]$, i.e. the seller's expected payoff at the beginning of the game when $k = 1$, is weakly higher under H if $\Pi(v)$ is weakly convex.

[26] Alternatively, the event can represent an arrival of information that reveals the value of the buyer.

Similarly, if $W(v)$ is weakly convex then so is $B(v)$, which makes the average buyer payoff, $E[B(v)]$ also higher under H. Intuitively, a higher dispersion of values has two effects for total surplus: there is a gain from early trade with high types and a loss from later trade with low types. But since the surplus from high types is higher, the total surplus on average goes up.

One may expect results (ii) and (iii) to be more robust, holding even when the post-arrival payoffs depend on the distribution of values. We discuss this issue further in Section III.B.

III Applications

We now turn to three examples to demonstrate how the general model can be adapted to different applications and used to derive additional predictions.

A Arrival of New Traders with Common Value

Suppose that the event represents two possibilities: either a second seller with an identical good arrives or a second buyer with identical valuation arrives (we call it *the common value* case).[27] The arrival rates are λ_s and λ_b respectively with $\lambda = \lambda_b + \lambda_s$. Upon arrival there is Bertrand competition on the long side of the market (for example, the agent on the short side of the market runs an English auction). As a result, the expected payoffs conditional on arrival in this case are:

$$W(v) = \frac{\lambda_s}{\lambda} v \ , \ \Pi(v) = \frac{\lambda_b}{\lambda} v \ .$$

and clearly they satisfy Assumption 1 (and the assumptions in Proposition 3).[28] Using the equilibrium conditions (10) and (13) we can calculate the limit $P(k)$ and $T(v) = K^{-1}(v)$ in a closed form:

$$P(v) = \frac{\lambda_b}{\lambda + r} v \ , \qquad T(v) = -\frac{\lambda_b}{r(\lambda + r)} \ln v \ . \tag{16}$$

and combining these two we get that the seller reduces the asking price over time according to:

$$p(t) = \frac{\lambda_b}{\lambda + r} \exp\left(-rt\frac{\lambda + r}{\lambda_b}\right)$$

The corresponding value functions are:

[27] It can be also described as private values with perfect correlation, since the buyers know their valuations and are not concerned with the winner's curse.

[28] These reduced-form payoffs are analogous in a model where the arrival means information arrives to the market that reveals the buyer value, and upon revelation they split the surplus efficiently, giving the seller a λ_b/λ share.

$$V(k) = \frac{\lambda_b}{\lambda + r} \int_0^k v \frac{f(v)}{F(k)} dv \ , \tag{17}$$
$$B(v) = \frac{\lambda_s v + r v^{\frac{\lambda_b + r}{r}}}{\lambda + r}.$$

Using the results in the previous section, we can observe that in the limit of stationary equilibria:

1) Market Tightness: Keeping $\lambda = \lambda_b + \lambda_s$ constant (the sum of arrival rates of the second buyer and seller), a decrease in the ratio $\frac{\lambda_b}{\lambda_s}$, implies a shorter equilibrium time on the market, a lower seller's expected payoff and a higher buyer's payoff. In the limit, as $\frac{\lambda_b}{\lambda_s} \to 0$ we get immediate trade with the buyer capturing all the surplus.

2) Market Thickness: Keeping the ratio $\frac{\lambda_b}{\lambda_s}$ fixed, delay is non-monotonic in the sum $\lambda = \lambda_b + \lambda_s$. It converges to zero as $\lambda \to \infty$ and also as $\lambda \to 0$, while it is greater than zero for intermediate values.

The first result shows that trade is faster when it is a buyers' market. This is because the higher the likelihood of arrival of the second seller, the more impatient the current seller gets, which makes him offer lower prices. In the limit, if only new sellers can arrive then trade takes place immediately and the buyers capture all the surplus as in FLT or GSW.

The second result, which follows from Proposition (1), shows that the delay is non-monotonic in the liquidity of the market. In the limit as we approach perfect competition ($\lambda_b + \lambda_s \to \infty$) arrivals and hence trade takes place immediately. Trade is also immediate when there is a bilateral monopoly with no possibility of arrival. But when we have a thin market there is some delay in trade.

Since $\Pi(v)$ and $W(v)$ are independent of the distribution $F(v)$, Proposition 3 applies and the equilibrium $P(k)$ and $K(t)$ are independent of the distribution. Does it mean that the distribution of values has no impact on the expected trade dynamics? No. In fact, as a corollary to Proposition 3 we get additional two observations:

3) Weak markets and time on the market: Consider two distributions of buyer's values F and H such that F first order stochastically dominates H. The expected time to trade is longer if the distribution of values is H (and average prices are lower).

4) Dispersion of values and efficiency of trade: Consider two distributions of values F and H such that F second order stochastically dominates H. Then the ex-ante expected sum of payoffs is higher under distribution of values H but the expected time to trade is lower under F.

These results can be derived directly from the expressions (16) and (17) by noting that $T(v)$ is decreasing and convex, the $V(1)$ depends only on the average v and $B(v)$ is convex in v.[29]

These results point to an interesting finding that trade takes longer in markets with weaker distributions of valuations. This could help explain some of the cyclical patterns in real estate

[29] The only new claim is that the expected time to trade is longer when the distribution of values is more dispersed. It follows from $t(v)$ being convex.

markets and in labor markets.

A.1 Impatience: Arrivals vs. Discounting

In Ariel Rubinstein's (1982) bargaining model the relative discounting of the buyer with respect to the seller is critical in determining the price at which the object is traded. In our model beyond discounting there is an additional source of impatience: the probability of having the arrival of a competing trader on your side of the market. In this Section we study how arrivals compare with discounting in determining the properties of the equilibrium.

Let r_s be the interest rate faced by the seller and r_b the one faced by the buyer. The time at which each type trades and the prices are given by:

$$P(v) = \frac{\lambda_b}{\lambda + r_s} v \quad ; \quad T(v) = -\frac{\lambda_b}{r_b(\lambda_s + r_s) + r_s \lambda_b} \ln v$$

We can see that, in contrast to Rubinstein's model, the seller's discount rate is the only one determining the path of prices. A higher buyer discount rate has no impact on prices. Also note that the seller's two sources of impatience have identical effects for determining prices. Hence having more fear of competition through higher likelihood of arrivals of sellers or a higher discount rate are identical sources of seller's impatience. This is not true on the buyer's side. The buyer discount has a direct effect in the time to trade. This is because the more he discounts the future the faster the seller can lower the prices without violating the buyer's indifference condition. The fear of arrivals of competing buyer's has two effects. A direct effect like the discount factor and an indirect effect via its effect on prices.

Furthermore, if r_s and r_b are much smaller than λ, then the prices paid by different types depend mostly on λ_s and λ_b and very little on r_s and r_b and $\frac{r_s}{r_b}$ affects the equilibrium only via delay by influencing which of the arrival rates is more important for the speed at which prices drop over time:

$$\begin{aligned} P(v) &\approx \frac{\lambda_b}{\lambda} v \ ; \\ T(v)(r_b + r_s) &\approx -\frac{\lambda_b}{\lambda_s \frac{r_b}{r_b + r_s} + \frac{r_s}{r_b + r_s} \lambda_b} \ln v \end{aligned}$$

Hence in thick markets what matters in terms of bargaining power is driven a lot by the relative arrival rates and much less by the relative rates of time discount.[30]

[30] The importance of arrival rates of buyers and sellers in bargaining has been analyzed before in models without asymmetric information, see for example Rubinstein and Asher Wolinsky (1985) or Curtis Taylor (1995). This section complements this literature by looking at a market with information frictions.

B Arrival of a New Buyer with Private Values

Consider now a case where only one additional buyer can arrive and his value $v \sim F(v)$ is independent of the current buyer's value (but comes from the same distribution). Also, upon arrival we assume the seller runs a second price auction with no reserve to allocate the good. In this environment we can calculate:

$$\begin{aligned} W(v) &= F(v)\, v - \int_0^v x f(x)\, dx \\ \Pi(v) &= \int_0^v x f(x)\, dx + (1 - F(v))\, v \end{aligned}$$

We can combine these with (10) and (13) to fully characterize the equilibrium in this environment:[31]

$$P(v) = \frac{\lambda}{\lambda + r} \left(\int_0^v x f(x)\, dx + (1 - F(v))\, v \right),$$

$$-\dot{K} = (\lambda + r) \frac{r}{\lambda} \frac{K(t)}{1 - F(K(t))}.$$

The expression for $K(t)$ is quite involved but its inverse is tractable:

$$T(v) = \frac{\lambda}{(\lambda + r)\, r} \int_v^1 \frac{1 - F(x)}{x} dx.$$

This environment is a good example of a situation where $\Pi(v)$ and $W(v)$ do depend on $F(v)$, so that Proposition 3 does not apply. In such situations, how do average prices and time to trade change as the distribution of types changes? Loosely speaking, there is a general tendency for weaker markets to have longer time to agreement, but it is not always the case. There are two counteracting effects. First, like in Proposition 3.*ii*, because $T(v)$ is decreasing, $E[T(v)]$ puts more weight on longer times to trade for weaker distributions. However, $T(v)$ is no longer independent of the distribution, which creates the second effect: a weaker distribution of values implies typically a lower $\Pi(v)$ (the new buyer is expected to provide weaker competition). In turn, that means lower asking prices for each cutoff and that the buyer has less incentives to wait for the prices to drop. So, even though we have more weak types, all types tend to trade faster and in general it is ambiguous how $E[T(v)]$ changes when the distribution weakens.

In particular, continuing with the model with private values, if the distribution changes in a way that $\frac{1-F(k)}{k}$ decreases for all k (which also means the distribution is weaker, albeit in a different sense) then *every type* trades faster. The expected time to trade conditional on no arrival is:

$$E[T(v)] = \frac{\lambda}{(\lambda + r)\, r} \int_0^1 \frac{(1 - F(v))\, F(v)}{v} dv \qquad (18)$$

[31] To simplify the equation for $\dot{K}$ we use that $\Pi(v) + W(v) = v$ and $\Pi'(v) = 1 - F(v)$.

and indeed first-order or second-order stochastic dominance of distributions are not sufficient to rank time to trade.

To provide some intuition, consider the example of a class of distributions $F(v) = v^a$ (with a higher a representing a stronger distribution). In this case the expected time to trade conditional on no arrival simplifies to:

$$E[T(v)] = \frac{\lambda}{2a(\lambda + r)r}.$$

That implies that as we move to a weaker distribution the effect of having more weak types dominates the effect that each type trades faster. Hence, at least within this family of distributions, time to trade is shorter for stronger distributions.

As far as dispersion of values and delay are concerned (to compare the results to observation 4 in the previous section), if the distribution of values is symmetric around $\frac{1}{2}$ a mean-preserving spread in the distributions leads to a longer average time on the market, as can been seen from a direct inspection of (18) (but in general, the effect of second-order stochastic dominance is ambiguous).

Finally, in terms of expected payoffs and dispersion of values (to compare with 3.*iii*), $B(v)$ is strictly convex in v (which can be shown using the proof of 3.*iii* and noting that $W(v)$ is convex in the private values case). So, comparing two distributions F and H such that H is a mean-preserving spread of F, on average the buyer is strictly better off under H. On the other hand, the ex-ante expected payoff of the seller is $V(1) = \frac{\lambda}{\lambda+r}\Pi(1) = \frac{\lambda}{\lambda+r}E[v]$, and it does not change with the mean-preserving spread.

B.1 Auction Format and Time Consistency.

So far we have assumed that the seller runs an English auction with no reserve price upon arrival of the second buyer. Modeling the impact of different auction formats is somewhat delicate because in general optimal bids depend on the belief the second buyer has about the value of the first buyer. In turn, the belief depends on what the second buyer observes about prior bargaining (and if what he observes is not common knowledge, the bidding depends also on the higher-order beliefs.) The analysis is tractable if we assume that upon arrival at least the lowest offer made so far is common knowledge. However, since the first and the second buyer are not symmetric at the beginning of the auction, different auction formats will yield different expected revenues. In particular, with i.i.d. ex-ante distributions of the two buyers, the first buyer is going to have a weaker (truncated) distribution.[32]

In general, to increase the competition faced by the stronger bidders, optimal auctions usually treat weaker bidders more favorably. On the other hand, one could be concerned that treating the first bidder more favorably in the auction would make him more stubborn during the bargaining phase and hence hurt the seller. Can that lead to time-inconsistency of the optimal auction choice

[32] The asymmetry arises endogenously because in our setup beliefs about the first buyer value are updated during the bargaining phase.

of the seller (i.e. that he would like to choose one format ex-ante and another one ex-post)?

We argue that in fact no such time-inconsistency would arise.[33] The reason is that given any auction format during the bargaining phase the seller's payoff is equal simply to his outside option. Therefore, if the seller chooses ex-post the auction format that maximizes ex-post revenues, he will at the same time maximize ex-ante payoffs as well.[34]

C Taste Diversity and Time on the Market.

In many markets it is natural to think that there are different groups of potential buyers of the asset, and that even though valuations within a group can be very similar, they would differ across groups quite a bit. For example, families with school age children could be one group with similar valuations for a given house. The group of retirees, on the other hand, could value the same house differently. The first group would put more weight on the quality of the school district while the latter care more about the quality of the walking paths. Similarly, if a firm is being sold, there are different groups of potential buyers such as competing firms and private equity funds that have different motives for purchasing the target.

To illustrate the effects of diverse taste groups of potential buyers on the bargaining dynamics, we parameterize the problem as follows. Assume there are n different groups of buyers. All members of a given group share the same valuation but valuations across groups are $i.i.d.$ according to $F(v)$. Now, when the second buyer arrives, with probability $\gamma = \frac{1}{n}$ he belongs to the same group (and has the same valuation) as the current buyer (and this is common knowledge). Otherwise, with probability $(1-\gamma)$, he belongs to a different group and his value is independent of the first buyer value. Therefore, a larger γ stands for a less diverse market place. In either case an English auction is used to allocate the good. For simplicity assume $\lambda_s = 0$.

In this case the expected payoffs conditional on arrival are:

$$\begin{aligned} W(v_1) &= (1-\gamma) F(v_1)(v_1 - E[v_2|v_2 \leq v_1]) \\ \Pi(v_1) &= \gamma v_1 + (1-\gamma)(F(v_1) E[v_2|v_2 \leq v_1] + (1-F(v_1)) v_1) \end{aligned}$$

Applying the general analysis above, we can establish the following comparative statics with respect to the taste diversity:

The limit of equilibria has the following comparative statics with respect to an increase in the number of groups, $n \uparrow (\downarrow \gamma)$:

(i) The expected time to trade decreases.

(ii) The payoff of the seller falls.

(iii) For any t the price offered is lower.

[33] At least for auction formats such that the resulting $\Pi(v)$ and $W(v)$ satisfy Assumption 1.

[34] Interestingly, the optimal auction format will be changing over time, since it is going to be dependent on current belief, k.

Part (i) follows from noting that $\frac{\partial \Pi'(v_1)}{\partial \gamma} = F(v_1) > 0$ and using the result from Proposition 2. (ii) and (iii) follow from noting that $\Pi(v_1)$ is decreasing in n (since the second term of $\Pi(v_1)$ is smaller than v_1) and using equations (9) and (10) which respectively characterize the seller's value and prices.

This result suggests that sellers would benefit more from specializing in a narrow market, intensively targeting a given group of potential buyers rather than casting a very wide net. Although we do not model it here, this benefit of specialization must be balanced against the potential drop in the contact frequency, λ_b.

IV Multiple Arrivals

In many markets the seller can wait for more than one additional buyer. That leads us to a natural extension of the model to multiple arrivals. Unfortunately, a general model in which the seller can bargain with multiple buyers at the same time and have more and more of them arrive is not tractable. To gain some intuition (and to demonstrate that some of the economics we described above are robust), we instead analyze a simpler, more stylized model.

In particular, we assume that there is a constant arrival rate of new buyers, λ, and the buyers have independent private values all drawn from the same distribution $F(v)$ with support $[0,1]$ (there are potentially infinitely many buyers that can arrive). Throughout this section we assume that $F(v)$ satisfies the downward-sloping marginal revenue condition, that is we assume that $v - \frac{1-F(v)}{f(v)}$ is strictly increasing.[35]

When a new buyer arrives we assume that the seller makes a last take-it-or-leave-it offer to the old buyer. If it is accepted, the game is over. If it is rejected, the new buyer replaces the old buyer and the bargaining starts from scratch until the next arrival. These assumptions are a combination of the setups in Fudenberg, Levine and Tirole (1987) (who allow for take-it-or-leave-it offers with replacement of buyers upon rejection, but have an infinite supply of buyers standing by available for immediate replacement) and Inderst (2008) who has Poisson arrivals but does not allow for final offers. Nonetheless, the resulting equilibrium dynamics are very different from the ones in either of those papers.

We now sketch the characterization of the continuous time limit of stationary equilibrium of this model. Note that after the old buyer rejects a final offer, the game starts afresh. Stationarity is crucial for tractability, since it allows us to keep track of only one state variable, k, the cutoff of the currently bargaining buyer.

Denote by $V(k)$ the value of the seller (i.e. his expected equilibrium payoff) when he is bargaining with one buyer with a cutoff belief k. Let $V^* = V(1)$. This is the seller's expected value at the beginning of the game and also his expected continuation payoff after the old buyer rejects his final

[35] Myerson (1981) calls this condition increasing virtual valuation, or the regular case.

take it or leave it offer. For now, we will take V^* as given.

Let $V_A(k)$ be the expected payoff of the seller upon arrival (before the current buyer responds to the take it or leave it offer). To find it, note that upon arrival the seller will chose the final offer $p_A(k)$ to maximize:

$$p_A(k) = \arg\max_p \left(\frac{F(k) - F(p)}{F(k)}\right) p + \frac{F(p)}{F(k)} V^* \tag{19}$$

and the expected payoff upon arrival will satisfy:

$$F(k) V_A(k) = \max_p (F(k) - F(p)) p + F(p) V^* \tag{20}$$

From the envelope condition we have:

$$\frac{\partial}{\partial k}(F(k) V_A(k)) = f(k) p_A(k) \tag{21}$$

To pin down the equilibrium we will also use a technical lemma:

Lemma 1 *For any V^* and $k > V^*$ there is a unique and strictly increasing $p_A(k) \in [V^*, k)$ that solves* (19). *For $k \leq V^*$ an optimal strategy is $p_A(k) = V^*$ and no trade with probability* 1.

The proof, which can be found in the Appendix, makes use of our regularity assumption on F.

Therefore, given a V^*, the above equations determine uniquely $V_A(k)$, and $p_A(k)$. Now, take $V_A(k)$ and $p_A(k)$ as given. How does the equilibrium in the one-on-one bargaining phase look like?

As long as the seller gradually (i.e. without atoms) screens down the demand function (which he will do if $P(k)$ is strictly decreasing) then the seller's problem is as in the base model of Section 3:

$$rV(k) = \max_{\dot{K} \in (-\infty, 0]} \lambda (V_A(k) - V(k)) + (P(k) - V(k)) \frac{f(k)}{F(k)} \left(-\dot{K}\right) + V'(k) \dot{K}$$

as before, in equilibrium we need the coefficients on $\dot{K}$ to add up to zero, which gives:

$$\begin{aligned} f(k) P(k) &= \frac{\partial}{\partial k}[V(k) F(k)] \\ V(k) &= \frac{\lambda}{\lambda + r} V_A(k) \end{aligned}$$

Therefore, using (21) we pin down the equilibrium prices (for the range that the seller smoothly screens down the demand):[36]

$$P(k) = \frac{\lambda}{\lambda + r} p_A(k)$$

[36]Note that we get this very simple expression for equilibrium prices even though we no longer use $V_A(k) = E[\Pi(v) | v \leq k]$.

Lemma (1) implies that $P(k)$ is strictly increasing for $k > V^*$. This guarantees no atoms over that range.

The buyer's local IC constraint for types $k > V^*$ still is:

$$(r+\lambda)(k - P(k)) = \lambda W(k) - P'(k)\dot{K} \tag{22}$$

Where the payoff upon arrival of the current cutoff type is:

$$W(k) = k - p_A(k).$$

because the current cutoff type trades for sure upon arrival (since $p_A(k) < k$ for $k > V^*$). Note as well that $k - W(k) = p_A(k)$ is strictly increasing (which implies that the local IC (22) is still sufficient).

Substituting the equilibrium $P(k)$ into the buyer's indifference condition we get:

$$-\dot{K} = \frac{\lambda + r}{\lambda} \frac{rK(t)}{p_A'(K(t))}$$

For example, if values are distributed uniformly we get simple expressions:

$$\begin{aligned} p_A(k) &= \frac{k + V^*}{2} \\ -\frac{\dot{K}}{K(t)} &= \frac{\lambda + r}{\lambda} 2r \Longrightarrow T(v) = -\frac{\lambda}{(\lambda + r)2r} \ln(v) \end{aligned}$$

The next step is to pin down V^*. Note that:

$$V_A(1) = (1 - F(p_A(1)))\, p_A(1) + F(p_A(1))\, V^*$$

and, as we argued, in equilibrium:

$$V(1) = V^* = \frac{\lambda}{\lambda + r} V_A(1)$$

Combining, we get an expression for $V_A(1)$:

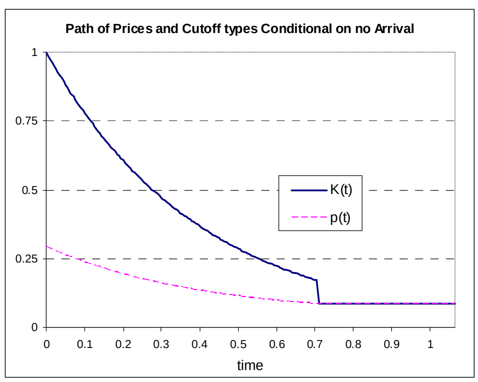
$$V_A(1) = \max_p (1 - F(p))\, p + F(p) \frac{\lambda}{\lambda + r} V_A(1) \tag{23}$$

which can be shown to have a unique solution, which implies a unique V^* .

The only part left is to figure out whether the seller is going to smoothly screen down the demand function through all the types or if he is going to stop at some type. Note that the seller would never trade with types below $\frac{\lambda}{\lambda+r}V^*$, since this is the expected payoff he gets by rejecting the current buyer even before the new buyer arrives and restarting the game empty-handed.

That leads to the following equilibrium dynamics: The seller runs smoothly down the demand function up to type $k^* = V^*$ (and the equilibrium $P(k)$, $K(t)$ and V^* are defined above, with the boundary condition $K(0) = 1$). But once he reaches k^*, since $p_A(k) = V^*$ for all $k \in \left[\frac{\lambda}{\lambda+r}V^*, k^*\right]$, the seller reaches the price $P(k^*) = \frac{\lambda}{\lambda+r}V^*$ and never decreases the price again. As a result, that price is immediately accepted by all types $v \in [P(k^*), k^*]$. In other words, the equilibrium reservation price of all types in this range is the same, $P(v) = \frac{\lambda}{\lambda+r}V^*$ (and all types below $\frac{\lambda}{\lambda+r}V^*$ have reservation prices equal to their types, $P(v) = v$, but the seller never trades with them).

Note that Assumption 1 is violated in this model[37] and it introduces a flat part in the reservation price function, $P(v)$ and a corresponding atom at the end of bargaining with the current buyer (the atom is consistent with the equilibrium since after it the seller does not drop the prices any more). Below we plot the path of offered prices and types trading in equilibrium conditional on no arrival for the case $\lambda = r = 1$. This parameterization implies $V^* = 0.172$. Prices fall until they reach $P(k^*) = \frac{\lambda}{\lambda+r}V^* = \frac{0.172}{2} = 0.086$ and then they remain flat at this level. This induces an atom of trade at time $t = 0.705$ when types between 0.172 and 0.086 accept the price $P(k^*) = 0.086$. Prices will not be reduced further since the seller would rather wait to start over than sell at lower prices.

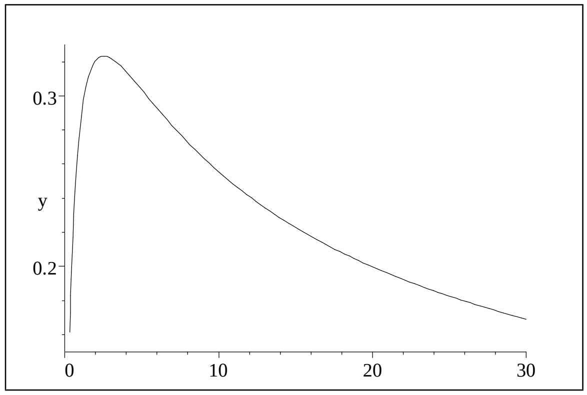


Example with uniform distribution, $\lambda = r = 1$.

Finally, we comment on how the time to trade changes with the frequency of arrivals, λ. When $\lambda > 0$ the expected time to trade is strictly positive. But what happens as $\lambda \to 0$ or $\lambda \to \infty$? We argue that in these cases the time to trade converges to zero.

First, in case $\lambda \to 0$, $P(k) \to 0$ for all k and hence the buyer finds it optimal not to delay trade. This is the classic Coase conjecture result. In the second case, $\lambda \to \infty$, we have that $V(1) \to 1$. That high expected payoff is achievable only if the expected time to trade converges to 0, since the transaction prices are bounded uniformly by 1. In the example with $v\tilde{}U[0,1]$, normalizing $r = 1$, we get that the expected time to trade as a function of λ is:

[37] Most parts of the Assumption 1 are violated: it is not efficient for all types to trade immediately, $\Pi(v)$ is flat for low types and $\Pi(0) > 0$. The only types that trade are $v \in [\frac{\lambda}{\lambda+r}V^*, 1]$ and we can think of $\frac{\lambda}{\lambda+r}V^*$ as the endogenous lowest buyer type that has value higher than the seller's opportunity cost.



Expected time to trade as a function of λ

In this example, for $\lambda > 3r$, the expected time to trade is decreasing in λ.

V Conclusions

When bargaining takes place in the context of a thin market, in which other traders might show up, trade will no longer take place immediately with the informed party capturing all the rents. Although many other explanations have been proposed for the observed delay in bargaining, we believe this to be a very natural one. It shows that delay is to be expected outside the extreme cases of perfect competition or bilateral monopolies.

Nonetheless, the Coasian dynamics are still useful in thinking about such markets, because the lack of commitment drives the seller's value down to his outside option of waiting for an arrival. This is what connects the characteristics of the market to the bargaining dynamics. For example, a higher ratio of buyers in the market leads to higher prices and longer times to trade. This, in turn, could affect the decision of traders to enter the market in the first place. The present model does not allow us to capture this general equilibrium effect since the arrival rates are exogenous in the model. Modeling endogenous entry of agents into the market is necessary to further our understanding of such markets.

VI Appendix

The main goal of this appendix is to prove Theorems 1 and 2 that state that for any sequence of games indexed by the period lengths that asymptotically decrease to 0 and any selection of stationary equilibria of these games $\{\kappa(p;\Delta), \ P(k;\Delta)\}$ and the corresponding sequence $\{V(k;\Delta), \ K(t;\Delta)\}$, as $\Delta \to 0$, these equilibria converge to the unique limit described in Theorem 1.

We start with a series of lemmas that lead up to the proofs. Recall that to keep track of the dependence of the game and equilibrium on Δ we use notation $V(k;\Delta)$ etc. We use $k_+ = \kappa(P(k;\Delta);\Delta)$ to denote next-period cutoff given current state k and the current equilibrium price $P(k;\Delta)$. Along the equilibrium path, for periods 2 onward, we let k_- to denote the previous period cutoff (if the current time is t, on the equilibrium path $k_- = K(t-\Delta;\Delta)$). Similarly, we let $k_{++} = K(t+2\Delta;\Delta)$ denote the cutoff two periods from now (again, given that we are on the equilibrium path).

Lemma 2 (No Quiet Period) *For all $\Delta > 0$, all stationary equilibria must have trade with positive probability in every period.*

Proof. Suppose that there exists an equilibrium in which after a cutoff type k^* is reached, there is a period in which the probability of trade is zero. That implies that next period cutoff type is also k^* and hence (by definition of stationary equilibrium) the price the seller sets in this and all future periods is simply $P(k^*;\Delta)$ and there is no trade till the end of the game.

The seller's expected continuation payoff is then simply the expected present value of the payoff upon arrival:

$$V(k^*;\Delta) = \frac{1-e^{-\Delta\lambda}}{1-e^{-\Delta(r+\lambda)}} V_A(k^*)$$

Suppose that the seller deviates to a price $p' = \frac{1-e^{-\Delta\lambda}}{1-e^{-\Delta(r+\lambda)}} V_A(k^*) + \varepsilon$ for some $\varepsilon > 0$. If this price is accepted by some types (i.e. $\kappa(p';\Delta) < k^*$) then, (for any ε) we have a contradiction, since the seller payoff would then be greater than $V(k^*;\Delta)$ (no matter what the cutoff type $k' = \kappa(p';\Delta)$ is, the seller can guarantee himself at least $\frac{1-e^{-\Delta\lambda}}{1-e^{-\Delta(r+\lambda)}} V_A(k')$ from the remaining types, exactly as in the original equilibrium, but obtains a strictly higher payoff from types (k', k^*)).

Suppose that this price is rejected for sure for every $\varepsilon > 0$. It implies that in the continuation game $k = k^*$ and hence the seller returns to $P(k^*;\Delta)$ forever. As a result, the buyer expected discounted continuation payoff is $\frac{e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)}{1-e^{-\Delta(r+\lambda)}} W(k^*)$. But since $\frac{e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)}{1-e^{-\Delta(r+\lambda)}} \left(W(k^*) + \Pi(k^*)\right) < k^*$, there exists an $\varepsilon > 0$ such that types close to k^* would be strictly better off accepting p', a contradiction.
■

Lemma 3 (Prices Don't Jump) *For all stationary equilibria (of any sequence of games with $\Delta \to 0$), there exist bounds A, B such that uniformly for all Δ and $k \in \Omega_\Delta$, $A \leq \frac{P(k;\Delta)-P(k_+;\Delta)}{\Delta} \leq B$. As a result, for every $\varepsilon > 0$, there exists a $\Delta' > 0$ such that for all $\Delta < \Delta'$ and all $k \in \Omega_\Delta$, $P(k,\Delta) - P(k_+,\Delta) < \varepsilon$*

Proof. Since there is trade in every period with positive probability (by the previous lemma), $\frac{P(k;\Delta)-P(k_+;\Delta)}{\Delta}$ are bounded for every Δ and k. The only issue is if this expression can be bounded uniformly for every Δ and k.

Recall the buyer's optimality:

$$k_+ - P(k;\Delta) \ = e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right) W(k_+) + e^{-\Delta(r+\lambda)}\left(k_+ - P(k_+;\Delta)\right))$$

We now re-group the terms:

$$k_+\left(1-e^{-\Delta(r+\lambda)}\right) = e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right) W(k_+) + \left(P(k;\Delta) - P(k_+;\Delta)\right) + \left(1-e^{-\Delta(r+\lambda)}\right) P(k_+;\Delta)$$

divide by Δ :

$$k_{+}\frac{\left(1-e^{-\Delta(r+\lambda)}\right)}{\Delta}=e^{-\Delta r}\frac{1-e^{-\Delta\lambda}}{\Delta}W\left(k_{+}\right)+\frac{P\left(k;\Delta\right)-P\left(k_{+};\Delta\right)}{\Delta}+\frac{1-e^{-\Delta(r+\lambda)}}{\Delta}P\left(k_{+};\Delta\right)$$

and take the limit:

$$-\frac{P\left(k;\Delta\right)-P\left(k_{+};\Delta\right)}{\Delta}\rightarrow\lambda W\left(k_{+}\right)+\left(\lambda+r\right)\left(\lim_{\Delta\rightarrow0}P\left(k_{+};\Delta\right)-k_{+}\right)$$

Now, if $\lim_{\Delta\rightarrow0}P\left(k_{+};\Delta\right)$ exists for all $k_{+}\in\Omega_{\Delta}$, then the bounds are simply

$$\begin{aligned}A &= \inf_{k}\lambda W\left(k\right)+\left(\lambda+r\right)\left(\lim_{\Delta\rightarrow0}P\left(k;\Delta\right)-k\right)\\ B &= \sup_{k}\lambda W\left(k\right)+\left(\lambda+r\right)\left(\lim_{\Delta\rightarrow0}P\left(k;\Delta\right)-k\right)\end{aligned}$$

If $\lim_{\Delta\rightarrow0}P\left(k_{+};\Delta\right)$ does not exist, then since $P\left(k_{+};\Delta\right)$ is bounded (because there is trade with positive probability in every period, as we have proven in Lemma 2), we can replace $\lim_{\Delta\rightarrow0}P\left(k;\Delta\right)$ in the expressions for A and B,by $\liminf_{\Delta\rightarrow0}P\left(k;\Delta\right)$ and $\limsup_{\Delta\rightarrow0}P\left(k;\Delta\right)$.

The last claim follows immediately by taking $\Delta'=\varepsilon/B$. ■

Lemma 4 (No Atoms) *For all $\varepsilon>0$ there exists a $\Delta>0$ such that for all $\Delta'<\Delta$, and all stationary equilibria of the Δ' game, the probability of trade in any period $t>0$ (after the initial period) is $<\varepsilon$. In other words, for all $t>0$, as $\Delta\rightarrow0$, $F\left(K\left(t;\Delta\right)\right)-F\left(K\left(t+\Delta;\Delta\right)\right)\rightarrow0$.*

Proof. Suppose the claim is false. Then, there must exist some $k^{h}>k^{l}$ with $F\left(k^{h}\right)-F\left(k^{l}\right)\geq\varepsilon'=\varepsilon/4>0$ and some sequence of games and equilibria with period lengths $\{\Delta_{i}\}_{i=1}^{\infty}$ such that $\Delta_{i}\rightarrow0$, such that for each equilibrium along that sequence there exists a time $\tau_{i}>0$ such that $k_{i}=K\left(\tau_{i};\Delta_{i}\right)\geq k^{h}$, $k_{i+}=K\left(\tau_{i}+\Delta_{i};\Delta_{i}\right)\leq k^{l}$. That is, there exists a time τ_{i} the cutoffs jump from above k^{h} to below k^{l} and the probability of trade at τ_{i} is least ε'. [38]

This implies that for all Δ_{i} there exists a buyer with value $\bar{k}=\frac{k^{h}+k^{l}}{2}$ that trades at the same time as type k^{l}.

If type k_{i} is the lowest type willing to trade at price $P\left(k_{i-};\Delta_{i}\right)$ (note that since $t>0$, and $k\in\Omega_{\Delta}$, k_{i-} is well-defined) and type k_{i+} is the lowest type willing to trade at price $P\left(k_{i};\Delta_{i}\right)$, then

[38] To see that if the lemma is false we can find such k^{h} and k^{l}, divide the range $[0,1]$ into regions $\{[v_{n+1},v_{n}]\}$ such that $F\left(v_{n}\right)-F\left(v_{n+1}\right)<\varepsilon/3$. Then if for all ε there is no sequence of games and equilibria such that along the sequence at least one of these regions is "jumped over" in some periods of the equilibrium, then the lemma is true for ε. Finally, since there is a finite number of these regions, we can find a subsequence such that one of these regions is "jumped over" in every equilibrium in this sequence.

buyer's optimality requires:

$$\begin{aligned} k_i - P(k_{i-};\Delta_i) &= e^{-\Delta_i r}\left(1-e^{-\Delta_i\lambda}\right)W(k_i) + e^{-\Delta_i(r+\lambda)}(k_i - P(k_i;\Delta_i)) \\ k_{i+} - P(k_i;\Delta_i) &= e^{-\Delta_i r}\left(1-e^{-\Delta_i\lambda}\right)W(k_{i+}) + e^{-\Delta_i(r+\lambda)}(k_{i+} - P(k_{i+};\Delta_i)) \end{aligned}$$

Note that

$$\begin{aligned} & v - e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)W(v) - ve^{-\Delta(r+\lambda)} \\ = \; & v\left(1-e^{-\Delta r}\right) + e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)(v - W(v)) \end{aligned}$$

is strictly increasing in v (because $v - W(v)$ is strictly increasing), uniformly for all Δ. Given that if k_{i+} weakly prefers to trade at $P(k_i;\Delta_i)$, then $\bar{k} > k_{i+}$ strictly prefers to trade; and if k_i weakly prefers to wait at $P(k_{i-};\Delta_i)$, then $\bar{k} < k_i$ strictly prefers to wait, $\varepsilon > 0$ implies that there exists $\delta > 0$ (uniform for all i) such that:

$$\begin{aligned} \bar{k} - P(k_i;\Delta_i) &> e^{-\Delta_i r}\left(1-e^{-\Delta_i\lambda}\right)W\left(\bar{k}_i\right) + e^{-\Delta_i(r+\lambda)}\left(\bar{k} - P(k_{i+};\Delta_i)\right) + \delta\left(1-e^{-\Delta_i(r+\lambda)}\right) \\ \bar{k} - P(k_{i-};\Delta_i) &< e^{-\Delta_i r}\left(1-e^{-\Delta_i\lambda}\right)W\left(\bar{k}_i\right) + e^{-\Delta_i(r+\lambda)}\left(\bar{k} - P(k_i;\Delta_i)\right) - \delta\left(1-e^{-\Delta_i(r+\lambda)}\right) \end{aligned}$$

Rearranging:

$$\begin{aligned} & \frac{e^{-\Delta_i r}\left(1-e^{-\Delta_i\lambda}\right)}{1-e^{-\Delta_i(r+\lambda)}}\left(W\left(\bar{k}\right) + P(k_i;\Delta_i) - P(k_{i+};\Delta_i)\right) + P(k_{i+};\Delta_i) + \delta \\ < \; & \bar{k} < \frac{e^{-\Delta_i r}\left(1-e^{-\Delta_i\lambda}\right)}{1-e^{-\Delta_i(r+\lambda)}}\left(W\left(\bar{k}\right) + P(k_{i-};\Delta_i) - P(k_i;\Delta_i)\right) + P(k_i;\Delta_i) - \delta \end{aligned}$$

Now in the limit as $\Delta_i \to 0$, using $P(k_i;\Delta_i) - P(k_{i-};\Delta_i) \to 0$ (uniformly for all Δ and $k \in \Omega_\Delta$, as proven in Lemma 3) and $\frac{e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)}{1-e^{-\Delta(r+\lambda)}} \to \frac{\lambda}{r+\lambda}$, we get:

$$\frac{\lambda}{r+\lambda}W\left(\bar{k}\right) + \lim_{\Delta_i\to 0} P(k_{i+};\Delta_i) + \delta \le \bar{k} \le \frac{\lambda}{r+\lambda}W\left(\bar{k}\right) + \lim_{\Delta_i\to 0} P(k_i;\Delta_i) - \delta$$

Which is a contradiction for small enough Δ_i (because there cannot exist such $\bar{k}$ for small enough Δ_i). Therefore on the equilibrium path, in the limit there cannot be a mass of buyers trading in any period after the initial period.[39] ■

Lemma 5 (Payoffs Converge) *For all stationary equilibria (of any sequence of games with* $\Delta \to 0$*),* $\lim_{\Delta\to 0} V(k;\Delta) = V(k) = \frac{\lambda}{r+\lambda}V_A(k)$ *for all* $k \in (0, k_{0+}]$ *where* $k_{0+} = \lim_{\Delta\to 0} K(\Delta;\Delta)$ *is the equilibrium cutoff after the first period.*

[39] If $\lim_{\Delta_i\to 0} P(k_{i+};\Delta_i)$ does not exist, we can before taking the limit subtract $P(k_{i+};\Delta_i)$ from both sides of the inequality and use Lemma 3 to obtain the same contradiction.

Proof. First, we can bound the seller's payoff from below by considering a deviation to (completely) slow down the trade. Since the seller can always choose to wait for the arrival of an event, his value must at least be equal to the expected discounted payoff upon arrival. That is, in all stationary equilibria and for all $\Delta > 0$ the seller's value $V(k;\Delta)$ must satisfy:

$$V(k;\Delta) \geq \frac{1-e^{-\Delta\lambda}}{1-e^{-\Delta(r+\lambda)}} V_A(k)$$

As $\Delta \to 0$ the RHS converges to $\frac{\lambda}{r+\lambda} V_A(k)$.

Second, we can bound the seller's payoff from above by considering a deviation to speed up trade. In particular, suppose that the highest remaining type is k and suppose that the seller deviates and instead of asking for $P(k;\Delta)$ he asks for $P(k_+;\Delta)$ (note that this is a deviation to prices which occur on the equilibrium path, so it is easy to calculate the continuation payoffs). For this not to be a profitable deviation, in all stationary equilibria and for all $\Delta > 0$ the seller's payoff must satisfy:

$$P(k;\Delta)[F(k)-F(k_+)] + e^{-\Delta r}U(k_+;\Delta) \geq P(k_+;\Delta)[F(k)-F(k_{++})] + e^{-\Delta r}U(k_{++};\Delta) \quad (24)$$

where to simplify notation we used $U(k;\Delta) \equiv F(k)V(k;\Delta)$.

By definition of $V(k;\Delta)$ we can write,

$$\begin{aligned} U(k_+;\Delta) &= \left(1-e^{-\Delta\lambda}\right) V_A(k_+) F(k_+) + \\ & e^{-\Delta\lambda}\left[P(k_+;\Delta)(F(k_+)-F(k_{++})) + e^{-\Delta r}U(k_{++};\Delta)\right] \end{aligned}$$

Substituting it to (24) and rearranging terms we get:

$$\begin{aligned} & [P(k;\Delta)-P(k_+;\Delta)][F(k)-F(k_+)] - P(k_+;\Delta)[F(k_+)-F(k_{++})]\left(1-e^{-\Delta(r+\lambda)}\right) \\ \geq & -e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)V_A(k_+)F(k_+) + e^{-\Delta r}\left(1-e^{-\Delta(r+\lambda)}\right)U(k_{++};\Delta) \end{aligned}$$

Divide by Δ :

$$\begin{aligned} & \frac{P(k;\Delta)-P(k_+;\Delta)}{\Delta}[F(k)-F(k_+)] - P(k_+;\Delta)[F(k_+)-F(k_{++})]\frac{1-e^{-\Delta(r+\lambda)}}{\Delta} \quad (25) \\ \geq & -e^{-\Delta r}\frac{1-e^{-\Delta\lambda}}{\Delta}V_A(k_+)F(k_+) + e^{-\Delta r}\frac{1-e^{-\Delta(r+\lambda)}}{\Delta}U(k_{++};\Delta) \end{aligned}$$

Now, recall from Lemma 3 that $\frac{P(k;\Delta)-P(k_+;\Delta)}{\Delta} \to O(const)$ and from Lemma 4 that $F(k) - F(k_+) \to 0$. Taking the limit $\Delta \to 0$ of both sides of (25) we get:[40]

$$\lambda V_A(k_+) F(k_+) \geq (r+\lambda) \lim_{\Delta\to 0} U(k_+;\Delta) =$$

[40] We have used here that $U(k_+;\Delta) \to U(k;\Delta)$, as $\Delta \to 0$. This is true since $U(k;\Delta)$ is continuous and $k \to k_+$.

This implies the upper bound for all $t > 0$:

$$\lim_{\Delta \to 0} V(k; \Delta) \leq \frac{\lambda}{r + \lambda} V_A(k)$$

Combining it with the opposite bound (that we obtained in the first step) yields the result:

$$V(k; \Delta) \underset{\Delta \to 0}{\to} V(k) = \frac{\lambda}{r + \lambda} V_A(k)$$

■

Lemma 6 (Prices Converge) *For all stationary equilibria (of any sequence of games with $\Delta \to 0$), as $\Delta \to 0$, $P(k; \Delta)$ converges to $P(k) = \frac{\lambda}{r+\lambda} \Pi(k)$ for all $k \in (0, k_{0+}]$ where $k_{0+} = \lim_{\Delta \to 0} K(\Delta; \Delta)$ is the equilibrium cutoff after the first period.*

Proof. First, Lemmas 3 and 4 and that $P(k; \Delta)$ is increasing in k, imply that for any $k \in (0, k_{0+}]$ and every sequence of equilibrium $P(k; \Delta_i)$ (with $\Delta_i \to 0$) there exist sequences $\varepsilon_j \to 0$, $\varepsilon_i \to 0$ and a sequence of $k_{ij} \to k$ such that:

$$\lim_{\varepsilon_j \to 0} \left(\lim_{i \to \infty} P(k_{ij}; \Delta_i) - P(k_{ij} + \varepsilon_j + \varepsilon_i; \Delta_i) \right) = 0$$

in other words, $P(k; \Delta)$ converge to a continuous function (note that we cannot pick k_{ij}, ε_j, ε_i arbitrarily, but only corresponding to some $k_{ij} = K(t_{ij}; \Delta_i)$ and $k_{ij} + \varepsilon_j + \varepsilon_i = K(t_{ij} + n_{ij}\Delta_i; \Delta_i)$ for some positive integer n_{ij}).

Hence, if there is a sub-sequence of equilibrium pricing rules $P(k; \Delta)$ converging to something different than $P(k)$, they must differ from $P(k)$ in an open interval. So suppose that there exists a converging sub-sequence of equilibrium pricing rules $P(k; \Delta)$ such that, as $\Delta \to 0$, $P(k; \Delta) \to \tilde{P}(k) \neq P(k)$ for $k \in (\underline{k}, \overline{k})$.

Consider first the case $\tilde{P}(k) > P(k) = \frac{\lambda}{\lambda+r} \Pi(v)$ for $k \in (\underline{k}, \overline{k})$. Such prices could not be part of an equilibrium because then the expected seller's value would exceed $\frac{\lambda}{r+\lambda} V_A(k)$, contradicting Lemma 5. To see this note that the payoff to the seller at the first cutoff lower than $\overline{k}$, k_0, from following $P(k; \Delta)$ would be:

$$\begin{aligned} \widehat{V}(k_0; \Delta) &= \sum_{n=0}^{N-1} e^{-n\Delta r} \left(\left(1 - e^{-\Delta\lambda}\right) e^{-n\Delta\lambda} \frac{F(k_n)}{F(k_0)} V_A(k_n) + e^{-(n+1)\Delta\lambda} P(k_n; \Delta) \frac{F(k_n) - F(k_{n+1})}{F(k_0)} \right) \\ &\quad + \frac{F(k_N)}{F(k_0)} e^{-N\Delta(r+\lambda)} \widehat{V}(k_N; \Delta) \end{aligned}$$

where N is the number of periods for which $k \in (\underline{k}, \overline{k})$ and $\{k_n\}$ is the sequence of equilibrium cutoff types (with k_0 the first cutoff type in this range and k_N the last one).

To bound $\widehat{V}(k_0; \Delta)$ suppose that the seller instead gets prices $\frac{\lambda}{\lambda+r} \Pi(k_n)$ (from the same trades types) and obtains continuation payoff $\frac{\lambda}{\lambda+r}V_A(k_N)$ instead of $\widehat{V}(k_N;\Delta)$. Both are lower bounds, since $P(k_n;\Delta) > \frac{\lambda}{\lambda+r}\Pi(k_n)$ uniformly for all small Δ, and $V(k_N;\Delta)$ converges to $\frac{\lambda}{\lambda+r}V_A(k_N)$ from above.

Call payoffs calculated by that substitution $V_L(k_0;\Delta)$. We get:

$$\lim_{\Delta\to 0}\widehat{V}(k_0;\Delta) > \lim_{\Delta\to 0}V_L(k_0;\Delta) = \frac{\lambda}{\lambda+r}V_A(k_0)$$

The equality follows since conditional on any type k, if the buyer deviated to always reject the offer, then the seller's expected payoff in the limit as $\Delta\to 0$ would be $\frac{\lambda}{\lambda+r}\Pi(k)$. Thanks to the stationarity of the Poisson process, this would be in fact the expected payoff at any moment of time. Moreover, given that the transaction prices are $\frac{\lambda}{\lambda+r}\Pi(k)$ and trade happens only conditional on the event not arriving yet, when the buyer accepts this price, the seller gets the same payoff from that type as he would if the buyer rejected forever.

That establishes that $\tilde{P}(k) > P(k)$ would allow the seller to earn even in the limit strictly more than $V(k)$, a contradiction.

Next, suppose there exists a sequence of equilibrium pricing rules $P(k;\Delta)$ such that, as $\Delta\to 0$, $P(k;\Delta)\to\tilde{P}(k) < P(k)$ for $k\in(\underline{k},\overline{k})$. These pricing rules cannot be part of an equilibrium sequence either, since after an analogous substitution (prices $\frac{\lambda}{\lambda+r}\Pi(k_n)$ and continuation payoff $\frac{\lambda}{\lambda+r}V_A(k_N)$), we would get a strictly higher payoff in the limit, implying that

$$\lim_{\Delta\to 0}\widehat{V}(k_0;\Delta) < \frac{\lambda}{\lambda+r}V_A(k_0)$$

contradicting Lemma 5 again.

Therefore, to satisfy Lemma 5 all equilibrium pricing rules $P(k;\Delta)$ have to converge to $P(k)$. ■

Lemma 7 (No Atom at t=0) *For all stationary equilibria (of any sequence of games with $\Delta\to 0$), as $\Delta\to 0$ there cannot be an atom of trade at $t=0$, that is $K(\Delta;\Delta)\to 1$. Moreover, $V(1;\Delta)\to V(1)$ and $P(1;\Delta)\to P(K(\Delta;\Delta);\Delta)\to P(1)$.*

Proof. Suppose that in equilibrium there exists some $\bar{k} < 1$ such that all types $v\geq\bar{k}$ trade at $t=0$. Then we claim that the seller payoffs, $(1-F(\bar{k}))P(\bar{k})+F(\bar{k})V(\bar{k})$, would be strictly less than $\frac{\lambda}{\lambda+r}V_A(1)$, contradicting that he can achieve that payoff by simply asking very high prices.

To see this note that:

$$
\begin{array}{rcl}
\Pi(\bar{k}) & < & \Pi(k) \text{ for all } k > \bar{k} \\
& \Downarrow & \\
\left(1-F\left(\bar{k}\right)\right)\Pi\left(\bar{k}\right)+F\left(\bar{k}\right)\int_0^{\bar{k}}\frac{\Pi(v)f(v)}{F\left(\bar{k}\right)}dv & < & \\
\left(1-F\left(\bar{k}\right)\right)\int_{\bar{k}}^1\frac{\Pi(v)f(v)}{1-F\left(\bar{k}\right)}dv+F\left(\bar{k}\right)\int_0^{\bar{k}}\frac{\Pi(v)f(v)}{F\left(\bar{k}\right)}dv & = & V_A(1) \\
& \Downarrow & \\
\left(1-F\left(\bar{k}\right)\right)\frac{\lambda}{\lambda+r}\Pi\left(\bar{k}\right)+F\left(\bar{k}\right)\frac{\lambda}{\lambda+r}V_A\left(\bar{k}\right) & < & \frac{\lambda}{\lambda+r}V_A(1) \\
& \Downarrow & \\
\left(1-F\left(\bar{k}\right)\right)P\left(\bar{k}\right)+F\left(\bar{k}\right)V\left(\bar{k}\right) & < & \frac{\lambda}{\lambda+r}V_A(1)
\end{array}
$$

Since along the sequence of equilibria, there is trade in equilibrium in the first period (by Lemma 2), it must be that $P(1;\Delta) \to P(K(\Delta;\Delta);\Delta) \to P(1)$ (otherwise the types that trade at 1 would be strictly better off to wait till time Δ). Since the probability of trade at time 0 converges to zero and the price is uniformly bounded, it must be that the total expected payoff $V(1;\Delta) \to V(K(\Delta;\Delta);\Delta) \to \frac{\lambda}{\lambda+r}V_A(1) = V(1)$. ■

Lemma 8 (Path of Types Converges) *Consider a sequence of stationary equilibria (of any sequence of games with $\Delta \to 0$). Let the equilibrium path of cutoff types be defined by $K(0;\Delta) = 1$, $K(t;\Delta) = \kappa(P(K(t-\Delta;\Delta);\Delta);\Delta)$ for $t \in \{0, \Delta, 2\Delta ...\}$. Moreover, extend the $K(t;\Delta)$ function to any $t \in (n\Delta, (n+1)\Delta)$ (where $n \in \mathbb{N}$) by setting $K(t;\Delta) = K(n\Delta;\Delta)$. That is, the $K(t;\Delta)$ function is a decreasing step function changing value at times that the seller makes offers.*
Then, as $\Delta \to 0$, $\frac{K(t+\Delta;\Delta)-K(t;\Delta)}{\Delta} \to \dot{K}(t)$ and $K(t;\Delta) \to K(t)$.

Proof. Recall that k and k_+ are defined as $K(t;\Delta)$ and $K(t+\Delta;\Delta)$, respectively. Recall the buyer optimality condition:

$$
k_+ - P(k;\Delta) \;= e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)W(k_+) + e^{-\Delta(r+\lambda)}\left(k_+ - P(k_+;\Delta)\right)
$$

Subtracting $e^{-\Delta(r+\lambda)}(k_+ - P(k;\Delta))$ from both sides, dividing by Δ and taking $\Delta \to 0$ we get (using that $P(k;\Delta)$ converges to $P(k)$ and that there are no atoms in the limit):

$$
(\lambda+r)\lim_{\Delta\to 0}\left(k_+ - P(k;\Delta)\right) \;= \lambda W(k_+) + \lim_{\Delta\to 0}\underbrace{\frac{\left(P(k;\Delta)-P(k_+;\Delta)\right)}{k-k_+}}_{\to P'(k)}\frac{k-k_+}{\Delta}
$$

so that in the limit we get the optimality condition for the equilibrium limit:

$$(\lambda + r)(K(t) - P(K(t))) = \lambda W(K(t)) - P'(k)\dot{K}(t)$$

so indeed $\frac{K(t+\Delta;\Delta)-K(t;\Delta)}{\Delta} \to \dot{K}(t)$. Finally, from Lemma 7 we have that $K(0) = 1 = K(0;\Delta) = \lim_{\Delta\to 0} K(\Delta;\Delta)$. Because $K(t;\Delta)$ is bounded and the derivative $\dot{K}(t)$ is bounded, we can use the fundamental theorem of calculus to claim that since derivative of the limit of $K(t;\Delta)$ converges to $\dot{K}(t)$, and $K(0;\Delta) = K(0)$, $K(t;\Delta)$ converges to $K(t)$ for all $t \geq 0$. ■

Proof of Theorem 1. Lemmas 5, 6 and 7 show that in the limit as $\Delta \to 0$ all discrete time equilibria deliver the same value to the seller and the same transaction prices given a current cutoff type. Lemma 8 then shows that how the cutoff types change through time also converges to $K(t)$. ■

Proof of Theorem 2. The fact that equations (10) and (13) together with the boundary condition $K(0) = 1$ characterize an equilibrium is discussed in detail in Section 3. Uniqueness follows from noting that only necessary conditions were used to characterize this equilibrium. ■

Proof of Proposition 2. (i) Conditions $\Pi_1(0) = \Pi_2(0)$ and $\Pi_1'(v) \geq \Pi_2'(v)$ imply that $\Pi_1(v) \geq \Pi_2(v)$. So in both cases ($\Pi_1(v) + W_1(v) = \Pi_2(v) + W_2(v)$ or $W_1(v) = W_2(v)$) in the numerator of equation (13) $\Pi_1(v) + W_1(v) \geq \Pi_2(v) + W_2(v)$. Since the denominators are ranked $\Pi_1'(v) > \Pi_2'(v)$, for any t such that $K_1(t) = K_2(t)$, we can rank $-\dot{K}_2 > -\dot{K}_1$. Since $K_1(0) = K_2(0) = 1$, we get that for almost all t, $K_1(t) > K_2(t)$ i.e. buyers with the same valuation trade faster in the environment with $\Pi_2(v)$.

(2) As $\Pi'(v) \to 0$, $\Pi(v) \to 0$ as well, which implies the seller's value: $V(k) = \frac{\lambda}{\lambda+r} V_A(k) \to 0$ and prices are also converging to zero $P(k) = \frac{\lambda}{\lambda+r}\Pi(k) \to 0$. Trade on the other hand is taking place faster since $-\dot{K} \to \infty$ therefore there will be no delay in trade and the buyer will capture the entire surplus. ■

Proof of Proposition 3. (*i*) Inspecting equations (10) and (13) we can see that if $\Pi(v)$ and $W(v)$ are independent of $F(v)$ then $P(k)$ and $\dot{K}$ are independent of $F(v)$ and therefore the equilibrium is independent of $F(v)$.

(*ii*) As argued above, $P(k)$ and $\dot{K}$ are the same under both distributions. The result follows simply from the fact that since $F(v)$ first order stochastically dominates $H(v)$ it is more likely that the realized v is higher. Since higher types trade earlier and at higher prices we get that the average time to trade is longer and the average prices are lower with H.

(*iii*) The total expected ex-ante surplus is $\underbrace{\frac{\lambda}{\lambda + r} E[\Pi(v)]}_{V(1)} + E[B(v)]$. If $\Pi(v)$ is weakly convex, the first term is weakly higher under H. We now argue that $W(v)$ being weakly convex implies $B(v)$ is strictly convex, so that the second term is strictly higher under H, establishing the claim.

Recall equation (15) :

$$B'(v) = e^{-(r+\lambda)T(v)} + \frac{\lambda}{\lambda + r}\left(1 - e^{-(r+\lambda)T(v)}\right)W'(v)$$

differentiating we get:

$$B''(v) = \underbrace{-T'(v)\,Y\left(r + \lambda\left(1 - W'(v)\right)\right)}_{>0} + \underbrace{\frac{\lambda}{\lambda + r}(1 - Y)\,W''(v)}_{\geq 0}$$

where $Y = e^{-(r+\lambda)T(v)} \in (0,1)$. So indeed $B''(v) > 0$. ■

Proof of Claims in Section III.C:.

$\Pi(v_1)$ can be re-written as:

$$\Pi(v_1) = \gamma v_1 + (1-\gamma)\left(\int_0^{v_1} x f(x)\, d(x) + (1 - F(v_1))\, v_1\right)$$

Hence,

$$\begin{aligned}
\Pi'(v_1) &= \gamma + (1-\gamma)\left(v_1 f(v_1) + (1 - F(v_1)) - f(v_1)\, v_1\right) \\
&= \gamma + (1-\gamma)(1 - F(v_1)) \\
&= 1 - F(v_1) + F(v_1)\,\gamma
\end{aligned}$$

Therefore:

$$\frac{\partial \Pi'(v_1)}{\partial \gamma} = F(v_1) > 0$$

Therefore, the larger γ the larger $\Pi'(v)\ \forall v$ and from Proposition (2) this implies that delay is decreasing in the number of different buyer classes. (ii) and (iii) follow from noting that $\Pi(v_1)$ is decreasing in n since the second term of $\Pi(v_1)$ is smaller than v_1 and using equations (2) and (3) which respectively characterize the seller's value and prices.■

Proof of Lemma 1. For $k > V^*$, $p_A(k)$ is a solution to the F.O.C.:

$$p - \frac{(F(k) - F(p))}{f(p)} = V^*$$

Now, the LHS is decreasing in k.[41] We claim that it is increasing in p if the marginal revenue is downward sloping. The derivative of the LHS with respect to p is:

$$1 - \frac{-f^2(p) - (F(k) - F(p))\, f'(p)}{f^2(p)} = 2 + \frac{(F(k) - F(p))\, f'(p)}{f^2(p)}$$

[41] Hence, if $p_A(k)$ is strictly increasing, the problem (19) is supermodular in k and p, guaranteeing that the F.O.C. is sufficient.

which if $f'(p) > 0$ is positive for all k and if $f'(p)$ is < 0 it is the smallest for $k = 1$, but then this expression is positive by assumption.

Hence the LHS of the F.O.C. is increasing in p for all k and decreasing in k, which implies that $p_A(k)$ is strictly increasing.

For $k \leq V^*$ the seller cannot get more than V^*, which he can guarantee by offering $p_A(k) = V^*$ and trading with probability 0. ■

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