HOMEOWNERSHIP AS A CONSTRAINT ON ASSET ALLOCATION

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ABSTRACT

Personal preferences and financial incentives make homeownership desirable for most families. Once a home is purchased transactions costs and/or preferences make it impractical (costly) for families to frequently change their ownership of residential real estate. Thus, by deciding how much home to buy, a family constrains their ability to adjust their asset allocations between residential real estate and other assets. To analyze the impact of this constraint on consumption, welfare, and post-retirement wealth we first investigate a representative individual's optimal asset allocation decisions when they are subject to a "homeownership constraint." Next, we perform a "thought experiment" where we assume the existence of a market where a homeowner can sell, without cost, a fractional interest in their home. Now the housing choice decision does not constrain the individual's asset allocations. By comparing these two cases, we estimate the differences in post-retirement wealth and the welfare gains potentially realizable if asset allocations were not subject to a homeownership constraint. For realistic parameter values we find that a representative homeowner would require between a 2 and 25 percent increase in total net worth to achieve the same level of utility as would be achievable if the choice of a home could be separated from the asset allocation decisions.

1. Introduction

Owning a home is an important part of the "American Dream." A broad range of government initiatives, from the income tax deductibility of mortgage interest payments to FHA loan guarantees, are designed to encourage homeownership.[1] These incentives, in conjunction with personal preferences, make owning a home desirable for most families. Nationally, the rate of homeownership exceeds 68 percent.[2]

When a home is purchased the owner's interest in the property is equivalent to a portfolio comprised of the property and a mortgage loan.[3] Once bought, transactions costs and/or personal preferences make it impractical (costly) for a family to frequently change their ownership of residential real estate. Thus, by deciding how much home to buy, a family constrains their ability to adjust their asset allocations between residential real estate and other assets. As long as the home is owned, the value of the family's investment in residential real estate is exogenously determined by market conditions.[4] We refer to this phenomenon as a "homeownership constraint." Borrowing out "excess equity" following a run-up in home values does not reduce the owner's exposure to housing market risk; it only increases the mortgage debt in the family's portfolio of assets. In this paper, we analyze the impact of the "homeownership constraint" on an individual's consumption, welfare, and post retirement wealth. Specifically, we analyze the asset allocation problem faced by an individual at the portion of their life (mid-career) when most of their non-human capital wealth is accumulated.

The problem we investigate is one of constrained asset allocation, not the determination of the demand for housing. To abstract from the housing choice (demand) decision we assume that a representative individual has irreversibly chosen (bought) the home in which he will live until retirement. In the language of real estate, the individual has just bought a "move-up" home. What is relevant for our analysis is the constraint

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[1] Glaeser and Shapiro (2002) investigate the effect of the income tax deductibility of home mortgage interest payments on homeownership. Gyourko and Sinai (2003) estimate the incidence of these benefits.
[2] There are relatively large inter-state variations in homeownership, for example for West Virginia and Minnesota the rate exceeds 77 percent and for California and New York the rate is less than 58 percent. Excepting Canada, international variation in institutions and customs result in lower homeownership rates
[3] If the mortgage is non-recourse the portfolio can be though of as including an option that gives the owner the right to put the property back to the lender at a price equal to the loan balance
[4] In practice, a family's investment in housing can be marginally increased by investing in home improvement or disinvestment can occur by not maintaining the property.

imposed on asset allocation, not the fact that the home cannot be sold (e.g., if an individual moves, they move to an equally valuable house). Conditional upon owning the home, the individual's objective is to maximize, through asset allocation, the present value of their expected utility from pre-retirement consumption and post-retirement wealth. We contend that by ignoring the decision as to how much home to buy (i.e., by ignoring the rent/buy decision), and by constraining the individual to own the same home until retirement, our model is limited in one dimension, but this simplicity enables us to investigate a richer set of investment opportunities.[5] Thus, we obtain a more realistic picture of the impact of homeownership on the asset allocation problem faced by families during a critical portion of their economic life. Our introduction of human capital to the asset allocation problem is particularly important, because, frequently when a region's economy is subject to a large shock, personal income (through hours of work and/or incidence of unemployment) and real estate values change in tandem. A homeowner facing such conditions can see the values of their two most important assets decline.

We formulate the allocation problem in terms of a stochastic dynamic programming problem with three state variables: total net worth, the market value of the home, and the individual's labor income. First, we solve the optimal asset allocation problem when the individual is subject to a homeownership constraint. Next, we perform a "thought experiment" where we assume the existence of a market where a homeowner can sell, without cost, a fractional interest in his home. Now the housing choice decision does not constrain the individual's asset allocations. By comparing these two cases, we estimate the differences in post-retirement wealth and the welfare gains potentially realizable if asset allocations were not subject to a homeownership constraint. In addition, by comparing the asset allocations over time we are able to investigate how individuals compensate for the homeownership constraint.

For realistic parameter values we find that a representative individual would require a 6% increase in total net worth to achieve the same level of expected utility as would be obtained if the choice of a residence could be separated from the asset allocation decisions. This compensation ranges between 2 and 25% of net worth over the

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[5] The own/rent decision would be a function of the loss in expected utility attributable to the homeownership constraint.

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state space considered. As would be expected, we find that homeownership results in an individual shifting their asset allocation away from equities. The homeownership constraint becomes less binding as the individual ages and financial wealth is accumulated. Even though residential real estate tends to have lower financial returns than stocks, it can be purchased with higher leverage, which can increase the expected return on the total portfolio. We also find evidence that the allocation to real estate in the *unconstrained* case includes a small but important hedging component that is consistent with the model of Sinai and Souleles (2002). This contrasts with the *constrained* case, where the individual purchases the entire home. Here the owner is fully hedged against increases in the price of housing services. This is true regardless of the level of debt used to finance the acquisition.

The framework we develop can also be used to consider the case where the individual is *constrained* to have a zero investment in the house. This case corresponds to the situation where the investor is forced to rent the house forever. This situation, however, does not capture the endogenously determined buy-rent decision. For the levels of wealth and income that we consider, it is never optimal to invest zero in housing.

The remainder of this paper is divided into sections that: (1) review the relevant literature; (2) describe the asset allocation problems faced by a representative individual; (3) numerically evaluate the effect of the constraint; and (4) summarize our findings and suggest directions for further research.

2. Literature and Extensions

Despite the vast literature on optimal portfolio choice, most papers do not consider the impact on asset allocation of an individual's investment in a home (i.e., a durable consumptions good). Grossman and Laroque (1991) present the first exception as they develop a theoretical model with a single illiquid durable consumption good (e.g., a house) from which an infinitely lived investor derives utility. The illiquidity derives from the fact that transaction costs are born when the good (house) is sold. In addition to the durable good the individual can invest in a risk free asset and a set of risky financial assets. At each time, the individual must decide whether to acquire a larger (smaller) house and how to allocate his or her remaining wealth among financial assets. Grossman-Laroque show that it is optimal for the individual to wait for large increases

(decreases) in wealth to increase (decrease) their consumption of the durable consumption good. In addition, they conclude that transaction costs cause the individual to allocate a smaller portion of their financial wealth to risky assets than would occur if the individual could adjust homeownership continuously. Marshall and Parekh (1999) calibrate the Grossman-Laroque model and conclude that it can explain one half the observed equity premium.

In another paper designed to explain the equity premium puzzle Chetty and Szeidl (2004) show, in a two good model, one of which is a durable consumption good, that a "consumption commitment" (e.g., for a house), will result in individuals acting as if they are more risk averse.[6] These authors conclude that their model can fully resolve the equity premium puzzle. Alternatively Piazzesi, Schneider and Tuzel (2004), Yugo (2003) and Siegel (2004) consider the effect of "composition risk" on asset pricing. Here the representative individual's utility function is not separable between the consumption of a durable good and a non-durable good. Composition risk is exogenously determined and variations in the consumption of the durable good relative to the consumption of other goods can help to explain time variations in the equity premium. Again the representative individual *acts* as if they are more risk averse. In comparison to the above papers we address a different but related question, that is, what are the effects of a "consumption commitment" or homeownership constraint on consumption, welfare, and post retirement wealth.

Flavin and Yamashita (2002) and Yamashita (2003) study the impact of the portfolio constraint imposed by the consumption demand for housing on an individual's optimal holdings of financial assets. In addition to a house, the individual can invest in T-Bills, T-Bonds, stocks, and borrow through a mortgage loan. They use PSID data to explore the life cycle impact of the "housing constraint" (as reflected by the ratio of housing to net worth) on the individual's optimal holding of financial assets. Flavin-Yamashita use mean-variance analysis to characterize optimal portfolios of financial asset over the life cycle. They conclude that an exogenous increase in the value of the house owned results in a relatively large shift from equities to bonds in a mean-variance

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[6] In Chetty and Szeidl model the individual's preferences are separable between the commitment and non-commitment good.

optimal portfolio. Kullmann and Siegel (2003) investigate empirically the relationship between real estate ownership and a households' financial asset allocation. Based upon an analysis of PSID data they find that the extent of homeownership strongly affects households' allocation to risky financial assets. Thus they conclude that asset allocation models and prescriptions can be improved by considering an individual's exposure to the risk associated with homeownership.

Cocco (1999) develops an empirically parameterized model of consumption and portfolio choice when there is an illiquid durable consumption good (a house). In his paper the individual purchases a home for the consumption services it provides. The value of the home is not stochastic. The individual has a stochastic income and can invest in two financial assets: a stock fund and Treasury bills. Cocco's portfolio optimization model is then used to predict the cross-sectional pattern of variation in the composition of wealth by age and net worth.

Bertaut and Haliassos (1997), Guiso, Jappelli, and Terlizzesse (1996), Heaton and Lucas (1999), and Poterba and Samwick (1997) have empirically investigated the portfolio choice of households. The goal of this research was to understand how labor income affects the allocation of wealth among financial assets. Issues of data availability, specifically the absence of data that describes the risk and expected return to investments in housing, have retarded the development of an appropriate treatment of the investment in real estate. Bertaut and Halliassos ignore the effect of housing altogether. Heaton and Lucas recognize the importance of housing and include them as regression variables.

These papers do not provide theoretical justification for either excluding the consideration of real estate all together or including it in a partial equilibrium regression model.

We extend previous work in several ways:
- We consider both uncertain income from human capital and risky investments in residential real estate whose returns may be correlated. As noted earlier, this is particularly important because a home and human capital are typically the individual's two most important assets.

- We implement a continuous-time long-term strategic asset allocation
  optimization subject to a number of realistic investment constraints.
- We explicitly model the main characteristics of investment in a specific house.
- We explicitly model the borrowing constraints associated with mortgage
  loans.

3. Model

Consider a representative individual who has a known working life, *T,* and post-retirement life span, *R.*[7] The individual's objective is to maximize the present value of his expected utility from pre-retirement consumption and post-retirement wealth. The individual has just purchased a home for a price *H.* The acquisition was financed with a non-amortizing, fixed interest rate, full recourse mortgage. The mortgage's contracted rate of interest equals the sum of the ***real*** mortgage rate, *r*, and the constant known inflation rate *i*.[8] To capture the essence of a homeownership constraint, we assume that the representative individual cannot sell the house until retirement at time *T.* Thus, the individual is committed to consuming a fixed flow of real housing services until retirement. While the "consumption commitment" imposed by homeownership in our model may be considered extreme (i.e., no adjustment for 20 years) we will argue that it captures the essence of the constraint faced by many mid-career families (i.e., they can't afford to step up and are reluctant to step down in terms of their family home). In addition, to assure that our results are not attributable to the consumption commitment associated with homeownership, both the constrained and unconstrained optimization problems are formulated so that the representative individual is committed to consume the same real level of housing services throughout their life.

3.1 The Assets

The individual can invest in two types of financial assets: a stock index and a money market fund. Let S_t denote the ***real value*** of the individual's investment in the non-dividend paying stock index fund at time t.[9] The dynamics of S_t are given by:

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[7] A fixed and known working life is useful for tractability.
[8] We assume the loan is full recourse to eliminate consideration of the value of the option to default.

$$dS = (r + \mu_S)Sdt + \sigma_S Sdz_S \qquad (1)$$

where μ_S represent the risk premium over the **_real mortgage interest rate_**, r, and σ_S is the volatility of the index fund, both of which are positive constants. For analytic convenience the equity risk premium is expressed relative to the constant real mortgage rate, rather than the risk free rate of interest. We will assume that investments in the stock fund can be neither bought on margin nor shorted.[10]

The money market fund pays a non-stochastic rate of interest equal to the sum of the _real_ mortgage interest rate, r and the constant known inflation rate i. Analytically, deposits in the money market fund are equivalent to partial repayment of the mortgage loan.[11] Consequently, the individual's mortgage loan balance and investments in the money market fund can, and will be, consolidated in the analysis. Thus, a mortgage loan is a large negative balance in the money market fund.

Although, in the constrained case, the homeowner cannot sell their house until retirement, there is a housing market where similar properties are sold. This market provides information about the value of the housing portion of the individual's portfolio. Let H_t denote the _real_ value of the house at time t, which is assumed to evolve according to the stochastic process:

$$dH = \mu_H Hdt + \sigma_H Hdz_H \qquad (2)$$

where μ_H and σ_H are positive constants. Note that the _real_ flow of housing services (h) received from homeownership is independent of the owner's equity position in the house. The correlation between the stock fund and home values is ρ_{SH}, with $\sigma_{SH} = \rho_{SH}\sigma_S\sigma_H$ denoting the covariance between the processes.

The individual's _real_ income from human capital, y_t, is stochastic and evolves according to the following process (for $t < T$):

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[9] Allowing the stock fund to pay dividends would not qualitatively change of our results
[10] The individual, however, can borrow out "excess" home equity to finance the purchase of stock.

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$$\frac{dy}{y} = \mu_y dt + \sigma_y dz \tag{3}$$

Note that a positively sloped age earnings profile occurs when $\mu_y > 0$. After retiring, $t \geq T$, the individual does not receive income from human capital ($y_t = 0$), but does receive a non-stochastic Social Security payment (y_s) for the remaining R years of their life. Let ρ_{yH} denote the correlation between income and the rate of home value appreciation, with $\sigma_{yH} = \rho_{yH} \sigma_y \sigma_H$ representing the corresponding covariance. The correlation between income and the return on the stock market is denoted by ρ_{yS}, with $\sigma_{yS} = \rho_{yS} \sigma_y \sigma_S$ being the corresponding covariance.

3.2 The Utility Function

The representative individual has an inter-temporal utility function, $U(h, C)$, defined over the consumption of housing, *h,* and non-housing services, *C.* For tractability, the utility function is assumed to be both time-separable and separable between the consumption of housing and non-housing services. By living in the house, the owner receives a constant flow of ***real*** housing services that are independent of the market value of the home.[12] Because the individual lives in the house until retirement, the ***real*** consumption of these services is fixed until retirement. Therefore, the consumption of housing services does not enter the asset allocation optimization. This simplicity is bought at a price. Piazzesi, Schneider and Tuzel (2003) and Siegel (2004) show that if, in fact, the utility function is not separable, exogenous variation in the expenditure shares between housing and non-housing services can result in the individual acting as if they are more risk adverse. To compensate for this issue we will assume that that the representative individual has relatively high risk aversion.[13]

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[11] This assumption is made to simplify the analysis and avoid the possibility of arbitrage between the money market fund and the mortgage loan.

[12] The real flow of services *h* depends upon the physical characteristics of the house (e.g., the number of bedrooms and/or bathrooms).

[13] Chetty and Szeidl (2004) obtain similar results with a separable utility function by working through the individual's budget constraint (i.e., by having a consumption commitment).

3.3 Constrained Asset Allocation

As noted earlier, the individual's objective is to maximize, by the choice of the level of consumption and asset allocations, the sum of the present value of the expected utility from consumption until retirement and the expected utility of post-retirement wealth.[14] Assuming isoelastic utility, the optimization at time t is:

$$V(W_t, H_t, y_t, t) = \max E_t [\int_t^T e^{-kx} U(C_x) dx + e^{-k(T-t)} U_T(W_T)] \qquad (4)$$

where

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x \geq 0$$
$$= -\infty, \quad x < 0,$$

C denotes consumption of non-housing services, k is the homeowner's rate of time preference, and γ is the risk aversion parameter.

The representation of the utility of post-retirement wealth, $U_T(W_T)$, is designed to reflect the trade off between pre and post retirement consumption (including the possibility of a bequest). To capture the essence of aging, while maintaining computational tractability, we assume the individual has a known time horizon (life), R, after retirement, (e.g., 15 years).   At retirement, the individual sells his or her house and cashes-in their investments in the stock index fund. For simplicity we assume that the individual is able to avoid capital gains taxation on the sale of the home and the liquidation of stock index fund (e.g., through withdrawals from a 401k when labor income is zero).   The proceeds of these transactions, along with the individual's money market fund balance, are then used to purchase: (1) a retirement or "trade down house" whose value is a fraction, λ, of the value of their home at retirement; and (2) an annuity

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[14] As noted above, we assume the utility function is separable between the consumption of housing services and non-housing services, consequently we ignore the housing services in the optimization problem.

that provides a fixed level of consumption, $C^R = \dfrac{r(W_T - \lambda H_T)}{1 - e^{-Rr}}$, for the remaining R years

of the individual's life.[15] Finally, after retiring the individual receives a non-stochastic

tax-free Social Security payment of y_s. The terminal utility, U_T, then becomes:

$$U_T(W_T) = \int_0^R e^{-kx} U(C^R + y_s) dx = \frac{1 - e^{-Rk}}{k} * U(C^R + y_s) \qquad (5)$$

Next, the evolution of wealth over time is considered. Let ϕ_t denote the proportion of

wealth invested in the stock market and η_t denote the proportion of wealth invested in the

house. In the constrained case the individual owns a particular house until retirement and

the allocation to real estate, $\eta_t = \dfrac{H_t}{W_t}$, is stochastic but not under his control. It follows

that $1 - \phi_t - \eta_t$ is the proportion invested in the "money market fund." In general, this

proportion will be negative, because real estate purchases are typically highly leveraged.

Thus, the evolution of **real after tax total wealth**, W_t, is given by:

$$\frac{dW}{W} = \phi \frac{dS}{S} + \eta \frac{dH}{H} + (1 - \phi - \eta)(r(1 - \tau) - i\tau)dt + \frac{(y(1 - \tau) - C)dt}{W} \qquad (6)$$

where τ denotes the tax rate and r is the **real before-tax** interest rate. Note that even

though all assets, wealth, and income are denominated in real terms, inflation affects the

evolution of real wealth (i.e., through $-i\tau$) because taxes are paid on nominal interest

income and mortgage interest expenses are tax deductible. Equation (6) reflects the

assumption that the individual is able to avoid taxation on home value and stock market

appreciation.

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[15] The trade down home can be thought as providing housing services until death and as a bequest after death. To simplify the problem we assume that λ exogenously determined.

The Bellman equation is:

$$\underset{C,\phi}{Max}[e^{-kt}U(C) + E(dV)/dt] = 0 \qquad (7)$$

Using Ito's Lemma we derive the following expression for the evolution of the value function:

$$dV = V_t dt + V_W dW + \frac{1}{2}V_{WW}(dW)^2 + V_H dH + \frac{1}{2}V_{HH}(dH)^2 + V_y dy + \frac{1}{2}V_{yy}(dy)^2 +$$
$$+ V_{WH} dWdH + V_{yH} dydH + V_{yW} dydW \qquad (8)$$

Substituting into the Bellman Equation (7), taking expectations, and using $\eta_t = \dfrac{H_t}{W_t}$ we obtain:

$$0 = \max[e^{-kt}U(C) + V_t + (\frac{1}{2}\sigma_S^2\phi^2 W^2 + \frac{1}{2}\sigma_H^2 H^2 + \phi HW\sigma_{SH})V_{WW} + \frac{1}{2}\sigma_H^2 H^2 V_{HH} +$$
$$+ \frac{1}{2}V_{yy}y^2\sigma_y^2 + V_{WH}(\phi HW\sigma_{SH} + \sigma_H^2 H^2) + yH\sigma_{yH}V_{yH} +$$
$$+ V_{yW}(\phi Wy\sigma_{yS} + Hy\sigma_{yH}))dt + V_H H\mu_H + V_y y\mu_y +$$
$$+ V_W W(\phi(\mu_S + r) + \mu_H \frac{H}{W} + (1 - \phi - \frac{H}{W})(r(1-\tau) - it)) + V_W(y(1-\tau) - C)] \qquad (9)$$

Differentiating equation (9) with respect to C and ϕ gives the following first-order conditions:

$$e^{-kt}U_C - V_W = 0 \qquad (10)$$

$$\sigma_S^2 WV_{WW}\phi + \sigma_{SH}HV_{WW} + H\sigma_{SH}V_{WH} + V_W(\mu_S + (r+i)\tau) + \sigma_{yS}yV_{yW} = 0 \qquad (11)$$

The optimal controls are:

$$C^* = (e^{kt}V_W)^{-\frac{1}{\gamma}} \tag{12}$$

$$\phi^* = -\frac{\sigma_{SH}HV_{WW} + H\sigma_{SH}V_{WH} + V_W(\mu_S + (r+i)\tau) + \sigma_{yS}yV_{yW}}{\sigma_S^2WV_{WW}} \tag{13}$$

Inspection of Equation (13) reveals how investments in residential real estate can affect the optimal allocation to the stock fund, specifically: (1) through the correlation between stock and home values; and (2) through V_W, V_{WH} and V_{WW}. It follows that even when home values and stock value are uncorrelated the optimal allocation to stocks depends upon the return characteristics of residential real estate (through V). Returning to our optimization problem, by substituting the optimal controls (12) and (13) into Equation (9) we obtain a partial differential equation for the value function. This PDE is subject to the following boundary conditions:

$$\phi W + (1-\theta)H \leq W \quad \text{for all } t$$
$$V(W_t, H_t, y_t, t) = U_T(W_T) = \frac{1-e^{-Rk}}{k} * U(C^R + y_s) \quad \text{at } t = T \tag{14}$$
$$\phi \geq 0 \quad \text{for all } t$$

where θ denotes the maximum allowable loan-to-value ratio. The first constraint states that the investment in stocks plus the minimum home equity cannot exceed the total wealth of the individual. This implies that a decline in home values may trigger a "margin call" where the individual has to redeem part of his investment in the stock fund so that the maximum loan-to-value requirement is satisfied. The second equation defines the utility of post-retirement wealth and the third constraint excludes short selling of the stock fund.

3.4 Unconstrained Asset Allocation

In the unconstrained case, the housing consumption decision is separated from the real estate investment decision. In both the constrained and unconstrained cases the

representative individual lives in the same house, but now there is a market where fractional interests in a home can be sold. Thus, the individual's investment in residential real estate is a choice variable that can be separated from the level of housing services consumed. To adjust the actual to the desired level of investment the representative individual sells an interest in their house equal to $H - \eta W$, where η denotes the proportion of wealth invested in the home. The individual still consumes all of the services provided by the house, but must pay "rent" for the services provided by the portion of the home that is not owned (i.e., $H - \eta W$). Formally, the individual pays "rent" of δ per dollar of value of the house that is sold, resulting in a total "rent" payment of $\delta(H - \eta W)$. While this exercise is somewhat unrealistic, it does allow us to investigate the costs associated with the housing constraint.

In our model, the individual can be forced to rent by constraining $\eta = 0$, but this does not capture the essence of the **voluntary** decision to rent rather than buy a home. Thus, comparing the total expected utility of this forced rent case with the constrained or unconstrained cases discussed above, does not provide us with an insight into the endogenously determined rent versus buy decision. Nonetheless, our analysis (not shown) indicates that, for the levels of income and wealth that we consider, owning the entire home is far more desirable then renting forever.

As was previously the case, at retirement the individual liquidates all of their investments. To maintain comparability with the constrained case, we assume that the individual places the same amount, λH_T, aside (invests at the risk-free rate or buys a "trade down" house) to provide post-retirement housing services. The surplus is then used to purchases an annuity that provides a fixed level of consumption for the remaining years of the individual's life.[16]

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[16] The two cases are not completely comparable because in the unconstrained case the quantity of real housing services that could be purchased after retirement would be a function of the price of these services, where as, in the constrained case the individual would be fully hedged against changes in the price of these services.

In the unconstrained case that evolution of wealth is given by:

$$\frac{dW}{W} = \phi \frac{dS}{S} + \eta \frac{dH}{H} +$$

$$+ (1 - \phi - \eta)(r(1 - \tau) - i\tau)dt + \frac{\left(y(1 - \tau) - C - \delta(H - \eta W)\right)dt}{W}. \tag{15}$$

Here the allocation to residential real estate in unconstrained and the "rent" payment for the portion of the house not owned is subtracted from disposable income. The Bellman equation becomes:

$$0 = \max[e^{-kt}U(C) + V_t + (\frac{1}{2}\sigma_S^2\phi^2 W^2 + \frac{1}{2}\sigma_H^2\eta^2 W^2 + \phi\eta W^2\sigma_{SH})V_{WW} + \frac{1}{2}\sigma_H^2 H^2 V_{HH} +$$

$$+ \frac{1}{2}V_{yy}y^2\sigma_y^2 + V_{WH}(\phi WH\sigma_{SH} + \eta WH\sigma_H^2) + yH\sigma_{yH}V_{yH} + \tag{16}$$

$$+ V_{yW}(\phi Wy\sigma_{yS} + \eta Wy\sigma_{yH}) + V_H H\mu_H + V_y y\mu_y +$$

$$+ V_W W(\phi(\mu_S + r) + \eta\mu_H + (1 - \phi - \eta)(r(1 - \tau) - it)) + V_W(y(1 - \tau) - C - \delta(H - \eta W))]$$

Differentiating equation (16) with respect to C, ϕ and η gives the following first-order conditions:

$$e^{-kt}U_C - V_W = 0 \tag{17}$$

$$\sigma_S^2 W V_{WW}\phi + \eta\sigma_{SH}W V_{WW} + \sigma_{SH}H V_{WH} + V_W(\mu_S + (r + i)\tau) + \sigma_{yS}y V_{yW} = 0 \tag{18}$$

$$\eta W\sigma_H^2 V_{WW} + \phi\sigma_{SH}W V_{WW} + \sigma_H^2 H V_{WH} + V_W(\mu_H + \delta - r(1 - \tau) + i\tau) + \sigma_{yH}y V_{yW} = 0 \tag{19}$$

To simplify the expressions for the optimal controls, we adopt the following notation:

$$a = \sigma_S^2 W V_{WW}$$

$$b = \sigma_{SH} W V_{WW}$$

$$c = V_W(\mu_S + (r+i)\tau) + \sigma_{yS} y V_{yW} + \sigma_{SH} H V_{WH}$$

$$d = \sigma_{SH} W V_{WW}$$
(20)

$$f = \sigma_H^2 W V_{WW}$$

$$g = \sigma_H^2 V_{WH} + V_W(\mu_H + \delta - r(1-\tau) + i\tau) + \sigma_{yH} y V_{yW}$$

Equations (18) and (19) can then be expressed in matrix form as:

$$\begin{pmatrix} a & b \\ d & f \end{pmatrix} \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} -c \\ -g \end{pmatrix}$$
(21)

The optimal controls C^*, ϕ^* and η^* are:

$$C^* = (e^{kt} V_W)^{-\frac{1}{\gamma}}$$
(22)

$$\phi^* = \frac{-cf + bg}{af - bd}$$
(23)

$$\eta^* = \frac{cd - ag}{af - bd}$$
(24)

The above optimization is subject to the following constraints:

$$\phi W + \eta W(1-\theta) \leq W \quad \text{for all } t$$

$$V(W_t, H_t, y_t, t) = U_T(W_T) = \frac{1 - e^{-Rk}}{k} * U(C^R + y_s) \quad \text{at } t = T$$
(25)

$$\phi \geq 0 \quad \text{for all } t$$

$$\eta \geq 0 \quad \text{for all } t$$

The first constraint states that the investment in stocks plus the minimum home equity cannot exceed the individual's total wealth. The second relationship defines the utility of terminal wealth. The final constraints exclude short selling of stock or ownership interest in residential real estate.

4. The Effect of a Homeownership Constraint

In this section we present the results of the calibration of the constrained and unconstrained asset allocation problems. When an individual purchases a home, they are buying a specific property, located in a specific real estate market. Their investment is subject to both market and property specific risks. Our analysis is directed at such an investment. While the PSID provides a wealth of data about individual's asset holding, including housing, it is not well suited to estimating the risk and returns to investing in a specific house.

Even though Los Angeles County is far from typical, we will assume the representative individual has just purchased a "typical" Los Angeles County home. We chose this point of reference because the data set developed by Cauley and Pavlov (2002) allowed us to estimate the risk and return profile for a "typical" house bought in this market. Their data includes every sale of a Los Angeles County single family house between 1985 and 2000. Following Cauley and Pavlov (2002), this data was used to estimate the expected return, variance and the covariance with the CRSP value weighted stock market index of an investment (purchase of) in a typical Los Angeles County house.[17] These values, along with the other default values for the parameters are presented in Table 1. The default values were chosen to provide a realistic representation of the situation for a mid-career individual who has just purchased (in 2002) a typical Los Angeles County house. In the base case the representative individual is assumed to: (1) have an initial family income of $100,000; (2) just purchased a home for $320,000; (3) financed the purchase with a mortgage loan of $256,000; and (4) has $120,000 of total wealth (including home equity but excluding human-capital).[18] In what follows, all quantities are expressed in hundreds of thousands of dollars. To the best of our knowledge no data set is currently available that would allow us to estimate the parameters associated with the individual's income, specifically: the volatility of income at the individual level; the correlation between the income and home values; and the

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[17] The data includes the very rapid appreciation in home values that occurred during the second-half of the 1980s but does not include the appreciation since 2000.

[18] Kullmann and Siegel (2003) report mean PSID data for a sample of homeowners (1985-1999): age 48 years, H=$178,435, W=$143,552, I=$39,900 and a market LTV of 57%. As will be seen below the asset allocations we derive for the constrained case at time equivalent to an age of 48 are remarkable similar to the means obtained from PSID.

correlation between income and equity returns. Our choice of these parameters is intuitively plausible but arbitrary. As noted earlier a homeownership constraint can be modeled as a consumption commitment (Chetty and Szeidl, 2004) or a consumption compensation constraint (Piazessi, Schneider, and Tuzel, 2003) in both cases the individual acts as if they were more risk averse than they actually are. For that reason we assume the representative individual has a relatively large risk aversion coefficient (γ) of 5. We use a finite difference approximation to solve the optimization. Appendix discusses the details of the numerical solution technique.

4.1 Evolution of the Optimal Asset Allocation, Consumption and Wealth

Figure 1 depicts the evolution, over the individual's working life, of the optimal asset allocations and state variables for the base case (H_0 = \$3.2, W_0 = \$1.2) assuming *stocks and real estate appreciate at their expected real rate of returns*. The starting point for each solution depicted in this figure is determined by the default values for the parameters presented in Table 1. For example, Panel A depicts the asset allocations to stocks and housing as a fraction of non-human capital wealth, *W*. The allocation to the money market fund is the complement of the sum of the allocations to stocks and housing. The vertical axis is the fraction of total wealth allocated to each asset. For example, initially in the constrained case the weight for the investment in a house is 2.67 (3.2/1.2 or 267% of wealth) and the weight for the stock fund is .47 (.56/1.2 or 47% of wealth). Thus, the home is financed with a mortgage (negative money market balance) that is equivalent to a weight of -2.13 (213% of wealth), this translates into a mortgage loan of approximately \$256,000. Initially, the sum of the allocations to stocks and real estate exceeds one. These allocations involve substantial borrowing (e.g., a negative allocation to the money market). Since the allocation to real estate is constrained, the evolution of *H/W* only reflects the growth of home value and wealth through time. The allocation to stocks reflects the total borrowing constraint through year 12, after which the optimal allocation drops as retirement approaches. While not strictly comparable it is interesting to compare the results presented in Figure 1 with the average ratio of housing value to wealth reported by the PSID. The average age of a homeowner as reported by Kullmann and Siegel (2003) was 48 years of age. This would correspond to time 8 or 9

in our problem. From Figure 1 we can see that the constrained allocation to housing as a fraction of non-human capital wealth is approximately 140 percent at time 8. This compares with an average value for the PSID data is 124 percent. While the average value of *H/W* derived from the PSID data need not corresponded to the average conditional upon the homeowner being 48 years of age, the closeness of these statistics tends to support the empirical relevance of our analysis.

Panel B depicts the optimal asset allocation in the unconstrained case. While the allocation to residential real estate is lower relative to the constrained case, it still exceeds the individual's wealth. The allocation to stocks is driven by the total investment constrained until year 13, after which it starts to drop as retirement approaches. Panels C and D depict the evolutions of consumption and total wealth in the two cases. These evolutions are similar because, while real estate has lower expected return, it can be purchased with higher leverage. This makes un-levered investment in stocks and in highly levered investment in housing nearly equally desirable. While consumption and the evolution of total wealth are similar in the two cases, the allocation among the assets is very different. In the constrained case, the allocations change substantially over the twenty year work life. The allocations in the unconstrained case are less extreme and more consistent through time. Furthermore, the effect of a homeownership constraint is largest at the beginning, i.e., for young households who have smaller net worth relative to current income. As individuals accumulate wealth, the homeownership constraint becomes less binding and the asset allocations in the two cases become similar.

In Los Angeles, like many other major metropolitan areas, there has been a secular decline in the "affordability" of homes. That is, homes prices have increased relative to a typical family's income and wealth. The data used to generate Figure 2 represents such a situation. Income and wealth are the same as the base case but the price of the home is now $450,000 and the home is "just affordable" for the family. In this example, the homeownership constraint has a larger effect on asset allocations and we can see that the unconstrained allocation to real estate is positively related to the value of the home being bought. As the price of the home increases the rent paid on the portion of the house not owned increases. Thus, the desired level of investment in the unconstrained case includes a larger demand to hedge against future increases in the cost

of housing services. Inspection of Panels C and D show that such a large investment in residential real estate has an important effect on the individual's consumption and post retirement wealth. Specifically, the housing constraint results in the individual consuming less non-housing services when they are young and having more post-retirement wealth.

4.2 Optimal Initial Asset Allocation and Consumption

Table 2 presents the optimal initial asset allocation and consumption for a number of different starting values of the state variables. The first three columns depict the starting value of each state variable. The computation and interpretation of the forth column is discussed in the following section. Columns 5 through 8 depict the optimal initial asset allocations and consumption for the constrained case. The unconstrained case is presented in columns 9 through 12.

The Table's first line depicts the base case using the parameters of Table 1. The following three lines hold real estate and income constant (H=$3.2 and Y=$1.0) and allow wealth to increase. The allocation to real estate in the constrained case, (H/W) is by construction falling for higher levels of total wealth. Notice that the allocation to real estate in the unconstrained case is also falling for higher levels of wealth. This suggests that a portion of the demand for real estate in the unconstrained case is driven by the desire to hedge against future increases of real estate values as discussed above. Another interesting result is that the allocation to real estate is actually slightly higher in the unconstrained case for extremely high levels of wealth. This suggests that a homeownership constraint may be limiting not only because it forces individuals to hold too much real estate, but, in some circumstances, because it forces them to hold too little. Notice that consumption is at least as large in the unconstrained case as it is in the constrained case.

The next three rows of Table 2 hold wealth and income constant (W=$1.2 and Y=$1.0) but allow home values to increase. Not surprisingly, the difference between the asset allocations in the constrained and unconstrained cases increases. The allocation to real estate in the unconstrained case increases with home prices even though the individual can freely reduce their investment in real estate. This is due to the hedging

demand for housing discussed above. The difference between consumption levels in the two cases also increases with home values.

The final three rows of the Table hold total wealth and real estate constant (W=$1.2 and H=$3.2) and allow income to grow. The constrained allocations are unaffected by income since the investment in real estate is fixed, and the investment in stocks is limited by the total borrowing constraint. Nonetheless, consumption increases with income. The unconstrained allocation to real estate exhibits an interesting pattern. As income grows, more wealth is allocated to real estate. Beyond a certain level of income, however, the hedging benefit of real estate ownership becomes less important and a shift towards higher-return, higher-risk stocks becomes optimal.

To present a more complete picture of how the optimal allocations and consumption varies with the initial state variables, we graphically expand Table 2. Figure 3 is constructed to further examine the relationship between the value of the investment in residential real estate, asset allocations and consumption. Three levels of initial non-human capital wealth are considered: $120,000, $200,000 and $300,000 ($W_0 =$ 1.2, 2, and 3). The horizontal axis for each of the graphs represents the level of investment in residential real estate, H. Dashed lines represent the values for the constrained case and solid lines represent the corresponding values for the unconstrained case. Not surprisingly, for all of the variables considered, the greater the value of the investment in residential real estate relative to the individual's wealth the larger the initial effect of the home ownership constraint. Interestingly, the allocation to stocks (house) is not uniformly higher (lower) in the unconstrained case. For very low house values, the allocation to real estate is sub-optimally low in the constrained case. For realistic house values, however, the unconstrained case allows for higher allocation to stocks and lower allocation to real estate. The intersection of the two lines in either figure depicts a constrained initial allocation to real estate that coincides with unconstrained allocation. Nonetheless, the two cases are still distinct, as future adjustments are not allowed in the constrained case. Inspection of Figure 3 also reveals that the ratio of home value to wealth has to be relatively large before the homeownership constraint has a substantial effect on the initial level of consumption.

Figure 4 examines the relationship between asset allocations, consumption and wealth. Three levels of investments in residential real estate are considered ($200,000, $320,000 and $450,000, i.e., $H_0 = 2$, 3.2, and 4.5). As was the case in Figure 3, the larger the investment in residential real estate the greater the effect of the housing constraint. The allocation to stocks depicted in these Figures exhibits an interesting pattern. For low levels of wealth, the allocation is constrained by the total investment limit. For higher levels of wealth (beyond the range depicted here), the allocation is not constrained. Unlike Merton's (1971) continuous time model, the unconstrained optimal allocation to stocks is not constant because of two reasons. First, human capital is not included in total wealth yet we include the income from human capital in the optimization. This is particularly important for low levels of total wealth (excluding human capital). Individuals in this situation have high income, but low net worth. According to our model, it is optimal for such individuals to invest as much as possible in the stock index fund. The second reason for the dependence of the allocation to stocks on the state variables in the unconstrained case is that the value of real estate, H, determines the "rent" payments the owner needs to make if they sell a fractional interest in their home. Higher real estate values result in higher rent, which, in turn, reduces the allocation of wealth to stocks. This effect is also evident in the allocation to real estate in the two cases. By construction, the allocation to real estate in the constrained case is equal to the value of the house, H, where as, the allocation in the unconstrained case can be adjusted. Nonetheless, the individual chooses to invest higher proportion of total wealth into housing in face of high real estate values. This choice reflects the hedging benefits of real estate ownership as discussed by Sinai and Souleles (2002). We will see below that this hedging demand for real estate disappears as retirement approaches.

Finally, for all levels of total wealth, W, consumption declines marginally for higher levels of H in both cases. For the constrained case, this effect is intuitive because higher forced investment in real estate increases the marginal value of wealth, which, in turn, reduces the optimal consumption level. For the unconstrained case, higher value of real estate increases the "rent" payments required if a portion of the house is sold, which, in turn, induces the individual to consume less and save more.

4.3 The Homeownership Constraint and Welfare.

To estimate the total welfare loss due to the homeownership constraint we compute the required percent increase in wealth required to make an individual indifferent between the two cases. This is similar to computing a certainty equivalent amount, except that we have two expected utility functions and we compute the required wealth increase so that the two functions take the same value. The fourth column of Table 2 presents the compensation as the percent increase in wealth that makes the constrained utility as high as the utility absent the homeownership constraint. This is the increase in wealth required to compensate for the homeownership constraint.

An individual whose total wealth, home value, and income take the values of the base case, ($120,000, $320,000, and $100,000, respectively), requires a nearly 6% increase in total wealth to be indifferent between the constrained and unconstrained cases.[19] Not surprisingly, the required compensation falls with higher wealth levels, as suggested by rows 2 through 4 of Table 2. In other words, the homeownership constraint is less binding for wealthier individuals.

An increase in real estate values increases the required compensation, as evident from rows 5 through 7. For the "just affordable" home case of $H_0 = 4.5$, the required compensation is over 25%. In other words, an individual who owns a home that is 4.5 times his or her income, would give up a quarter of their total wealth to be able to freely adjust their investment in real estate. While this example may seem extreme, it is not uncommon for Los Angeles County home buyers to be in this situation. Finally, higher levels of income reduce the required compensation, as any investment misalignment has a relatively lower impact on long-term consumption levels.

Figure 5 extends the evidence presented by the forth column of Table 2 by depicting the required compensation as a function of two of the state variables – total wealth and home value. As already discussed, higher home values relative to total wealth result in substantially higher required compensation. In other words, the homeownership constraint is more binding for higher home values. Any increase in wealth reduces the impact of the constraint, both because the asset allocation misalignment is reduced and

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[19] Transaction cost associated with changing homeownership would tend to decrease the utility loss associated the homeownership constraint.

also because the required compensation is expressed as a proportion of total wealth. Interestingly, even if the ratio of home value to wealth is the same as in the unconstrained case, the required compensation is positive and at least 0.84%.

Figure 6 depicts the required compensation as a function of home value and income. Not surprisingly, higher income reduces the required compensation for any home value. A consistent picture has emerged regarding the relationship between the individual's investment in residential real estate, wealth and asset allocations: the larger the investment in the house, relative to wealth, the greater the effect of the homeownership constraint and the more extreme the individual's initial asset allocations.[20] Because the asset allocation is more extreme, the volatility of wealth would be expected to be higher. Thus the homeownership constraint would be expected to result in a decrease in expected utility.

The analysis summarized in Table 2 strongly suggests that when housing is "affordable" the loss in utility resulting from the housing constraint is small but when "affordability" is a major problem (e.g., Los Angeles or New York City) it may be large. This finding strongly suggests the importance of a detailed examination of the impact of inter-regional differences in home prices on asset allocation and post-retirement wealth. The risk and return for investments in residential real estate used in this analysis were estimated from data for Los Angeles County (a high risk and expected return locality). Our analysis, assumes that the risk and rate of return to investments in residential real estate are independent of the price of a home. This may in fact not be true for all regions of this country. In this case our results may underestimate the burden of homeownership constraint when the price of the home is relatively low. Thus, we conclude that it is important to investigate inter-regional difference in the burden of a homeownership constraint.

5. Conclusion

Buying a home is a lumpy investment that places a constraint on the owner's asset allocation decisions. In this paper we analyze the impact of this constraint on a representative individual's consumption, welfare and post retirement wealth. For realistic

parameter values we find that a homeownership constraint has a relatively large effect on the representative individual's asset allocation. Specifically the individual invests a larger than optimal fraction of his or her wealth in a home when investments in homes are lumpy and constrained. We conclude in a base case that homeowners would require a 6% increase in total net worth to achieve the same utility level as an individual not facing the asset allocation constraint. As would be expected, the required compensation and the differences in asset allocation between the two cases diminish with time.

We find that the evolution of wealth and consumption, if all assets receive their expected rate of return, is similar in the two cases. Even though real estate tends to have lower total return then stocks, it can be purchased with higher leverage that can potentially increase the expected return on investment. In a stochastic environment, however, the evolution of wealth and consumption can be very different in the two cases, as reflected in the total loss of utility due to a homeownership constraint.

We further show that demand for real estate has a small but important hedging component. Specifically, individuals invest more in real estate then predicted by the risk-return characteristics of the pure investment asset. The desire to hedge against future increases in rent payments generates this additional demand. Nonetheless, for reasonable parameter values, the total investment in real estate is substantially lower if the individual can freely choose their asset allocation.

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[20] As wealth is accumulated over the individual's working life the constrained and unconstrained asset allocation converge.

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Figure 1: Evolution of Asset Allocation, Consumption and Wealth (H_0=\$3.2 and W_0=\$1.2)
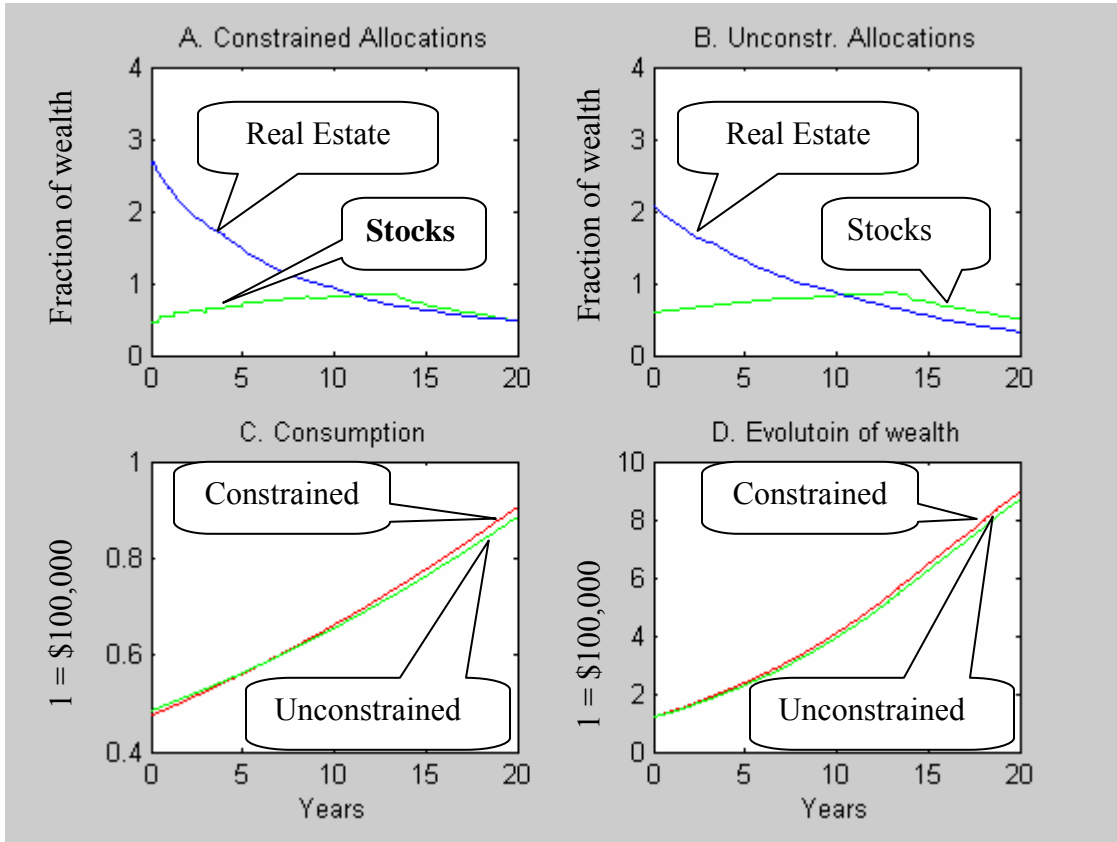

Figure 1 depicts the evolution of asset allocations, consumption, and wealth in the constrained and unconstrained cases assuming all returns equal their expectation. Some figures are not smooth because the solution jumps through the grid lines of the finite difference method as wealth is accumulated.

Figure 2: Evolution of Asset Allocation, Consumption and Wealth (H=$4.5 and W=$1.2)
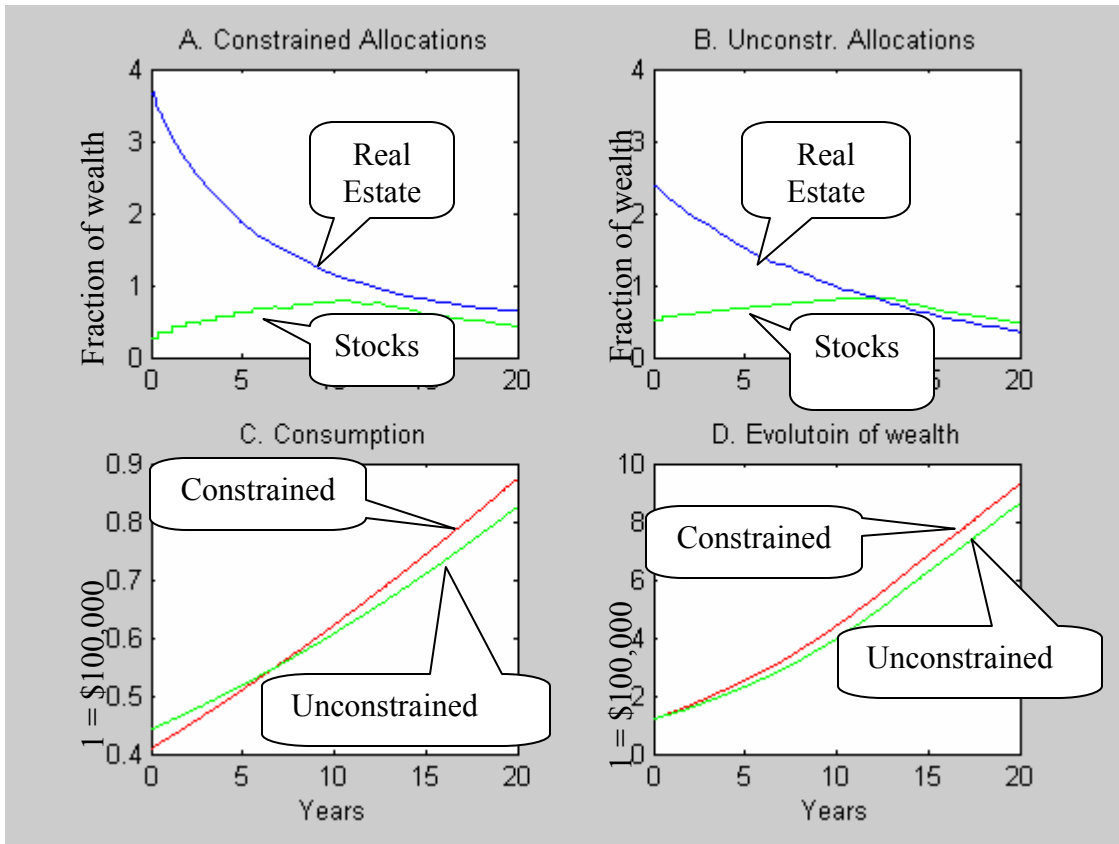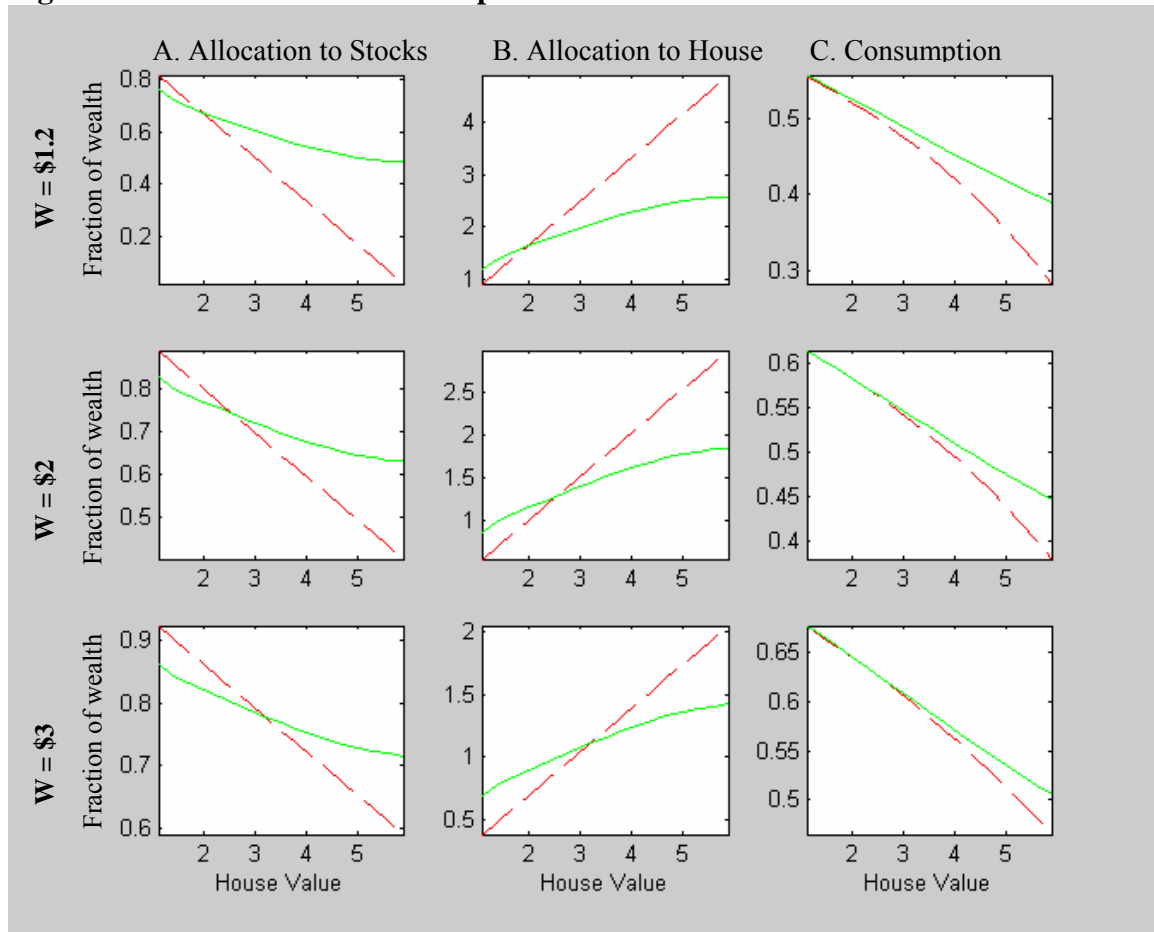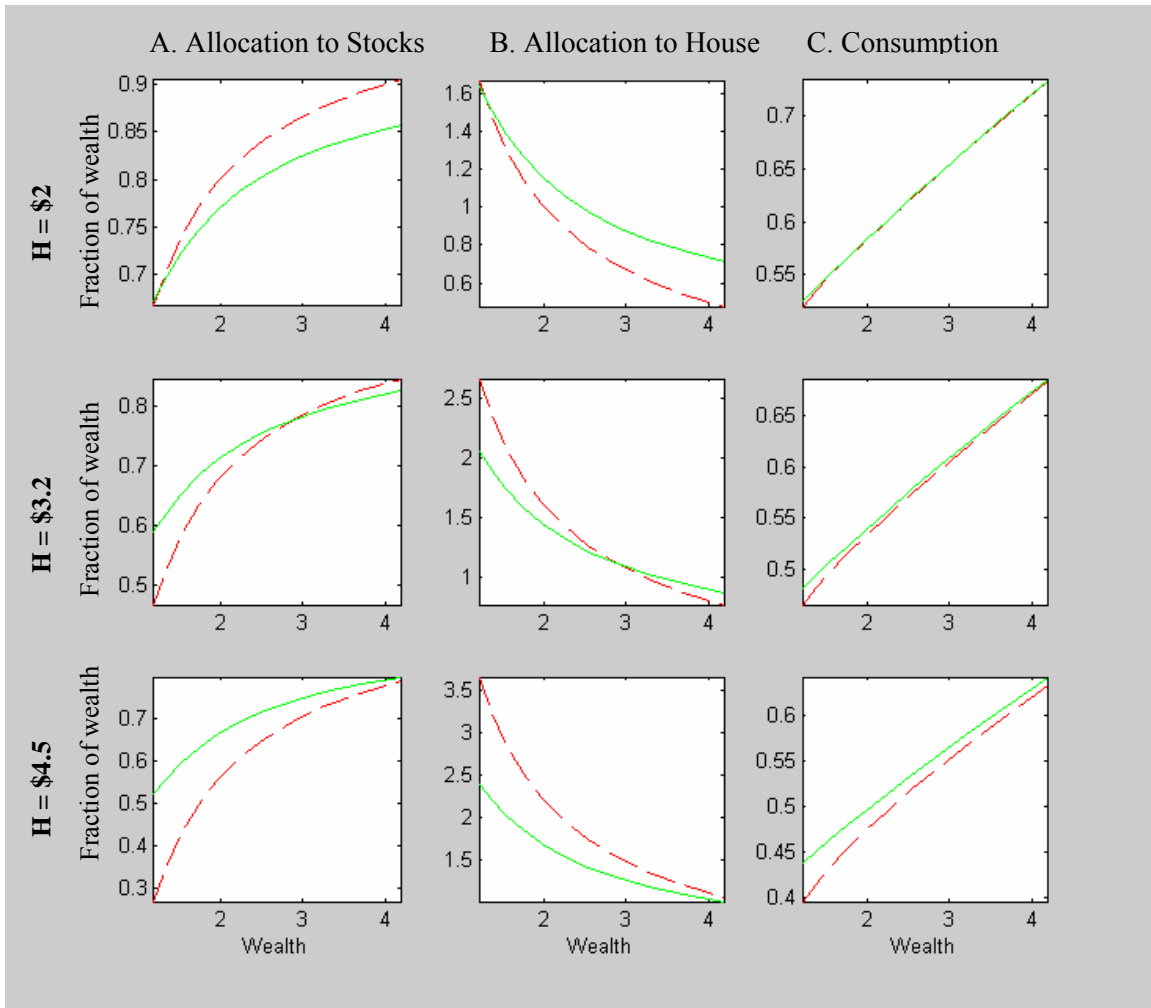

Figure 2 depicts the evolution of asset allocations, consumption, and wealth in the constrained and unconstrained cases assuming all returns equal their expectation. Some figures are not smooth because the solution jumps through the grid lines of the finite difference method as wealth is accumulated.

Figure 3: Allocations and Consumption as a Function of House Value

The figure depicts the allocation to stocks, house, and consumption as a function of house value, *V*, for three levels of total wealth, 1.2, 2, and 3. Dashed line depicts the constrained case and solid line depicts the unconstrained case.

Figure 4: Allocations and Consumption as a Function of Total Wealth

The figure depicts the allocation to stocks, house, and consumption as a function of total wealth, *W*, for three levels of house value, 2, 3.2, and 4.5. Dashed line depicts the constrained case and solid line depicts the unconstrained case.

Figure 5: Compensation for the Constraint: Percent of TOTAL Wealth
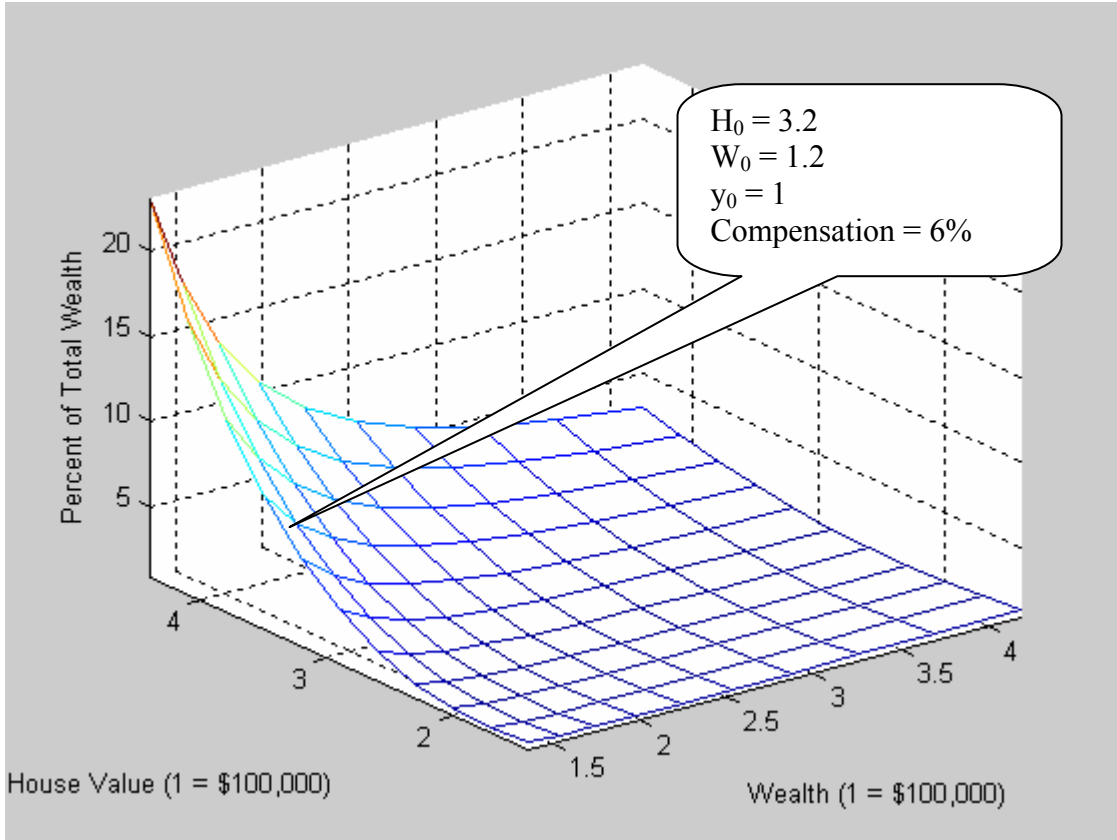

Figure 5 shows the compensation for the home ownership constraint as a function of initial wealth, W, and initial house value, H. We compute the compensation as the percent increase in wealth that makes the constrained utility function as high as the unconstrained one. Not surprisingly, higher home values require higher compensation. Higher wealth levels require lower compensation.

Figure 6: Compensation for the Constraint: Percent of TOTAL Wealth

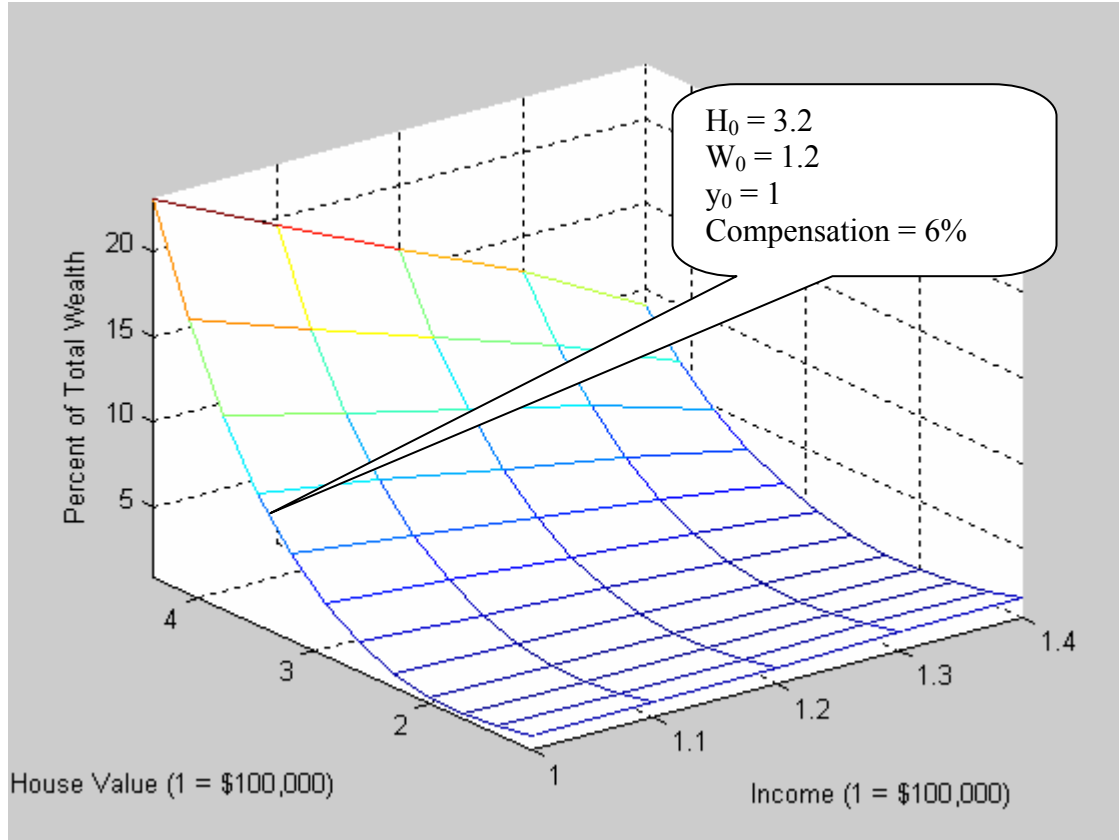

Figure 6 shows the compensation for the home ownership constraint as a function of initial annual income, y, and initial house value, H. We compute the compensation as the percent increase in wealth that makes the constrained utility function as high as the unconstrained one. Not surprisingly, higher home values require higher compensation. Higher income levels require lower compensation.

Table 1: Parameters of the Model

Parameter	Description	Default value
r	Mortgage rate, real, before tax, annualized, continuously comp.	6%
γ	Coefficient of risk aversion	5
k	Rate of time preference	3%
y_0	Starting annual family income level, real, before tax, before mortgage payment	$100,000
y_s	Social Security payment[*]	$17,700
H_0	Starting value of the house	$320,000
W_0	Starting wealth level (excluding human-capital)	$120,000
μ_S	Risk premium for stocks **over the mortgage rate**, r, annual	4%
μ_H	Expected rate of home value appreciation, real, not taxed	1.5%
μ_y	Expected rate of growth of income, real, before tax	1.9%
σ_S	Volatility of stock prices, annual	16.7%
σ_H	Volatility of home prices, annual	13%
σ_y	Volatility of income, annual	5%
ρ_{yH}	Correlation between real estate and income	.1
ρ_{yS}	Correlation between stocks and income	.1
ρ_{HS}	Correlation between stock and home prices	.2
i	Inflation	2.5%
τ	Tax rate	37%
δ	Annual service flow from housing (proportion of home value)	4%
T	Time horizon until retirement	20 years
R	Time horizon after retirement	15 years

[*]More than one member of family receives benefits.

Table 2: Initial Compensation and Allocations for Different Starting Points

| \multicolumn{3}{Starting Values} | | | Constrained Allocations | | | | Unconstrained Allocations | | | |
W	H	Y	Comp	Stocks	House	Money Market	Consumption	Stocks	House	Money Market	Consumption
$1.20	$3.20	$1.00	5.97	0.47	2.67	-2.13	0.47	0.59	2.06	-1.64	0.48
$1.20	$3.20	$1.00	5.97	0.47	2.67	-2.13	0.47	0.59	2.06	-1.64	0.48
$2.00	$3.20	$1.00	2.78	0.68	1.60	-1.28	0.53	0.71	1.43	-1.15	0.54
$3.00	$3.20	$1.00	1.58	0.79	1.07	-0.86	0.61	0.78	1.08	-0.86	0.61
$1.20	$2.00	$1.00	1.29	0.67	1.67	-1.33	0.52	0.67	1.64	-1.31	0.53
$1.20	$3.20	$1.00	5.97	0.47	2.67	-2.13	0.47	0.59	2.06	-1.64	0.48
$1.20	$4.50	$1.00	25.79	0.25	3.75	-3.00	0.39	0.52	2.41	-1.93	0.43
$1.20	$3.20	$1.00	5.97	0.47	2.67	-2.13	0.47	0.59	2.06	-1.64	0.48
$1.20	$3.20	$1.20	4.22	0.47	2.67	-2.13	0.54	0.58	2.10	-1.68	0.56
$1.20	$3.20	$1.40	2.29	0.47	2.67	-2.13	0.66	0.60	2.02	-1.61	0.67

Table 2 reports the compensation required (column 4) for having to hold a fixed level of investment in housing along with the allocations and consumption in the constrained and unconstrained cases for different initial values of the state variables. As expected, higher level of initial wealth reduces the required compensation. Higher house values increase the required compensation. Higher income also slightly reduces compensation.

Table 3: Parameters of the Finite Difference Solution Method

Parameter	Description	Value Employed
Grid size	Size of the solution grid (W x H x y)	25x17x5
Grid resolution	Spacing between the grid lines. This is variable, with highest density for low levels of W	1 on average
Time step	Length of the time step in years	.05 years = 12.5 days
Time horizon	Number of time steps	20 years

Appendix: Details of the Numerical Solutions

The numerical solution of the optimization problems utilizes an explicit finite difference approximation to solve differential equations (9) and (16) subject to the respective optimal controls and boundary conditions. Unlike most continuous time asset allocation models, wealth cannot be excluded from the model. In fact, the relationship between wealth and home value is a crucial element of the investigation. However, including wealth in the solution poses a number of computational difficulties. First, the typical boundary conditions of setting the first and second derivatives to zero or constants cannot be used in our setting because some of them enter the denominator of the expressions for the optimal controls. Instead, we set the ratio of the first and second derivatives, $V_w / W V_{ww}$, to a constant at the boundary. This provides for very smooth and stable solutions. Furthermore, the higher order derivatives of the utility function that is employed are non-zero and change substantially over the range of total wealth considered. For instance, the third derivative is very large in absolute value for low levels of wealth, and is numerically indistinguishable from zero for high levels of wealth. The error of the first or second order numerical derivatives is sensitive to the higher order derivatives, which introduces a consistent bias in the optimization. To remedy this potentially serious problem we employ an exponentially variable grid in W. For low levels of wealth, the grid is very dense, which reduces the numerical error in the first and second derivatives. For larger levels of wealth, the grid is sparse, which allows for numerically tractable computation of the derivatives. Employing a variable grid is equivalent to a transformation in the state variables and the utility function. To the extent that such transformation could potentially introduce further numerical errors and would make presentation of the results more difficult, we prefer the variable grid approach. Table 3 shows the parameters used in the finite difference approximation.

As with most economic optimizations, we compute the value function at time of retirement, T, given the terminal conditions (14) and (25). This value function (and its numerical derivatives) provides the optimal controls given by equations (12), (13), (22), (23), and (24). The derivatives with respect to wealth at the borders are computed by

setting $V_w \big/ WV_{ww}$ to a constant. The derivatives with respect to home value and income are computed by setting the second derivative to zero at the borders. Substitute these derivatives and the optimal controls into Equations (9) and (16) to obtain the value function on the grid points at time *T-1*. We then repeat this procedure until time zero. This is an explicit procedure, as the controls and partial derivatives from period *t+1* are used to compute the value function at time *t*.