How Auctions Amplify House-Price Fluctuations

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Abstract

I develop a dynamic search model of the housing market where house prices are determined in auctions rather than by Nash bargaining as in the housing search model from the literature. The model with auctions generates fluctuations between booms and busts. During the boom multiple buyers compete for one house, while in the bust buyers are choosing among several houses. The model improves on the performance of the model with Nash bargaining by producing highly volatile house prices which helps to solve the puzzle of excess volatility of house prices. Higher volatility arises because of the competition between buyers with heterogeneous values. With heterogeneous valuations, the price determination becomes important for the quantitative properties of the model. With Nash bargaining, the buyer is chosen randomly among all interested buyers. Then the average of buyers' house values determines the house price. In the auction model, the buyer is chosen by the maximum bid among all interested buyers, so the highest value determines the house prices. During the boom, the highest values increase more than the average values, making the sales price more volatile. This high volatility is constrained efficient since the equilibrium in the model decentralizes the solution of the social planner problem, constrained by the search frictions.

Keywords: housing, real estate, volatility, search and matching, pricing, liquidity, Nash bargaining, auctions, bidding wars

JEL codes: E30, C78, D44, R21, E44, R31, D83

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1 Introduction

House prices fluctuate between booms and busts, and are volatile relative to fundamentals, such as rents and income in the local housing market[1]. The top panel in Figure 1 shows the monthly house price growth in the Los Angeles Metropolitan Statistical Area from 1996 to 2016. The bottom panel of the same Figure 1 shows a simulation of the house price growth from the calibrated benchmark search model with Nash bargaining, currently employed in the literature. The benchmark model produces less volatility of the house prices than observed in the data.

My hypothesis is that the housing search model cannot explain this volatility, because the sales mechanism does not account for competition between buyers. Specifically, in the benchmark housing search model a seller is bargaining one-to-one with a randomly selected buyer to determine the house price. In reality, especially during booms, the seller is dealing with multiple buyers and sells to the highest bidder. I show that the volatility of the house prices is quantitatively higher if the model takes into account the competition between buyers.

The competition between buyers, often referred to as a bidding war, does happen in the housing markets in the US and other countries. In the US the seller puts the house on the market, and holds an open house, usually during the weekend. Then during the first weekdays of the next week buyers submit their offers, and the seller usually sells to the highest bidder. For example, the Boston Globe reviews the sale of the condo in Brookline, Massachusetts[2], where "three hundred people came through the open house, 25 made offers, and the bidding war lasted eight rounds and four days". So bidding wars actually occur in local housing markets.

Bidding wars are not only real phenomenon, they are also common, in particularly during the housing booms. The *New York Times* writes on August, 13, 1997, "Bidding wars are no longer uncommon, especially in affluent areas of northern New Jersey, Los Angeles, the San Francisco Bay area and Boston."[3] On June 10, 2015, Trulia echoes "those bidding wars - oh, those bidding wars... when inventory is low, those bidding wars can escalate into a kamikazelike battle with 17 other buyers..."[4]

Bidding wars are common, but how often do they happen quantitatively? Han and Strange (2014) show that in the US the frequency of bidding wars rose to 30% in some markets between 1995 and 2005. The bidding wars continued to be frequent, as can be seen from Figure 2. Figure 2 plots the bidding wars index from the Redfin, a real estate brokerage firm in the US from 2009 to 2015. When a Redfin client places an offer on a house, Redfin records whether there was at least one competing offer. The graph shows the percentage of offers that faced competition from other buyers. On average half of the offers faced competition in the US. Similarly, in England there are multiple buyers making offers on the same house leading to de facto auctions, see Merlo and Ortalo-Magné (2004) and Merlo, Ortalo-Magné, and Rust (2015)[5].

Hence, bidding wars are real, common and frequent. However, the literature has been focusing

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[1] See, for example, Davis and Heathcote (2005), Davis and Nieuwerburgh (2015), Piazzesi and Schneider (2016).

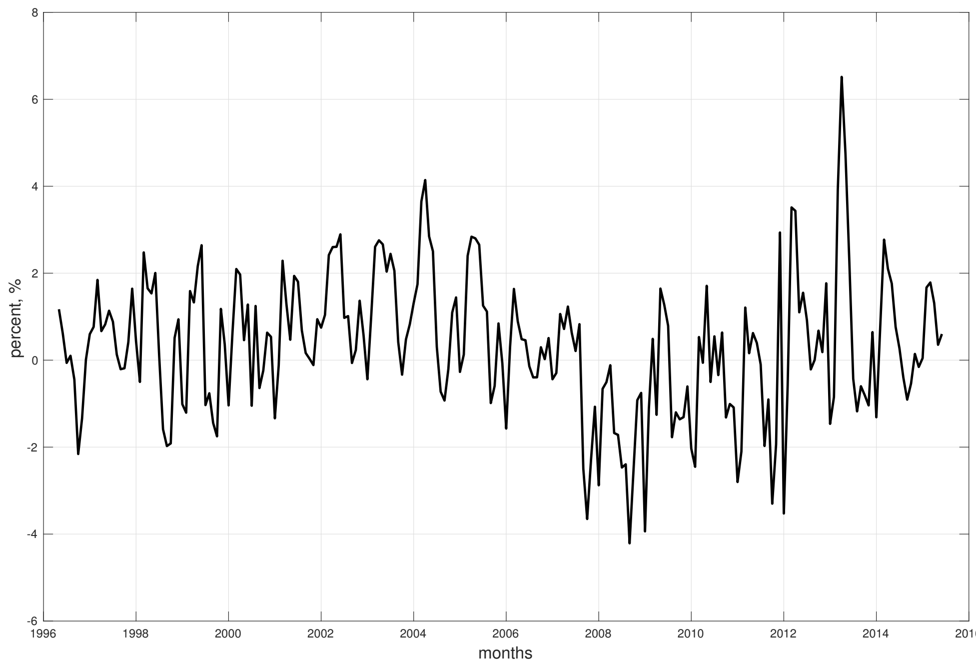
[2] https://www.bostonglobe.com/business/2015/03/30/forget-location-location-location-some-realtors-have-new-mantra-bidding-war/3gI6wnpNnf82QMvpjN3UWJ/story.html.

[3] http://www.nytimes.com/1997/08/13/business/home-buyers-find-the-bidding-wars-are-back.html?pagewanted=all

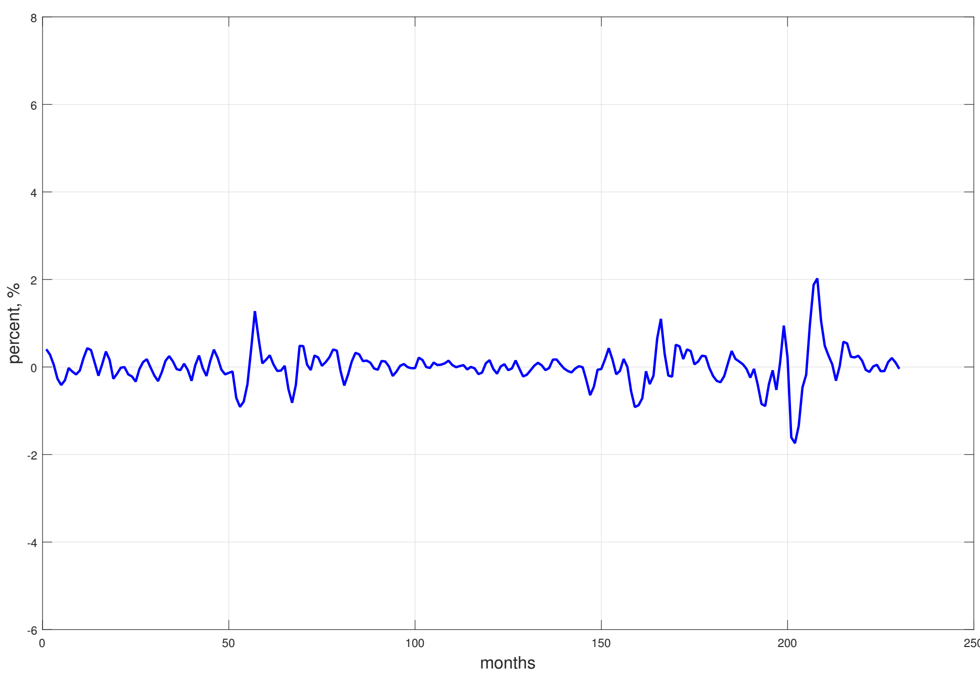
[4] http://www.trulia.com/blog/7-crazy-things-about-buying-in-a-sellers-market/

[5] The average number of buyers who are making an offer is 1.44 for the full sample from 1995 to 1998, and 1.26, 1.62, 1.53 for Yorkshire 1995-1998, London 1995-1996, London 1997-1998 subsamples, correspondingly.

Figure 1: The house price growth in the data and in a simulation of the benchmark housing search model with Nash bargaining



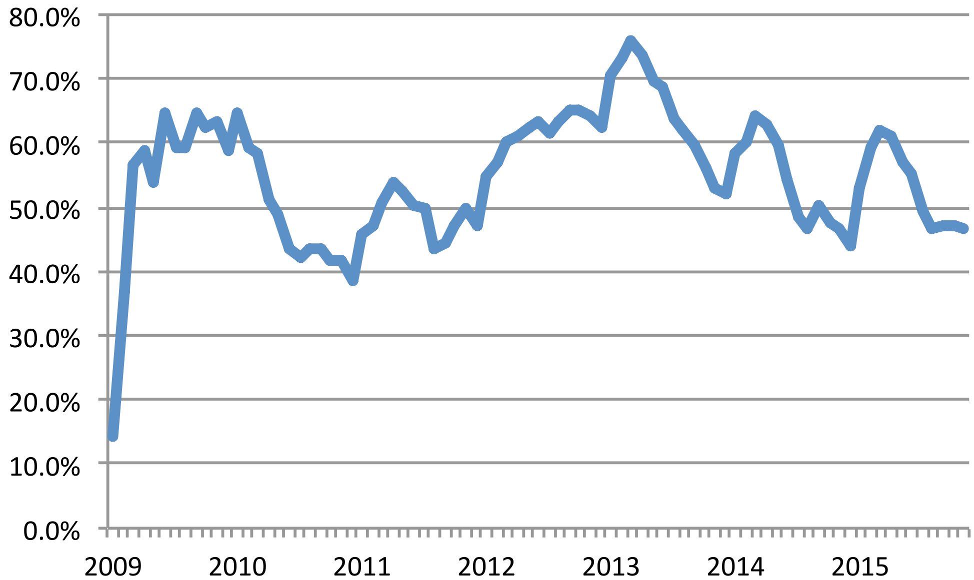
(a) Los Andgeles MSA data



(b) Benchmark model

Notes: The monthly data on the house prices comes from Zillow. Zillow applies the Henderson Moving Average filter and then STL filter to produce the seasonally adjusted series of the house price growth, see `http://www.zillow.com/research/zhvi-methodology-6032/`.

Figure 2: Percent of offers, facing competition



Source: Redfin, the real estate brokerage firm. Redfin is based in Seattle, services 3% of market, operates in nineteen local housing markets in the US. The data is monthly, not seasonally adjusted, average over the markets.

on the Nash bargaining price determination mechanism, where a seller bargains one-to-one with a randomly selected buyer. In practice, during the booms the buyers compete for the same house, and house is sold to the buyer with the highest offer. A natural way to model this sales mechanism is an auction model[6].

The auction model is not only a natural representation of this process, it also helps explain high volatility of the house prices, observed in Figure 1a. When house prices are determined in an auction instead of Nash bargaining, house prices fluctuate more in response to the demand shocks generated by the influx of buyers. The influx of buyers provokes the competition between buyers. The competition between buyers is important for the volatility of the house prices because of the heterogeneity in the house values. With heterogeneous values, the method of choosing the buyer among all the interested buyers becomes important for the quantitative properties of the model. In the benchmark model with Nash bargaining, the buyer is chosen randomly among all interested buyers. Then the average house values of buyers determine the house price. In the auction model the buyer is chosen by the maximum bid among all interested buyers, so the highest value, or more specifically the second highest value, determines the house prices. During housing booms, the highest values increase more than the average values, making the sales price

[6]Auctions is a popular way to sell distressed properties. The auctions of distressed properties, for example, foreclosure auctions, often are official and take standardized forms. The auction model can be applied directly in this case. However, the auction model also describes the non-foreclosure sales of houses where the bidding war between several buyers, for example, by means of escalation clauses, is an unofficial de-facto auction, portrayed in this paper.

more volatile.

To demonstrate this, I build a dynamic search model of the local housing market with auctions. Then I compare this model with the benchmark model with Nash bargaining. These models differ only in how the prices are determined. I calibrate the models based on the data from the Los Angeles Metropolitan Statistical area[7], and show that the volatility in the auction model is higher than in the Nash bargaining model, and is similar to the volatility, observed in the data which is the main result of my paper. Then I solve the problem of the social planner constrained by the search frictions, and show that the auction model with directed search decentralizes the solution of the social planner problem. In this sense high volatility, produced by the auction model with directed search, is efficient.

The rest of the paper is organized as follows. Section 2 reviews the literature. Section 3 introduces the framework of the models and provides an example to highlight the differences of the benchmark model and the model with auctions. Section 4 compares the steady-states of these models, illustrates how the shocks are amplified in the auction model as compared to the Nash bargaining model, then calibrates the model to the data and finally discusses the quantitative results. Section 5 solves the problem of the social planner constrained by the search frictions, and shows the efficiency of the equilibrium in the auction model with directed search. Section 7 concludes.

2 Literature review

My paper aims to explain the volatility of house prices observed in the data, and contributes to several strands of literature on microstructure of housing markets, house price dynamics and applied theory.

The model in the paper builds on the growing literature of microstructure of housing markets that considers housing search and matching with bargaining and auctions. This literature is summarized by Han and Strange (2015), who observe that the literature on real-estate auctions, especially on the theoretical side, is sparse. My paper fills this gap by building a tractable dynamic model of the housing market with auctions and comparing this model with the prevalent housing search model with bargaining.

The theoretical approach in my paper is closes to that of Head, Lloyd-Ellis, and Sun (2014) and Albrecht, Gautier, and Vroman (2016). Head, Lloyd-Ellis, and Sun (2014) consider a random search and matching model of the housing market with bargaining[8]. In addition to the bargaining model similar to that of Head, Lloyd-Ellis, and Sun (2014), I construct the auction models with random and directed search to show the importance of bidding wars in amplifying the house price volatility. Albrecht, Gautier, and Vroman (2016) build a static auction model with

[7]The LA MSA is chosen as an example of the MSA with highly volatile house prices.

[8]See also Arnott (1989), Wheaton (1990), Salant (1991), Yavas (1992), Yavas and Yang (1995), Berkovec and Goodman (1996), Arnold (1999), Krainer (2001), Ford, Rutherford, and Yavas (2005), Albrecht, Anderson, Smith, and Vroman (2007), Piazzesi and Schneider (2009), Novy-Marx (2009), Caplin and Leahy (2011), Carrillo (2012), Genesove and Han (2012), Anenberg (2013), Anenberg and Bayer (2013), Burnside, Eichenbaum, and Rebelo (2013), Díaz and Jerez (2013), Carrillo (2013), Ngai and Tenreyro (2014), Piazzesi, Schneider, and Stroebel (2014), Head, Lloyd-Ellis, and Sun (2014), Head, Lloyd-Ellis, and Stacey (2014), Guren (2015), Guren and McQuade (2013), Carrillo, de Wit, and Larson (2015), Moen, Nenov, and Sniekers (2015), Hedlund (2016a), Hedlund (2016b), Garriga and Hedlund (2016), Ngai and Sheedy (2016).

directed search to study the role of the asking price in the housing market. By contrast, my paper considers a dynamic auction model, and aims to isolate the qualitative and quantitative implications of the auction price-finding process against the Nash bargain for the house price dynamics. The dynamic framework in this paper makes it possible to take into account the option values of buying or selling the house later that propagate the housing market shocks, absent in the previous papers on housing auctions[9].

The literature on house price dynamics struggles to explain the observed house price volatility within a fully rational framework, even when it considers various amplification mechanisms, such as amplification of the income shocks through borrowing constraints [10], so it turns to extrapolative expectations, speculation and bubbles[11]. Giglio, Maggiori, and Stroebel (2016) tests the existence of housing bubble in the UK and Singapore, and finds no evidence of classic rational bubbles even during the period of the recent boom-bust episode. My paper contributes to this literature by producing the house price volatility that arises endogenously in a model with rational expectations and no bubbles due to the bidding wars of buyers with heterogeneous valuations.

My paper is related to papers on the selling institutions and search frictions in labor, asset and retail markets. For instance, Julien, Kennes, and King (2000) examine the labor market in which employees auction their labor services to firms. They produce the wage dispersion in the steady-state equilibrium of the search model. My paper studies the housing market and

[9]Mayer (1995) also considers the performance of negotiations versus auctions in the steady-state of the model with costless search and perfect matching technology. My paper adds the dynamics and search frictions to study the time-series volatility of house prices. Quan (2002) studies the endogenous choice of agents between the two separated housing markets in a static search model. In the first market the prices are determined in negotiations, and in the second – by auctions. I am not considering the endogenous choice between these two markets, but compare the house price volatility in the dynamic housing search model with bargaining and the dynamic housing auction model with auctions. Similarly, Genesove and Hansen (2014) compare the prices from negotiations and auctions in the dynamic setting both empirically and theoretically. This paper complements Genesove and Hansen (2014) paper in comparing the house prices from auctions and negotiations, but it explicitly considers the option value to buy and sell in the dynamic setting and incorporates search frictions to highlight importance of the ratio of buyers to sellers in intermediating large movements in transaction prices. Merlo, Ortalo-Magné, and Rust (2015) study the dynamic problem of the home seller who potentially can face multiple offers. In my paper I additionally model the bidding behavior of buyers as well as the process of price determination in an equilibrium model in the presence of search frictions. Han and Strange (2016) study the role of the asking price in a model where lower asking price attracts more buyers which produces bidding wars. In the model in the present study the seller chooses the reservation price, rather than the asking price. The reservation price differs conceptually from the asking price, since the seller never sells below the reservation price in the auction. In contrast, the seller can sell below the asking price in the data. However, lowering the reservation price has similar effect to that of Han and Strange (2016)'s of increasing the number of buyers who visit leading to bidding wars. Adams et al. (1992) study the choice of seller between the auctions and posted prices in a continuous-time search model. In their model the arrival of buyers is governed by the Poisson process with constant arrival rate. But, due to the continuous time, the probability of arrival of more than one buyer in their model is zero. Hence, the seller's optimal strategy reduces to choosing the posted price and waiting for the first buyer willing to accept it. In my paper the probability of the arrival of several buyers is positive, allowing the seller to run an auction with more than one buyer. Moreover, this probability depends on the ratio of buyers to sellers through the search frictions. During the housing booms the ratio of buyers to sellers increases making houses more liquid which boosts prices.

[10]For example, Stein (1995), Ortalo-Magné and Rady (2006).

[11]For example, Glaeser, Gyourko, and Saiz (2008), Bayer, Geissler, Mangum, and Roberts (2011), Haughwout, Peach, Sporn, and Tracy (2012), Case, Shiller, and Thompson (2014), Glaeser and Nathanson (2015), Chinco and Mayer (2016), Nathanson and Zwick (2016), Gao, Sockin, and Xiong (2016), Giacoletti and Westrupp (2017).

the time-series dispersion of prices. In retail markets, Einav, Farronato, Levin, and Sundaresan (2016) look at the choice of sellers between auctions and posted prices in online markets. In the asset markets, many papers study the information percolation and information assymetries in the dynamic over-the-counter markets with search (Duffie, Malamud, and Manso (2009), Glode and Opp (2016)) and double-auctions (Duffie, Malamud, and Manso (2014)), correspondingly. Hendershott and Madhavan (2015) study the choice between auctions and bilateral search in the OTC market. I add to this literature by comparing the quantitative performance of these two mechanisms within the search environment, albeit, in the housing market. The paper is also related to the theoretical literature on the choice of the selling institutions[12]. These papers usually compare auctions with sequential search. Here I compare auctions with very specific Nash bargaining, used in the housing search and matching literature.

In order to focus on the implications of the sales mechanism and make models tractable, I abstract from some features of the housing market. First, I assume that once a homeowner has moved into a house, she is never separated from the house. The possibility of resale and turnover of housing stock is potentially an important channel that is discussed, for example, in Anenberg and Bayer (2013), Head, Lloyd-Ellis, and Sun (2014), Piazzesi, Schneider, and Stroebel (2014), Moen, Nenov, and Sniekers (2015). Second, I do not model the mortgage market, which is extensively studied in the housing literature, see, for example, Favilukis, Ludvigson, and Nieuwerburgh (2015) and Landvoigt (2014). The interaction of the search frictions and credit constraints is explored in Guren and McQuade (2013) and Hedlund (2015).

3 Models

In this section I describe the models of the local housing market that I am comparing. The models have the same building blocks, except for the way the prices are determined.

3.1 Elements Common to Both Models

Time is discrete $t \in \{0, 1, ...\}$. There are infinitely-lived risk neutral agents, buyers and sellers, who discount future at the common fixed discount factor β, and use rational perfect foresight expectations. There are two goods in the economy, consumption, taken as numeraire, and housing; and two markets corresponding to these goods. The consumption is frictionless, while the housing market has search frictions.

By going through the search process a buyer can purchase one house that provides a flow of housing services x forever[13]. When the buyer searches for a house, she[14] visits the house and finds out the value of flow services x. A visit includes both viewing photos and information online as well as visiting a property. The value of flow services $x \geq 0$ is distributed independently over buyers and time[15] with the cumulative density function $F(.)$, probability distribution function $f(x) > 0$ with weakly increasing hazard rate $f(x)/(1 - F(x))$[16]. Buyers rent at exogenous rental

[12]For example, Wang (1993), Arnold and Lippman (1995), Bulow and Klemperer (2009).

[13]In the model there is no resale of house, and I impose the transversality conditions to eliminate bubbles.

[14]The buyers are referred to as "she", and sellers as "he" throughout the text.

[15]The case of affiliated values, potentially very important for owner-occupied housing, can be considered in future research after the basic model with independent values has been analyzed.

[16]Increasing hazard rate is sufficient for existence of solution to the seller's problem.

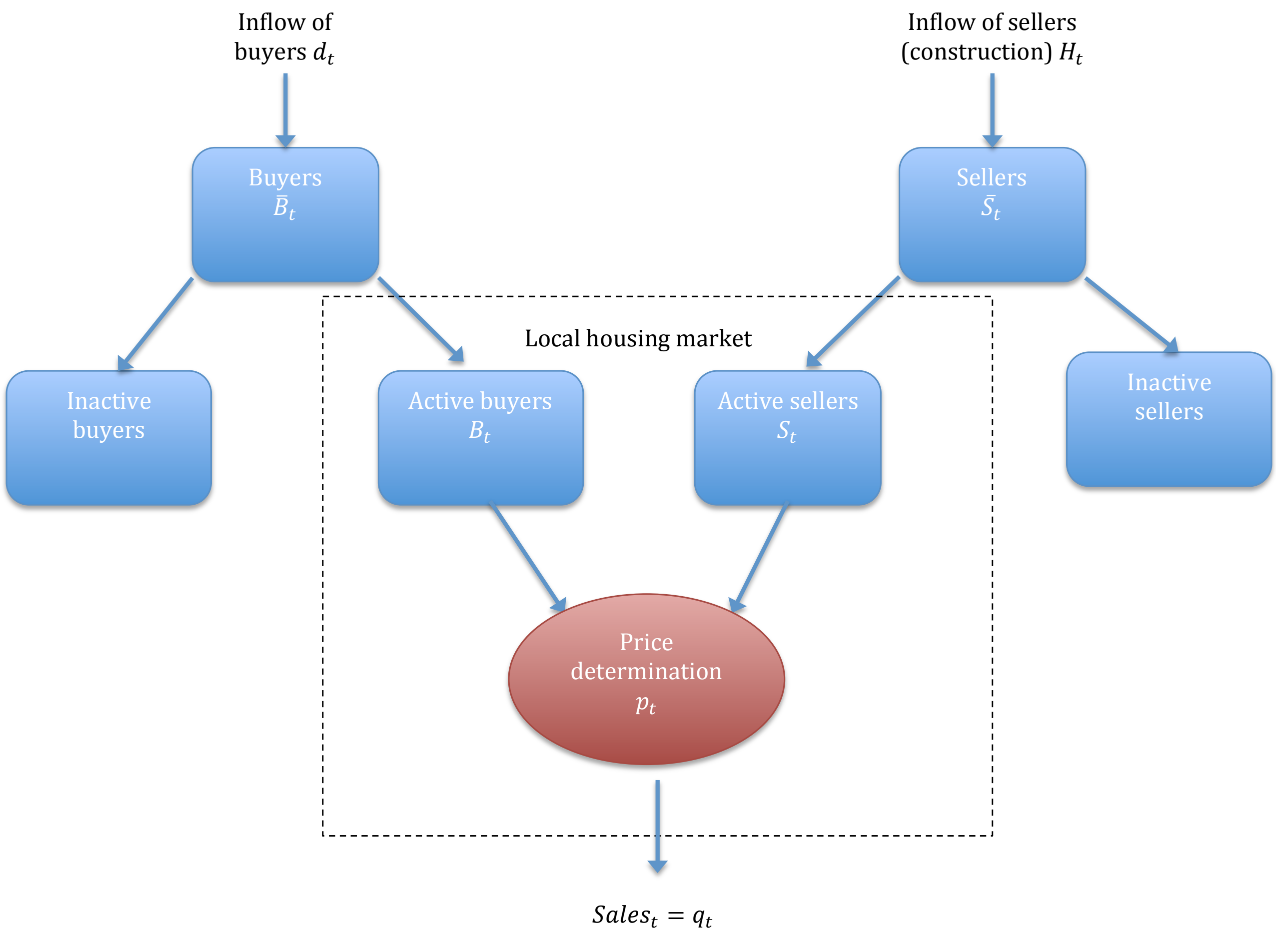
rate w until they buy and move in a house. The service flow from the rental housing is normalized to zero.

Both buyers and sellers decide whether to participate in the housing market. If they do, they have to pay fixed search cost, respectively, c^B and c^S per period[17]. This allows buyers and sellers to time their participation in the market. The new sellers enter endogenously, and have increasing marginal costs of entry. The supply is assumed to be endogenous, because it has been shown to be an important determinant of volatility of house prices, see Glaeser, Gyourko, and Saiz (2008), Saiz (2010). Following Glaeser, Gyourko, and Saiz (2008), the marginal costs of supplying h_t homes are assumed to be linear and increasing $MC(h_t) = c^h + \psi h_t$, where c^h, ψ are parameters. The new buyers enter the market at the exogenous rate d, for example, due to job relocation. The housing demand changes faster than the supply, so the driving force of fluctuations in the model are the demand shocks due to this exogenous influx of buyers.

Figure 3 shows how the models work in period t. Each period starts with $\bar{B}_t$ buyers and $\bar{S}_t$ sellers. They decide whether to actively search in the local housing market. If they decide to actively search, they pay search costs to participate, and become active buyers and sellers. The numbers of active buyers and active sellers are B_t and S_t, respectively. Active buyers and sellers search for each other in a local housing market, randomly meet, and determine the house price. The price determination mechanism is the only building block where the two models are different. I describe the search frictions and price setting in the auction models and Nash bargaining model in Section 3.2.

[17] In the calibration the buyer's search cost is zero. In the model it is nonzero for generality.

Figure 3: The models in period t



The search frictions and price determination influence the house prices[18] and sales $q_t = \pi_t S_t$[19] through the probability of sale π_t. The transacted buyers and sellers leave the market, and new buyers and sellers arrive. The number of new buyers is d_t and new sellers is h_t. I assume that the influx of buyers d_t is governed by $AR(1)$ process.

The transition of the state of the economy from period t to $t+1$ is summarized by three state variables: the number of buyers $\bar{B}_t$, sellers $\bar{S}_t$ and influx of buyers d_t. The dynamics of the state $\mathcal{S}_t = (\bar{B}_t, \bar{S}_t, d_t)$ is

$$\bar{B}_{t+1} = \bar{B}_t + d_t - q_t = \bar{B}_t + d_t - \pi_t S_t, \tag{1}$$
$$\bar{S}_{t+1} = \bar{S}_t + h_t - q_t = \bar{S}_t + h_t - \pi_t S_t, \tag{2}$$
$$d_t = \rho d_{t-1} + (1-\rho) d_0 + \varepsilon_t, \tag{3}$$

[18] The house prices p_t denotes the cross-section of transactions in period t

[19] The quantity sold q_t is the product of the number of active sellers and the probability of sale, i.e. $q_t = \pi_t S_t$, by the law of large numbers

where d_0 is the unconditional mean and ρ is the persistence parameter of the process for the influx of buyers d_t with shocks $\varepsilon_t \sim iidN(0, \sigma^2)$. In equation (1) the total number of buyers $\bar{B}_t$ increases by the influx of buyers d_t and decreases by the outflow, equal to the number of sales q_t. Similarly, equation (2) shows the dynamics of sellers, where the number of sellers increases through endogenous entry h_t of sellers and decreases by the number of sold homes q_t. In this system, the search frictions and price determination process affect the dynamics of buyers and sellers through the probability of sale π_t and sales q_t.

3.2 Search Frictions and Price Determination

In this section I discuss the microstructure of the local housing market, in particular, search frictions and price determination mechanism. In the benchmark housing search model the search is random, where each active buyer visits one randomly chosen active seller per period[20]. Then the number of active buyers that visit each individual active seller is distributed with Poisson distribution[21] with the mean that is equal to the ratio of active buyers to active sellers, $\theta_t = B_t/S_t$. This ratio is called tightness, with the interpretation that the tightness is high, the housing market is in boom, because there are many buyers per seller, and vice versa for the cold market.

In the benchmark housing search model, after the buyers arrive, the seller picks one buyer at random out of all buyers who visited him. The value x of the buyer becomes common knowledge, and then the price splits the joint surplus from the trade according to fixed weights equal to bargaining powers of the seller, $\alpha \in (0, 1)$, and the buyer, $1 - \alpha$.

I am proposing two versions of the auction model. In the first version the search is random, similarly to the benchmark Nash bargaining model. However, after buyers have visited a seller, he runs an ascending bid auction with the reservation price among all buyers who visited him. The seller does not know the home values of buyers, and chooses and commits to the reservation price before the auction. The auction starts at this price, and the price increases until only one buyer is left. The buyer pays the price at which the auction stops, which is the maximum of the second highest bid and reservation price.

In the second version of the auction model the search is directed. In the directed search

[20]The search process in the residential real estate market is asymmetric: a seller posts a house for sale, and buyers visit potentially suitable houses. The assumption about ability to visit one seller is not restrictive for the per period payoffs, since those are similar to the case when buyers are allowed to visit multiple sellers as in Albrecht, Gautier, and Vroman (2016). The question, model and results, however, differ from Albrecht, Gautier, and Vroman (2016), see Section 2.

[21]The Poisson approximation can be motivated in two ways. First, if each active buyer is picking a seller at random, then the number of buyers that visit a particular seller is distributed binomially. When the number of sellers goes to infinity $S \to \infty$, holding the ratio of number of active number to sellers $\theta = B/S$ constant, by the Poisson limit theorem the distribution of active buyers that visit a particular seller is Poisson with the mean equal to the ratio of active buyers to active sellers.

For the second motivation assume that homes are equally spaced across a given area of a local housing market. Suppose that buyers search randomly, so as to be located across the area according to the uniformly-intensity spatial Poisson distribution. Each buyer visits the home nearest to his or her location. Then in the limit, as the number of active buyers and sellers goes to infinity, the total number of buyers visiting each home would be approximately distributed by Poisson, iid across homes, with a mean equal to the market-wide ratio θ of the number of buyers per number of active sellers. See, for example, Badderley, Barany, Schneider, and Weil (2007). I thank Darrell Duffie for pointing this out.

model with auctions the seller posts the reservation price $\bar{p}_t$ that starts the ascending auction. Active buyers observe the posted reservation prices and decide which seller to visit. The model is a directed search model because all buyers observe all posted prices and direct their search to the sellers who post attractive reservation price and where they expect little competition from other buyers.

To summarize, the auction and Nash bargaining models differ in terms of the informational structure, the rule for selecting the winning buyer and the division of the surplus for buyer and seller. The next section provides an example to illustrate these differences.

Example

The model is dynamic, and both buyers and sellers can time their participation in the housing market which is summarized by their value functions. The value functions of buyer and seller in period t are V_t^B and V_t^S, respectively. These value functions are endogenously determined in the model. However, I take the value functions of the buyer V_{t+1}^B and seller V_{t+1}^S exogenously in this example to illustrate the differences between the Nash bargaining model with random search, auction model with random search and auction model with directed search. The value functions are endogenized in the next sections.

Table 1: The auction and Nash bargaining price determination examples

Buyers

Present value	100	200	300
Value to wait	100	100	100

Seller

Expected present value	100
Value to wait	50

Auction with random search: sales price 150

Optimal bid	0	100	200	Reservation price	150

Auction with directed search: sales price 100

Optimal bid	0	100	200	Reservation price	50

Nash bargaining with power 0.5: sales price 75

Joint surplus for average buyer = 50

Sale price = Value to wait for seller + 0.5 × Joint surplus = 75

Table 1 shows the example of how prices are determined using the Nash bargaining model and the auction models. In this example an active seller, who has paid search costs, meets a random number of active buyers. This seller is lucky, and he is visited by three active buyers. All buyers and seller have an option to postpone the transaction till the next period. The seller can sell tomorrow, and he values this option at $V_{t+1}^S = 50$. Similarly, buyers can wait till tomorrow which allows them to buy a house tomorrow but requires them to pay rent today, both of those are summarized in the buyer's value function V_{t+1}^B which is equal to 100 in this example.

When the buyers visit the house, they observe the benefits from living in this house x per period with the present value $PV_x = x/(1-\beta)$. To simplify calculations, assume the exponential distribution for x. The exponential distribution simplifies the expressions allowing to use the present value of housing services for the bidding and pricing. In this example, the present value of housing services are $PV_x^1 = 100$ for the first buyer, $PV_x^2 = 200$ for the second buyer, and $PV_x^3 = 300$ for the third buyer.

First, consider the auction price determination with the random search. The present values from the house in the auction models are private valuations of buyers. Before the auction the seller does not observe the realization of x, but he knows the distribution of x and can compute the expected present value of housing services, which is assumed to be $EPV_x = 100$. The seller decides on the optimal reservation price before the auction starts and commits to this price. Then it follows from Proposition 2 that the optimal reservation is $\bar{p}_t = V_{t+1}^S + EPV_x = 50 + 100 = 150$. All sellers are homogenous, so all sellers post the same reservation price. The buyers formally do not observe the reservation price, but they can compute the optimal reservation price by solving the seller's problem. Once the buyers observed their values, the auction starts from the reservation price, and price increases until only one buyer is left standing.

What is the auction price here? To answer this question, I need to know the optimal drop-out prices, or bids, of buyers. This is a standard ascending bid auction with the dominant strategy to bid the value of the object. The value of the house is $PV_x - V_{t+1}^B$, because if a buyer buys a house, he gets the present value of housing services, but loses the value function representing the value of waiting and buying a house later. Hence, the dominant strategy is drop-out of the auction when the price exceeds $b_t^*(x) = PV_x - V_{t+1}^B$.

The value function of the buyer appears in the bid because the buyer takes into account the option value to search in the next period, which shades the bid by their value function. This makes the bid endogenously rise during the booms. If market is currently in the boom, the next period it will be in the boom with high probability because of the search frictions. Hence, given the same realization of the present value of housing services PV_x, the bids will be higher during the boom because of the low option value to buy in the next period[22].

The optimal drop-out prices, or in other words, maximum bids, $b_t^*(x)$ for the three buyers are 0, 100, 200, respectively. The third buyer wins the auction, and she pays the price at which the auction stopped. Here, the drop-out prices of the first and second buyers are lower than the reservation price, so that the auction stops at the reservation price, so the house is sold for 150.

Now consider the auction price determination with the directed search. The process differs from the random search, described in the previous paragraphs, by the way that the seller determines the reservation price and how buyers arrive to the seller. The seller starts by posting the reservation price in the ascending bid auction. The buyers then observe all the reservation prices, and decide which seller to visit. Hence, if the seller hikes up the reservation price as compared to other sellers, he loses buyers. This drives the reservation price down the competitive level, i.e. $\bar{p}_t = V_{t+1}^S = 50$, eliminating the monopoly distortion in the auction model with random search, see discussion of the constrained social optimum allocation in Section 5. In this example, the auction starts at 50, and the second buyer drops out at 100. Hence, the third buyer is the winner, and she pays the second highest drop-out price, 100, which is the sales price in the auction model with directed search.

[22] The shading of the bids due to the option value of participating in the future auctions has been recently discussed in the context of the online auctions, see Zeithammer (2006), Ingster (2009), Said (2011), Said (2012), Backus and Lewis (2012), Hendricks, Onur, and Wiseman (2012), Hopenhyan and Saeedi (2015), Coey, Larsen, and Platt (2016)

Finally, consider the benchmark housing search model with Nash bargaining. In the Nash bargaining the seller selects one buyer at random to negotiate with. In our example, the seller can potentially choose any buyer with equal probability, so that the sales price reflects the average values. In the example, assume that the seller randomly picked the second buyer. Once the seller picks the buyer, the realization of the present value of housing services PV_x become common knowledge, and they compute the joint surplus from the trade to determine the sales price. The joint surplus from the trade is the difference of what they gain from the deal, which is the present value of housing services PV_x, and what they will lose from the deal, which is the buyer's and seller's continuation values, V^B_{t+1} and V^S_{t+1}. The joint surplus in our example is then $PV^2_x - V^B_{t+1} - V^S_{t+1} = 200 - 100 - 50 = 50$. This joint surplus is then split according the the bargaining weights of the seller, α, and the buyer, $(1-\alpha)$.

In this example I will use $\alpha = 0.5$, but in the calibration the bargaining power of the seller in the Nash bargaining model is calibrated to the ex-ante expected share of surplus of the seller in the auction model. With $\alpha = 0.5$, the Nash bargaining price is $\hat{p}_t = V^S_{t+1} + \alpha(PV^2_x - V^B_{t+1} - V^S_{t+1}) = 50 + 0.5 \times 50 = 75$, so the seller is compensated for his option value to sell and gets half of the joint surplus from the deal.

3.3 Price determination by Nash bargaining

In this section I close the Nash bargaining model by writing down explicitly the pricing mechanism and dynamics of the value functions of the buyer and the seller.

In the beginning of period t the problem of each buyer and seller is to decide whether to participate in the local housing market, and if they do, they have to pay search costs. Denote a^B_t and a^S_t the dummies for the buyer's and seller's participation so that a^i_t equals one if agent i participates in the local housing market in period t, and zero otherwise. After the participation decisions has been made, the Nash bargaining model prescribes how the active buyers and sellers meet and how the price is determined.

The first component of the Nash bargaining model is the meeting technology. In the models I assume that the number of active buyers that visit the seller is distributed by Poisson distribution with the mean, equal to the ratio of buyers to sellers $\theta_t = B_t/S_t$. In the Nash bargaining model the seller picks one buyer out of all who visited him at random, hence the probability of a meeting is $1-\exp(-\theta_t)$. To produce this meeting technology, I use the urn-ball meeting function $M(B_t, S_t) = S_t(1 - (1 - B_t/S_t)^{B_t})$, which gives the number of meetings M from the number of active buyers B_t and sellers S_t. This number of meetings occurs if buyers reach out to sellers and, if the seller gets more than one buyer, he selects the buyer at random. If the number of buyers and sellers is large, the meeting function can be well approximated by

$$M(B_t, S_t) = S_t(1 - \exp(-B_t/S_t)) = S_t(1 - \exp(-B_t/S_t))$$

The probability of meeting a buyer to a seller is $q^s(\theta_t) = 1 - \exp(-\theta_t)$, which is the same as the probability of a seller meeting at least one buyer in the auction model, which is $P(\text{seller meets exactly one buyer}) = 1 - \exp(-\theta_t)$. For the buyer, the probability of of meeting a seller in the Nash bargaining model is then $q^b(\theta_t) = q^s(\theta_t)/\theta_t = (1 - \exp(-\theta_t))/\theta_t$.

In the Nash bargaining model, the price is determined using the Nash bargaining solution where the seller and buyer meet and bargain over the price. Following the standard search model

Genesove and Han (2012), the transaction only occurs if the joint surplus $PV_x - V^B_{t+1} - V^S_{t+1}$ from the sale is positive. In the models I assume that the buyer and seller sign an agreement in period t, but the settlement, transfer of the house and payment occur in period $t+1$ to simplify notation. Because of this timing assumption, the present value of housing services and the value functions of buyer and seller tomorrow are of the same time period, $t+1$, in the expression for the joint surplus. The sale will occur only if the joint surplus is positive, that is if the realized value of the housing services x is higher than the threshold value, denoted $\bar{x}$, i.e. $x \geq \bar{x}_t = \max\{(1-\beta)(V^S_{t+1} + V^B_{t+1}), 0\}$. The threshold value $\bar{x}$ is the value of the housing services for the marginal buyer. The marginal buyer is a buyer who is just indifferent between buying or not buying this house, and this buyer prices the house.

Similarly, the probability of buying for the buyer is $\pi(\bar{x}_t, \theta_t)/\theta_t$, where $\pi(\bar{x}_t, \theta_t) = (1 - \exp(-\theta_t))(1 - F(\bar{x}_t))$ is the overall probability of sale. If the sale occurs, the transaction house price is then[23] $\hat{p}_t = V^S_{t+1} + \alpha(\frac{x}{1-\beta} - V^B_{t+1} - V^S_{t+1})$. If I take the expectation of the house prices $\hat{p}_t$ over all cross-section of transactions in a period, expected price can be computed as

$$p_t = E[\hat{p}_t|\text{Sale}] = E[\hat{p}_t|x \geq \bar{x}_t] = V^S_{t+1} + \alpha(\frac{E[x|x \geq \bar{x}_t]}{1-\beta} - V^B_{t+1} - V^S_{t+1}), \tag{4}$$

Given the search and price-setting process, I endogenize the value functions of buyer and sellers. Let V^B_t be the value function of the buyer in the beginning of the period before the decision of searching or not searching is made. The value function of the buyer is the option value to buy minus the expected present value of rent. In the equilibrium the value function of the buyer today V^B_t is the sum of the discounted value tomorrow βV^B_{t+1} and the value of participating in the local housing market today minus the rent w. The value of participating in the market is zero if the buyer sits out of the market, and is the expected surplus from buying a house net of search costs c^B, when the buyer participates. The ex-ante expected surplus in turn is $\frac{E[x|x \geq \bar{x}_t]}{1-\beta} - p_t - V^B_{t+1}$. The buyer gets this surplus with probability π_t/θ_t. A similar reasoning applies to the seller. The value function of the seller is just the option value to sell. The value function of the buyer is the option value to buy net of the present value of rent. Proposition 1 summarizes the dynamics of the buyer's and seller's value functions and the expected house price.

Proposition 1. In the equilibrium of the Nash bargaining model with random search the value function of the buyer V^B_t and seller V^S_t, the threshold match quality $\bar{x}_t$ and expected cross-section

[23] The buyer's surplus is $BS_t = \beta(\frac{x}{1-\beta} - V^B_{t+1} - \hat{p}_t)$, the seller's surplus is $SS_t = \beta(\hat{p}_t - V^S_{t+1})$, and the price maximizes the weighted product of the surpluses $SS^\alpha_t BS^{1-\alpha}_t$, the first order condition is $SS_t = \alpha(BS_t + SS_t)$ or $\hat{p}_t - V^S_{t+1} = \alpha(\frac{x}{1-\beta} - V^B_{t+1} - \hat{p}_t + \hat{p}_t - V^S_{t+1})$

prices p_t satisfy

$$V_t^B = \beta V_{t+1}^B - w + \max_{a_t^B \in \{0,1\}} (\beta(1-\alpha)\frac{\pi_t}{\theta_t}(\frac{E(x|x \geq \bar{x}_t)}{1-\beta} - V_{t+1}^B - V_{t+1}^S) - c^B)a_t^B, \tag{5}$$

$$V_t^S = \beta V_{t+1}^S + \max_{a_t^S \in \{0,1\}} (\beta\alpha\pi_t(\frac{E(x|x \geq \bar{x}_t)}{1-\beta} - V_{t+1}^B - V_{t+1}^S) - c^S)a_t^S, \tag{6}$$

$$\bar{x}_t = \max\{(1-\beta)(V_{t+1}^B + V_{t+1}^S), 0\}, \tag{7}$$

$$p_t = V_{t+1}^S + \alpha(\frac{E(x|x \geq \bar{x}_t)}{1-\beta} - V_{t+1}^B - V_{t+1}^S), \tag{8}$$

where $\pi_t = (1 - \exp(-\theta_t))(1 - F(\bar{x}_t))$ is the probability of sale.

Proof. See Appendix A.

3.4 Price determination by auctions with random search

This section explains how the price is determined in the auction model with random search and closes the model by endogenizing the value functions of the buyer and seller. The search process starts in the same way as in the Nash bargaining model, that is each buyer and seller decide whether to incur search costs and participate in the local housing market. If they do, they become active buyers and sellers. Then each active buyer visits one active seller. When a buyer visits a seller, she draws a match-specific value of housing services x. From this point on, the auction process differs from the Nash bargaining process. The value of the house for buyer is private information that is unobservable for the seller. Let N be a random variable representing the number of active buyers that have visited the seller. As mentioned earlier, $N \sim \text{Poisson}(\theta)$. If the seller has no visiting buyers, then $N = 0$, the seller keeps the house with an option to sell it the next period. If there is at least one buyer, $N \geq 1$, the seller runs the ascending bid auction with the reservation price $\bar{p}$. The seller chooses the reservation price before observing how many buyers will visit and commits to the chosen price[24] The buyer does not observe the reservation price before the auction. During the auction if all buyers dropped out at the reservation price, the seller keeps the house. Otherwise, the price increases until only one buyer is left. This buyer gets the house and pays the price at which the auction stopped.

In the auction model the problem of the buyer and seller is to decide whether to be active or not, similarly to the Nash bargaining model. However, in addition to this decision each buyer decides on the optimal drop-out price $b_t(x)$, and seller decides on the optimal reservation price $\bar{p}_t$. As it has been discussed in Section 3.2, the optimal drop-out price of the buyer with value x is $b_t(x) = \frac{x}{1-\beta} - V_{t+1}^B$.

The optimal reservation price $\bar{p}_t$ can be derived using the parallel between the seller's problem in the auction and the monopolist problem, as in Bulow and Roberts (1989). The optimal reservation price equalizes the marginal revenue from a buyer and the marginal costs of serving this buyer. The marginal costs of serving the buyer is foregoing the option value to sell tomorrow V_{t+1}^S. The marginal revenue from a buyer is the virtual value $v(b) = b - \frac{1-G(b)}{g(b)}$, where b is the

[24] If the seller cannot commit to the reservation price, then he has incentives to revise the price after an unsuccessful auction, driving the price down to the competitive level as in the auction model with directed search, see Section 3.5.

value of the object and $G(.)$ and $g(.)$ are the cdf and pdf of b, correspondingly. In this model the marginal revenue is $v_t(x) = \frac{x}{1-\beta} - V^B_{t+1} - \frac{1-F(x)}{(1-\beta)f(x)}$. The threshold value of housing services $\bar{x}_t$ that makes the buyer's drop-out price solves $v_t(\bar{x}_t) = \frac{\bar{x}_t}{1-\beta} - V^B_{t+1} - \frac{1-F(\bar{x}_t)}{(1-\beta)f(\bar{x}_t)} = V^S_{t+1}$. If the threshold value $\bar{x}_t$ that solved this equation is negative, it means that for any value $x \geq 0$ the seller should sell the house at the reservation price $b_t(0) = -V^B$ even if only one buyer with zero value shows up, hence $\bar{x}_t = \max\{\frac{1-F(\bar{x}_t)}{f(\bar{x}_t)} + (1-\beta)(V^B_{t+1} + V^S_{t+1}), 0\}$. For the exponential distribution of x from Section 3.2, the threshold value $\bar{x}_t = \max\{Ex + (1-\beta)(V^B_{t+1} + V^S_{t+1}), 0\}$. The corresponding optimal reservation price is $\bar{p}_t = b_t(\bar{x}_t) = \frac{\bar{x}_t}{1-\beta} - V^B_{t+1}$.

The marginal buyer in the auction model with random search has the value of housing services $\bar{x}_t = \max\{(1-\beta)(V^B_{t+1} + V^S_{t+1}) + \frac{1-F(\bar{x}_t)}{f(\bar{x}_t)}, 0\}$, which differs from the threshold value of housing services in the Nash bargaining model $\bar{x}_t = \max\{(1-\beta)(V^B_{t+1} + V^S_{t+1}), 0\}$. Given the same value functions of the buyer and seller, the marginal buyer values the house more in the auction model, which will be reflected in higher house prices. Another way to look at this difference is to compare the reservation prices of the seller in the auction and Nash bargaining models. In the auction model the reservation price is $\bar{p}_t = V^S_{t+1} + \frac{1-F(\bar{x}_t)}{(1-\beta)f(\bar{x}_t)}$, while in the Nash bargaining model it is V^S_{t+1}, since the seller is forced to sell the house as long as the surplus from the trade is positive even if it is optimal to wait. Higher reservation price in the auction model with random search reflects the monopoly behavior of the seller due to costly search of buyers[25] as well as the ability to commit to the reservation price. In a random search model, once a buyer has paid search costs and has visited a seller, the seller becomes a local monopolist. If the seller is forced to compete with other sellers by allowing buyers to direct their search to seller with certain reservation price, then the optimal reservation price will be V^S_{t+1}, see Section 3.5 for the auction model with directed search. If the seller could not commit to the reservation price, he has an incentive to sell as long as a buyer offers anything higher than his continuation value V^S_{t+1}.

Due to the dynamic nature of the model, the bids and reservation prices depend on the value functions of buyers and sellers, and are all jointly determined in the equilibrium. The value function of buyer today V^B_t must be equal to the discounted value function tomorrow βV^B_{t+1} net of rent w plus the maximum of her expected payoff from participating in the market and zero payoff from staying off the market. If the buyer with value of housing services x participates, the probability that she has the highest value is[26] $\exp(-\theta_t(1 - F(x)))$. The surplus from buying a house is the difference between the value of the object $b_t(x)$ and the marginal revenue $v_t(x)$[27].

[25] As in the Diamond's (1971) paradox despite the competition between sellers, the existence of the search costs of observing the price charged by the seller, the seller is able charge the monopoly price. The Coase (1972) conjecture that the monopolist selling a durable good to buyers over time has to set competitive price due to competition with his future self does not apply here, because the seller is facing a different pool of buyers every period, and buyers with high values cannot hang on to this particular seller until the seller changes its pricing. In the directed search model, buyers can costlessly observe all reservation prices before visiting the seller, and the seller has to set the reservation price at the competitive level avoiding the Diamond paradox.

[26] The probability that the buyer has the highest valuation is the expectation over the number of buyers who visited this seller N of the probability to be the highest bidder, $E_N F^{N-1}(x) = \exp(-\theta_t(1 - F(x)))$, where $N \sim \text{Poisson}(\theta_t)$ is the number of active buyers per active seller.

[27] See Bulow and Roberts (1989) and Bulow and Klemperer (2009). Bulow and Klemperer (2009) compare the efficiency and optimality of the auctions and sequential mechanism with endogenous seller entry which is similar to this paper. The search model with Nash bargaining is different from the sequential mechanism discussed in their paper, because the buyers are selected randomly upon entry rather than based on their bids (and hence house values). This paper quantifies the differences in the standard Nash bargaining model with search and the

Hence, the ex-ante expected surplus of the buyer is

$$\int_{\bar{x}_t}^{\infty}(b_t(x)-v_t(x))e^{-\theta_t(1-F(x)))}f(x)dx=\frac{1}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))e^{-\theta(1-F(x))}dx$$

For the exponential distribution, it simplifies to $\frac{\pi_t}{\theta_t}EPV_x$, where $\pi_t/\theta_t=(1-e^{-\theta_t(1-F(\bar{x}_t))})/\theta_t$ is the probability to buy a house.

The option value of the seller is determined similarly. The ex-ante expected surplus of the seller is the difference of the marginal revenue $v_t(x)$ and the marginal costs V_{t+1}^S for all the cases when the transaction happens $x\geq\bar{x}_t$ for each buyer, multiplied by the expected number of buyers:

$$\theta_t\int_{\bar{x}_t}^{\infty}(v_t(x)-V_{t+1}^S)e^{-\theta_t(1-F(x))}f(x)dx$$

The sales price in the auction[28] is either the reservation price $\bar{p}_t$ or the drop-out price, which is the second-highest bid $b_{(2)t}$,

$$\begin{aligned}p_t&=\frac{E_N[\bar{p}_tP(\text{Sale at }\bar{p}_t)+Eb_{(2)t}P(\text{Sale at }b_{(2)t})]}{P(\text{Sale})}=\\&=\bar{p}_t+\frac{1}{1-\beta}\frac{\int_{\bar{x}_t}^{\infty}(1-e^{-\theta_t(1-F(x))}-\theta_t(1-F(x))e^{-\theta_t(1-F(x))})dx}{\pi_t}\end{aligned}$$

where $\pi_t=1-e^{-\theta(1-F(\bar{x}_t))}$ is the probability of sale.

If the distribution of x is exponential and in the equilibrium the threshold value $\bar{x}_t>0$, then the expected revenue of the seller is $\varphi(\theta_t(1-F(\bar{x}_t))EPV_x$, where $\varphi(\theta_t(1-F(\bar{x}_t)))=\int_{\bar{x}_t}^{\infty}\pi(x,\theta_t)dx/Ex=\int_0^{\theta_t(1-F(\bar{x}_t))}\frac{1-e^{-t}}{t}dt$ where $\int_0^{z_t}\frac{1-e^{-t}}{t}dt$ is the Euler integral[29]. The function $\varphi(z)$ is increasing and concave in the adjusted tightness[30] z. The expected revenue of the seller depends on the adjusted tightness $z_t=\theta_t(1-F(\bar{x}_t))$ which is the ratio of "serious" buyers out of all active buyers per seller. These buyers are "serious" in the sense that they are willing to pay higher than the reservation price $\bar{p}_t$, because their value x is higher than the threshold value $\bar{x}_t$ of the marginal buyer. The adjusted tightness z_t, as compared to the tightness θ_t, takes into account that the house values are heterogeneous and not each buyer-seller pair is a good match. Similarly, the expression $e^{-\theta(1-F(\bar{x}_t))}$ is the probability that there are zero serious buyers who showed up at the open house, and hence $\pi_t=1-e^{-\theta(1-F(\bar{x}_t))}$ is the probability of sale. Given the revenue function $\varphi(.)$, the expected price for the exponential distribution simplifies to

$$p_t=V_{t+1}^S+\frac{\varphi(\theta_t(1-F(\bar{x}_t)))}{\pi_t}EPV_x \tag{9}$$

auction models with search.

[28] The proof for the price equation is proved in Appendix A.

[29] The general formula for the expected house prices for the exponentially distributed x is $p=V^S+\frac{\varphi(\theta(1-F(\bar{x}))}{\pi(\bar{x},\theta)}EPV_x+\frac{1}{1-\beta}(\bar{x}-Ex-(1-\beta)(V^B+V^S))$ where the last term drops out as long as $\bar{x}>0$ in an equilibrium of the auction model with random search.

[30] See Figure 9a in Appendix B.

where the ratio $\varphi(z_t)/\pi(z_t)$ is increasing in the adjusted tightness z_t, see Figure 9b in Appendix B.

Proposition 2 summarizes the dynamics of the value functions of buyers and sellers and the price-setting equation and the threshold value $\bar{x}_t$ for the auction model with random search.

Proposition 2. In the equilibrium of the auction model with random search the value function of the buyer V_t^B and seller V_t^S, the threshold match quality $\bar{x}_t$ and expected cross-section prices p_t satisfy

$$
\begin{aligned}
V_t^B &= \beta V_{t+1}^B - w + \max_{a_t^B \in \{0,1\}} \Big(\frac{\beta}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))e^{-\theta_t(1-F(x))}dx - c^B\Big)a_t^B \\
V_t^S &= \beta V_{t+1}^S + \max_{a_t^S \in \{0,1\}} \Big(\frac{\beta\theta_t}{1-\beta}\int_{\bar{x}_t}^{\infty}\Big(x - \frac{1-F(x)}{f(x)} - (1-\beta)(V_{t+1}^B + V_{t+1}^S)\Big)e^{-\theta_t(1-F(x))}f(x)dx - c^S\Big)a_t^S \\
\bar{x}_t &= \max\Big\{\frac{1}{\lambda(\bar{x}_t)} + (1-\beta)(V_{t+1}^B + V_{t+1}^S), 0\Big\} \\
p_t &= \frac{\bar{x}_t}{1-\beta} - V_{t+1}^B + \frac{1}{1-\beta}\frac{\int_{\bar{x}_t}^{\infty}(1-e^{-\theta_t(1-F(x))} - \theta_t(1-F(x))e^{-\theta_t(1-F(x))})dx}{\pi_t} = \\
&= V_{t+1}^S + \frac{\theta_t}{(1-\beta)\pi_t}\int_{\bar{x}_t}^{\infty}\Big(x - \frac{1-F(x)}{f(x)} - (1-\beta)(V_{t+1}^B + V_{t+1}^S)\Big)e^{-\theta_t(1-F(x))}f(x)dx
\end{aligned}
$$

where $\theta_t = B_t/S_t$ - tightness, $\lambda(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of distribution of values $F(x)$ and $\pi_t = 1 - e^{-\theta_t(1-F(\bar{x}_t))}$ is the probability of selling a house.

Before I define a symmetric dominant strategy perfect foresight equilibrium in the auction model with random search, to close the model the influx of sellers continues until the marginal costs of supplying homes equalizes with the marginal benefit of supplying homes, V_{t+1}^S. The same is done for the other two models.

Definition 1. *For given values of the initial state*[31]*, $\mathcal{S}_0 = (\bar{B}_0, \bar{S}_0, d_0)$, a discrete-time perfect foresight stationary equilibrium is a set of time-invariant value functions $V_t^B = V^B(\mathcal{S}_t)$ for a buyer and $V_t^S = V^S(\mathcal{S}_t)$ for a seller, and a set of policy functions $b_t(x) = b(x, \mathcal{S}_t)$ for a buyer and $\bar{p}_t = \bar{p}(\mathcal{S}_t)$ for a seller, an influx of sellers $h(\mathcal{S}_t)$ with the linear increasing marginal costs $MC(h_t)$ and a law of motion $\mathcal{S}_{t+1} = \Gamma(\mathcal{S}_t)$ such that*

1. *the value functions V_t^B, V_t^S satisfy the Bellman equations,*
2. *buyers follow the weakly dominant strategy $b_t(x)$,*
3. *sellers choose the optimal reservation price $\bar{p}_t$,*
4. *free entry for sellers $MC(h_t) = \beta V_{t+1}^S$ hold,*

[31] See the end of Section 3.1 for the description of the state of the economy.

5. *the law of motion for the state is consistent with the individual behavior:*

$$\bar{B}_{t+1} = \bar{B}_t + d_t - \pi_t S_t,$$
$$\bar{S}_{t+1} = \bar{S}_t + h_t - \pi_t S_t,$$
$$d_t = \rho d_{t-1} + (1-\rho) d_0 + \varepsilon_t,$$

6. *Transversality conditions hold:* $lim_{t\to\infty}\beta^t V_t^B$ *and* $lim_{t\to\infty}\beta^t V_t^S$ *are finite*

The definition of an equilibrium in the Nash bargaining model is similar, but drops the requirement on the buyers choosing the optimal bidding strategy and sellers setting the optimal reservation price. The definition of an equilibrium the auction model with directed search requires the buyers to optimally select the seller to visit.

3.5 Price determination by auction with directed search

In this section I consider the model in which the home sales prices are set in the process of directed search with auction. In the auction model with directed search each active seller posts a reservation price $\bar{p}_t$ that starts the ascending bid auction. Active buyers observe the posted reservation prices of all sellers and decide which seller to visit.

Buyer's problem

Consider a buyer, who has an opportunity to search tomorrow which gives him expected utility V_{t+1}^B. She decides whether to be active or not in the beginning of period t. If the buyer decides to be active by paying search costs, she can visit one submarket with the reservation price $\bar{p}(\bar{x}_t)$. In each submarket with the reservation price $\bar{p}_t = \bar{p}(\bar{x}_t)$, or equivalently with the threshold value $\bar{x}_t$, the ratio of active buyers per active sellers is $\theta(\bar{x}_t)$. Then the surplus of the buyer in a submarket $\bar{x}_t$ is $\frac{\beta}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))e^{-\theta_t(1-F(x))}dx$, see Section 3.4. Once a buyer visits the seller, she gets a realization of the value of housing services x, and participates in the auction. When a buyer searches for a seller, in equilibrium she is indifferent between a submarket $\bar{x}_1$ and $\bar{x}_2$ if the expected surplus from buying a house is the same in these two submarkets, i.e.

$$\int_{\bar{x}_{1t}}^{\infty}(1-F(x))e^{-\theta_t(1-F(x))}dx = \int_{\bar{x}_{2t}}^{\infty}(1-F(x))e^{-\theta_t(1-F(x))}dx \tag{10}$$

where equation (10) implicitly defines the equilibrium tightness $\theta(\bar{x}_t)$ with the slope

$$\theta'(\bar{x}_t) = -\frac{(1-F_t)(1-\pi_t)}{\int_{\bar{x}_t}^{\infty}(1-F(x))^2 e^{-\theta_t(1-F(x))}dx} \tag{11}$$

where the probability of sale is $\pi_t = \pi(\bar{x}_t, \theta(\bar{x}_t))$, $F_t = F(\bar{x}_t)$. The interaction of the buyer and the seller after the meeting occurred is the same as in the random search model of Section 3.4.

Seller's problem

An active seller has an option value to sell tomorrow V^S_{t+1}, and he is deciding on the optimal reservation price $\bar{p}_t$ to post. Each reservation price $\bar{p}_t = \bar{p}(\bar{x}_t) = b^*(\bar{x}_t) = \frac{\bar{x}_t}{1-\beta} - V^B_{t+1}$ corresponds to the threshold value of housing services $\bar{x}_t$, so we can think of the seller choosing the threshold $\bar{x}_t$ directly. For each $\bar{x}_t$, the number of active buyers that he can expect to visit is distributed by Poisson with expectation $\theta(\bar{x}_t)$. It can be shown that the expected payoff of the seller from the auction is

$$e^{-\theta(\bar{x}_t)(1-F(\bar{x}_t))}V^S_{t+1} + (1 - e^{-\theta(\bar{x}_t)(1-F(\bar{x}_t))})(\frac{\bar{x}}{1-\beta} - V^B_{t+1}) +$$
$$+ \frac{1}{1-\beta}\int_{\bar{x}_t}^{\infty}(1 - e^{-\theta(\bar{x}_t)(1-F(x))} - (1-F(x))\theta(\bar{x}_t)e^{-\theta(\bar{x}_t)(1-F(x))}dx$$

and maximizing this with respect to $\bar{x}_t \geq 0$ and rearranging gives the first-order condition

$$-\frac{d\pi_t}{d\bar{x}_t}V^S_{t+1} + \frac{d\pi_t}{d\bar{x}_t}\bar{p}_t + \frac{1}{1-\beta}\pi_t + \frac{1}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))^2(1-\pi(x,\theta(\bar{x}_t)))\theta(\bar{x}_t)\theta'(\bar{x}_t)dx = 0$$

Using the slope of the tightness $\theta'(\bar{x}_t)$ from the buyer's indifference condition (11), gives

$$-\frac{d\pi_t}{d\bar{x}_t}V^S_{t+1} + \frac{d\pi_t}{d\bar{x}_t}\bar{p}_t + \frac{1}{1-\beta}\pi_t +$$
$$\frac{1}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))^2(1-\pi(x,\theta(\bar{x}_t))\theta(\bar{x}_t)\frac{(1-F(\bar{x}_t))(1-\pi(\bar{x}_t,\theta(\bar{x}_t))))}{\int_{\bar{x}_t}^{\infty}(1-F(x))^2(1-\pi(x,\theta(\bar{x}_t)))dx}dx = 0$$

which simplifies to $\bar{p}_t = V^S_{t+1}$ or $\bar{x}_t = \max\{(1-\beta)(V^S_{t+1} + V^B_{t+1}), 0\}$. The seller sets the reservation price at the competitive level due to the endogenous search of buyers. Otherwise, the Bellman equations for the value functions of buyers and sellers as well as the expected house price are the same to the auction model in the random search, and are summarized in Proposition 3.

Proposition 3. In the equilibrium of the auction model with directed search the value functions of the buyer V^B_t and seller V^S_t, the threshold match quality $\bar{x}_t$ and expected cross-section prices p_t satisfy

$$V^B_t = \beta V^B_{t+1} - w + \max_{a^B_t \in \{0,1\}}(\frac{\beta}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))e^{-\theta_t(1-F(x))}dx - c^B)a^B_t$$
$$V^S_t = \beta V^S_{t+1} + \max_{a^S_t \in \{0,1\}}(\frac{\beta\theta_t}{1-\beta}\int_{\bar{x}_t}^{\infty}(x - \frac{1-F(x)}{f(x)} - (1-\beta)(V^B_{t+1} + V^S_{t+1}))e^{-\theta_t(1-F(x))}f(x)dx - c^S)a^S_t$$
$$\bar{x}_t = \max\{(1-\beta)(V^B_{t+1} + V^S_{t+1}), 0\}$$
$$p_t = \frac{\bar{x}_t}{1-\beta} - V^B_{t+1} + \frac{1}{1-\beta}\frac{\int_{\bar{x}_t}^{\infty}(1 - e^{-\theta_t(1-F(x))} - \theta_t(1-F(x))e^{-\theta_t(1-F(x))})dx}{\pi_t} =$$
$$= V^S_{t+1} + \frac{\theta_t}{(1-\beta)\pi_t}\int_{\bar{x}_t}^{\infty}(x - \frac{1-F(x)}{f(x)} - (1-\beta)(V^B_{t+1} + V^S_{t+1}))e^{-\theta_t(1-F(x))}f(x)dx$$

where $\theta_t = B_t/S_t$ - tightness, $\lambda(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of distribution of values $F(x)$ and $\pi_t = 1 - e^{-\theta_t(1-F(\bar{x}_t))}$ is the probability of selling a house.

The properties of the equilibrium of the auction models with random and directed search, however, are different due to the nature of search. The seller in the auction model with directed search is competing with other sellers for interested buyers by luring them with lower reservation price, or in other words, by lower $\bar{x}_t$. For example, for the exponential distribution of housing services x and $\bar{x}_t > 0$, the expected prices are

$$p_t = V^S_{t+1} + \frac{\varphi(\theta_t(1 - F(\bar{x}_t))) - \pi_t}{\pi_t} EPV_x$$

As compared to the auction model with random search in equation (9), is the addition $-\pi_t$ term in the numerator of second term. This term appears because of the competition between sellers. Because of the competition between seller, the equilibrium allocation of the auction model with the directed search solves the problem of the social planner constrained by the search frictions, see Section 5.

In the next Section I compare these models and provide the intuition on why the auction models, with random and directed search, produce higher volatility than the Nash bargaining model.

4 Comparison of the Nash bargaining and auction models

In the previous section I have discussed the dynamic search models of the local housing market, the Nash bargaining model, prevalent in the housing literature, and the auction models, that I am proposing. This section compares the equilibrium outcomes of these models. First, in Sections 4.1 and 4.2 I illustrate how the main housing market variables are determined and differ in the steady-states of the models if I use the same parameter values for these models and the housing services are distributed exponentially. I find that the house prices are higher, there are more houses on the market, and the houses stay longer in the Nash bargaining model as compared to the auction model with random search.

Second, in Section 4.3 I illustrate how the shocks to the influx of buyer can be amplified in the auction model. Lastly, in Section 4.4 I calibrate the models to match the moments from the data to the model, solve them numerically and compare their quantitative performance. The baseline numerical solution is done for the exponential distribution of the housing services x, which has a constant hazard rate, and redone for another distribution with increasing hazard rate to provide a robustness check. In Section 4.5 I demonstrate that the house price growth is more volatile in the auction models than in the Nash bargaining model, and are closer to the volatility of the house prices observed in the data.

4.1 The steady state equilibrium

In order to illustrate the determination of the steady state equilibrium, I consider the auction model with directed search[32] and assume that $MC(d)/\beta > w/(1-\beta)$ and $c_b = 0$. These

[32]In the auction model with random search the free entry condition is the same as in the auction model with directed search, and the price setting condition behaves similarly to the price setting condition in the auction model with directed search, because the inverse of the hazard rate $(1 - F(x))/f(x)$ is decreasing in $\bar{x}$. The illustration for the Nash bargaining is standard, see Pissarides (2000)

restrictions on the parameters hold in the calibration, and guarantee that the the housing market is active with $\bar{x} > 0$, $V^S > 0$ and $V^B > -w/(1-\beta)$. The system of equilibrium equations reduces to the system in $(\bar{x}, \theta)$:

$$(1-\beta)MC(d)/\beta = \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(\pi(x,\theta) - \theta(1-F(x))(1-\pi(x,\theta)))dx - c^S \tag{FE}$$

$$\bar{x} = (1-\beta)MC(d)/\beta - w + \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(1-F(x))e^{-\theta(1-F(x))}dx \tag{PS}$$

where the equation (FE) is the requirement on the option value to sell to be equal to the marginal cost of entering the market for the seller and is called the Free Entry condition (FE), and the equation (PS) is the equation on the optimal threshold value of the seller $\bar{x} = \max\{(1-\beta)(V^S_{t+1} + V^B_{t+1}), 0\}$, called the Price Setting (PS) equation.

Figure 4: The free entry and price setting conditions for the steady state of the auction model with directed search

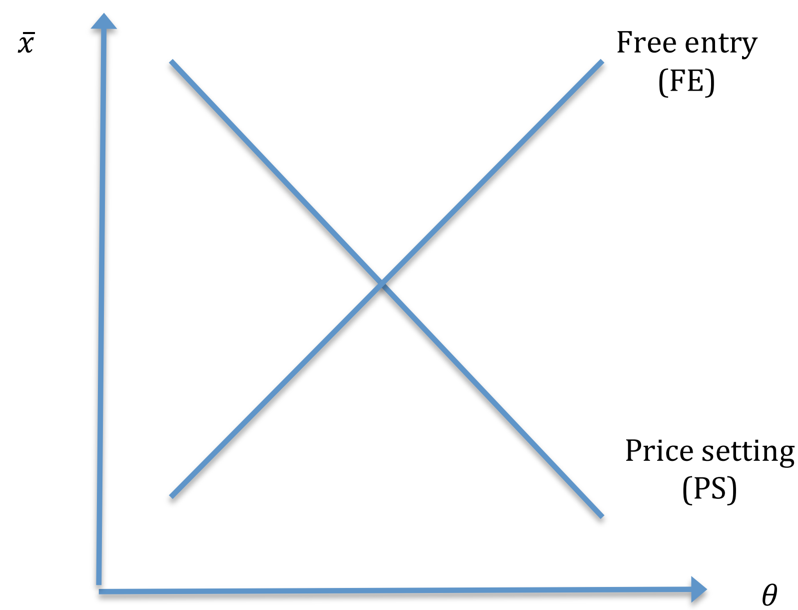


Figure 4 show the free entry and price setting conditions in the $(\bar{x}, \theta)$ axis. The free entry condition is increasing, because for higher levels of tightness θ, the threshold value of housing services $\bar{x}$ has to be higher to decrease the probability of sale which keeps the option value to sell tied to the marginal costs of entry. The price setting is decreasing because for higher levels of tightness θ, the threshold value of housing services has to be lower to keep the buyers interested in buying at the low price despite low probability of buying. Appendix A.3 proves that the positive slope of (FE) and negative slope of (PS). The curvature of the lines depends on the distribution. For example, for the exponential distribution the FE line is concave and the PS line is concave for small levels of tightness θ and convex for high levels of tightness θ[33], so Figure 4 plots these lines schematically around the steady state. When the influx of buyers d rises, both lines shifts to the right which increases the steady state tightness. The effect on the steady

[33]The steady-state is on the concave part of this curve for the calibrated parameter values.

state threshold $\bar{x}$ depends on the relative magnitude of these shifts. The transitional dynamics of the system is addressed in Section 4.5 where I show that the auction models produce higher volatility than the standard Nash bargaining model.

4.2 Comparison of the steady states

To illustrate how the mechanisms can produce different predictions, I consider the case when the housing services x are distributed exponentially, and $MC(d)/\beta > \frac{w}{1-\beta}$ holds[34]. The second condition intuitively means that the marginal costs of entering for sellers are higher than the rent saving of the buyers in the steady state, which guarantees that the seller choices non-trivial (non-zero) value of the threshold housing services $\bar{x}$.

Proposition 4 (Comparison of the steady-states in the Nash bargaining and auction model with random search)**.** *If $MC(d)/\beta > w/(1-\beta)$, the housing services x are distributed exponentially and the parameters are the same across models, then in the auction model with random search as compared to the Nash bargaining model*

1. *the expected prices p are higher,*
2. *the number of houses for sale S is greater,*
3. *the probability of house sale π is lower and the time on the market for sellers T^S is higher,*
4. *the sales q, the value function of a seller V^S and the entry of sellers h are the same.*

Proof. See Appendix A. □

The result on the higher house prices in the auction model with random search as compared to the Nash bargaining model is intuitive, and is similar to the static result in Bulow and Klemperer (1996). The seller acts as a monopolist and maximizes the expected revenue by manipulating the reservation price that influences the final sales price. Because the seller prices the houses so high, he has to stay on the market for longer and enjoy lower probability of sale similarly to how the monopolist hikes up the prices by decreasing the production. The pool of houses for sale is bigger since the sellers are sitting on the market and waiting for the high value buyer to show up. The sales, the value function of the seller and the entry of sellers because the number of buyers and sellers have to be stabilized by selling exactly as many houses as the number of buyers or sellers enters the market. In Section 5 I consider the social planner problem constrained by the same search frictions as in the auction models, and find that the auction model with random search is inefficient due to the local monopoly of the seller.

Proposition 5 (Comparison of the steady-states in the auction model with random search and the auction model with directed search)**.** *If $MC(d)/\beta > w/(1-\beta)$, the housing services x are distributed exponentially and the parameters are the same across models, then in the auction model with random search as compared to the auction model with directed search*

1. *the adjusted tightness $z = \theta(1 - F(\bar{x}))$ is lower*

[34]This condition is satisfied for all models for the calibrated parameter values in Section 4.4

2. the probability of sale π is lower

3. the threshold match-specific value $\bar{x}$ is higher

4. the number of houses on the market S is bigger

5. the sales q, the value function of a buyer V^S, the influx of sellers h are the same

Proof. See Appendix A

□

4.3 Amplification of shocks in the auction models

The intuition of the amplification of shocks in the auction models as compared to the Nash bargaining models can be gained from solving the arbitrage equations for the price forward for each model taking the expected path of the probability of sale $\{\pi_t\}_{t=0}^{\infty}$. To illustrate how expected prices depend on the probability of sale, assume that the value of housing services x is distributed exponentially, $MC(h_t)/\beta > w/(1-\beta)$, and that the seller actively searches in the market and has positive probability of sale and positive expected surplus from being active. Then the price in the Nash bargaining is given by

$$p_t = V_{t+1}^S + \alpha EPV_x, \tag{12}$$

where the option value to sell V_{t+1}^S evolves according to

$$V_{t+1}^S = \beta V_{t+2}^S + \alpha\beta\pi_{t+1} EPV_x - c^S. \tag{13}$$

Solving equation (13) forward gives the solution for the price in the Nash bargaining

$$p_t = \alpha EPV_x \sum_{i=1}^{\infty} \beta^i \pi_{t+i} - \frac{c^S}{1-\beta} + \alpha EPV_x. \tag{NB}$$

In the auction model with random search the price setting and the dynamics of the option value to sell are

$$\begin{aligned} p_t &= V_{t+1}^S + \frac{\varphi_t}{\pi_t} EPV_x, \\ V_{t+1}^S &= \beta V_{t+2}^S + \beta\varphi_{t+1} EPV_x - c^S, \end{aligned}$$

where

$$\begin{aligned} \varphi_t &= \int_0^{-\log(1-\pi_t)} \frac{1-e^{-t}}{t} dt \\ \varphi_t(-\log(1-\pi_t)) &= \int_0^{-\log(1-\pi_t)} \frac{1-e^{-t}}{t} dt \end{aligned}$$

is an increasing convex function in π_t[35].

[35] See Lemma 3 in Appendix A.

The forward solution for the prices is

$$p_t = \sum_{i=1}^{\infty} \beta^i \varphi_{t+i} EPV_x - \frac{c^S}{1-\beta} + \frac{\varphi_t}{\pi_t} EPV_x. \tag{RA}$$

Similarly, for the auction model with directed search

$$p_t = V_{t+1}^S + \frac{\varphi_t - \pi_t}{\pi_t} EPV_x, \tag{14}$$

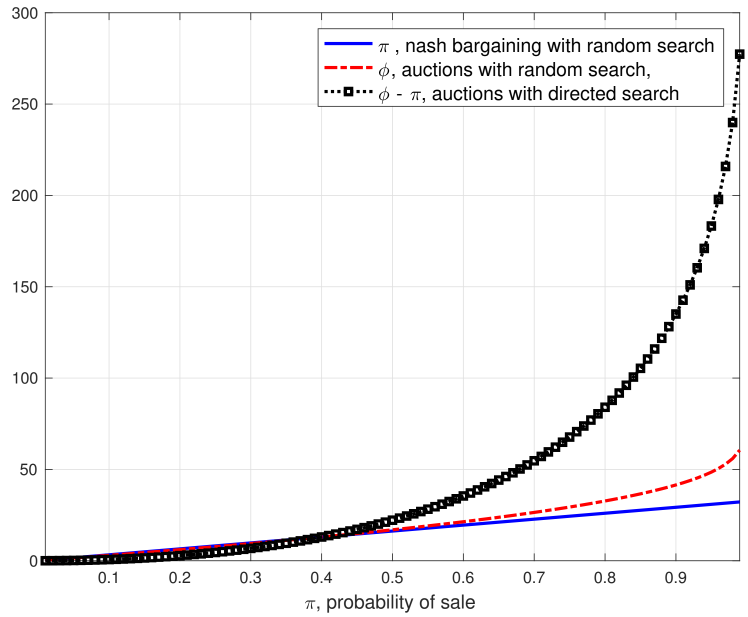
$$V_{t+1}^S = \beta V_{t+2}^S + \beta(\varphi_{t+1} - \pi_{t+1}) EPV_x, \tag{15}$$

and the forward solution for the prices is

$$p_t = \sum_{i=1}^{\infty} \beta^i (\varphi_{t+i} - \pi_{t+i}) EPV_x - \frac{c^S}{1-\beta} + \frac{\varphi_t - \pi_t}{\pi_t} EPV_x, \tag{DA}$$

The dependence of the house prices on the expected probability of sale is similar to the dependence of the stock prices on the expected dividends. The change in the expected probability of sale induces change in the house prices. The magnitude of this influence is determined by the slope $\partial p_t / \partial \pi_{t+i}$ from the price setting mechanisms in (NB), (RA), (DA). Figure 5 shows one representative term from (NB), (RA), (DA) using the parameters from the calibration in Section 4.4 for all parameters, except the bargaining power of the seller α. The bargaining power of the seller in the Nash bargaining model is set to one, $\alpha = 1$, to show the highest possible slope for that model. Specifically, the plot shows $\alpha EPV_x \beta \pi_{t+1}$ with $\alpha = 1$ for the Nash bargaining, $\beta EPV_x \varphi(-\log(1-\pi_{t+1}))$ for the auctions with random search and $\beta EPV_x(\varphi(-\log(1-\pi_{t+1})) - \pi_{t+1})$. These terms represent the influence of π_{t+i} on p_t for $i \geq 1$ (case $i = 0$ is considered in a paragraph). The calibration fits the time on the market $1/\pi$ for the seller to 2.5 month, making the probability of sale $\pi = 1/2.5 = 0.4$ in the steady state for all models, so that all the graphs intersect in the steady state.

Figure 5: Dependance of the expected prices on the probabilities of sale for the Nash bargaining model with random search from equation (NB), auction model with random search from (RA) and auction model with directed search from (DA)



The prices increase linearly with the probability of sale for the Nash bargaining model and similarly to the exponent in the auction model. Because buyers can get the extreme realizations of the house values, and the prices reflect these extreme values in the sales prices, the house prices can shoot up significantly in the auction models as opposed to the Nash bargaining. In the Nash bargaining the increase of prices is limited even during the housing booms when the houses sell. Hence, based on this graph, given the volatility of the probability of sale, the volatility of the prices is expected to be the highest in the auction model with directed search, and higher in the auction model with random search than in the Nash bargaining model, which is shown in the Section 4.5 based on the numerical simulations.

The comparison for $i = 0$ is similar, since the Nash bargaining prices are flat with respect to π_t. The slope of the prices with respect to the probability of sale in the auction model with random search and directed search depends on the slopes $\varphi(-\log(1-\pi_t))/\pi_t$ and $(\varphi(-\log(1-\pi_t)) - \pi_t)/\pi_t$. Both of these functions are increasing and convex in π, see lemma 4 in Appendix A, and graphically look similar to the $\varphi(-\log(1-\pi_t))$ and $\varphi(-\log(1-\pi_t)) - \pi_t$, depicted in Figure 5. So the comparison of the models is the same for $i = 0$ as for $i \geq 1$, i.e. the auction model with directed search is the most sensitive to the changes in the probability of sale and the Nash bargaining model is the least sensitive.

4.4 Calibration

In this section I calibrate the three models from Section 3 by matching the moments from the model and the data, and solve these models to study the fluctuations of house prices over time.

The strategy of the moments matching calibration is the following. First, set the basic parameters, i.e. the discount factor β, the influx of buyers d and the rent w, and the search costs c^B, c^S, to be equal to the same basic values for all models. Then choose the remaining parameters, i.e. parameters of the marginal costs of entry of sellers c^h, ψ, the parameterization of the distribution for the steady state of each the models separately to fit the housing market statistics. The Nash bargaining model has one additional parameter, which is the bargaining power of a seller α. It is set to the expected share of surplus of seller in the auction model with random search. The data moments are computed from the house prices in the Los Angeles MSA. The level of analysis is set to the MSA level, since the MSA is a natural housing market because residents commute inside MSA. Moreover, the house prices swing between booms and busts and are volatile in the Los Angeles MSA is used as an example to calibrate the values of rents, prices, sales and time on the market for buyers and sellers.

The values for calibrated parameters are shown in Table 2. The period in the model is taken to be a month. The discount factor β is set to produce the discount rate of 6% annually. The calibrated rent w and influx of buyers d equals the mean of monthly seasonally adjusted real rent and sales for 2010:M2-2015:M5[36], correspondingly, from Zillow.com in Los Angeles MSA. In the steady state the influx of buyers d has to be equal to the sales to keep the number of buyers constant, so the average sales are used to calibrate the influx of buyers. The buyer's search costs c^B are calibrated to zero, since the buyer's search costs are negligible as compared to the sellers costs of marketing and putting the house on sale. The seller's search costs c^S are calibrated to 6% of the sales price which represent the typical commission and closing costs.

To parametrize the distribution, I consider the exponential cdf $F(x) = 1 - e^{-\mu x}$ as the leading example, where μ is the inverse of the expectation of x. The exponential distribution has constant hazard rate. Since the distribution has to have weakly increasing hazard rate and positive support $x \geq 0$, some other standard distributions, for example, Pareto, normal, uniform[37] cannot be used for the robustness check. I use a slightly modified cdf $F(x) = (1 - e^{-\mu x})^2$ that satisfies the increasing hazard rate condition as the robustness exercise. The parameter μ is calibrated together with other parameters to match the data moments.

The parameter of distribution μ, the parameters of the marginal costs of entry of sellers c^h, ψ are jointly calibrated to match mean prices the $p = \$453,580$, the time on the market for sellers $T^S = 2.5$, the elasticity of housing supply $\varepsilon_p^{H^S} = 0.63$ (Saiz (2010)). The time on the market for seller moment is based on the median time on the market from the National Association of Realtors. The autocorrelation and standard deviation of the influx of buyers d are set to reproduce the volatility of the house sales in the data $\hat{\sigma}^{\text{Data}}_{dlogSales}$.

[36]The longest available series for sales. Longer series are available for the rent and price, but the series are picked to represent the same period.

[37]The Pareto distribution has decreasing hazard rate. Positive support is used in proofs where Fubini theorem is used, and the normal and uniform distributions do not have positive support.

Table 2: Moments-matching calibration

Parameter	Symbol	RA	DA	NB	Moment
discount factor, per annum	β	0.94	0.94	0.94	6% return
rent	w	2076	2076	2076	mean real rent
inflow of buyers	d	8022	8022	8022	sales
bargaining power of seller	α	-	-	0.8	≈ share of surplus in random auctions
seller's search costs \$	c^s	9072	9072	9072	6% of sale price
mean housing services \$	Ex	160	1382	259	calibrated jointly to match
level of MC \$	c^h	-292135	-292135	-296380	$p = \$450K$, $T^S = 2.5$,
angle of MC \$	ψ	89.29	89.29	89.29	$\varepsilon_p^{H^S} = 0.63$ in LA MSA, Zillow
marginal prod cost \$	MC	424131	424131	419886	
standard deviation of d_t	σ_d	-	-	-	fit $\hat{\sigma}^{\text{Data}}_{dlogSales}$

Notes: This table shows the calibrated parameters for the auction model with random/directed search in columns "RA/DA" and Nash bargaining model in column "NB". Each model is individually calibrated to match the same moments, observed in the data. MC stands for the marginal costs. The standard deviation σ_d of the influx of buyers d_t fits the volatility of sales $\hat{\sigma}^{\text{Data}}_{dlogSales}$ in each of the models.

4.5 Quantitative results

In this section I compare the volatilities, generated by the Nash bargaining model and the auction models, in response to the shocks of the influx of buyers d_t, with the moments in the data. The data comes from monthly Zillow house price index for Los Angeles MSA from 1996:M4 to 2015:M6. To compare the moments from the data and models, each model is simulated 100 times to produce a time-series of $T = 231$ months (to match the length of data), and the average moments from these experiments are reported in Tables 3 and 4 for the exponential and non-exponential distributions, respectively. All simulations start from the steady-state of the corresponding model. Zillow applies the Henderson filter to the raw data and then uses a seasonal-trend decomposition (STL) procedure to remove seasonality[38], I apply the same filter to the simulated data.

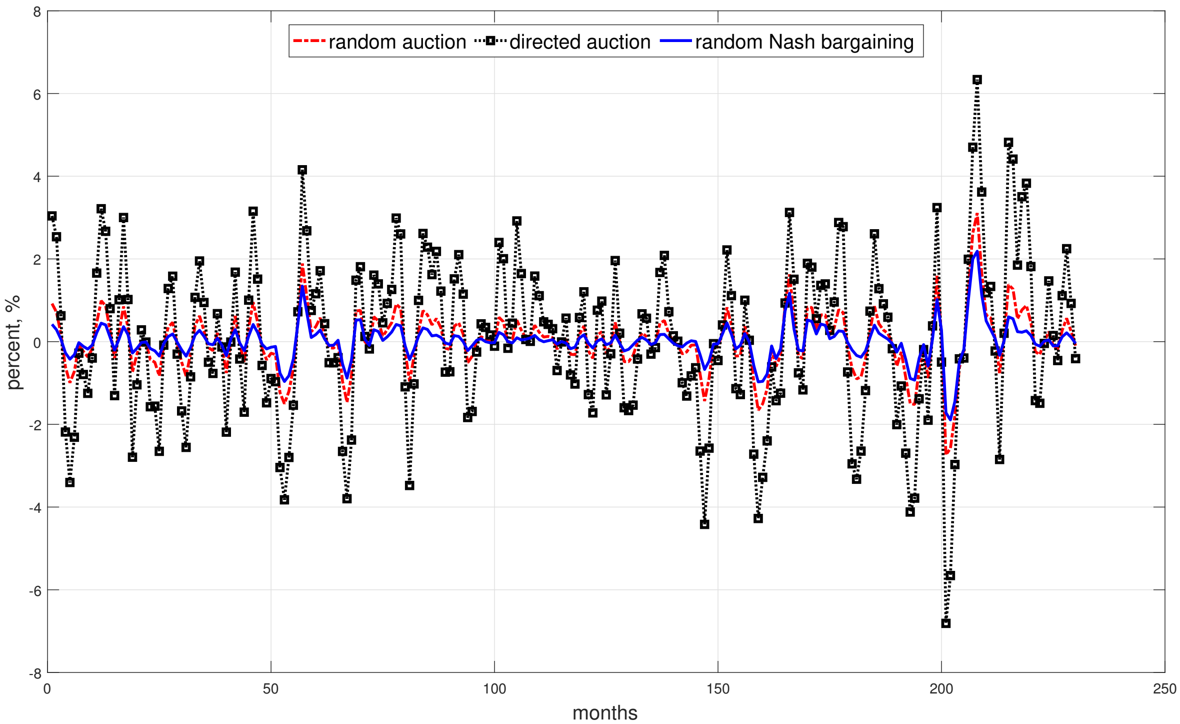
Figures 6 and 7 illustrate simulations of the house price growth for each other model in response to the same series of shocks. Visually the volatility of the home prices in the auction models is higher than in the Nash bargaining models, and the auction model with directed search produces the highest volatility. It is confirmed by the average moments from the experiments in Tables 3 and 4. This is the main quantitative result of the paper.

To gauge how the volatility in the model compares with the volatility in the data, I plot both the graph of the house price growth in the LA MSA from Section 1 and the simulations of the

[38]Zillow ZHVI methodology: `http://www.zillow.com/research/zhvi-methodology-6032/`

models with non-exponential distribution in the same figure, see Figure 8. The auction model with directed search produces the volatility quantitatively similar to the volatility in the data.

Figure 6: The volatility of the simulated prices in the auction models is higher than in the Nash bargaining model, example of exponential distribution



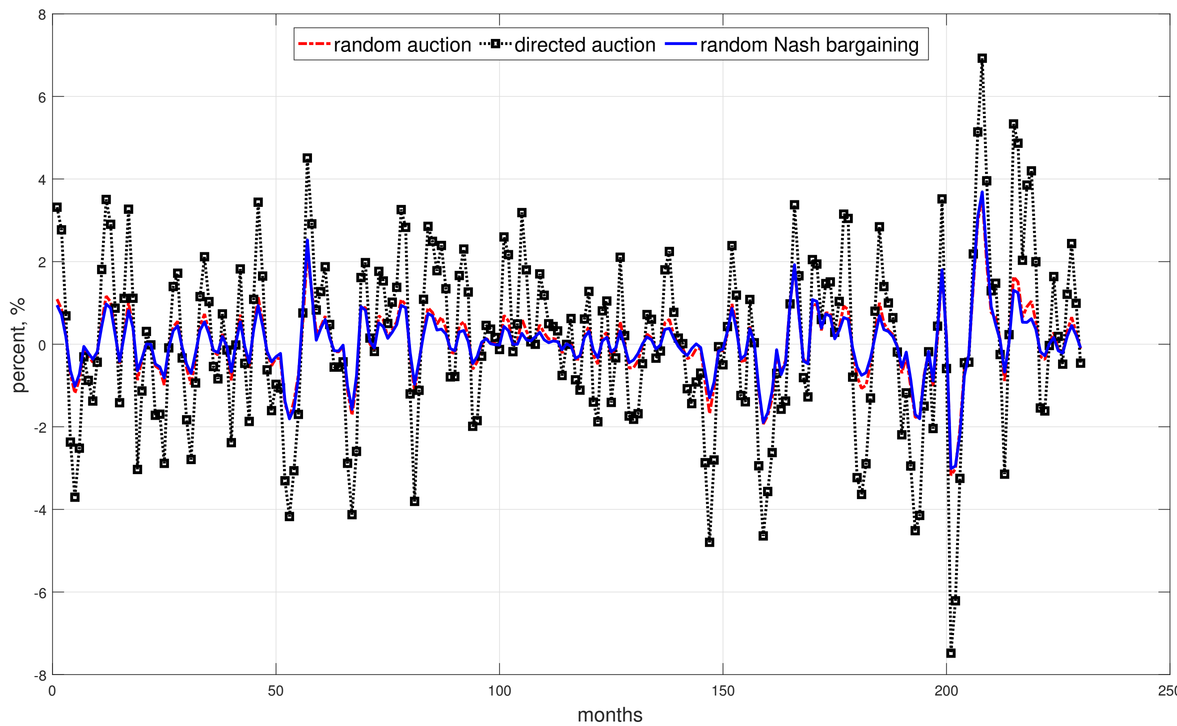
Notes: This graph shows an example of simulated monthly series of the house price growth in percent from the auction model with random search in dashed red line, from the auction model with directed search in dashed dotted black line, from the Nash bargaining with random search in the solid blue line. In each model the housing market is subject to the same series of shocks, fixed with the seed, and the same exponential distribution of the values, $x \sim F(x) = (1 - e^{-\mu x})$, where μ is calibrated to fit the data moments, see text.

Table 3: The auction house prices are more volatile in the auction models as compared to the benchmark Nash bargaining model, example of exponential distribution

	Data	RA	DA/SP	NB
$\sigma_{\Delta logp}$ monthly	**0.0160**	**0.0127**	**0.0232**	**0.0095**
$\rho_{\Delta logp}$ monthly	0.5814	0.5706	0.5759	0.5627
$\sigma_{\Delta logp}$ quarterly, last	0.0388	0.0293	0.0540	0.0220
$\rho_{\Delta logp}$ quarterly, last	0.3830	0.0730	0.0996	0.0500
$\sigma_{\Delta logp}$ quarterly, average	0.0370	0.0258	0.0477	0.0194
$\rho_{\Delta logp}$ quarterly, average	0.4221	0.2429	0.2743	0.2159
$\sigma_{\Delta logp}$ annual, last	0.1026	0.0639	0.1227	0.0470
$\rho_{\Delta logp}$ annual, last	0.6662	0.0243	0.1395	-0.0437
$\sigma_{\Delta logp}$ annual, average	0.1016	0.0533	0.1050	0.0388
$\rho_{\Delta logp}$, annual, average	0.7225	0.2047	0.3272	0.1295

Notes: This table shows the moments based on Zillow house price growth data in column "Data", average moments from 10,000 simulations of the auction model with random and directed search in column "RA" and "DA/SP", correspondingly, and random Nash bargaining model in column "NB". The "SP" name of the column refers to the social planner solution that can be decentralized by the auction model with directed search. $\sigma_{\Delta logp}$ and $\rho_{\Delta logp}$ stand for standard deviation and autocorrelation of the change in log prices. The distribution of values x is exponential $F(x) = (1 - \exp(-\mu x))$. I have applied the Henderson filter and STL filter for seasonal adjustment to the simulated series from the models to make them comparable to the data series from Zillow, see `http://www.zillow.com/research/zhvi-methodology-6032/`. The labels "average" and "last" refer to the method of computing the quarterly and annual series from the monthly data. The quarterly series that are computed as the prices at the last month in the quarter referred to as "last", or the average monthly prices referred to as "average". Similarly, for the annual series.

Figure 7: The volatility of the simulated prices in the auction models is higher than in the Nash bargaining model, example of non-exponential distribution



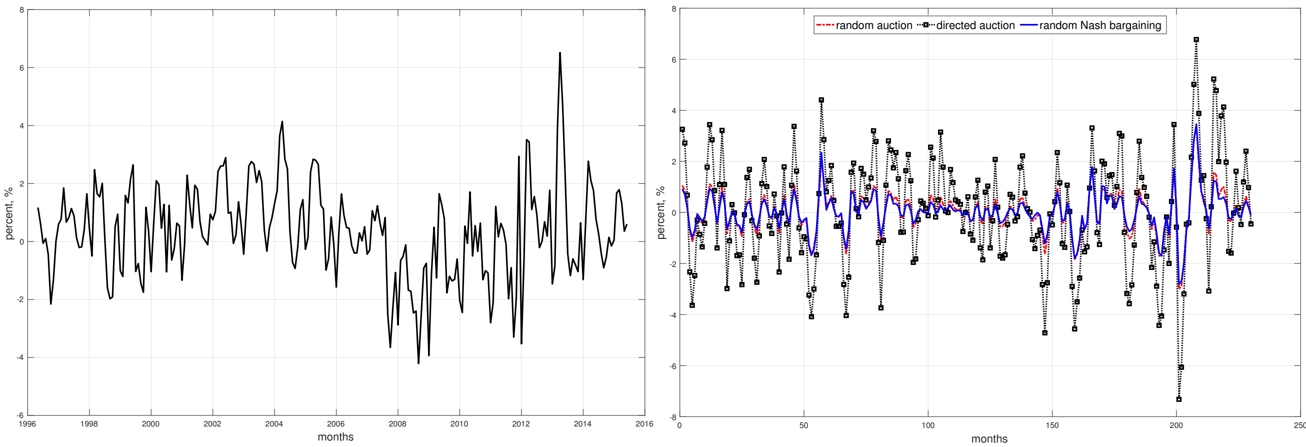
Notes: This graph shows an example of simulated monthly series of the house price growth in percent from the auction model with random search in dashed red line, from the auction model with directed search in dashed dotted black line, from the Nash bargaining with random search in the solid blue line. In each model the housing market is subject to the same series of shocks, fixed with the seed, and the same distribution of the values, $x \sim F(x) = (1 - e^{-\mu x})^2$, where μ is calibrated to fit the data moments, see text.

Table 4: The auction house prices are more volatile in the auction models as compared to the benchmark Nash bargaining model, example of non-exponential distribution

	Data	RA	DA/SP	NB
$\sigma_{\Delta logp}$ monthly	**0.0160**	**0.0147**	**0.0249**	**0.0127**
$\rho_{\Delta logp}$ monthly	0.5814	0.5711	0.5759	0.5674
$\sigma_{\Delta logp}$ quarterly, last	0.0388	0.0341	0.0579	0.0293
$\rho_{\Delta logp}$ quarterly, last	0.3830	0.0749	0.1002	0.0624
$\sigma_{\Delta logp}$ quarterly, average	0.0370	0.0300	0.0511	0.0258
$\rho_{\Delta logp}$ quarterly, average	0.4221	0.2448	0.2750	0.2295
$\sigma_{\Delta logp}$ annual, last	0.1026	0.0745	0.1314	0.0631
$\rho_{\Delta logp}$ annual, last	0.6662	0.0284	0.1392	-0.0207
$\sigma_{\Delta logp}$ annual, average	0.1016	0.0622	0.1124	0.0523
$\rho_{\Delta logp}$, annual, average	0.7225	0.2093	0.3269	0.1556

Notes: This table shows the moments based on Zillow house price growth data in column "Data", average moments from 10,000 simulations of the auction model with random and directed search in column "RA" and "DA/SP", correspondingly, and random Nash bargaining model in column "NB". The "SP" name of the column refers to the social planner solution that can be decentralized by the auction model with directed search. $\sigma_{\Delta logp}$ and $\rho_{\Delta logp}$ stand for standard deviation and autocorrelation of the change in log prices. The distribution of values x is $F(x) = (1 - \exp(-\mu x))^2$. I have applied the Henderson filter and STL filter for seasonal adjustment to the simulated series from the models to make them comparable to the data series from Zillow, see `http://www.zillow.com/research/zhvi-methodology-6032/`. The labels "average" and "last" refer to the method of computing the quarterly and annual series from the monthly data. The quarterly series that are computed as the prices at the last month in the quarter referred to as "last", or the average monthly prices referred to as "average". Similarly, for the annual series.

Figure 8: The house price growth from the data and from the simulation in the models



(a) Los Andgeles MSA data

(b) Models

Notes: This figure shows the house price growth in panel (a) and the simulation of the models, where "random auction" is the auction model with random search, "directed search" is the auction model with directed search, "random Nash bargaining" is the Nash bargaining with random search" in panel (b). In each model the distribution of house values $x \sim \text{Exp}$, and each model is hit by the same series of shocks. The line "data" shows the house price growth from Zillow Los Angeles MSA.

5 Constrained Socially Optimal Allocation

The literature[39] on competing mechanisms established that if the seller meets several buyers at a time, the weakly preferred mechanism for the seller is the second-price auction. Section 1 argues that in housing markets the seller is frequently visited by several buyers and the seller is using the sales mechanism that is essentially the second price auction with the reservation price. Section 4.5 shows that if this sales mechanism is built into the search and matching model, the quantitative predictions of the model differ substantially from the frequently employed Nash bargaining model.

What is the socially efficient level of house price volatility? This section tackles this questions through the lens of the search theory. The standard random search models usually are not constrained efficient, except for the knife-edge case in which the Hosios (1990) condition holds. In the random search model with auctions from the first part of the paper the seller runs an optimal auction that may not be socially efficient in the presence of the search frictions. In contrast, a well-known result from the literature on the directed search in the labor market (Moen (1997), Shimer (1996)) is that allowing the sellers to compete for buyers, – by posting the trading mechanism – eliminates the inefficiencies that are present in the random search models.

However, these results may fail to extend if a more general setting is considered. For example, Guerrieri (2008) shows that the equilibrium in the dynamic directed search model of the labor market is not efficient once the workers have private information, unless the economy starts from the steady-state[40]. However, the private information coupled with the free entry of the sellers

[39]McAfee (1993), Peters (1997)

[40]Geromichalos (2012) shows that when the seller has several goods to sell and capacity constrains the efficiency

leaves the efficiency results intact in the static model of the housing search with auctions, as demonstrated by Albrecht, Gautier, and Vroman (2014). In this section I extend this result to a dynamic setting by building a dynamic equilibrium model of directed search with auctions and shows that it delivers the socially efficient allocation, constrained by search frictions.

This result is shown in several steps. I start with the framework of Section 3 to find the socially efficient allocation, constrained by the search frictions with many-to-one matches, and show that the equilibrium allocation in the random search auction model from Section 3.4 is not constrained efficient. The reason for the failure of efficiency is a monopoly behavior of seller in the optimal auction. The monopoly arises when the buyer randomly visits a seller without knowing the expected terms of the trade. The seller then becomes a monopolist, because if the buyer fails to transact with the current seller, the buyer has to wait till the next period and go through the search frictions, which is costly. If the sellers are allowed to advertise and commit to the terms of the trade beforehand and the buyers to direct their search to sellers after observing the promised trading mechanism as in the auction model with directed search, I demonstrate that the equilibrium model of directed search decentralizes the constrained efficient allocation.

The constrained social optimum allocation is a solution of a social planner problem, constrained by the search frictions. In particular, given the current state of the economy $\mathcal{S}_t = (\bar{B}_t, \bar{S}_t, d_t)$, the social planner decides how many new sellers h_t enter the market, how many buyers B_t and sellers S_t are active out of the pool of all buyers $\bar{B}_t$ and sellers $\bar{S}_t$. Then, due to the search frictions, each active buyer is sent to an active seller according to the same Poisson process as in the decentralized auction and Nash bargaining models. Upon meeting a seller, a buyer draws a realization of the match-specific value of housing services x. After observing these realizations, the social planner must decide whether to distribute a house from the active seller today or wait till tomorrow, and, if the house is distributed, which buyer gets the house.

The efficient distribution of the house prescribes the house to be awarded to the highest value buyer. Given that, the question is whether to distribute the house to the highest value buyer this period or keep the house and potentially distribute it the next period. Since the values for a house are independently and identically distributed over time and over buyers, the solution is be characterized by the threshold value of housing services $\bar{x}_t$, such that if the highest draw x of housing services exceeds $\bar{x}_t$, the house is distributed. The threshold $\bar{x}_t$ is determined endogenously and can vary over time.

Hence, the social planner chooses a sequence of influx of seller, number of active sellers and active buyers, the threshold value for distributing a house $\{h_t, S_t, B_t, \bar{x}_t\}_{t=0}^{\infty}$, given the initial total number of buyers, total number of sellers and influx of buyers $\mathcal{S}_0 = (\bar{B}_0, \bar{S}_0, d_0)$, to maximize the

of the directed search can also break down.

present discounted flow of housing services from distributed house.

$$\max_{\{H_t,S_t,B_t,\bar{x}_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [S_t \pi(\bar{x}_t,\theta_t) \frac{\beta E[x_{(1)t}|x_{(1)t} \geq \bar{x}_t]}{1-\beta} - wB_t - c^s S_t - c^b B_t - TC(h_t)] \tag{16}$$
$$\bar{B}_{t+1} = \bar{B}_t + d_t - \pi(\bar{x}_t,\theta_t)S_t$$
$$\bar{S}_{t+1} = \bar{S}_t + h_t - \pi(\bar{x}_t,\theta_t)S_t$$
$$d_{t+1} = \rho d_t + (1-\rho)d_0 + \varepsilon_{t+1}$$
$$0 \leq S_t \leq \bar{S}_t, 0 \leq B_t \leq \bar{B}_t, \bar{x}_t \geq 0$$
$$\bar{B}_0 > 0, \bar{S}_0 \geq 0, d_0 \text{ given}$$

where $S_t\pi(\bar{x}_t,\theta_t)$ is the number of distributed homes. The total costs of supplying new homes are quadratic $TC(h_t) = c^h h_t + \frac{\psi}{2}h_t^2$, as before, $\pi(\bar{x}_t,\theta_t)$ is the probability of distributing the house and $E[x_{(1)t}|x_{(1)t} \geq \bar{x}_t]$ is the expectation of the highest value of housing services, conditional on value exceeding the threshold $\bar{x}_t$. The expectation in $E[x_{(1)t}|x_{(1)t} \geq \bar{x}_t]$ is taken both over the number visitors N of the house and over the realized values x of those visitors.

Denote the value function of the social planner in the beginning of period t by Ω_t. Let $V_t^S = \frac{\partial \Omega_t}{\partial \bar{S}_t}$ be the increase in the social welfare function Ω_t, produced by adding one more seller $\bar{S}_t$, the option of adding a seller. Similarly, let $V_t^B = \frac{\partial \Omega_t}{\partial \bar{B}_t}$ be the value of adding a buyer. Then the proposition 6 summarizes the optimality conditions for the social planner.

Proposition 6. The option value to add a seller $V_t^S = \frac{\partial \Omega_t}{\partial \bar{S}_t}$, the option value to add a buyer $V_t^B = \frac{\partial \Omega_t}{\partial \bar{B}_t}$, the threshold value of housing services to distribute the house $\bar{x}_t$, that solve the social planner problem, satisfy

$$V_t^B = \beta V_{t+1}^B - w + \max_{a_t^B \in \{0,1\}} (\frac{\beta}{1-\beta} \int_{\bar{x}_t}^{\infty} (1-F(x))e^{-\theta_t(1-F(x))}dx - c^B)a_t^B$$
$$V_t^S = \beta V_{t+1}^S + \max_{a_t^S \in \{0,1\}} (\frac{\beta\theta_t}{1-\beta} \int_{\bar{x}_t}^{\infty} (x - \frac{1-F(x)}{f(x)} - (1-\beta)(V_{t+1}^B + V_{t+1}^S))e^{-\theta_t(1-F(x))}f(x)dx - c^S)a_t^S$$
$$\bar{x}_t = \max\{(1-\beta)(V_{t+1}^B + V_{t+1}^S), 0\}$$

where $\theta_t = B_t/S_t$ - tightness, $\lambda(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of distribution of values $F(x)$ and $\pi_t = 1 - e^{-\theta_t(1-F(\bar{x}_t))}$ is the probability of selling a house.

Corollary 6.1. The auction model with directed search decentralized the solution of the social planner problem constrained by the search frictions.

The comparison of the dynamics of the buyer's and seller's value functions from Proposition 1, 2 and 6 suggests that generally the Nash bargaining with random search and auction models with random search are not socially efficient. The Nash bargaining model with random search is not constrained efficient because the search frictions in the social planner problem allow for many-to-one matches while the search frictions in the standard Nash bargaining model do not. To gain intuition on why the auction model is not socially efficient, compare the steady-states of the auction model and social planner solution for the exponential distribution.

The models disagree on how the tightness θ and the threshold value for the housing services $\bar{x}$ are determined. Specifically, in the auction model tightness and the threshold value of housing services are found as the solution of a pair of equations

$$\varphi(\theta(1-F(\bar{x}))) = \frac{(1-\beta)MC(d)/\beta + c^s}{\beta EPV_x} \tag{17}$$

$$\bar{x} = Ex + \frac{\beta\pi(\bar{x},\theta)}{\theta}EPV_x + \frac{1-\beta}{\beta}MC(d) - c^b - w \tag{18}$$

while the social planner would solve

$$\varphi(\theta(1-F(\bar{x}))) - \pi(\bar{x},\theta) = \frac{(1-\beta)MC(d)/\beta + c^s}{\beta EPV_x} \tag{19}$$

$$\bar{x} = \frac{\beta\pi(\bar{x},\theta)}{\theta}EPV_x + \frac{1-\beta}{\beta}MC(d) - c^b - w \tag{20}$$

to determine θ and $\bar{x}$. In the social planner equation for the tightness has an extra $-\pi(\bar{x},\theta)$ term. This term represents the monopoly distortion. The social planner chooses higher adjusted tightness $z = \theta(1-F(\bar{x}))$ than the adjusted tightness that emerges in the equilibrium of the auction model, which follows from equations (17) and (19). Because the adjusted tightness is lower in the auction model, the probability of sale $\pi(\bar{x},\theta) = 1-\exp(-z)$ is also lower.

The expression for the threshold value in the auction model has additional Ex term, making the threshold value in the auction model higher as compared to the socially efficient allocation. This comparison is not immediate, because the equations include other endogenously determined variables, but can be proved. Higher threshold value $\bar{x}$ in the auction model is a consequence of the static inefficiency of the optimal auction. The seller in the auction model behaves as a monopolist, which leads to higher prices, and hence higher threshold value $\bar{x}$ for distributing the house. This static inefficiency in the auction model has dynamic consequences. Because the seller has higher threshold value $\bar{x}$, he keeps the house on the market longer as compared to what the social planner would choose. In other words, the seller suboptimally chooses to exercise this option value to sell too late. Board (2007) finds a similar prediction, although in a different setup.

6 Discussion

What have we learned so far? The choice of the price determination mechanism is important for qualitative and quantitative properties of house prices. Bidding wars between buyers in auctions produce more volatile house prices than the benchmark Nash bargaining model, which is closer to the data. If search in auctions is directed, then an equilibrium allocation in the auction model decentralizes the solution of the social planner problem, constrained by search frictions. Hence, high volatility in the auction model with directed search in this sense is efficient.

There are many other ways to interpret these results. First, there are competing mechanisms for selling houses. The seller or society can choose among these mechanisms. The model in the paper shows the differences in the outcomes depending on the type of mechanism employed in the dynamic search environment.

Second, the models can be used to study the time-series properties of house prices in a local housing market. One way to accomplish this is to study similar houses sold via auctions and via bargaining. Then the models are informative on the time-series properties of prices determined in auctions and bargaining. In the time series we observe fluctuations between booms and busts. One can argue that houses can be sold using auctions during booms and using bargaining during busts. However, both booms and busts can be captured by the auction model. During the booms, inflow of many buyers spurs bidding wars, described by auctions, and during the busts, the auction model works as the price posting. The seller posts the reservation price, and if a buyer with high enough valuation visits the seller, the seller sells at the reservation price to this buyer. In this case the auction model behaves similar to the bargaining model in which the seller makes take-or-leave it offer without knowing the valuation of the buyer.

Third, the models can quantify differences in the housing market statistics in the cross-section of local housing markets, for example, across neighborhoods, cities, or metro areas. Some markets can have higher incidence of auctions due to attractive local amenities, for instance great schools, or short supply due to geographical or zoning restrictions (Saiz (2010)). In these cases the auction model can be applied to both types of markets. In hot markets, such as Palo Alto, CA, the model can be calibrated to have higher ratio of buyers to seller, as compared to the cold markets, such as Detroit, MI, with lower ratio of buyer to sellers.

Another way to think about the price determination mechanism is to use bargaining for standardized houses and auctions for unique houses. But both types of houses can potentially attract multiple interested buyers, and auctions is a useful way of modeling the price formation in these situations. If a house is sold through bargaining and multiple buyers arrive, then it is possible to apply models of multilateral bargaining, in which the outside option in the current negotiation is the payoff from bargaining with the next interested buyer. Using models of multilateral bargaining allows to account for the competition between buyers, which is the main idea of the paper. Technically, it is easier to use auctions to model this process of competition between buyers.

The auction model can be extended in several dimensions to accommodate other important aspects of the housing market. In the current setup the house values are private and independent, however, common values are potentially important in the housing market. If one buyer has high value for a particular house, that probably means that other buyer also has high value. The current setup can be applied to a homogenous set of houses or a segment of a local housing market where differences between houses are idiosyncratic. If the common values are used instead of private values, then the details of the auction protocol matter for the seller's revenue and house prices, because the revenue equivalence theorem might not apply. In this case a researcher has to make a judgement call on what is the appropriate auction format[41]. The auction model can also be extended to accommodate the risk-averse agents and budget constraints, in which case the model may lose tractability, but allow to understand the bidding pattern of risk-averse buyers who are pre-approved to borrow up to a certain limit. Finally, the model can be used for structural estimation to recover the distribution of house values to inform housing policies[42].

[41]The English ascending auction is used frequently in housing markets, which can be a starting point.

[42]For instance, given the distribution of house values, what is the welfare cost and benefit of mortgage interest rate deduction.

7 Conclusion

Auctions are widely employed in housing markets. In hot markets sellers are confronted with multiple interested buyers and run informal auctions, inviting bids and rebids until a single buyer remains. In some cases, notably in Australia, UK, New Zealand, Singapore[43] and in US for foreclosed properties[44], the auctions take standardized forms. This paper studies the role of auctions in housing markets, comparing a model with auctions to the standard model, where only one-on-one bargaining determines prices.

During the booms each seller attracts multiple buyers, an auction is highly effective at selecting the buyer with the highest valuation. Optimal selection results in higher prices for the seller in the auction model. In contract, in the Nash bargaining model a seller picks one of many interested buyers at random and negotiate only with that buyer. This price-finding process is not optimal for the seller, because the randomly selected buyer is probably not the buyer who places the highest value on the house, so prices are lower. During the busts it is common for only a single buyer to be interested in a house, so the seller picks a reservation price and the buyer decides whether to buy at that price or not.

There are alternative price-finding processes that arise in the housing market that are beyond the scope of this paper, but deserve further attention from researchers. First, other auction formats may be used to sell houses. If the buyers are risk neutral and their values are independently and identically distributed over time and over bidder, then, by the revenue equivalence, the expected revenue for the seller is the same. But studying housing auctions with affiliated and correlated over time housing valuations and risk-averse buyers could impact the implications of the search model. Second, in cold markets, where many houses are available to a buyer without competition from other buyers, the buyer effectively runs an auction by considering the prices of the suitable houses that are currently on the market and picking the lowest one. Third, another alternative-price finding process is alternating-offer bargaining. In setting where one-on-one bargaining occurs, it takes the alternating-offer form. Not only this process is seen in the real world, its game-theoretic foundations are stronger than the Nash bargain and proved to change the implications of the job search model (Hall and Milgrom (2008)). Finally, auctions and bargaining can be combined. In the housing market, the seller first picks the buyer with the highest valuation and bargains with this buyer one-on-one. In used-car auctions, it is common for the winning bid to fall short of the seller's hidden reserve price. In that case, the winning bidder and the seller engage in bargaining to see if the seller will agree to a price below the earlier reserve or the buyer will agree to a price above his winning bid (Larsen (2015)).

This paper focuses on the price-finding and is stripped down in other respects. It makes no claim to do justice to all the complexities of the housing market. Rather, it points out that the model used for price-finding has important consequences for the volatility and responsiveness of the house prices to exogenous shocks. The amplification of the housing market shocks in the auction model as compared to the Nash bargain model comes from the heterogeneity in the house values and rule for selecting the winning buyer. In the Nash bargaining, the buyer is chosen randomly so the sales price is determined by the average house values. In the auction models

[43]See Maher (1989), Lusht (1996), Genesove and Hansen (2014) for Australia, Dotzour, Moorhead, and Winkler (1998) for New Zealand, Merlo and Ortalo-Magné (2004), Merlo, Ortalo-Magné, and Rust (2015) for UK, Chow, Hafalir, and Yavas (2011) for Singapore

[44]Mayer (1998), Quan (1994)

the buyer is chosen as the highest bidder so the sales price is determined by the second highest value. During the booms, when there are many buyers, the highest values increase significantly as compared to the average values which contributes to the higher volatility of the house prices, helping to resolve the puzzle of the excess volatility of the prices in the housing markets.

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A Proofs

A.1 Proof of Proposition 1

Proof of Proposition 1. See the proof of the price equation in footnote 23.

The seller's payoff today is

$$(\pi_t \beta p_t + (1-\pi_t)\beta V_{t+1}^S - c^S)a_t^S + \beta V_{t+1}^S(1-a_t^S)$$

where the first two terms show the payoff of the seller if he participated in the local housing market in period t, i.e. $a_t^S = 1$, where $\pi_t = (1-\exp(-\theta_t))(1-F(\bar{x}_t))$ is the probability of sale. With probability π_t, the seller sells the house and tomorrow gets payment p_t, and, with probability $(1-\pi_t)$, he does not sell the house and gets the option value to sell tomorrow V_{t+1}. The search costs c^S are deducted from his payoff. If the seller does not participate in the local housing market, i.e. $a_t^S = 0$, then he enjoys the option value to sell tomorrow, i.e. V_{t+1}^S. Rearranging and maximizing over the participation decision a_t^S gives the Bellman equation

$$V_t^S = \beta V_{t+1}^S + (\beta\pi_t(p_t - V_{t+1}^S) - c^S)a_t^S$$

Using the Nash bargaining price equation delivers the Bellman equation in the text:

$$\begin{aligned} V_t^S &= \beta V_{t+1}^S + \max_{a_t^S\in\{0,1\}} (\beta\pi_t(V_{t+1}^S + \alpha(\frac{E[x|x\geq \bar{x}_t]}{1-\beta} - V_{t+1}^S - V_{t+1}^B) - V_{t+1}^S) - c^S)a_t^S = \\ &= \beta V_{t+1}^S + \max_{a_t^S\in\{0,1\}} (\pi_t\alpha\beta(\frac{E[x|x\geq \bar{x}_t]}{1-\beta} - V_{t+1}^S - V_{t+1}^B) - c^S)a_t^S \end{aligned}$$

The Bellman equation for buyers is proved similarly, using the probability of buying a house, equal to the probability of meeting a seller $(1-\exp(-\theta_t))/\theta_t$ times the probability of matching given the meeting $(1-F(\bar{x}_t))$, altogether $(1-\exp(-\theta_t))/\theta_t \times (1-F(\bar{x}_t)) = \pi_t/\theta_t$.

The threshold house quality $\bar{x}_t$ is such that the joint surplus from the trade, i.e. $x/(1-\beta) - V_{t+1}^B - V_{t+1}^S$ is zero, or in other words $\bar{x}_t = \max\{(1-\beta)(V_{t+1}^B + V_{t+1}^S), 0\}$. The no-negativity restriction comes from non-negative values $x \geq 0$. □

A.2 Proof of Proposition 2

Proof of Proposition 2. The expected price is

$$\frac{E_N[\bar{p}_t P(\text{Sale at } \bar{p}_t) + Eb_{(2)t}P(\text{Sale at } b_{(2)t})]}{P(\text{Sale})} = \frac{E_N[\bar{p}_t NG(\bar{p}_t)^N(1-G(\bar{p}_t)) + Eb_{(2)t}P(\text{Sale at } b_{(2)t})]}{\pi_t}$$

Let $M(y,N) = 1 - G(y)^N - NG^{N-1}(y)(1-G(y))$, then the cdf of the second order statistic conditional on selling higher than the reservation price is $\frac{M(\bar{p}_t,N)-M(y,N)}{M(\bar{p}_t,N)}\mathbb{1}_{\{y\geq\bar{p}\}}$, hence the conditional pdf is $-\frac{\partial M(y,N)}{\partial y}/M(\bar{p},N)$. Now the last term from the numerator of the price equation becomes

$$M(\bar{p}_t,N)\int_{\bar{p}_t}^{\infty} y(-\frac{\partial M(y,N)}{\partial y}/M(\bar{p},N))dy = \int_{\bar{p}_t}^{\infty} yd(-M(y,N))$$

The numerator of the expected price is then

$$
\begin{aligned}
&E_N[\bar{p}_t N G(\bar{p}_t)^N (1-G(\bar{p}_t)) - \int_{\bar{p}_t}^{\infty} y dM(y,N)] = \\
&= E_N[\bar{p}_t N G(\bar{p}_t)^N (1-G(\bar{p}_t)) - yM(y,N)|_{\bar{p}_t}^{\infty} + \int_{\bar{p}_t}^{\infty} M(y,N)dy] = \\
&= E_N[\bar{p}_t N G(\bar{p}_t)^N (1-G(\bar{p}_t)) + \bar{p}_t M(\bar{p}_t,N) + \int_{\bar{p}_t}^{\infty} M(y,N)dy]
\end{aligned}
$$

where the last equality uses that $\lim_{y\to\infty}(y(1-NG(y)^{N-1}(1-G(y))-G(y)^N)=0$, which holds for the distributions used in the text.

Rearranging, using $E_N(1-G(\bar{p}_t)^N) = P(\text{Sale}) = \pi_t$ and the Fubini's theorem for exchanging the integral and expectation, the last expression simplifies to

$$
\bar{p}_t\pi_t + \int_{\bar{p}_t}^{\infty} E_N M(y,N)dy
$$

where $E_N M(y,N) = 1-e^{-\theta_t(1-G(y))}-(1-G(y))\theta e^{-\theta_t(1-G(y))}$ by applying the Poisson pdf formula.

Then the expected price is

$$
p_t = \frac{\bar{p}_t\pi_t + \int_{\bar{p}_t}^{\infty}(1-e^{-\theta_t(1-G(y))}-(1-G(y))\theta_t e^{-\theta_t(1-G(y))})dy}{\pi_t}
$$

where $G(y) = F((1-\beta)(y+V_{t+1}^B))$, hence,

$$
p_t = \bar{p}_t + \frac{1}{1-\beta}\frac{\int_{\bar{x}_t}^{\infty}(1-e^{-\theta_t(1-F(x))}-(1-F(x))\theta_t e^{-\theta_t(1-F(x))})dx}{\pi_t}
$$

□

A.3 The steady state equilibrium

Two curves in $(\bar{x},\theta)$ determine the steady state equilibrium:

(1) Free entry (FE)

$$
FE(\bar{x},\theta) = -(1-\beta)MC(d)/\beta + \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(\pi(x,\theta)-\theta(1-F(x))(1-\pi(x,\theta)))dx - c^S
$$

(2) Price setting (PS)

$$
PS(\bar{x},\theta) = -\bar{x} + (1-\beta)MC(d)/\beta - w + \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(1-F(x))e^{-\theta(1-F(x))}dx
$$

The slope of the free entry:

$$\frac{\partial FE(\bar{x},\theta)}{\partial \bar{x}} = -\frac{\beta}{1-\beta}(\pi(\bar{x},\theta) - \theta(1-F(\bar{x}))(1-\pi(\bar{x},\theta)))$$

$$\frac{\partial FE(\bar{x},\theta)}{\partial \theta} = \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(\frac{\partial \pi(x,\theta)}{\partial \theta} - [(1-F(x))(1-\pi(x,\theta)) + \theta(1-F(x))(-1)\frac{\partial \pi(x,\theta)}{\partial \theta}])dx] =$$

$$= \frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}\theta(1-F)^2(1-\pi)dx$$

$$\frac{\partial \bar{x}}{\partial \theta}|_{FE} = -\frac{\partial FE/\partial\theta}{\partial FE/\partial\bar{x}} = \frac{\int_{\bar{x}}^{\infty}\theta(1-F(x))^2(1-\pi(x,\theta))dx}{\pi(\bar{x},\theta) - \theta(1-F(\bar{x}))(1-\pi(\bar{x},\theta))} > 0$$

$$\frac{\partial FE(\bar{x},\theta)}{\partial d} = -(1-\beta)\psi/\beta$$

$$\frac{\partial \bar{x}}{\partial d}|_{FE} = -\frac{(1-\beta)\psi/\beta}{\frac{\beta}{1-\beta}(\pi(\bar{x},\theta) - \theta(1-F(\bar{x}))(1-\pi(\bar{x},\theta)))} < 0$$

where the denominator is positive, because $\pi(x,\theta)$ is concave in θ so $\pi(x,\theta) \geq \theta\frac{\partial\pi(x,\theta)}{\partial\theta} = \theta(1 - F(x))(1-\pi(x,\theta))$.

The slope of the price setting:

$$\frac{\partial PS(\bar{x},\theta)}{\partial \bar{x}} = -1 - \frac{\beta}{1-\beta}(1-F(\bar{x}))(1-\pi(\bar{x},\theta))$$

$$\frac{\partial PS(\bar{x},\theta)}{\partial \theta} = -\frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(1-F)^2(1-\pi)dx$$

$$\frac{\partial \bar{x}}{\partial \theta}|_{PS} = -\frac{-\frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(1-F)^2(1-\pi)dx}{-(1+\frac{\beta}{1-\beta}(1-\bar{F})(1-\bar{\pi}))} = -\frac{\frac{\beta}{1-\beta}\int_{\bar{x}}^{\infty}(1-F)^2(1-\pi)dx}{(1+\frac{\beta}{1-\beta}(1-\bar{F})(1-\bar{\pi}))} < 0$$

$$\frac{\partial PS(\bar{x},\theta)}{\partial d} = (1-\beta)\psi/\beta$$

$$\frac{\partial \bar{x}}{\partial d}|_{PS} = -\frac{(1-\beta)\psi/\beta}{-1-\frac{\beta}{1-\beta}(1-F(\bar{x}))(1-\pi(\bar{x},\theta))} > 0$$

A.4 Proof of Proposition 4

Proof of Proposition 4. In the steady state the entry of sellers h has to be equal to the sales q and the influx of buyers d to keep the number of sellers and buyers constant. Moreover, the steady state of the value function of the seller has to equal to the marginal costs of entry, so that $\beta V^S = MC(d)$. This proves part 4 of the proposition.

To proceed, consider the equilibrium threshold for the housing services $\bar{x}$ in the steady-state. In the Nash bargaining model it solves

$$\bar{x} = \max\{(1-\beta)V^S + (1-\beta)V^B, 0\}$$

where $(1-\beta)V^S = (1-\beta)MC(d)/\beta$ and $(1-\beta)V^B = -w + (\beta\alpha\frac{\pi(\bar{x},\theta)}{\theta}EPV_x - c^B)^+$, or

$$\bar{x} = \max\{(1-\beta)MC(d)/\beta - w + (\beta\alpha\frac{\pi(\bar{x},\theta)}{\theta}EPV_x - c^B)^+, 0\}$$

If $(1-\beta)MC(d)/\beta - w > 0$, then the solution for $\bar{x}$ exists and is positive. The solution exists because the left-hand side is increasing in $\bar{x}$ while the right-hand side is either a positive constant or decreasing in $\bar{x}$. Hence, $\bar{x} > 0$. The same argument applies to show that the solution for $\bar{x}$ is positive in both auction models.

Given that $\bar{x} > 0$ in the steady state of all models, the expected house prices in the Nash bargaining model p_{NB} and the auction model with random auctions p_A are

$$\begin{aligned} p_{NB} &= V^S + \alpha EPV_x \\ p_A &= V^S + \frac{\varphi(\theta_A(1-F(\bar{x}_A)))}{\pi(\bar{x}_A,\theta_A)} EPV_x = V^S + \frac{\varphi(\theta_A(1-F(\bar{x}_A)))}{\pi(\bar{x}_A,\theta_A)} EPV_x \end{aligned}$$

To compare those, I use lemma 1 that follows this proof below. By this lemma $\frac{\varphi(\theta_A(1-F(\bar{x}_A)))}{\pi(\bar{x}_A,\theta_A)} > 1 \geq \alpha$, and hence the expected prices are higher in the auction model with random search rather than in the Nash bargaining, which proves the first statement of the proposition.

To compare the probabilities of sale, time on the market and probability of sale, consider the option value to sell in each of the models from the Bellman equations

$$\begin{aligned} V^S_{NB} &= \frac{1}{1-\beta}(\beta\alpha\pi_{NB}(\bar{x}_{NB},\theta_{NB})EPV_x - c^S)^+ \\ V^S_A &= \frac{1}{1-\beta}(\beta\varphi(\theta_A(1-F(\bar{x}_A)))EPV_x - c^S)^+) \end{aligned}$$

where $\pi_{NB}(\bar{x}_{NB},\theta_{NB}) = (1-F(\bar{x}_{NB})(1-\exp(-\theta_{NB}))$.

As shown before, the value functions of the seller are the same in both models. Moreover, since $V^S = MC(d)/\beta > w/(1-\beta) > 0$, they are positive. Hence, it must be the same that

$$\pi_{NB}(\bar{x}_{NB},\theta_{NB}) > \alpha\pi_{NB}(\bar{x}_{NB},\theta_{NB}) = \varphi(\theta_A(1-F(\bar{x}_A))) > \pi(\bar{x}_A,\theta_A)$$

so that the probability of sale in the auction model with random search is lower than in the Nash bargaining. The time on the market for sellers $T^S = 1/\pi$ is the reciprocal of the probability of sale, which makes is higher in the auction model.

Finally, the number of sellers in the steady state is $S = \min(\bar{S}, d/\pi)$ because the sales $S\pi$ have to equal to the influx of buyers d, unless he hit the maximum number of active buyers $\bar{S}$. Because the probability of the sale is lower in the auction model with random search, it has to be $S_A = \min(\bar{S}, d/\pi_A) \geq S_{NB} = \min(\bar{S}, d/\pi_{NB})$.

□

Lemma 1. *In the auction model $\varphi(z) > \pi(z)\ \forall z > 0$*

Proof. Let the adjusted tightness $z = \theta(1-F(\bar{x})) = \theta e^{\bar{x}/Ex}$ and $\varphi(z) \equiv \int_0^z \frac{1-e^{-y}}{y}dy$. We can now express the probability of sale in auction as $\pi(\bar{x},\theta) = \pi(z) = 1-\exp(-z)$ and conclude that

$$\lim_{z\to 0}\frac{\varphi(z)}{\pi(z)} = \lim_{z\to 0}\frac{(1-\exp(-z))/z}{\exp(-z)} = \lim_{z\to 0}\frac{\exp(z)-1}{z} = \lim_{z\to 0}\frac{\exp(z)}{1} = 1 \quad (21)$$

$$\varphi(0) = \pi(0) = 0 \quad (22)$$

$$(\varphi(z)-\pi(z))' = \frac{1-e^{-z}}{z} - e^{-z} = \frac{1-(1+z)e^{-z}}{z} > 0 \quad (23)$$

Hence, $\varphi(z)/\pi(z) > 1$.

□

A.5 Proof of Proposition 5

Proof of Proposition 5. The sales q and inflow of new sellers h are equal to the influx of buyers d in both models, and the option value to sell V^S are pinned down by the marginal costs of entering and $\bar{x} > 0$ by the same argument as in the proof of Proposition 4 from above.

To compare the ratio of serious buyers $z = \theta(1 - F(\bar{x}))$ in the two models, consider the value functions of the seller in the steady-state, i.e.

$$(1-\beta)V_{RA}^S = (\beta\varphi(z_{RA})EPV_x - c^S)^+$$
$$(1-\beta)V_{DA}^S = (\beta(\varphi(z_{DA}) - (1 - e^{-z_{DA}}))EPV_x - c^S)^+$$

where RA (DA) denotes the steady-state value of the variable in the auction model with random search (directed search).

Because $V_{RA}^S = V_{DA}^S = MC(d)/\beta > w/(1-\beta) > 0$, both option values to sell are positive, and we express $\varphi(z)$ as

$$\varphi(z_{RA}) = \frac{(1-\beta)V_{RA}^S + c^S}{\beta EPV_x}$$
$$\varphi(z_{DA}) - (1 - e^{-z_{DA}}) = \frac{(1-\beta)V_{DA}^S + c^S}{\beta EPV_x}.$$

We now can compare z_{RA} and z_{DA}, using

$$\varphi(z_{RA}) = \varphi(z_{DA}) - (1 - e^{-z_{DA}}) < \varphi(z_{DA})$$

where the last inequality follows from noticing that $\theta > 0$, because otherwise $z_{DA} = 0$ and $V_{DA}^S = 0$. Because the function $\varphi(.)$ is increasing, $z_{RA} < z_{DA}$. It also immediately follows that $\pi_{RA} < \pi_{DA}$ because $\pi(z) = 1 - e^{-z}$ is increasing in z. The number of active sellers is $S = \min\{d/\pi, \bar{S}\}$ so that $S_{RA} = \min\{d/\pi_{RA}, \bar{S}\} \geq S_{DA} = \min\{d/\pi_{DA}, \bar{S}\}$.

The threshold value in the auction model with random search and directed search, hence

$$\bar{x}_{RA} = Ex + \frac{\beta\pi(z_{RA})}{z_{RA}} e^{-\bar{x}_{RA}/Ex} + V^S \tag{24}$$
$$\bar{x}_{DA} = \frac{\beta\pi(z_{DA})}{z_{DA}} e^{-\bar{x}_{DA}/Ex} + V^S \tag{25}$$

Hence, the difference satisfies

$$\bar{x}_{RA} - \bar{x}_{DA} = Ex + \frac{\beta\pi(z_{RA})}{z_{RA}} e^{-\bar{x}_{RA}/Ex} - \frac{\beta\pi(z_{DA})}{z_{DA}} e^{-\bar{x}_{DA}/Ex}$$

where the function $\pi(z)/z$ is decreasing [45] in z, and $\pi(z_{RA})/z_{RA} > \pi(z_{DA})/z_{DA}$.

We are ready to prove by contradiction. Assume that $\bar{x}_{DA} > \bar{x}_{RA}$, then

$$\bar{x}_{RA} - \bar{x}_{DA} = Ex + \frac{\beta\pi(z_{RA})}{z_{RA}} e^{-\bar{x}_{RA}/Ex} - \frac{\beta\pi(z_{DA})}{z_{DA}} e^{-\bar{x}_{DA}/Ex} < 0.$$

[45] $(\frac{\pi(z)}{z})'_z = \frac{\pi'(z)z - \pi(z)}{z^2} = \frac{e^{-z}z - (1-e^{-z})}{z^2} = \frac{e^{-z}(z+1) - 1}{z^2} < 0$

Since $Ex > 0$, it has to be the case true that

$$\frac{\beta\pi(z_{RA})}{z_{RA}}e^{-\bar{x}_{RA}/Ex} - \frac{\beta\pi(z_{DA})}{z_{DA}}e^{-\bar{x}_{DA}/Ex} < 0.$$

Moreover, $e^{-\bar{x}_{DA}/Ex} < e^{-\bar{x}_{RA}/Ex}$ so

$$\frac{\beta\pi(z_{RA})}{z_{RA}}e^{-\bar{x}_{DA}/Ex} - \frac{\beta\pi(z_{DA})}{z_{DA}}e^{-\bar{x}_{DA}/Ex} < \frac{\beta\pi(z_{RA})}{z_{RA}}e^{-\bar{x}_{RA}/Ex} - \frac{\beta\pi(z_{DA})}{z_{DA}}e^{-\bar{x}_{DA}/Ex} < 0.$$

with $\bar{x} > 0$ for both models it implies that $\pi(z_{RA})/z_{RA} < \pi(z_{DA})/z_{DA}$, contradicting the previous finding that $\pi(z_{RA})/z_{RA} > \pi(z_{DA})/z_{DA}$. Hence, $\bar{x}_{DA} < \bar{x}_{RA}$.

□

A.6 Proofs for the constrained social planner problem

Lemma 2. Assume the hazard rate $\lambda(x) = f(x)/(1 - F(x))$ is weakly increasing, then the expectation of the maximum value of the housing services, conditional on this value exceeding the threshold $\bar{x}_t$, is $E[x_{(1)}|x_{(1)} \geq \bar{x}_t] = \bar{x}_t + \frac{\int_{\bar{x}_t}^{\infty}\pi(x,\theta_t)dx}{\pi(\bar{x}_t,\theta_t)}$

Proof. The expectation and integral in the conditional expectation are interchanged by the Fubini's theorem. The expectation inside the integral is

$$E[x_{(1)t}|x_{(1)t} \geq \bar{x}_t] = \frac{E_N\int_{\bar{x}}^{\infty} xdF^N(x)}{E_N(1 - F(\bar{x})^N)} = \frac{\int_{\bar{x}_t}^{\infty} xE_N NF(x)^{N-1}f(x)dx}{\pi(\bar{x}_t, \theta)}$$

$$E_N NF(x)^{N-1} = e^{-\theta}\sum_{n=0}^{\infty} n\frac{\theta^n F(x)^{n-1}}{n!} \quad = \theta e^{-\theta}\sum_{n=1}^{\infty}\frac{(\theta F(x))^{n-1}}{(n-1)!} = \theta e^{-\theta}e^{\theta F(x)} = \theta e^{-\theta(1-F(x))}$$

$$E[x_{(1)t}|x_{(1)t} \geq \bar{x}_t] = \frac{\int_{\bar{x}_t}^{\infty} x\theta e^{-\theta(1-F(x))}dF(x)}{\pi(\bar{x}_t, \theta)}$$

The integral in the numerator is

$$\int x\theta e^{-\theta(1-F(x))}dF(x) = \int xe^{-\theta(1-F(x))}d(-\theta + \theta F(x)) =$$
$$= \int xde^{-\theta(1-F(x))} = -\int xd(1 - e^{-\theta(1-F(x))}) = -x(1 - e^{-\theta(1-F(x))}) + \int(1 - e^{-\theta(1-F(x))})dx$$

Hence,

$$\int_{\bar{x}_t}^{\infty} x\theta e^{-\theta(1-F(x))}dF(x) = -x(1 - e^{-\theta(1-F(x))})|_{\bar{x}_t}^{\infty} + \int_{\bar{x}_t}^{\infty}(1 - e^{-\theta(1-F(x))})dx$$

where by L'Hopital's rule

$$\lim_{x\to\infty} x(1-e^{-\theta(1-F(x))}) = \lim_{x\to\infty} \frac{x}{\frac{1}{(1-e^{-\theta(1-F(x))})}} = \lim_{x\to\infty} \frac{1}{-\frac{\theta f(x)}{(1-e^{-\theta(1-F(x))})^2}} = \lim_{x\to\infty} \frac{(1-e^{-\theta(1-F(x))})^2}{\theta f(x)},$$

the last one can be rewritten using the hazard rate $\lambda(x) = f(x)/(1-F(x))$ and Taylor expansion for the exponent

$$\lim_{x\to\infty} \frac{(1-e^{-\theta(1-F(x))})^2}{\theta\lambda(x)(1-F(x))} = \lim_{x\to\infty} \frac{(\theta(1-F(x)))^2}{\lambda(x)\theta(1-F(x))} = \lim_{x\to\infty} \frac{\theta(1-F(x))}{\lambda(x)} = 0,$$

where the last equality is due to the property of the cdf $\lim_{x\to\infty} F(x) = 1$ and weakly increasing hazard rate $\lambda(x)$.

Thus,

$$\int_{\bar{x}_t}^{\infty} x\theta e^{-\theta(1-F(x))} dF(x) = \bar{x}(1-e^{-\theta(1-F(\bar{x}))}) + \int_{\bar{x}_t}^{\infty} (1-e^{-\theta(1-F(x))}) dx.$$

Then the original expectation of the first order statistics given that it is higher than the threshold value is

$$E[x_{(1)t}|x_{(1)t} \geq \bar{x}_t] = \bar{x}_t + \frac{\int_{\bar{x}_t}^{\infty} \pi(x,\theta_t)dx}{\pi(\bar{x}_t,\theta_t)}.$$

□

A.7 Proof of Proposition 6

Proof of Proposition 6. The recursive formulation of the social planner problem constrained by the search frictions is

$$\Omega_t(\bar{B}_t,\bar{S}_t) = \max_{h_t,\bar{x}_t,S_t\in[0,\bar{S}_t],B_t\in[0.\bar{B}_t]} [\frac{\beta}{1-\beta} S_t\pi(\bar{x}_t,\theta_t)E[x_{(1)}|x_{(1)} \geq \bar{x}_t] - w\bar{B}_t - c^S S_t - c^B B_t + \tag{26}$$

$$\mu_t^S(\bar{S}_t - S_t) + \mu_t^B(\bar{B}_t - B_t) + \gamma_t^S S_t + \gamma_t^B B_t + \beta\Omega_{t+1}(\bar{B}_t + d - \pi(\bar{x}_t,\theta_t)S_t, \bar{S}_t + h_t - \pi(\bar{x}_t,\theta_t)S_t)], \tag{27}$$

where $\gamma_t^B, \mu_t^B, \gamma_t^S, \mu_t^S \geq 0$ are the Lagrange multipliers for the restrictions $0 \leq B_t \leq \bar{B}_t$ and $0 \leq S_t \leq \bar{S}_t$, correspondingly.

The first order condition with respect to h_t is $\beta V_{t+1}^S = MC(h_t)$.

Case 1. At least one buyer and seller are active, $B_t > 0$ **and** $S_t > 0$**.**

Using lemma 2, the problem is

$$\Omega_t(\bar{B}_t,\bar{S}_t) = \max_{h_t,\bar{x}_t\geq 0,S_t\in[0,\bar{S}_t],B_t\in[0.\bar{B}_t]} [\frac{\beta}{1-\beta} S_t\pi(\bar{x}_t,\theta_t)\bar{x}_t + \frac{\beta}{1-\beta} S_t \int_{\bar{x}_t}^{\infty} \pi(x,\theta_t)dx - w\bar{B}_t - c^S S_t - c^B B_t + \tag{28}$$

$$\mu_t^S(\bar{S}_t - S_t) + \mu_t^B(\bar{B}_t - B_t) + \gamma_t^S S_t + \gamma_t^B B_t + \beta\Omega_{t+1}(\bar{B}_t + d - \pi(\bar{x}_t,\theta_t)S_t, \bar{S}_t + h_t - \pi(\bar{x}_t,\theta_t)S_t)].$$

Let $V_t^S \equiv \frac{\partial \Omega_t}{\partial S_t}$ be the value of adding a seller and $V_t^B \equiv \frac{\partial \Omega_t}{\partial B_t}$ be the value of adding a buyer, then the envelope conditions are

$$\begin{aligned} V_t^B &= \beta V_{t+1}^B - w + \mu_t^B, \\ V_t^S &= \beta V_{t+1}^S + \mu_t^S, \end{aligned}$$

The first order condition with respect to $\bar{x}_t \geq 0$ is

$$\frac{\beta}{(1-\beta)} S_t \frac{\partial \pi(\bar{x}_t, \theta_t)}{\partial \bar{x}_t} \left(\bar{x}_t - (1-\beta)(V_{t+1}^B + V_{t+1}^S) \right) = 0,$$

with complementary slackness and $\partial \pi(\bar{x}_t, \theta_t)/\partial \bar{x}_t = -\theta_t f(\bar{x}_t)(1 - \pi(\bar{x}_t, \theta_t))$ so that

$$\frac{\beta}{(1-\beta)} B_t (1 - \pi(\bar{x}_t, \theta_t)) f(\bar{x}_t) \left(\bar{x}_t - (1-\beta)(V_{t+1}^B + V_{t+1}^S) \right) = 0. \tag{29}$$

First, consider buyers. The derivative of the right-hand size of (28) with respect to B_t is

$$\begin{aligned} &\frac{\beta}{1-\beta} (\bar{x}_t (1 - F(\bar{x}_t))(1 - \pi(\bar{x}_t, \theta_t)) + \int_{\bar{x}_t}^{\infty} (1 - F(x))(1 - \pi(x, \theta_t)) dx) \\ &\qquad - \beta (V_{t+1}^B + V_{t+1}^S)(1 - F(\bar{x}_t))(1 - \pi(\bar{x}_t, \theta_t)) - c^B - \mu_t^B + \gamma_t^B, \end{aligned} \tag{30}$$

which uses $\partial \pi(x, \theta_t)/\partial \theta_t = (1 - F(x))(1 - \pi(x, \theta_t))$.

Since $B_t > 0$, $\gamma_t^B = 0$. The threshold value $\bar{x}_t$ is determined from equation (29) as $\bar{x}_t = (1-\beta)(V_{t+1}^B + V_{t+1}^S)$, and the first-order condition is

$$\frac{\beta}{1-\beta} \int_{\bar{x}_t}^{\infty} (1 - F(x))(1 - \pi(x, \theta_t)) dx - c^B = \mu_t^B \geq 0,$$

which means that there is a non-negative surplus represented by the expression on the left-hand side to share. Then the value of the buyer then follows $V_t^B = \beta V_{t+1}^B - w + \mu_t^B$. Plugging the expression from the previous equation gives

$$V_t^B = \beta V_{t+1}^B - w + \frac{\beta}{1-\beta} \int_{\bar{x}_t}^{\infty} (1 - F(x))(1 - \pi(x, \theta_t)) dx - c^B.$$

Now consider sellers. The derivative of the right-hand size of (28) with respect to S_t is

$$\begin{aligned} &\frac{\beta}{1-\beta} (\bar{x}_t - (1-\beta)(V_{t+1}^B + V_{t+1}^S)) \pi(\bar{x}_t, \theta_t) + \frac{\beta}{1-\beta} \int_{\bar{x}_t}^{\infty} \pi(x, \theta_t) dx \\ &- \theta_t \frac{\beta}{1-\beta} \int_{\bar{x}_t}^{\infty} (1 - F(x))(1 - \pi(x, \theta_t)) dx - \frac{\beta}{1-\beta} (\bar{x}_t - (1-\beta)(V_{t+1}^B + V_{t+1}^S)) \theta_t (1 - F(\bar{x}_t))(1 - \pi(\bar{x}_t, \theta_t))) \\ &- c^S - \mu_t^S + \gamma_t^S. \end{aligned} \tag{31}$$

To compare with the equilibrium allocation of the auction model with directed search the first three terms can be rearranged using $\bar{x}_t\pi_t/\theta_t + \int_{\bar{x}_t}^{\infty} \pi(x,\theta_t)dx/\theta_t = \int_{\bar{x}_t}^{\infty} x(1-\pi(x,\theta_t))f(x)dx$ and $(V_{t+1}^B + V_{t+1}^S)\pi_t/\theta_t = \int_{\bar{x}_t}^{\infty}(V_{t+1}^S + V_{t+1}^B)e^{-\theta_t(1-F(x))}f(x)dx$ so that

$$\frac{\beta}{1-\beta}\theta_t \int_{\bar{x}_t}^{\infty}(x - \frac{(1-F(x))}{f(x)} - (1-\beta)(V_{t+1}^S + V_{t+1}^B))e^{-\theta_t(1-F(x))}f(x)dx - \tag{32}$$
$$-\frac{\beta}{1-\beta}(\bar{x}_t - (1-\beta)(V_{t+1}^B + V_{t+1}^S))\theta_t(1-F(\bar{x}_t))(1-\pi(\bar{x}_t,\theta_t))) - c^S - \mu_t^S + \gamma_t^S,$$

The second term in equation (32) is zero from the optimality of the threshold value $\bar{x}_t = (1-\beta)(V_{t+1}^S + V_{t+1}^B)$ for $B_t > 0$. The Lagrange multiplier $\gamma_t^S = 0$, because $S_t > 0$, hence

$$\frac{\beta}{1-\beta}\theta_t \int_{\bar{x}_t}^{\infty}(x - \frac{(1-F(x)}{f(x)} - (1-\beta)(V_{t+1}^S + V_{t+1}^B))e^{-\theta_t(1-F(x))}f(x)dx - c^S - \mu_t^S = 0. \tag{33}$$

If $\mu_t^S = 0$, then the surplus from transfer is zero and $V_t^S = \beta V_{t+1}^S$. If $\mu_t^S > 0$, then the surplus from transfer is zero and

$$V_t^S = \beta V_{t+1}^S + \frac{\beta}{1-\beta}\theta_t \int_{\bar{x}_t}^{\infty}(x - \frac{1-F(x)}{f(x)} - (1-\beta)(V_{t+1}^S + V_{t+1}^B))e^{-\theta_t(1-F(x))}f(x)dx - c^S.$$

Case 2. Either all buyers or all sellers are not active, i.e. $B_t = 0$ **or** $S_t = 0$**.**

The first term in (26), representing the surplus from the transfer of the house from seller to buyer, is zero, and, using the envelope condition, the dynamics of the value of adding a seller $V_t^S \equiv \frac{\partial \Omega_t}{\partial \bar{S}_t}$ and the value of adding a buyer $V_t^B \equiv \frac{\partial \Omega_t}{\partial \bar{B}_t}$ is

$$V_t^B = \beta V_{t+1}^B - w,$$
$$V_t^S = \beta V_{t+1}^S.$$

The tightness θ_t and the threshold value of housing services $\bar{x}_t$ are not defined.

Summarizing both cases, the conditions for the social optimum are

$$V_t^B = \beta V_{t+1}^B - w + \max\{\frac{\beta}{1-\beta}\int_{\bar{x}_t}^{\infty}(1-F(x))(1-\pi(x,\theta_t))dx - c^B, 0\},$$
$$V_t^S = \beta V_{t+1}^S + \max\{\frac{\beta}{1-\beta}\theta_t \int_{\bar{x}_t}^{\infty}(x - \frac{(1-F(x))}{f(x)} - (1-\beta)(V_{t+1}^S + V_{t+1}^B))e^{-\theta_t(1-F(x))}f(x)dx - c^S, 0\},$$
$$\bar{x}_t = \max\{(1-\beta)(V_{t+1}^B + V_{t+1}^S), 0\} \text{ when } B_t, S_t > 0,$$
$$MC(h_t) = \beta V_{t+1}^S,$$

in addition to the equations for the dynamics of the state, and resource constraints. □

Lemma 3. $\varphi(-\log(1-\pi))$ is an increasing convex function of π

Proof.

$$\frac{\partial \varphi(-\log(1-\pi))}{\partial \pi} = \varphi' \times \frac{\partial - \log(1-\pi)}{\partial \pi} = -\varphi' \frac{1}{1-\pi}(-1) = \varphi'/(1-\pi)$$

where

$$\varphi' = \frac{1 - e^{-(-\log(1-\pi))}}{-\log(1-\pi)} = \frac{1 - \exp(\log(1-\pi))}{-\log(1-\pi)} = -\frac{\pi}{\log(1-\pi)} > 0$$

$$\frac{\partial\varphi(-\log(1-\pi))}{\partial\pi} = \varphi'/(1-\pi) = -\frac{\pi}{(\log(1-\pi))(1-\pi)}$$

so that

$$(\frac{\partial\varphi(-\log(1-\pi))}{\partial\pi})'_{\pi} = -[\frac{[\frac{\pi}{1-\pi}]'\log(1-\pi) - \frac{\pi}{1-\pi}\frac{-1}{1-\pi}}{(\log(1-\pi))^2}]$$

where the numerator is

$$\frac{(1-\pi)-(-1)\pi}{(1-\pi)^2}\log(1-\pi) + \frac{\pi}{(1-\pi)^2} = \frac{\pi + \log(1-\pi)}{(1-\pi)^2}$$

where $\log(1-\pi) < -\pi$ so that the

$$\frac{\partial\varphi(-\log(1-\pi))}{\partial\pi} = \varphi'/(1-\pi) = -\frac{\pi}{(\log(1-\pi))(1-\pi)} = -\frac{\log(1-\pi)+\pi}{(1-\pi)^2(\log(1-\pi))^2} > 0$$

and the curve is convex in π.

$\square$

Lemma 4. $\varphi(-\log(1-\pi))/\pi$ is an increasing convex function of π.

Proof.

$$\frac{\partial\varphi(-\log(1-\pi))/\pi}{\partial\pi} = \frac{\frac{\partial\varphi}{\partial\pi}\pi - \varphi}{\pi^2}$$

Since $\varphi(-\log(1-\pi)) \equiv \varphi(\pi)$ is convex function in π, it is true that $\varphi(0) \geq \varphi(\pi) + \varphi'(\pi)(0-\pi)$. Hence, $\varphi(\pi) \leq \varphi'(\pi)\pi$, so

$$\frac{\partial\varphi(-\log(1-\pi))/\pi}{\partial\pi} = \frac{\frac{\partial\varphi}{\partial\pi}\pi - \varphi}{\pi^2} \geq 0,$$

which means that the function $\varphi(\pi)/\pi$ is increasing.

The function is also convex in π, because

$$\frac{\partial^2\varphi(-\log(1-\pi))/\pi}{\partial\pi^2} = \frac{\frac{\partial}{\partial\pi}(\frac{\partial\varphi}{\partial\pi}\pi - \varphi) - 2\pi(\frac{\partial\varphi}{\partial\pi}\pi - \varphi)}{\pi^4}.$$

The numerator can be further simplified as

$$\frac{\partial^2\varphi}{\partial\pi^2}\pi + \frac{\partial\varphi}{\partial\pi} - \frac{\partial\varphi}{\partial\pi} - 2\pi^2\frac{\partial\varphi}{\partial\pi} + 2\pi\varphi = \pi(\frac{\partial^2\varphi}{\partial\pi^2} - 2\pi(\frac{\partial\varphi}{\partial\pi}\pi - \varphi))$$

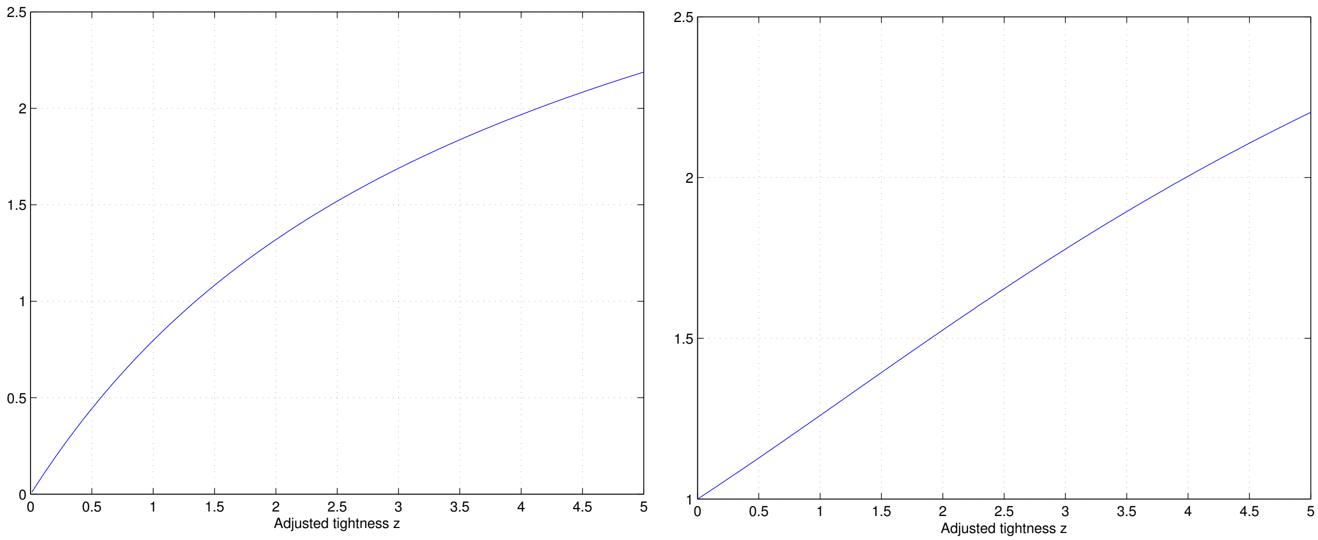
Since φ is convex, $\frac{\partial^2\varphi}{\partial\pi^2} \leq 0$ and $\frac{\partial\varphi}{\partial\pi}\pi - \varphi \geq 0$. Thus, $\frac{\partial^2\varphi(-\log(1-\pi))/\pi}{\partial\pi^2} < 0$ and φ/π is convex in π.

$\square$

B Figures

Figure 9: The functions $\varphi(z)$ and $\frac{\varphi(z)}{\pi(z)}$ as functions of adjusted tightness $z = \theta(1 - F(\bar{x}))$

(a) $\varphi(z)$ (b) $\frac{\varphi(z)}{\pi(z)}$



The parameters for the distribution function $F(.)$ are taken from the calibration of Section 4.4.