Firm Size and Capital Structure*

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First version: January 2005. This version: October 2005

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*We would like to thank Wouter Denhaan, Darrell Duffie, Mike Harrison, Kristian Miltersen and participants of seminars at Arizona State University, Stanford GSB, and the European Finance Association 2005 meeting in Moscow.
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Abstract

Firm size has been empirically found to be strongly positively related to capital structure. A number of intuitive explanations can be put forward to account for this stylized fact, but none have been considered theoretically. This paper starts bridging this gap by investigating whether a dynamic capital structure model can explain the cross-sectional size-leverage relationship. The driving force that we consider is the presence of fixed costs of external financing that lead to infrequent restructuring and create a wedge between small and large firms. We find four firm size effects on leverage. Small firms choose higher leverage at the moment of refinancing to compensate for less frequent rebalancings. But longer waiting times between refinancings lead on average to lower levels of leverage. Within one refinancing cycle the relationship between leverage and firm size is negative. Finally, there is a mass of firms opting for no leverage. The analysis of dynamic economy demonstrates that in cross-section the relationship between leverage and size is positive and thus fixed costs of financing contribute to the explanation of the stylized size-leverage relationship. However, the relationship changes the sign when we control for the presence of unlevered firms.

Keywords: Capital structure, leverage, firm size, transaction costs, default, dynamic programming, dynamic economy, refinancing point
JEL Classification Numbers: G12, G32

Firm size has become such a routine to use as a control variable in empirical corporate finance studies that it receives little to no discussion in most research papers even though not uncommonly it is among the most significant variables. This paper's goal is to provide rationale for one of the size relationships, that is between firm size and capital structure. Cross-sectionally, it has been consistently found that large firms in the U.S. tend to have higher leverage ratios than small firms. International evidence suggests that in most, though not all, countries leverage is also cross-sectionally positively related to size.[1] Intuitively, firm size matters for a number of reasons. In the presence of non-trivial fixed costs of raising external funds large firms have cheaper access to outside financing per each dollar borrowed.[2] Related, larger firms are more likely to diversify their financing sources. Alternatively, size may be a proxy for the probability of default, for it is sometimes contended that larger firms are more difficult to fail and liquidate, or, once the firm finds itself in distress, for recovery rate.[3] Size may also proxy for the volatility of firm assets, for small firms are more likely to be growing firms in rapidly developing and thus intrinsically volatile industries.[4] Yet another explanation is the extent of the wedge in the degree of information asymmetry between insiders and the capital markets which may be lower for larger firms, for example because they face more scrutiny by ever-suspicious investors.

All these explanations are intuitively appealing and it is very likely that all of them are, to smaller or larger extent, at work. All these explanations also remain in the realm of intuition: while economic theories have been preoccupied with the determinants of firm size and its optimality at least since Coase (1937), existing theories are silent on the firm size effect on the extent of external financing and, in particular, on quantitative implications that would provide rationale behind the observed size-leverage relationship. Empirical research has also not attempted so far to investigate and decompose the size effect and some observed stylized facts appear to be inconsistent with the proposed explanations. This state of affairs has made researchers admit that "we have to conclude that we do not really understand why size is correlated with leverage" (Rajan and

[1]Titman and Wessels (1988), Rajan and Zingales (1995), and Fama and French (2002) are among many others documenting cross-sectional evidence for the US. International evidence is documented in Rajan and Zingales (1995) for developed countries and Booth et al. (2001) for developing countries.

[2]Indirect evidence of fixed costs is provided by studies documenting infrequent restructuring and the mean reversion of leverage (see e.g. Leary and Roberts (2004)). Altinkilic and Hansen (2000) and Kim, Palia, and Saunders (2003) find costs of public debt issuance in the order of 1% of debt principal. In addition, Altinkilic and Hansen estimate fixed costs to constitute about 10% of issuance costs on average (they are, of course, relatively more important for small firms). Section III.2 provides further detailed discussion.

[3]As the cases of Chrysler in the US and Fiat in Italy demonstrate. Shumway (2001) finds that the size of outstanding equity is an important predictor of bankruptcy probability. Duffie, Saita and Wang (2005) report that a 10% increase in asset value decreases the probability of default by about 2% conditional on firm's financing arrangements.

[4]To this effect, Fama and French (2002), among others, justify the conditioning of cross-sectional leverage regressions on size. This and default probability explanations are related since, conditional on the financial structure, low volatility firms are less likely to default.

Zingales, 1995, p. 1457). In this paper, we make the very first step towards bridging this important gap in our knowledge. We choose, for clarity and relative simplicity, only one source of the size effect, namely the "fixed transaction cost" explanation, and attempt to see to what extent this explanation alone can provide economically reasonable relationship between size and leverage. The presence of external financing issuance costs suggests the dynamic nature of the problem. Thus, the exact question that we ask in this paper, is: *Can a theory of dynamic capital structure produce economically reasonable size effect due to fixed transaction costs of financing?*

This paper's contribution to the field is two-fold. First, we develop a theoretical framework and the solution method that can be applied to a wide class of dynamic financing problems. Second, by applying this framework to the problem at hand, we advance an intuition behind the firm size and leverage dynamic relationship both at the level of an individual firm and cross-section and investigate the cross-sectional relationship quantitatively. While we are ambivalent to the choice between a number of worthy theoretical ideas explaining leverage, we choose the trade-off model, which balances tax benefits of debt with various distress costs, as our workhorse. Not the least reason for our choice is that it is the only theory at present to produce viable dynamic models delivering quantitative predictions. The theoretical framework and, to some extent, a number of results are invariant to a particular driving force of capital structure decisions.

To understand the economic intuition and empirical predictions behind our first set of findings, consider an all-equity firm contemplating debt issuance for the first time. The firm will choose an optimal leverage ratio that will balance the trade-off between expected tax benefits of debt and distress costs. In the absence of fixed costs, the firm will find it optimal to lever up immediately and will subsequently increase its debt continuously as its fortunes improve to restore the optimal balance. With fixed costs, however, it is suboptimal to refinance too often. The timing decision of the next refinancing will now balance fixed debt issuance costs with benefits of having more debt. The infrequency of refinancings will lower expected tax benefits and, to compensate for that, at each refinancing the firm will take on more leverage. (Of course, very small firms will find it optimal to postpone their first debt issuance until their fortunes improve substantially relative to the costs of issuance and so the model also delivers the result that some firms are optimally zero-levered). The higher expected costs of future financing also imply that firms default sooner, at a higher level of asset value. As firm size increases, fixed costs become relatively less important and thus expected waiting times between refinancings are shorter and leverage at refinancings is closer to the no-fixed-cost case. In the limit, the firm's optimal decision coincides with the decision in the absence of fixed costs.

It follows from the above discussion that, if we consider the comparative statics of firm's optimal decision at the time of refinancing, the relationship between size and leverage will be negative, for smaller firms will have, conditional on issuing, an additional benefit of sustaining higher level of debt. This comparative statics result, which we call *the beginning-of-cycle effect*, is inconsistent with the observed empirical cross-sectional relationship that larger firms have higher leverage. However, firms are rarely at their refinancing points; they are more likely to be in the midst of a refinancing cycle, between two restructurings. This leads to two more effects on an individual firm level. The first one that we call *the within-cycle effect* says that, between any two consecutive restructurings, the market value of equity increases as the firm grows, decreasing the quasi-market leverage ratio and thus inducing the negative correlation between firm size and leverage. The second effect that we call *the end-of-cycle effect* implies that while smaller firms rely on issuing more debt, they issue less often and so longer waiting times lead on average to lower levels of leverage at the end of the refinancing cycle. With non-trivial waiting times, the presence of distress costs creates asymmetry between costs of being a low and a high levered firm and leads firms, in pursuit of decreasing expected costs of distress, to increase leverage at refinancing only slightly but wait longer. For an individual firm, the "total" comparative statics within one refinancing cycle is thus opposite of the static one: smaller firms tend on average to have lower leverage and so the relationship between firm size and capital structure is a positive one.

In any cross-section, firms are likely to be at different stages of their refinancing and it is impossible to predict the implications of the above two effects on cross-sectional behavior by considering only the dynamic comparative statics of an individual firm. To complicate the matter, cross-sectional results are likely to be contaminated by the presence of zero-leverage firms in the economy which also happen to be the smallest in size. Since they constitute a mass point, they may induce a positive relationship between size and firm. We call this purely cross-sectional effect *the zero-leverage effect*. To see the joint outcome of the four effects discussed above, we investigate the dynamic cross-sectional properties of the model to which our second set of results relates. In particular, we are interested in whether transaction costs can be the driving factor in the cross-sectional relationship between size and leverage and whether they also can be responsible for the mean reversion of leverage. For the benchmark case, replicating the standard empirical approach on artificially generated data, we find the strongly positive relationship between firm size and leverage, consistent with empirical findings. Thus, we find that transaction costs can, in theory, explain the sign of the leverage-size relationship. Quantitatively, the slope coefficient of size is very similar to the one found in the COMPUSTAT data set.

Interestingly, we also find that the positive relationship is an artefact of the presence of small unlevered firms in the economy. When we control for unlevered firms, the relationship between firm size and leverage becomes slightly but statistically significant negative. Whether controlling for the zero-leverage effect overturns the empirical result demonstrating that the beginning-of-cycle and within-cycle effects dominate the end-of-cycle effect is an open empirical question. This, for example, can be an explanation of the empirical finding that size has a negative impact on leverage in Germany (reported by Rajan and Zingales (1995)) for the German capital markets are less developed and only relatively large firms are publicly traded.

We also develop a number of empirical implications that can be derived from the cross-sectional dynamics of our model and which easily lend themselves to testing using standard corporate finance data sets. Small firms should restructure less frequently, add more debt at each restructuring, have higher likelihood of default and lower mean reversion. The slope coefficient of firm size in standard leverage level regressions is predicted to be negative for small levered firms, increasing with firm size and be insignificant for large firms.

It may seem surprising that the question we are investigating here has not been addressed before. A traditional dynamic capital structure framework, rooted in the trade-off explanation, has been developed and successfully applied to many problems in works by Fisher, Heinkel and Zechner (1989), Leland (1998), Goldstein, Ju and Leland (2001), Ju, Parrino, Poteshman and Weisbach (2003), Christensen, Flor, Lando, and Miltersen (2002) and Strebulaev (2004). However, this framework can not be applied for our purposes. The major modelling trick the above models utilize is the so-called "scaling feature" (or first-order homogeneity property) which implies that at rebalancing points firms are replicas of themselves, just *proportionally* larger. All dollar-denominated variables (such as the value of the firm) are scaled and all ratios (such as debt to equity ratio) are invariant to scaling. The necessity of such an assumption comes from difficulties associated with the dimensionality of the optimization problem. Essentially, the scaling assumption allows researchers to map a dynamic problem into a static problem, for the optimal firm behavior at one refinancing point is identical to its behavior at all other refinancing points. Since all ratios are invariant to size, there are really no fixed costs and therefore the true size of the firm never enters the equation. To avoid instantaneous readjustments of debt which will happen if costs are proportional to a marginal increase in debt, some of these models make an unrealistic but necessary assumption that refinancing costs are proportional to the total debt outstanding.[5,6]

To be able to characterize the solution for our question, we reconsider the existing framework and develop a new solution method, which is the second contribution of this paper. The modified framework enables us to solve for the optimal dynamic capital structure for truly fixed costs of external financing in a time-consistent rational way. By time consistency we mean that the optimal solution can be formulated as a certain rule at the initial date and that this rule will be optimal at

[5] Mauer and Triantis (1994) develop a framework, with the solution based on finite difference method, of dynamic financing and investment decisions that allows for fixed costs of refinancing. In their model, equityholders always maximize the value of the firm, i.e. there is no ex-post friction between equity- and debtholders where equity does not internalize the value of debt outstanding in making default or issuance decisions.

[6] Another approach is to introduce exogenously fixed intervals between rebalancings.

all subsequent dates. Importantly, our framework introduces both firm size and realistic marginal proportional costs which are now proportional only to new net debt issued. The solution method is both computationally feasible and intuitive and can potentially be applied to a wide array of problems of corporate finance featuring fixed costs, for example, the role of firm size in other corporate finance investigations.

The intuitive idea behind the solution method is as follows. Instead of the case when the firm pays fixed costs for every restructuring, assume that no fixed costs are to be paid after a (sufficiently large) number of restructuring. Right after the last "fixed-cost" restructuring, the model will then represent a standard dynamic capital structure model (as modelled e.g. by Goldstein, Ju and Leland (2001)). At the penultimate "fixed-cost" restructuring, the optimal solution will depend on the value of assets *at this* and *next* restructurings both of which are unknown at the initial date. We proceed by guessing the firm size (measured by asset value) at the last "fixed-cost" restructuring. The firm size summarizes all past history of the firm. Having guessed the asset value, we can find conditional optimal solution for previous restructuring. Using a recursive procedure, we find optimal firm sizes at all previous restructurings conditional on our initial guess. Finally we solve for the optimal size at the initial date. Since the initial firm value is known, we compare these two values. If our initial guess did not lead to indistinguishable values, we readjust the initial guessed value and repeat the recursive procedure until our guess and the initial firm value coincide. We show that this procedure converges to the optimal solution in a computationally friendly way. Since this framework features more realistic description of transaction costs that have been so prominent in dynamic financing models, one of course can make these models more lifelike by incorporating in them modified transaction costs and solving them using this method.

To investigate the dynamic properties of the model, we use the method developed for related capital structure issues in Strebulaev (2004) based on the simulation procedure of Berk, Green, and Naik (1999). In particular, we simulate a number of dynamic economies and, after calibrating for the initial distribution of firm size to match the distribution of firms in COMPUSTAT, we replicate the empirical analysis conducted by cross-sectional capital structure studies.

The remainder of the paper is organized as follows. Section I develops the theoretical framework and the solution algorithm. Section II applies the general framework to develop a dynamic trade-off model with fixed costs and analyzes the optimal solution and firm size effects in comparative statics. Section III contains the main dynamic analysis of the paper. Section IV discusses empirical implications of the model. Section V considers a number of extensions and Section VI concludes. Appendix A contains all proofs and Appendix B provides details of the simulation procedure.

I Theoretical framework

In this section, we first develop a general model of firm dynamic financing decisions, then provide the solution framework and, finally, apply the framework to the problem at hand by considering the dynamic trade-off model.

I.1 A general model of dynamic financing

Our model of firm dynamic financing is based on the following economic assumptions, a number of which can be relaxed with sacrifice to the simplicity of the exposition: (1) External financing is costly and includes a fixed component; (2) At every point in time, owners of the firm choose financing policy to maximize their wealth; (3) Markets are perfectly rational and foresee all future actions by the owners of the firm; (4) Investment policy is independent of financial policy. The crucial assumption to what follows is the first one. The second assumption creates the necessary agency costs between the owners of the firm and other claimants to firm assets and cash flows, for the owners do not internalize the ex-post value of other claimant's securities, but particulars can be relaxed. For example, by considering "owners" we abstract from equityholders-manager conflicts, which could be easily introduced. The third assumption, while standard in most dynamic financing models, can be relaxed by allowing, for instance, behavioral biases and restrictions on no-arbitrage conditions. The fourth is the standard Modigliani-Miller assumption.

The first assumption prevents the firm from "too frequent" changing the book composition of its financial structure. In the continuous framework we consider, the set of time points when the firm takes no active financial decisions has a full measure: most firms prefer to be passive most of the time. The times when firms decide to change their financial structure we call interchangeably "refinancings", "restructurings" or "recapitalizations" followed terminology used by Fischer, Heinkel, and Zechner (1989), Goldstein, Ju and Leland (2001) and Strebulaev (2004).

Other ingredients of the model are as follows. The owners of the firm choose between two types of external financing: common equity and debt. For simplicity, we abstract from considering more complex financial structures. Decision to refinance depends on changes in firm fortunes and we introduce dichotomous types of refinancings: default, when the future is bleak, and "upper" refinancing, when fortunes are excellent.[7]

The firm owns a profit-generating project with the present value of cash flows at any date t denoted by V_t. The random behavior of V stands for what we call "firm fortunes" and thus many variables measured relative to V are called V-adjusted. At any date t, the firm has gone through

[7]Other refinancing types can be introduced easily. For example, Fischer, Heinkel, and Zechner (1989) consider "lower" restructuring and Strebulaev (2004) considers liquidity-driven refinancing both of which are intermediate states between "upper" restructuring and default.

k upper restructurings, $k = 0, 1, ...$, and we will call the firm as being in the $(k+1)^{st}$ refinancing cycle (period) if it is between the k^{th} and $(k+1)^{st}$ restructurings. The firm starts at date t_0 as an all-equity firm and, if it decides to restructure immediately, zero restructuring takes place at date t_0. The presence of fixed costs, however, may lead the firm to postpone introducing debt. Zero restructuring, thus, is defined as the first restructuring when the firm changes its status from an all-equity firm to a partially debt-financed firm.

At any restructuring, owners' actions can be described in terms of changing the level of debt and the wealth redistribution associated with it. At the beginning of the k^{th} refinancing cycle, the firm adjusts its debt level by promising a constant V-adjusted (in units of V_{k-1}) coupon payment c_k to debtholders as long as the firm remains solvent. The owners also determine the new V-adjusted threshold $\psi_k V_k$ at which they choose to default. Finally, the owners decide when to restructure next time by finding $\gamma_k = \frac{V_k}{V_{k-1}}$. So, given the initial value of asset, V_{t_0}, the k^{th} restructuring takes place when the asset value size reaches $\Gamma_k V_{t_0}$, where

$$\Gamma_k = \prod_{m=0}^{k} \gamma_m. \tag{1}$$

The firm equityholders' claim to intertemporal cash flows during refinancing cycle k in units V_{k-1} is denoted by $e(x_k)$, $x_k = \{\gamma_k, c_k, \psi_k\}$, and similar claim of debtholders is defined as $d(x_k)$. For what follows it is important to observe that $e(x_k)$ and $d(x_k)$ are independent of V_{k-1}. Note that both $e(x_k)$ and $d(x_k)$ include the present value of default payouts. The combined value of debt and equity within the k^{th} refinancing cycle is $F(x_k)$ and it is the sum of $e(x_k)$ and $d(x_k)$ subtracting any *proportional* transaction costs associated with *period-k* debt or equity issuance. The sum of equity and debt intertemporal cash flows does not necessarily equal the total payout of the project due to issuance costs and the presence of other claims (e.g. government taxes). *Fixed* costs of refinancing paid at the moment of refinancing k are constant at q_k.

Let $p(x_k)$ be the refinancing date $k-1$ value of a claim that pays \$1 contingent on asset value reaching $V_k = \gamma_k V_{k-1}$ (before reaching the bankruptcy threshold $\psi_k V_{k-1}$). Owners' objective is to choose $x_k = (\gamma_k, \psi_k, c_k)$, $k = 0, 1, ...$, in order to maximize the present value of their claim, Ω, subject to limited liability:

$$\begin{cases} \Omega = V_{t_0} F(x_0) + \sum\limits_{k=0}^{\infty} P_k \left[V_{t_0} \Gamma_k F(x_{k+1}) - q_k \right] \to \max\limits_{x_0, x_1, \ldots} \\ s.t.\ B_k = \widetilde{p}(x_k) \left\{ \sum\limits_{m=k}^{\infty} \frac{P_m}{P_k} \left[V_{t_0} \Gamma_m F(x_{m+1}) - q_m \right] - V_{t_0} \Gamma_{k-1} D^0(x_k) \right\} + V_{t_0} \Gamma_{k-1} \widetilde{e}(x_k) = 0, \quad k > 0 \\ \quad \gamma_k \geq 1, \quad k \geq 0, \qquad c_0 = 0, \qquad \psi_0 = 0, \end{cases} \tag{2}$$

where

$$P_k = \prod_{m=0}^{k} p(x_m), \qquad \widetilde{p}(x_k) = \left. \frac{\partial p(V; x_k)}{\partial V} \right|_{V=\psi_k}, \qquad \widetilde{e}(x_k) = \left. \frac{\partial e(V; x_k)}{\partial V} \right|_{V=\psi_k}.$$

The "budget constraints" B_k are smooth-pasting conditions determining when the firm will default on its debt in corresponding refinancing cycles. More precisely,

$$B_k = \left. \frac{\partial E_k(V; x)}{\partial V} \right|_{V=\psi_k}, \tag{3}$$

where

$$E_k(V; x) = p(V; x_k) \left[\sum_{m=k}^{\infty} \left(\frac{\Gamma_m}{\Gamma_{k-1}} \frac{P_m}{P_k} F(x_{m+1}) - \frac{P_m}{P_k} q_k \right) - D^0(x_k) \right] + e(V; x_k) \tag{4}$$

is the date k-1 total equity claim for arbitrary asset value V during period k, and $D^0(x_k)$ is the V-adjusted called value of the k^{th}-period debt (if the debt issue is called). Constraint $\gamma_0 \geq 1$ specifies that firms can issue debt only starting at date t_0.

I.2 Solution method: Heuristic Approach

In this section we provide an heuristic description of the method that we have developed to solve our problem.

In the presence of fixed costs, firms restructure infrequently. Extant dynamic capital structure models with no fixed costs (e.g. Goldstein, Ju and Leland (2001)) incorporate infrequent restructuring by assuming that proportional costs of issuing debt are proportional to the total debt issued (rather than to its incremental amount). The absence of fixed costs simplifies the problem substantially since proportionality leads the firm at every refinancing be a scaled replica of itself. However, in addition to modelling proportional costs rather unrealistically, it does not allow to address any issues related to firm size. Technically, this method is equivalent to the dynamic programming approach in discrete time where the nominal value of the value function next period is the same as the total maximized value, which, in turn, allows to find the solution as a root of a stationary Bellman equation. If issuance costs have the component that is independent of firm size, the scaling property is lost and the above method does not work.

Our method to solve the problem can be intuitively described as follows. Imagine that, instead of the case when fixed costs are to be paid at every restructuring, as it would be the case in a truly dynamic model of financing we envision, the firm has to pay fixed costs only for finite number of restructurings K after very first levering up ("zero restructuring"). We assume that K is perfectly known by all market participants. For now, we make an additional temporary assumption (which will be relaxed below) that "zero restructuring" coincide with initial date t_0 (i.e. the firm will immediately lever up).

At the initial date t_0, the owners, who maximize the present value of their claim, determine conditions of all subsequent restructurings, in particular all subsequent debt levels and default conditions. After the K^{th} restructuring, the firm is back into the no-fixed-cost dynamic financing problem which can be easily solved using existing methods (since the scaling property holds).

The presence of fixed costs leads the firm's optimal decision to depend on the absolute level of asset value. For sake of exposition we will refer in this section to V as firm size, even though strictly speaking it is not true since V includes also such claims as government taxes and various costs that are not covered by firm size. Since firm value and asset value are highly positively related, this substitution is not of any material consequences.

The equityholders' decision depends on past financing of the firm which is an accumulation of information on all firm sizes at which the firm refinanced previously. The most important point to observe is that the information content of the firm size in the k^{th} refinancing cycle, represented by $\Gamma_{k-1}V_{t_0}$ (see equation (1)), is sufficient information relevant to the owners for making decisions. In other words, in restructuring space, firm size follows a Markov process. Intuitively, this holds because the owners care not about the "dollar value" of fixed restructuring costs, but about what may be called "firm-size adjusted" costs, which depend only on the current level of asset value: the larger the firm, the smaller the relative costs of refinancing. Intuitively, Γ_{k-1} can be thought of as a sufficient statistic in the k^{th} refinancing cycle.

With that intuition, we start with last "fixed-cost" restructuring K and guess firm size at this restructuring, $\Gamma_K V_{t_0}$. Knowing Γ_K and the solution right after the last "fixed-cost" restructuring (which is a "static" financing problem) we can solve for optimal decision at the $(K-1)^{st}$ restructuring, which includes coupon and default levels in the K^{th} cycle, as well as γ_K (this is because $\Gamma_{K-1} = \Gamma_K/\gamma_k$; in other words, conditional on knowing Γ_K, we can answer the question what the optimal firm size was at the previous restructuring). Recursively, we find optimal firm sizes at all restructurings $K-2, ..., 2$. For the first restructuring there are two ways to find γ_1. First, knowing all other γs and the initial firm value V_{t_0}), we can find γ_1^* directly. But also, given V_{t_0}, we can solve for optimal γ_1, which we denote γ_1^{**}, in the same way as for previous restructurings. If our guess of Γ_K was a correct one, these two methods of finding γ_1 will give the same solution. If $\gamma_1^* > \gamma_1^{**}$, then optimal Γ_K is smaller than our guess, and vice versa. Thus, if the two solutions are not equal, we refine our guess of Γ_K and repeat the procedure.

There are two additional economic features that have so far been left unsatisfactorily. The first, a finite number of restructurings with fixed-cost payments, is addressed by showing that the solution to the above problem converges to the original one as the number of refinancings with fixed costs, K, increases. Second, zero restructuring does not necessarily takes place at date t_0. For relatively small firms, in the presence of fixed costs, it is optimal to wait until the firm fortunes improve (and so firm-size adjusted costs decrease) rather than to issue debt immediately. Thus, for some initial period small firms will have zero leverage. Therefore, our definition of zero restructuring incorporates the constraint that this restructuring occurs when the firm levers up for the first time. It is sufficient to find the threshold level of q_0 that will separate small and large firms at date t_0 and thus condition for zero restructuring, γ_0.

This intuitive discussion has thus far been very imprecise about numerous technical details. The purpose of the next subsection is to accomplish this task.

I.3 Solution method

The original model we want to solve is problem (2) with fixed costs $q_k = \bar{q} = 1$.[8] We introduce an auxiliary model in which fixed costs q_k after the K^{th} restructuring are assumed to be zero:

$$q_k = \begin{cases} 1, & k \leq K, \\ 0, & k > K. \end{cases} \tag{5}$$

Intuitively, since the maximized function in (2) is restricted from the above by V_{t_0}, the present-value sum of fixed-cost payments for all restructurings is a convergent series and therefore, for sufficiently large K, its residual goes to zero. On the other hand, this residual, the present value of the sum of claims to the fixed costs paid for restructurings $K+1, K+2...$, is exactly the difference between the solutions to firm's problem with fixed costs given by (5) and the one with fixed costs equal to $q_k = 1$ for all restructurings.

In problem (2) with fixed costs given by (5), after the K^{th} restructuring we are back to the standard no-fixed-cost problem, for which the scaling property holds. Therefore, following, for example, Goldstein, Ju and Leland (2001), the solution to this problem can be represented (in units of V_K) as:

$$\begin{cases} \Omega_\infty = \max_x \left[\frac{F(x)}{1-p(x)\gamma}\right] \\ \\ s.t. \quad \widetilde{p}\,(x) \left[\gamma \frac{F(x)}{1-p(x)\gamma} - D^0\,(x)\right] + \widetilde{e}\,(x) = 0, \end{cases} \tag{6}$$

Defining the state variable s_k as the "size of the firm" in the beginning of the k^{th} refinancing cycle

$$s_k = V_{k-1} = \Gamma_{k-1} \times V_{t_0}, \tag{7}$$

[8]So that all claims are valued in the units of fixed-cost payments.

and the period-k transition function as

$$s_{k+1} = f(s_k, x_k) = s_k x_{1,k} = s_k \gamma_k, \tag{8}$$

we observe that in (2) the equityholders' decision about the k^{th} restructuring, x_k, depends on all past states and actions $\{s_0, x_0, ..., x_{k-1}, s_k\}$ only through the value of the period-k state s_k. This allows us to reformulate problem (2) as a Markovian infinite-dimensional dynamic programming problem.

More formally, define the state space $S = (0, +\infty)$ and the control space $X = [1, +\infty) \times [0, 1] \times [0, 1]$, so that $x = \{x_1, x_2, x_3\} = \{\gamma, \psi, c\}$. The value function $\Omega(s_k)$ is going to be the s_k-measured shareholder's wealth at date k. Note that, since $\lim_{x_1 \to \infty} p(x) = \lim_{x_1 \to \infty} \widetilde{p}(x) = 0$, imposing an additional condition $x_1 = \infty$ leads to the static (when no more restructurings are allowed) and thus independent of firm size solution Ω_0:

$$\begin{cases} \Omega_0 = \max_{x_2, x_3} \lim_{x_1 \to \infty} F(x) \\ s.t. \quad \lim_{x_1 \to \infty} \widetilde{e}(x) = 0. \end{cases} \tag{9}$$

Therefore, $\Omega(s) \geq \Omega_0$. In Appendix A we also show that a quite intuitive fact that the value function $\Omega(s)$ is bounded from above by the no-fixed-cost solution Ω_∞ as given by (6). This allows us to consider $B = \{\Omega : S \to [\Omega_0, \Omega_\infty]\}$ as the value function set.

Define the mapping T by

$$\begin{cases} T(\Omega)(s) = \max_x \left[F(x) - \frac{p(x)}{s} + p(x) x_1 \Omega(s x_1) \right] \\ s.t. \quad \widetilde{p}(x) \left[x_1 \Omega(s x_1) - \frac{1}{s} - D^0(x) \right] + \widetilde{e}(x) = 0, \end{cases} \tag{10}$$

and let $\Omega^*(s)$ be the solution to problem (2) (measured in units of $s = V_{t_0}$) assuming that date t_0 has already passed and so the firm has optimally chosen a certain debt level.

The following theorem shows that $\Omega^*(s)$ is the fixed point of the functional mapping T and that for sufficiently large K problem (2) with fixed costs given by (5) approximates the benchmark problem with fixed costs always equal to $q_k = 1$.

Theorem 1 *Assume that the functions $p(x)$, $F(x)$ and*

$$M(x) = D^0(x) - \frac{\widetilde{e}(x)}{\widetilde{p}(x)} \tag{11}$$

are continuously differentiable on X and that there exists the constant $K > 0$ such that either for

$I=2$ or for $I=3$, for every $x \in X$

$$\frac{\partial M(x)}{\partial x_I} \neq 0, \tag{12}$$

and

$$0 \leq p(x) + \left[\frac{\partial F(x)}{\partial x_I} + \frac{\partial p(x)}{\partial x_I} M(x)\right]\left(\frac{\partial M(x)}{\partial x_I}\right)^{-1} \leq \frac{K}{x_1}. \tag{13}$$

Then $\Omega^*(s)$ satisfies the Bellman functional equation given by

$$\Omega^*(s) = T(\Omega^*)(s). \tag{14}$$

Furthermore,

$$\Omega^*(s) = \lim_{K \to \infty} T^K(\Omega_\infty)(s), \tag{15}$$

and there exists a stationary optimal policy.

Proof. See Appendix A. ■

One could find $\Omega^*(s)$ and a stationary optimal policy in the limit (15) by means of the standard dynamic programming algorithm. Then the threshold firm size level s_R to restructure for the first time may be obtained as a solution to the following equation:

$$\arg\max_{\gamma}\left[F(\gamma,0,0) - \frac{p(\gamma,0,0)}{s_R} + p(\gamma,0,0)\gamma\Omega^*(s_R\gamma)\right] = 1. \tag{16}$$

Alternatively, we propose the following algorithm to solve firm's dynamic financing problem, when fixed costs are given by (5) and initial size of the firm is $s_0 = V_{t_0}$ (and K is sufficiently large). We define $G_K = \Omega_\infty$, and $H_K = 1$ and make guess $\widehat{s}_K$ about the size of the firm right before restructuring K.

Then, we solve the model recursively. In the beginning of the k^{th} refinancing cycle, $k = 1, ..., K-1$, we maximize the present V-adjusted value of equityholders' claims:[9]

$$\begin{cases} F(x_k) + p(x_k)\left[\gamma_k G_{k+1} - \frac{H_{k+1}}{s_k}\right] \to \max\limits_{x_k} \\ s.t. \quad \widetilde{p}(x_k)\left[\gamma_k G_{k+1} - \frac{H_{k+1}}{s_k} - D^0(x_k)\right] + \widetilde{e}(x_k) = 0, \end{cases} \tag{17}$$

[9] Note that according to our recursive procedure, in (17), H_{k+1} and G_{k+1} are given but not s_k (at date k, we know only $\widehat{s}_{k+1}$). In order to be able to solve maximization problem (17) correctly, we have to derive its first order conditions taking s_k as given, then replace abstract s_k by $\widehat{s}_{k+1}/\gamma_k$, and solve the system with respect to γ_k, c_k and ψ_k.

obtain optimal $x_k^* = (\gamma_k^*, \psi_k^*, c_k^*)$, and define recursively

$$
\begin{aligned}
G_k &= F(x_k^*) + \gamma_k^* p(x_k^*) \times G_{k+1}, && (18)\\
H_k &= 1 + p(x_k^*) \times H_{k+1}, && (19)\\
\widehat{s}_k &= \frac{\widehat{s}_{k+1}}{\gamma_k^*}. && (20)
\end{aligned}
$$

Finally, we find when it is optimal to issue debt for the first time:

$$
F(\gamma_0, 0, 0) + p(\gamma_0, 0, 0) \left[\gamma_0 G_1 - \frac{H_1}{s_0} \right] \to \max_{\gamma_0 \geq 1} \tag{21}
$$

Then, we check if $\widehat{s}_1$ and the solution to (21), $s_0 \gamma_0^*$, are sufficiently close numbers. If they are, the model is solved. Otherwise, we refine our guess about the size of the firm before the very last restructuring

$$
\widehat{s}_K^{new} \doteqdot \omega \times s_0 \prod_{m=0}^{K-1} \gamma_m^* + (1 - \omega) \times \widehat{s}_K, \qquad \omega \in (0, 1),
$$

and repeat the procedure.

II Firm size and leverage: Refinancing point analysis

II.1 A dynamic trade-off model

The dynamic trade-off framework lends itself easily to introducing firm size. For benchmark comparisons and for simplicity, we will here focus on the modification of the benchmark Goldstein, Ju and Leland (2001) model, though other frameworks also can be easily adapted.

The state variable in the model is the total time t net payout to claimholders, δ_t, where "claimholders" include both insiders (equity and debt) and outsiders (government and various costs). The evolution of δ_t is governed by the following process under pricing measure $\mathbb{Q}$[10]

$$
\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t \qquad \forall t \geq 0, \delta_0 > 0, \tag{22}
$$

where μ and σ are constant parameters and Z_t is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t \geq 0})$. Here, μ is the risk-neutral drift and σ is the instantaneous volatility of the project's net cash flow.

The default-free term structure is assumed flat with an instantaneous after-tax riskless rate r at

[10] Since we consider an infinite time horizon, some additional technical conditions on Girsanov measure transformation (e.g. uniform integrability) are assumed here. In addition, the existence of traded securities that span the existing set of claims is assumed. Thus, the pricing measure is unique.

which investors may lend and borrow freely. The marginal corporate tax rate is τ_c. The marginal personal tax rates, τ_d on dividends and τ_i on income, are assumed to be identical for all investors. Finally, all parameters in the model are assumed to be common knowledge.

All corporate debt is in the form of a perpetuity entitling debtholders to a stream of continuous coupon payments at the rate of c per annum and, in line with previous trade-off models, allowing equityholders to call the debt at the face value at any time. The main features of the debt contract are standard in the literature.[11] If the firm fails to honor a coupon payment in full, it enters restructuring. Restructuring, either a work-out or a formal bankruptcy, is modelled in reduced form. The absolute priority rule is enforced and all residual rights on the project are transferred to debtholders. However, distressed restructuring is costly and, in the model, restructuring costs are assumed to be a fraction α' of the value of assets on entering restructuring. In addition, debt contracts are assumed to be non-renegotiable and restrict the rights of equityholders to sell the firm's assets.

The fundamental driving force of the model is the inherent conflict of interest between the different claimholders since ex-ante (prior to the issuance of debt) and ex-post (after debt has been issued) incentives of equityholders are not aligned. Equityholders maximize the value of equity (that includes debt still to be issued), and thus do not internalize a debtholders claim in a default decision. Debtholders foresee the future actions by equityholders and value debt accordingly.

The only rationale for issuing debt in the model is the existence of tax benefits of debt. In the absence of debt cash flow to equityholders is $\delta_t(1-\tau_c)(1-\tau_d)$. The interest expense of c changes cash flow to equity and debtholders to $(\delta_t - c)(1-\tau_c)(1-\tau_d) + c(1-\tau_i)$. The maximum tax advantage to debt for one dollar of interest expense paid is thus $(1-\tau_i) - (1-\tau_c)(1-\tau_d)$. The trade-off of issuing debt are financial distress costs: in case of default proportion α of asset value upon default is lost.

Up to this point our model is similar to the Goldstein, Ju and Leland (2001) model. In our model, debt issuance costs consist of two components, proportional and fixed. The proportional cost, q', is proportional to the *marginal* amount of debt issued and thus represents what most would think as truly proportional costs. Fixed costs, q_0, are given in dollars measured as the fraction of all future net payouts at date t_0, V_{t_0}.

At every date t equityholders decide on their actions. Firms whose net payout reaches an upper threshold choose to retire their outstanding debt at par and sell a new, larger issue to take advantage of the tax benefits to debt. Refinancing thus takes the form of a debt-for-equity swap.

An additional feature of realism in which we follow Goldstein, Ju, and Leland (2001) is that the firm's financial decisions affect its net payout ratio. Empirically, higher reliance on debt leads

[11] The callability assumption is for technical convenience as long as only one seniority level is allowed. See Strebulaev (2004) for further discussion of debt contract features.

to a larger net payout. Here, for simplicity, we assume that the net payout ratio depends linearly on the after-tex coupon rate:

$$\frac{\delta}{V} = a + (1 - \tau_c)\frac{c}{V_0}. \tag{23}$$

Given the general description of our model in Section I.1, the only features remained to be elaborated are the values of $p(x)$, $e(x)$, $d(x)$, $F(x)$, and $D^0(x)$.

At any date t, the δ_t-measured values of equity and debt cash flows in one refinancing cycle, which starts, say, at date 0 and finishes either at $T_R = \inf\{u \geq 0 : V_u = \gamma V_0\}$ in case of upper restructuring or at $T_B = \inf\{u \geq 0 : V_u = \psi V_0\}$ in case of bankruptcy, are

$$e(x;t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^{T_R \wedge T_B} e^{-r(u-t)}(1-\tau_c)(1-\tau_d)\frac{\delta_u - c}{\delta_t}du\right], \tag{24}$$

$$\begin{aligned} d(x;t) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^{T_R \wedge T_B} e^{-r(u-t)}(1-\tau_i)\frac{c}{\delta_t}du\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T_B-t)}(1-\alpha)(1-\tau_c)(1-\tau_d)\frac{\delta_{T_B}}{\delta_t}|T_B < T_R\right], \end{aligned} \tag{25}$$

and

$$e(x) = e(x;0), \qquad d(x) = d(x;0). \tag{26}$$

The second term in equation (25) is the present value of recovery value debtholders expect at default.

Then, the date-t value of a δ_t-measured debt claim issued at date 0 is

$$D(x;t) = d(x;t) + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T_R-t)}D(x;0)|T_R < T_B\right] = d(x;t) + p(x;t)D(x), \tag{27}$$

where

$$D(x) = D(x;0) = \frac{d(x)}{1-p(x)} \tag{28}$$

is the par value of the debt claim, and

$$p(x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT_R}|T_R < T_B\right] \tag{29}$$

is exactly the value of a claim that pays \$1 at date 0 contingent on asset value reaching γV_0 before reaching the bankruptcy threshold ψV_0.

The combined value of debt and equity *within* the refinancing cycle, $F(x)$, is the sum of $e(x)$ and $d(x)$ subtracting the present value of this and next period transaction costs portions that include called value $D(x)$. The restructuring costs paid in the beginning of this period are $q'[D(x) - D^{prev}(x)]$, and the present value of those to be paid in the beginning of the next period is $q'p(x)\left[D^{next}(x) - D(x)\right]$. So, $F(x)$ is given by

$$F(x) = e(x) + d(x) - q'D(x) + q'p(x)D(x) = e(x) + \left(1 - q'\right)d(x). \tag{30}$$

In our case of marginal restructuring costs, $D^0(x)$, as defined through equation (4) in Section I.1, is not just the called value of this period debt, $D(x)$, but it also accounts for the presence of a portion of the next period restructuring costs in $F(x)$ (the fourth term in (30)):[12]

$$D^0(x) = D(x) - q'D(x) = \frac{(1 - q')\,d(x)}{1 - p(x)}. \tag{31}$$

An important result which allows us to formulate the model as a dynamic programming problem is that the right-hand sides in equations (26) and (28)–(31) are not functions of δ_0. Note that all the above expressions can also be applied to zero restructuring by substitution $x_0 = (\gamma_0, 0, 0)$ for $x = (\gamma, \psi, c)$.

II.2 Comparative statics analysis at the refinancing point

To understand the economic intuition behind our first set of findings on how fixed costs of financing effect the leverage decision of firms, it is worth to consider first the workings of the dynamic capital structure models such as Goldstein, Ju, and Leland (2001) and Strebulaev (2004) which, to allow for infrequent refinancing in the absence of truly fixed costs, the costs that are proportional to the *total* debt outstanding rather than *marginal* debt issuance are modelled. To this end we introduce a new state-space representation of these models shown in Figure 1. We can think of δ, which stands for asset's cash flow and depicted on the horizontal axis, as a proxy for firm size and c, which denotes the level of coupon payment, as a proxy for the extent of debt financing. Every firm size is associated with some optimal level of leverage as predicted by the dynamic trade-off model. At the time of refinancing, the firm will choose the optimal debt level corresponding to its size on the middle line (for example, c_1 for δ_0). Thus, the middle straight line (which we call here the beginning-of-cycle line) is the relationship between optimal leverage and firm size at refinancing points. Then the firm moves along the horizontal within-cycle line (the dashed line at c_1 in Figure 1). If its fortunes substantially improve and it reaches the lower straight line (which is often called the upper restructuring line and which we call the end-of-cycle line) the firm refinances again (from

[12] When proportional restructuring costs are proportional to the total debt issued, other things unchanged,

$$\begin{aligned} F(x) &= e(x) + d(x) - q'\frac{d(x)}{1 - p(x)}, \\ D^0(x) &= D(x) = \frac{d(x)}{1 - p(x).}. \end{aligned}$$

c_1 to c_2). If the firm's fortunes deteriorate materially enough, the firm will default when it reaches the upper-left default line. In dynamics, firms can be at any point in the shadow area. Notice that since all the three lines are straight, and firm size does not matter for any financing decisions and, in particular, optimal leverage is constant. Moreover, all firms optimally have debt and large firms refinance as frequently as do small firms.

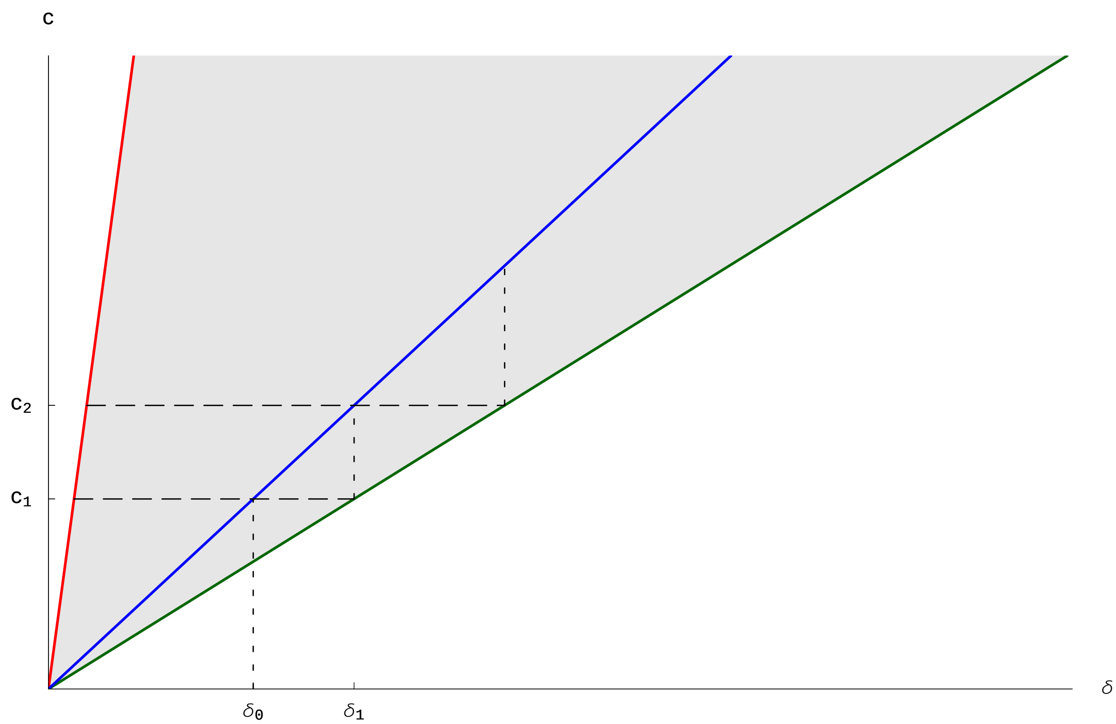

Figure 1: **Firm size and leverage: costs are proportional to total debt outstanding**

What happens in our model in the presence of fixed and marginal proportional costs? If there were no fixed costs and proportional costs were marginal, the firm would find it optimal to restructure every moment its asset value increases. Graphically, the beginning- and end-of period lines in Figure 1 would merge and the firm would restructure continuously. However, adding fixed costs prevents infinitesimal increases from being the optimal strategy. To start with, very small unlevered firms for which the present value of tax benefits is less than the cost of refinancing will abstain from issuing any debt. Such firms postpone debt issuance until their fortunes sufficiently improve. Figure 2 shows these firms on the horizontal zero-leverage segment between 0 and $\delta = \delta_0$ with the optimal coupon of 0. Once the threshold δ_0 is reached, the firm will issue debt up to the level of c_1. Fixed costs lead to the discontinuity in the firm's initial decision to use external debt financing. The larger the fixed costs, the less often the firm restructures and, to compensate for that in order to maximize on the present value of tax benefits, at each restructuring the firm takes on more leverage. Thus, coupon c_1 is optimal for a smaller firm than in the absence of fixed costs (geometrically (δ_0, c_1) is above the no-fixed-cost beginning/end-of-cycle threshold). Analogously, firms now defer the restructuring decision for longer so that the new end-of-cycle line is below the no-fixed-cost solid line. The evolution of the firm is now within the shadow area and has the following pattern: it either moves horizontally to the right along the within-cycle line (the present value of future cash flows is larger) until the new upper threshold is reached (in this case, δ_1) when the firm refinances and immediately moves vertically to the new optimal coupon level c_2. Or the firm moves horizontally to the left (its future prospects become less favorable) until it reaches the default threshold. The new default boundary is reached sooner than in the no-fixed-cost case, for the present value of the shareholders' claim of continuing to own the firm is decreased by the present value of expected future fixed costs.

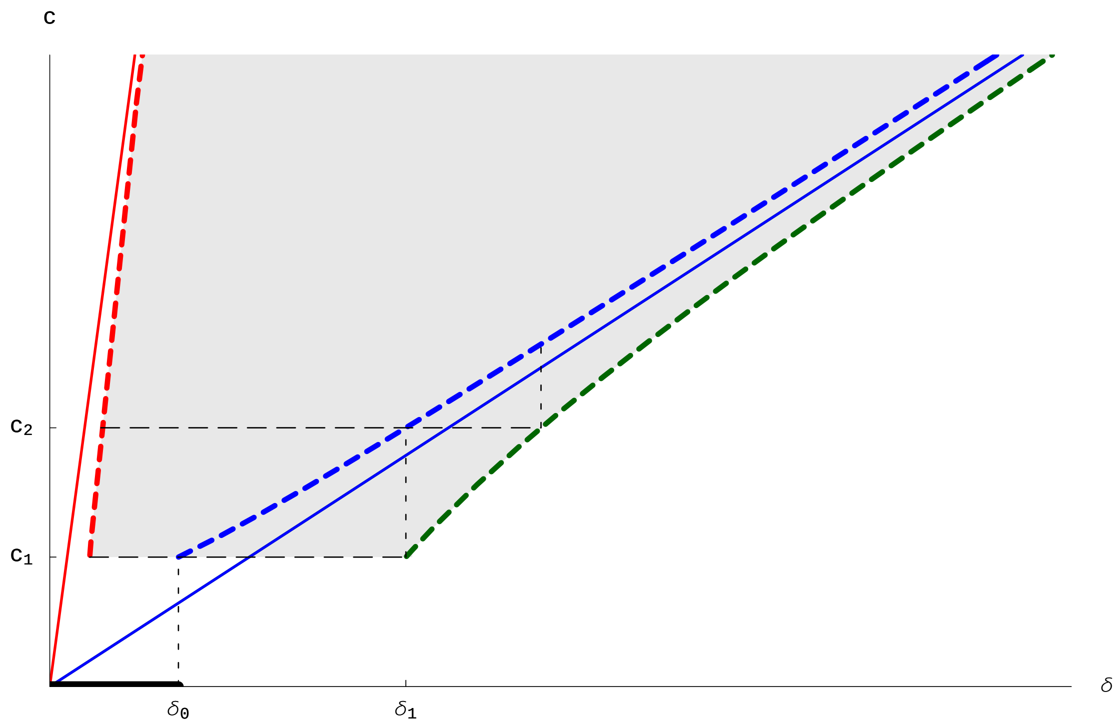

Figure 2: **Firm size and leverage: model with fixed and marginal proportional costs.**

As firm size increases, fixed costs become less and less important, which is evident in Figure 2 where all three optimal dashed curves approach the optimal decision solid lines of the no-fixed-cost case. The timing between subsequent refinancings decreases and attenuates to zero. In particular, the ratio of fixed to proportional costs in total issuance costs is larger for small firms and attenuates as firm size increases. There is another noteworthy pattern that emerges from Figure 2. If, upon default, the value of the firm is less than initial threshold δ_0, the firm emerges from bankruptcy restructuring optimally unlevered.

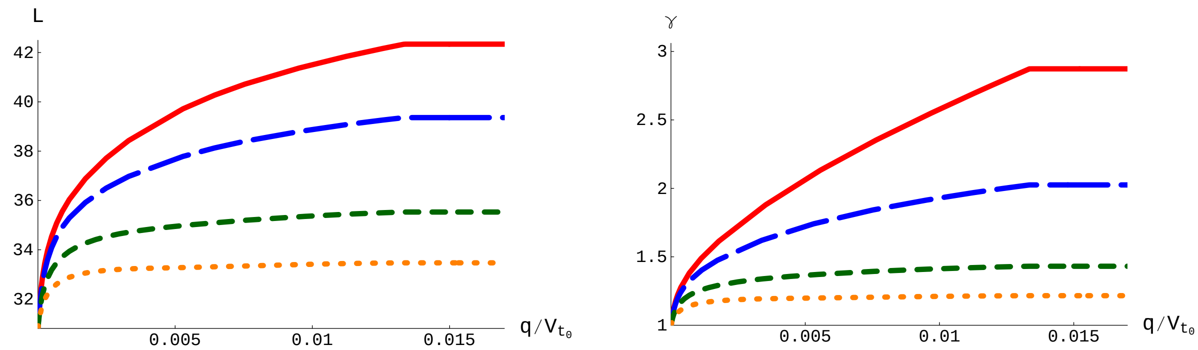

Figure 3: **Optimal leverage and next refinancing decision.**

Next, we investigate the dependence of optimal leverage decisions for our main model on the fixed costs of issuance. The upper solid curve in the left panel of Figure 3 shows how the leverage ratio, L_1, changes as fixed costs measured in units of initial firm's asset value increase at the very first restructuring. (We address the numerical properties of the solution in the next section.) The plot shows that conditional on issuance higher fixed costs lead to higher leverage. It also shows that, after some threshold level of fixed costs, leverage decision at the first restructuring is independent of fixed costs. This is because when facing too high fixed costs of issuance firms will defer issuance decision until some threshold level of firm size is reached and at that point, as Figure 2 shows (point δ_0^*), the leverage decision is invariant to initial conditions.

The three dashed curves in the left panel of Figure 3 show what happens to optimal leverage decisions at the second, fifth and tenth restructurings. Not surprisingly, given our previous discussion, at each subsequent refinancing, firm size-adjusted costs are smaller and so the leverage ratio decreases. The leverage curve attenuates to the horizontal line with the value of optimal leverage ratio for the no-fixed costs solution on the vertical axis.

The right panel of Figure 3 demonstrates by how much firm size should increase (the scale factor as represented by γ on the vertical axis) before the firm refinances again as a function of fixed costs. A similar pattern to optimal leverage at refinancing point emerges. The upper curve shows the scale factor at the first refinancing, γ_1. As firm size-adjusted costs decrease, the firm restructures more often as evident by the attenuation of γ-curves in the figure. In the no-fixed costs case γ is equal to one.

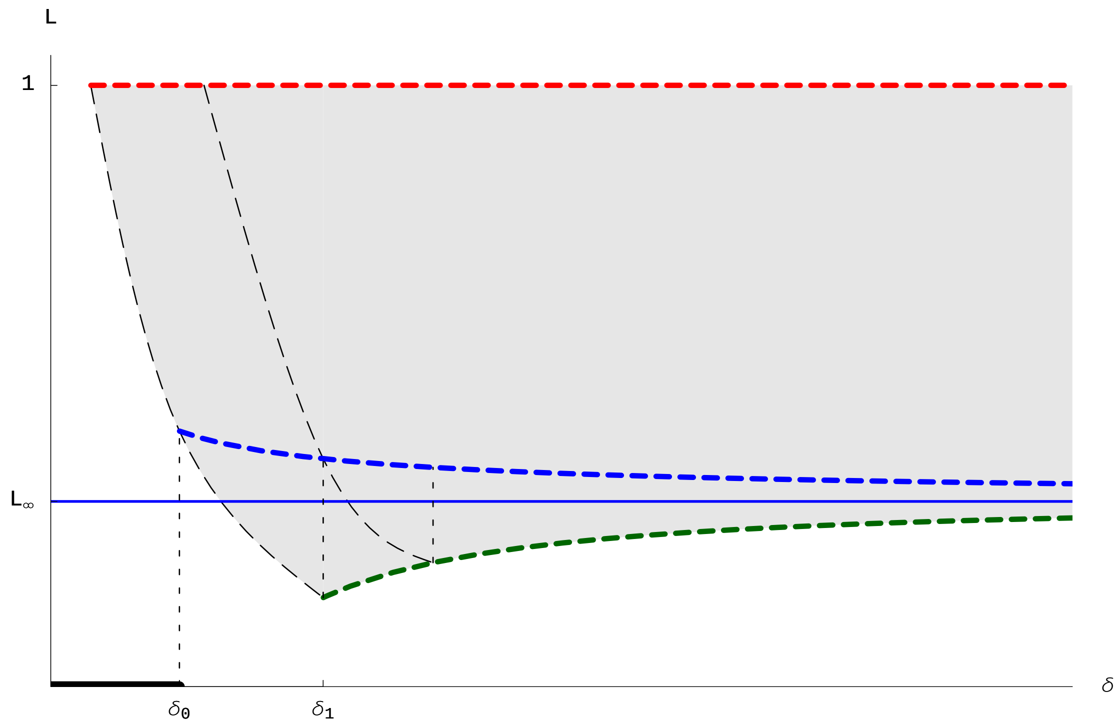

Figure 4: **Firm size and leverage: implications for dynamic analysis.**

III Firm size and leverage: Dynamic analysis

The objective of this section is to investigate the cross-sectional relationship between firm size and leverage. The cross-sectional relationship in dynamics is impossible to investigate by studying only the comparative statics of optimal leverage decisions at refinancing points; the failure of this comparative statics and, intrinsically, static analysis to explain the cross-sectional dependencies was investigated in capital structure context by Strebulaev (2004). To see the intuition of why it is impossible to proceed in this way with our problem, consider Figure 4 which is, roughly speaking, a reparametrization of Figure 2 of the form $\{\delta, c\} \rightarrow \{\delta, L = \text{``}c/\delta\text{''}\}$. In dynamics at any point in time firms can be either on the zero-leverage segment line (between 0 and δ_0 on the horizontal axis) or in the shadow area. When the firm's leverage ratio reaches the lower boundary, it optimally restructures (and the leverage ratio jumps to the middle beginning-of-cycle curve). At the time of default, firm's capital consists only of debt for equity is worthless. The figure shows the complex relationship between firm size and leverage. First, zero-leverage firms form a cluster that would tend to make the relationship positive. We call this *the zero-leverage effect*. Second, the end-of-cycle boundary, as an increasing function of size, also enhances the positive relationship (we call this *the end-of-cycle effect*). Third, the optimal leverage ratio at the refinancing point is a decreasing function of firm size (we call this *the beginning-of-cycle effect*). Finally, firm's path is, conditional on the same debt level, a decreasing function of size (as demonstrated for example by the left vertical within-cycle boundary) and so it may induce a negative relationship between firm size and leverage. We call this *the within-cycle effect*. The last three effects can be seen at the individual firm level while the zero-leverage effect is a purely cross-sectional one.

The dynamic relationship will depend on the distribution of firms in the zero leverage segment and the shadow area. To relate the model to empirical studies, it is necessary to produce within the model a cross-section of leverage ratios structurally similar to those which would have been studied by an empiricist. Since we do not know the distribution in dynamics, we can only follow the evolution of the economy from initially specified date until the distribution converges to the dynamic one and then study the resulting cross-section in dynamics.

Thus, we proceed to generate artificial dynamic economies from the model and then use the generated data to relate the leverage ratio to firm size. We use the method developed for corporate finance applications by Strebulaev (2004) based on the simulation method developed in the asset-pricing context by Berk, Green, and Naik (1999). Our first aim is to replicate a number of cross-sectional regressions used in empirical studies that produced stylized facts on the relation between leverage and firm size. The two questions which we wish to illuminate are whether fixed costs of issuance can produce results that are qualitatively similar to those found in empirical research, and, if so, whether the empirical estimates could have been generated by the model with reasonable probability under a feasible set of parameters.

III.1 Data generation procedure

This section describes the simulation procedure. Technical details are given in Appendix B.

To start with, observe that, while only the total risk of the firm matters for pricing and capital structure decisions (since each firm decides on its debt levels independently of others),[13] economy wide shocks lead to dependencies in the evolution of the cash flow of different firms. To model such dependencies, shocks to their earnings are drawn from a distribution that has a common systematic component. Thus, cross-sectional characteristics of leverage are attributable both to firm-specific characteristics and to dependencies in the evolution of their assets. Following Strebulaev (2004), we model the behavior of the cash flow process as

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma_I dZ^I + \beta\sigma_S dZ^S \qquad \forall t \geq 0, \delta_0 > 0. \tag{32}$$

Here, σ_I and σ_S are constant parameters and Z_t^I and Z_t^S are Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\geq 0})$. The shock to each project's cash flow is decomposed

[13] Thus, we assume that events ex ante are uncorrelated among firms and, say, default of firm i neither increases nor decreases firm's j chances of survival.

into two components: an idiosyncratic shock that is independent of other projects ($\sigma_I dZ^I$) and a systematic (market-wide or industry) shock that affects all firms in the economy ($\sigma_S dZ^S$). The parameter β is the systematic risk of the firm's assets, which we will refer to as the firm's "beta", and systematic shocks are assumed independent from idiosyncratic shocks. The Brownian motion dZ in equation (22) is thus represented as an affine function of two independent Brownian motions, $dZ = dZ^I + \beta dZ^S$, and the total instantaneous volatility of the cash flow process, now denoted as σ_A, is

$$\sigma_A \equiv (\sigma_I^2 + \beta^2 \sigma_S^2)^{\frac{1}{2}}. \tag{33}$$

At date 0 all firms in the economy are "born" and choose their optimal capital structure. Our benchmark scenario will be the case when all firms are identical at date 0 but for their asset value. This will allow us to concentrate on the firm size-leverage relationship since the only difference between firms will be firm size-adjusted fixed costs of external financing. For the benchmark estimation we simulate 300 quarters of data for 3000 firms. To minimize the impact of the initial conditions, we drop the first 148 observations leaving a sample period of 152 quarters (37 years). We refer to the resulting data set as one "simulated economy". Using this resulting panel data set we perform cross-sectional tests similar to those in the literature. The presence of a systematic shock makes cross-sectional relations dependent on the particular realization of the market-wide systematic component.[14] Therefore we repeat the simulation and the accompanying analysis a large number of times. This allows us to study the sampling distribution for statistics of interest produced by the model in dynamics.

In any period each firm observes its asset value dynamics over the last quarter. If the value does not cross any boundary, the firm optimally takes no action. If its value crosses an upper refinancing boundary (including the very first levering up), it conducts a debt-for-equity swap re-setting the leverage ratio to the optimal level at a refinancing point, and so starting a new refinancing cycle. If the firm defaults, bondholders take over the firm and it emerges in the same period as a new (scaled down) firm with the new optimal leverage ratio.

III.2 Parameter calibration

This section describes how firms' technology parameters and the economy-wide variables are calibrated to satisfy certain criteria and match a number of sample characteristics of the COMPUSTAT and CRSP data. An important caveat is that for most of parameters of interest, there is not much empirical evidence that permits precise estimation of their sampling distribution or even their range. Overall, the parameters used in our simulations must be regarded as ad hoc and approximate. There

[14] In the absence of the systematic shock, cross-sectional relations will be nearly identical in all simulations once economies reach their steady state.

are two ways we deal with this problem. First, whenever possible we will use parameters used elsewhere in the literature to simplify the comparison. Second, we perform a number of robustness checks. Table I summarizes the descriptive information for the parameters described below.

In the model the rate of return on firm value is perfectly correlated with changes in earnings. In calibrating the standard deviation of net payout we therefore use data on securities returns. Firms differ in their systematic risk, represented by β. A distribution of β is obtained by running a simple one-factor market model regression for monthly equity returns for all firms in the CRSP database having at least three years of data between 1965 and 2000 with the value-weighted CRSP index as the proxy for the market portfolio. The resulting β distribution is censored at 1% left and right tails and used as an estimate of the asset beta.

The volatility of firm assets, σ is chosen to be 0.25, to coincide with a number of previous studies. It is also very close to the mean volatility of assets found by Schaefer and Strebulaev (2004) in the cross-section of firms that issued public debt. The standard deviation of the systematic shock, σ_S, is estimated to 0.11. The volatility of idiosyncratic shocks, σ_I, is then chosen to be consistent with the distribution of total risk.

The proportional cost of restructuring in default, α, is equal to 0.05, consistent with a number of empirical studies. As was previously found, the model results are but slightly affected by the variation in distress costs.

All corporate taxes have the same value as in Goldstein, Ju, and Leland (2001) for the ease of comparisons. These values are also largely supported by empirical evidence. The corporate tax rate is equal to the highest existing marginal tax rate, $\tau_c = 0.35$. The marginal personal tax rate on interest income, τ_i, is estimated by Graham (1999) to be equal to 0.35 over the period of 1980–1994. The marginal personal tax rate on dividend payments, τ_d, is 0.2. Thus, the maximum tax benefit to debt, net of personal taxes, is $(1 - \tau_i) - (1 - \tau_c)(1 - \tau_d) = 13$ cents per one dollar of debt. The after-tax risk-free interest-rate is assumed to be 0.045 and the risk premium on the rate of return on firm assets is equal to 0.05. The net payout ratio increases with interest payments and the parameter a depends, ultimately, on firms' price-earnings ratios and dividend policy. Its value is taken to be 0.035 the same as used by Goldstein, Ju, and Leland (2001). When the net payout flow is very small, the firm starts losing part of its tax shelter. Since the remaining tax shelter depends on carry-forwards and carry-backs benefit provisions it is likely that firms lose a substantial part of the tax shield when current income is not sufficient to cover interest payments. In modelling the partial loss offset boundary, we follow Goldstein, Ju, and Leland (2001) and assume that firm starts losing 50% of its debt offset capacity if the ratio of earnings to debt is relatively small.

Proportional costs of marginal debt issuance, q', are assumed to be equal to 0.007. Fixed costs of restructuring, $\bar{q}$, are calibrated in such a way that the total costs on average in a dynamic economy are about 1% of the amount of debt issuance. Datta, Iskandar-Datta and Patel (1997) report total expenses of new debt issuance over 1976–1992 of 2.96%; Mikkelson and Partch (1986) find underwriting costs of 1.3% for seasoned offers and Kim, Palia and Saunders (2003), in a study of underwriting spreads over the 30-year period find them to be 1.15%. Altinkilic and Hansen (2000) also find that costs are in the order of 1% and that fixed costs on average constitute approximately about 10% of total issuance costs. To calibrate the ratio of total costs to debt issuance to be about 1% per each dollar of debt issued in the economy in the last 35 of 75 years of simulations, we choose the initial distribution of V. In addition, the standard deviation of the initial distribution of V is chosen to resemble the distribution of firms in COMPUSTAT. The benchmark scenario's initial V distribution is lognormal with mean of 7 and standard deviation of 3. Appendix B provides further details on calibration.

III.3 Preliminary empirical analysis

We now bring together the calibrated model with the results of comparative statics at the refinancing point and some empirical results from the literature. We use two definitions of leverage, both based on the market value of equity. The first, the market leverage ratio, can be defined as

$$ML_t = \frac{D(\delta_t; x)}{D(\delta_t; x) + E(\delta_t; x)}. \tag{34}$$

Typically, however, market values of debt are not available and book values are used. We therefore introduce a second definition, the quasi-market leverage ratio, defined as the ratio of the par value of outstanding debt to the sum of this par value and the market value of equity:

$$QML_t = \frac{D(x)}{D(x) + E(\delta_t; x)}, \tag{35}$$

where $D(x)$ is the book value of debt as defined in (28). Typically, the difference between ML and QML is very small. For financially distressed firms, however, it can be more substantial. Intuitively, these ratios reflect how the firm has financed itself in the past since both the par and market values of debt reflect decisions taken early in a refinancing cycle.

Table II summarizes the cross-sectional distribution of these various measures in a dynamic economy and at the initial refinancing point. The average leverage ratio at the initial refinancing point is 0.28, compared to, say, 0.37 in a model by Goldstein, Ju and Leland (2001). To gauge the reasons of such a stark difference consider the distribution of optimal leverage at the refinancing point. Notice that the one percentile of firms have zero leverage. In fact, for the reasons of discontinuity in leverage decisions, about 17% of firms at the initial refinancing point are unlevered.

If we exclude firms that do not have leverage, the average leverage ratio goes up from 0.28 to 0.36. This observation suggests that the low leverage puzzle (that refers to the stylized fact that average leverage in actual economies are lower that most of trade-off models would predict) can be driven to a large extent by unlevered firms.[15] Notice also that the cross-sectional variation at refinancing point is attributed solely to the differences in firm size. For the remainder of the section, we measure firm size as the natural logarithm of the sum of book debt and market equity, in line with empirical studies:

$$LogSize_t = \log\left(D(x) + E(\delta_t; x)\right). \tag{36}$$

An average firm has the initial value of about \$530mln and the value in dynamics of \$1.8bln. The distribution in dynamics is similar to the distribution of firm size in COMPUSTAT. Finally, credit spreads at optimal refinancing point are 162 basis points on average or 185 basis points, conditional on issuing debt.

Of more importance, however, are the descriptive statistics for dynamics. Means for dynamic statistics are estimated in a two-step procedure. First, for each simulated economy statistics are calculated for each year in the last 35 years of data. Second, statistics are averaged across years for each simulated economy and then over economies. To get a flavor of the impact of systematic shocks we also present minimum and maximum estimates over all economies. We begin by comparing the leverage statistics in the dynamic economy to those at the initial refinancing point where the impact of the dynamic evolution of firm's assets is ignored. Table II shows, in line with the results obtained by Strebulaev (2004), that leverage ratios in the dynamic cross-section are larger than at refinancing points. An intuition for this observation is quite general: unsuccessful firms tend to linger longer that successful firms who restructure fairly soon, especially so because firms who opt for higher leverage at refinancing points also choose lower refinancing boundary. In addition, firms that are in distress or close to a bankruptcy typically have leverage exceeding 70%, and these firms have a strong impact on the mean. The difference between Strebulaev (2004) and our results is that the dynamic cross-section in our model has a substantial number of firms that are unlevered. Table III shows, in particular that, on average, 11% of firms are unlevered at any point in time. Thus, our model is able to deliver low leverage for a large fraction of firms in cross-section and explain partially the low leverage puzzle *in dynamics*.[16]

What Table II also shows is that the distribution of all parameters of interest (leverage, firm size

[15] This fact is established empirically by Strebulaev and Yang (2005) who show that taking out all almost zero-leverage firms increases average leverage on a sample of COMPUSTAT firms between 1987 and 2003 from 25% to 35%.

[16] Between 1987 and 2003 about 11% of COMPUSTAT firm-year observations have zero leverage as measured by book interest-bearing debt (the sum of data items 9 and 34) as reported by Strebulaev and Yang (2005).

and credit spreads) is much wider and closer to the empirically observed distribution than at the point of refinancing. In summary, because firms at different stages in their refinancing cycle react differently to economic shocks of the same magnitude, the cross-sectional distribution of leverage as well as the other variables in Table II is drastically different in dynamics and at the initial refinancing point.

Panel (a) of Table III shows that the annual default frequency is around 110 basis points. Every year about 9% of firms experience liquidity-type financial distress (their interest expense is larger than their pre-tax profit) and they have to resort to equity issuance to cover the deficit (recall that asset sales are not allowed in the model). Around 19% of firms restructure every year. This statistics, of course, hides a substantial cross-sectional variation between large and small firms and also between years when firms in the economy were relatively small and years when firms were relatively large. To gauge the effect of size, panels (a) and (b) give the same statistics for 25% of smallest and largest firms (where the sample of firms is updated every year). Small firms default more often. While about 40% of the smallest firms in panel (b) are unlevered, the remaining have higher leverage ratio and lower default boundary than largest firms. Conditional on issuing debt, the likelihood of default within one year for these firms is about 2.10% compared to 0.7% for the 25% largest firms. Not surprisingly, small firms are also more likely to find themselves in financial distress (demonstrated by the higher fraction of small firms issuing equity).

Table III also shows that the largest firms restructure on average almost every second year, while small firms may wait for decades without refinancing. Finally, panel (a) also provides some insight on the importance of systematic shock by sketching the distribution of frequency of events across generated economies. The systematic shock of realistic magnitude can lead to substantial variation in quantitative results. For example, in the "best-performing" economy out of a thousand simulated on average only 2% of firms were unlevered at any point in time and in the "worst-performing" economy 29% of firms were unlevered. The results of all empirical studies are naturally based on only one realized path of systematic shock.

Table IV demonstrates the relative importance of fixed and proportional costs in simulated dynamic economies. The ratio of total costs to debt issuance, of 1.03%, was calibrated with the choice of initial distribution of V. The ratio of fixed to total costs is on average about 18%, somewhat higher than 10% reported by Altinkilic and Hansen (2000) but within a reasonable range. Panels (b) and (c) again report the same results for the subsamples of smallest and largest firms. Not surprisingly, smallest firms pay dearly to restructure with fixed costs being by far the largest component.

III.4 Cross-sectional regression analysis

This section examines the cross-sectional dynamic relationship between leverage and firm size. Recall that there are four general effects that have different effects in cross-section. Firstly, some firms are unlevered (the zero-leverage effect). Secondly, smaller firms tend to take on more debt at refinancing to compensate for longer waiting times (the between-refinancing cycle effect). Thirdly, an increase in firm size increases the value of equity and decreases leverage (the within-refinancing cycle effect). Fourthly, smaller firms wait longer before restructuring and the leverage ratio deviates more from the leverage at refinancing point (the end-of-cycle effect). Our first task is to investigate which of these four effects are likely to dominate in the cross-section by replicating standard empirical tests.

Recall that each simulated data set ("economy") consists of 3000 firms for 300 quarters. As described in section III.1, we simulate a large number of economies dropping the first half of the observations in each economy. For each economy we then conduct the standard cross-sectional regression tests. We choose the procedure used by Fama and French (2002) where they first estimate year-by-year cross-sectional regressions and then use the Fama-MacBeth methodology to estimate time-series standard errors that are not clouded by the problems encountered in both single cross-section and panel studies.[17] We run a Fama-MacBeth regression on the last 35 years of each simulated economy and then averaging these results across economies. In addition, to control for substantial autocorrelation (since leverage is measured in levels), we report the Newey-West adjusted t-statistics.[18]

Table V reports the results of this experiment. Panel (a) reports the results of a standard regression of leverage on a constant and firm size. The relationship for the whole sample is positive, consistent with the existing empirical evidence. Thus, a dynamic trade-off model of capital structure is able to produce qualitatively the relationship between firm size and leverage as observed in empirical studies. Another important observation is that quantitatively the size of the effect seems close to empirical regressions (e.g. Rajan and Zingales (1995) report a coefficient of 0.03 and Fama and French (2002) of 0.02-0.04 on $LogSize$).[19] The coefficient of 0.03 roughly means that a 1% increase in the value of assets increases leverage by 3 basis points.

The result demonstrates that the joint outcome of end-of-cycle and zero-leverage effects domi-

[17] Strebulaev (2004) demonstrates that other cross-sectional methods (e.g. those of Bradley, Jarrell, and Kim (1984) and Rajan and Zingales (1995)) produce the same results when applied to the generated data.

[18] Petersen (2005) shows that unadjusted Fama-MacBeth method can lead to understated standard errors.

[19] An important caveat is in order. Our definition of size is, while obviously highly correlated with theirs, is not a one-to-one mapping. Rajan and Zingales (1995) use the logarithm of annual sales and Fama and French (2002) use the logarithm of total book assets as their proxy for size while our proxy is the logarithm of quasi-market value of firm assets. To study the consequences of that, we investigated the empirical relationship between firm size (as measured by the sum of book debt and market value) and sales in the COMPUSTAT data. Not reported, they are significantly positively correlated as one would expect.

nate that of beginning-of-cycle and within-cycle effects. We therefore proceed to investigate to what extent our neglect of of firms with no leverage affects our results. both in simulated and actual economies. It is known econometrically that when a cross-section has mass in the extreme point, the results can be biased, and the geometric intuition behind Figure 4 suggests that in our case the bias can be substantial. Therefore, we proceed by estimating the same regression but controlling for the presence of unlevered firms by including a dummy that equals 1 if a firm is unlevered in a given year and 0 otherwise. Panel (b) reports the results of this regression. Unsurprisingly, the coefficient on the dummy is large and negative. Perhaps, surprisingly, incorporating the dummy changes the sign of the size effect. Thus, controlling for the presence of unlevered firms, the beginning-of-cycle and within-cycle effects dominate the end-of-cycle one. Interestingly, this finding can provide, for example, an explanation of why Rajan and Zingales (1995) find a positive firm size-leverage relation for most countries but Germany, for Germany has a less developed capital market with relatively larger firms publicly traded (and thus present in their data set).

Next, we study the relation between the size and the speed of mean reversion. [SECTION TO BE COMPLETED]

Next, we study the relation between firm size and the likelihood of default. [SECTION TO BE COMPLETED]

IV Empirical implications

In this section we summarize a number of empirical implications that can be derived from our model that easily lend themselves to testing using standard corporate finance data sets. A word of caution is in place: we use here “small” and “large” when talking about firms rather frivolously. The size of the firm is a relative term, here in particular relative to the value of fixed costs which can vary, for example, among industries, and this will have to be taken care of in empirical research.

The first set of predictions is directly related to zero-leverage firms. Obviously, the exclusion of zero leverage firms will tend to increase average leverage in the economy. In the benchmark model, the average leverage ratio in dynamics increases from 45% to 50%. More interestingly, the model predicts that a zero-leverage policy is more likely to be followed by small firms. Strebulaev and Yang (2005) find that unlevered firms are more likely to be the smallest firms in their industries, consistent with this result. Certainly, in reality there are a number of large firms that follow low-leverage and zero-leverage policy not explained by the model. The likely reason is that our model does not consider optimal cash policy of the firm. Large zero-leverage firms tend to have higher cash balances. Likewise, a number of small firms are levered. In addition to short-term bank debt that presumably is associated with lower costs, the financing decisions of these firms can be driven by their investment policy that we do not model.

Two testable empirical predictions are at the core of the difference between small and large firms. Small firms restructure less often. At the same time, conditional on restructuring, small firms issue relatively more debt (relative to the amount of debt outstanding). This leads to another prediction that the propensity to default is a negative function of firm size and small firms are more likely to fail. The observation that large firms will allow less deviations from optimal leverage ratios at restructuring implies that there is a negative relation between firm size and firm leverage volatility.

Related to the above argument, the model predicts that the economy with relatively higher fixed costs will experience the higher fraction of zero-leverage firms, lower frequency of restructurings and relatively larger amount of debt issuance conditional on restructurings taking place. A possibility to examine this empirically is to compare the data from the 1960-1970s to the data from the 1980-1990s, for it is often argued that external financing issuing costs had decreased substantially over the past decades (see e.g. Kim, Palia, and Saunders (2003)).

For the results of cross-sectional regressions, our simulation predicts that if we condition on firms that have no leverage, the slope of the logsize coefficient will become less significant and may become negative. To the extent that we do not expect to observe empirically a perfect watershed between small and large firms that the model produces, controlling for unlevered firms may produce a lesser effect. If empirical investigation finds that the coefficient is substantially less positive, it will also suggest that a number of other established results and stylized facts can be affected. We also conjecture that if regressions are run separately on the subsamples of small and large levered firms, the slope coefficient will be more negative in the subsample of small firms.

Finally, to the extent that systematic shock can be thought of as a proxy for a business cycle variable, we hypothesize that the fraction of unlevered firms increases in downturns and decreases in times of growth with the frequency of restructurings revealing the opposite tendency.

V Fixed costs of bankruptcy

Intuition, corroborated by empirical evidence, suggests that bankruptcy costs are also likely to consist of two components, proportional and fixed. To investigate whether adding the fixed component of bankruptcy changes our conclusions we modify the benchmark model by postulating that if the firm defaults in refinancing cycle k the fixed costs of size α_k have to be paid in addition to proportional costs, bringing the total bankruptcy costs to $\min\left[\alpha_k + \alpha' V, V\right]$. This lowers the recovery value debtholders receive at default so that the δ_{k-1}-measured value of debt cash flows in the k^{th} refinancing cycle defined earlier in (25) is now reduced by dollar-measured quantity $\alpha_k p_B\left(x_k\right)$, where

$$p_B(x_k) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT_B}\,|T_B < T_R\right] \tag{37}$$

is the value of a claim that pays \$1 at the moment of bankruptcy contingent on the firm defaulting in cycle k. Therefore, using (30) and (31), the shareholders' problem (2) is modified to

$$\begin{cases} \Omega = V_{t_0}F(x_0) + \sum\limits_{k=0}^{\infty} P_k\big[V_{t_0}\Gamma_k F(x_{k+1}) - q_k - (1-q')\alpha_k p_B(x_{k+1})\big] \to \max\limits_{x_0,x_1,\ldots} \\ s.t.\ \widetilde{p}(x_k)\left\{\sum\limits_{m=k}^{\infty}\frac{P_m}{P_k}\left[V_{t_0}\Gamma_m F(x_{m+1}) - q_m\right] - \frac{(1-q')\bar{\alpha}p_B(x_k)}{1-p(x_k)} - V_{t_0}\Gamma_{k-1}D^0(x_k)\right\} + V_{t_0}\Gamma_{k-1}\widetilde{e}(x_k) = 0, \\ \quad \gamma_0 \geq 1, \quad c_0 = 0, \quad \psi_0 = 0, \end{cases} \tag{38}$$

Again, to solve the model with $\alpha_k = \bar{\alpha} > 0$, for sufficiently large K we approximate that with the problem for which both fixed restructuring and bankruptcy costs are zero after the K^{th} restructuring. Noting that firm size (7) still summarizes all information about past restructuring that is necessary for the shareholders to make optimal decisions about next refinancings, we adopt a similar backward-induction approach to solve this approximating problem. The only difference with our solution procedure (17)–(21) is that at step k we now solve:

$$\begin{cases} F(x_k) - (1-q')\bar{\alpha}\frac{p_B(x_k)}{s_k} + p(x_k)\left[\gamma_k G_{k+1} - \frac{H_{k+1}}{s_k}\right] \to \max\limits_{x_k} \\ s.t. \quad \widetilde{p}(x_k)\left[\gamma_k G_{k+1} - \frac{H_{k+1}}{s_k} - \frac{(1-q')\bar{\alpha}p_B(x_k)}{[1-p(x_k)]s_k} - D^0(x_k)\right] + \widetilde{e}(x_k) = 0, \end{cases} \tag{39}$$

and for optimal x_k^* define the cycle-k value of all future fixed-cost payments as:

$$H_k = 1 + p(x_k^*)\,H_{k+1} + (1-q')\bar{\alpha}p_B(x_k^*)\,. \tag{40}$$

Introducing fixed bankruptcy costs raises the threshold value δ_0 at which very small firms restructure for the first time. For the case of the fixed component of bankruptcy costs five times larger than fixed debt issuance costs, $\bar{\alpha} = 5\bar{q}$, the fraction of unlevered firms in dynamic economy increases from 11% to 15%. Moreover, to reduce the likelihood of now costlier default firms issue less debt than in the benchmark case. As a result, the beginning-of-cycle curve (right above the flat no-fixed-cost line L_∞ in Figure 4) is no longer a monotonically decreasing function of firm size. It is always below its benchmark-case counterpart and first rises with firm size and at some point starts its decline attenuating to L_∞. The rationale for this behavior lies in the tension between the two effects: as firm size increases, fixed costs of both issuance and bankruptcy become less important with relatively reduced fixed costs of issuance decrease optimal leverage at refinancing and fixed costs of bankruptcy increase it. For small firms the latter effect dominates. The end-of-cycle curve is almost unchanged.

Having more unlevered firms in our simulated economies results in a higher (and more significant) firm size coefficient in our standard regression (the coefficient rises from 0.02 to 0.03). On the other hand, the non-monotonicity of the beginning-of-cycle curve is the main reason for an increase in the size coefficient when we control for the presence of unlevered firms (from -0.004 in the benchmark case to -0.002 once fixed bankruptcy costs are introduced).

[SECTION TO BE COMPLETED]

VI Concluding Remarks

This paper is the first to investigate in a formal and economically intuitive model the cross-sectional relationship between firm size and capital structure. We construct and find a solution for a general dynamic financing model with infrequent adjustment in the presence of fixed costs of external financing. Using an application of this model to the dynamic trade-off model of capital structure, we show that there are two effects on the relationship. Smaller firms issue more debt conditional on refinancing but they wait longer before restructuring again. We generate data that structurally resembles data used in empirical studies. In this way, the method allows us to compare the predictions of a capital structure model in "true dynamics" both to the findings of the empirical literature and to the comparative statics predictions of the same model. We find that the model results are consistent with empirically observed positive relationship between firm size and leverage. Whether the model can give rise to data that are consistent with this relationship quantitatively is so far an open question. But these findings provide a clear signal of the need for further research in this area.

A clear and principal direction which future work should take is to investigate other factors effecting firm size-leverage relationship. The framework developed here could usefully be extended to allow for other effects. Another important direction is to put to empirical investigation a number of cross-sectional and time-series predictions developed here.

Appendix A Proof of Theorem 1

Denote

$$x' = \begin{cases} (x_1, x_3), & \text{if } I = 2 \\ (x_1, x_2), & \text{if } I = 3 \end{cases} \tag{41}$$

and $X' = [1, +\infty) \times [0, 1]$.

From (12), by the global inverse function theorem, there exists the function $\varphi : X' \times [\Omega_0, \Omega_\infty] \to [0, 1]$ such that

$$\widetilde{p}[x', \varphi(x', \omega)] \left\{ x_1 \omega - D^0 \widetilde{p}[x', \varphi(x', \omega)] \right\} - \widetilde{e}[x', \varphi(x', \omega)] \equiv 0 \tag{42}$$

and

$$\frac{\partial \varphi}{\partial \omega} = x_1 \left(\frac{\partial M}{\partial x_I} \right)^{-1}. \tag{43}$$

Define

$$\begin{aligned} R(x', \omega) &= F[x', \varphi(x', \omega)] + p[x', \varphi(x', \omega)] x_1 \omega, &(44)\\ \Pi(s, x', \Omega) &= R[x', \Omega(sx_1) - 1/(sx_1)]. &(45) \end{aligned}$$

For any function $\nu : S \to X'$, let the mapping T_ν be

$$T_\nu(\Omega)(s) = \Pi(s, \nu(s), \Omega). \tag{46}$$

Then

$$\Omega^*(s) = \sup_{\nu_0, \nu_1, \ldots} \lim_{K \to \infty} (T_{\nu_0} \ldots T_{\nu_K})(\Omega_\infty)(s). \tag{47}$$

Assumption (13) implies that

$$0 \leq \frac{\partial R}{\partial \omega} \leq K, \tag{48}$$

and thus the monotonicity property for the return function Π holds: for every $s \in S$ and $x' \in X'$

$$\Pi(s, x', \Omega_1) \leq \Pi(s, x', \Omega_2), \qquad \text{if} \quad \Omega_1 \leq \Omega_2. \tag{49}$$

Moreover, since Ω_∞ solves

$$\Omega_\infty = \sup_{x'} R(x', \Omega_\infty), \tag{50}$$

it is immediate that for every $s \in S$ and $x' \in X'$

$$\Pi(s, x, \Omega_\infty) \leq \Omega_\infty. \tag{51}$$

Therefore, Assumptions I, I.1 and I.2 in Bertsekas and Shreve (1996) (see chapter 5, p. 70-71) hold (the latter holds because of the upper bound in (48)). Thus, by Proposition 5.2 (p. 73) $\Omega^*(s)$ satisfies Bellman equation (14).

Since we are not interested in the values of $\Omega^*(s)$ for very small firm size s, since in the true model (2) the firm is initially unlevered (see also (16)), and since $f(s, x) \geq s$, we can restrict the values of the state variable s to be greater than certain small but strictly positive constant $\underline{s}$. This, in turn, bounds from above the values of the control variable $x_1 = \gamma$. Therefore, the assumptions of Proposition 5.10 (Bertsekas and Shreve, p. 86) are satisfied so that dynamic programming equation (15) holds and there exists a stationary optimal policy. This completes the proof of the theorem.

Appendix B Details of Simulation Analysis

The details are similar to those reported in Strebulaev (2004). The process for δ is discretized using the following approximation:

$$\delta_t = \delta_{t-\Delta t} e^{\left(\mu_A - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} z_t}, \tag{52}$$

where Δt is one quarter, z_t is a standard normal variable and μ_A is the growth rate of the net payout ratio under the physical measure. The benchmark simulation is for 300 quarters and 3000 firms. Note that while we discretize the model for the purpose of simulation, firms still operate in a continuous environment. In particular, it must happen now that firms will sometimes "overshoot" over boundaries and make their financial decisions not exactly at the prescribed optimal times. Unreported robustness checks show that increasing the frequency of observations does not at all change the results.

To choose the number of observations that will be dropped to minimize the impact of initial conditions the following ad-hoc procedure has been implemented. We simulate the panel dataset for 3000 firms with the benchmark set of parameters in the absence of systematic shock 250 times. For each economy the average leverage ratio is calculated. The rolling sum of the first differences in average leverage ratios (quarter by quarter) over the last 10 quarters is estimated. The stopping rule is for this variable to be less than 0.5% in absolute magnitude (for comparison, the average value of this variable in the first 10 quarters is 5%). The economy is then defined as converged to its steady state. The resulting distribution of steady state stopping times across all economies has the mean of 30 quarters, 95% percentile value of 50 quarters and the maximum of 76 quarters. For a conservative estimate we double the maximum. Since this procedure is largely ad-hoc, we check the result by simulating 20 economies for 1000 quarters and confirm that there is no difference in the average leverage ratio behavior for the last 900 quarters by investigating rolling sums over the entire period.

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Table I
Parameter Values for Benchmark Simulations

Listed are the values and sampling distributions chosen for all parameters required to simulate the benchmark case of the model. V_0 is the initial present value of future cash flows. σ_A is the instantaneous volatility of the cash flow process. q' is the proportional restructuring costs (as a fraction of marginal debt issuance). $\bar{q}$ is the fixed costs of debt issuance. RP_A is the asset risk premium. τ_c is the marginal corporate tax rate, τ_d and τ_i are the marginal personal tax rates on dividends and on income, respectively. r is an instantaneous after-tax riskless rate.

	Distribution	Mean	Std.dev
$\log(V_0)$	normal	7	3
σ_A	constant	0.25	
q'	constant	0.007	
$\bar{q}$	constant	1	
α'	constant	0.05	
$\bar{\alpha}$	constant	5	
RP_A	constant	0.05	
τ_c	constant	0.35	
τ_i	constant	0.35	
τ_d	constant	0.20	
r	constant	0.045	

Table II
Descriptive Statistics

The table reports descriptive statistics for the following variables: market leverage (ML), quasi-market leverage (QML), LogSize and credit spreads (CS). Ref. point refers to the case when all firms are at their refinancing points. All other statistics are given for dynamics. 10 data sets are generated, each containing 75 years of quarterly data for 3000 firms. For each dataset the statistics are first calculated for each year in the last 35 years of data and then are averaged across years. Finally, they are averaged over data sets. Min and Max give the minimum and maximum over the 1000 data sets of the annual averages.

		percentiles						
	Mean	1%	50%	90%	95%	99%	st. dev.	N
Market leverage, ML								
Initial	0.28	0.00	0.34	0.40	0.41	0.42	0.14	3000
Average	0.44	0.00	0.43	0.72	0.82	0.95	0.22	3000
Min	0.35	0.00	0.37	0.59	0.69	0.87	0.15	3000
Max	0.50	0.00	0.49	0.83	0.91	0.98	0.29	3000
Quasi-market leverage, QML								
Initial	0.28	0.00	0.34	0.40	0.41	0.42	0.14	3000
Average	0.45	0.00	0.43	0.76	0.85	0.96	0.23	3000
Min	0.36	0.00	0.37	0.62	0.72	0.90	0.16	3000
Max	0.51	0.00	0.50	0.87	0.93	0.99	0.30	3000
Log Size								
Initial	6.27	-1.12	6.28	10.15	11.29	13.75	3.08	3000
Average	7.54	-0.00	7.48	11.94	13.19	15.56	3.38	3000
Min	5.16	-2.74	5.01	9.56	10.86	13.27	3.25	3000
Max	10.13	2.72	10.04	14.70	16.02	18.76	3.57	3000
Credit spreads, CS								
Initial	1.59	0.00	1.99	2.00	2.01	2.02	0.80	3000
Average	2.46	0.00	2.20	4.02	5.01	7.28	1.37	3000
Min	1.96	0.00	2.04	3.00	3.66	5.72	0.88	3000
Max	2.82	0.04	2.42	5.13	6.29	8.48	1.86	3000

Table III
Frequency of Events

The table reports the frequency of various events in the generated data sets. Restructure refers to restructuring at the upper boundary. Equity Issues refer to the situation when interest expense is larger than current cash flow. Unlevered is the fraction of firms that have zero leverage. 1000 data sets are generated, each containing 75 years of quarterly data for 3000 firms. For each dataset frequencies are computed across the last 35 years of data and then averaged over data sets. Min, 25%, 75% and Max give, correspondingly, the minimum, 25% percentile, 75% percentile and maximum annual averages over all data sets. All frequencies are annualized and given in percentages.

	Default	Restructure	Equity Issues	Unlevered
Panel (a): All firms				
Mean	1.10	18.53	9.41	10.26
Median	1.06	17.40	9.43	9.24
Std. Dev.	0.45	6.43	2.40	4.52
Min	0.24	6.42	3.40	1.81
25% quantile	0.77	13.84	7.67	6.93
75% quantile	1.38	21.87	11.13	12.78
Max	2.79	43.85	17.22	28.98
Panel (b): 25% smallest firms				
Mean	1.28	2.04	8.54	
Median	1.25	1.74	8.21	
Std. Dev.	0.42	1.22	4.24	
Panel (c): 25% largest firms				
Mean	0.73	43.68	6.89	
Median	0.67	41.98	6.00	
Std. Dev.	0.38	11.82	4.13	

Table IV
Fixed and Proportional Costs

The table reports the relative size of fixed and proportional costs of debt issuance in the generated data sets. Total costs is the sum of fixed and proportional costs. Firm Value is the sum of book debt and market equity. 1000 data sets are generated, each containing 75 years of quarterly data for 3000 firms. For each dataset frequencies are computed across the last 35 years of data and then averaged over data sets. Min, 25%, 75% and Max give, correspondingly, the minimum, 25% percentile, 75% percentile and maximum annual averages over all data sets. All ratios are given in percentages.

	$\frac{\text{Fixed}}{\text{Total Costs}}$	$\frac{\text{Total Costs}}{\text{Debt Issuance}}$	$\frac{\text{Total Costs}}{\text{Firm Value}}$
Panel (a): All firms			
Mean	18.66	1.03	0.11
Median	18.51	1.02	0.10
Std. Dev.	3.18	0.09	0.03
Min	9.74	0.83	0.05
25% quantile	16.52	0.97	0.08
75% quantile	20.78	1.08	0.12
Max	30.08	1.40	0.24
Panel (b): 25% smallest firms			
Mean	83.89	4.95	1.85
Median	85.14	5.05	1.85
Std. Dev.	4.79	0.97	0.59
Panel (c): 25% largest firms			
Mean	5.76	0.75	0.02
Median	5.28	0.74	0.02
Std. Dev.	2.65	0.03	0.00

Table V
Cross-sectional Regressions

The table reports the results of cross-sectional regressions on the level of the quasi-market leverage ratio, QML. Independent variable is firm size measured as the log of the sum of book debt and market equity. 1000 data sets are generated, each containing 75 years of quarterly data for 3000 firms. Coefficients and t-statistics are means over 1000 simulations. Fama-MacBeth (1973) method is used, with the regressions run over the last 35 years of each data set and then averaged. The last three columns report additional information on the regression: the standard deviation of coefficients and t-statistics, and the 10th and 90th percentile values of these coefficients across simulations. Standard errors are Fama-MacBeth (1973) with Newey-West correction.

		10%	90%	Std.dev.
Panel (a): Standard regressions				
Constant	0.31	0.22	0.37	0.05
	(51.73)	(35.32)	(68.57)	(12.94)
LogSize	0.02	0.01	0.03	0.01
	(30.92)	(21.53)	(38.37)	(7.79)
R^2	0.10	0.03	0.17	0.05
	(35)	(35)	(35)	(35)
N	3000	3000	3000	0
Panel (b): Regressions with unlevered dummy				
Constant	0.53	0.50	0.57	0.02
	(62.15)	(48.62)	(80.98)	(11.75)
LogSize	-0.004	-0.005	-0.003	0.001
	(-20.32)	(-30.94)	(-12.04)	(6.98)
Unlevered firms	-0.51	-0.55	-0.48	0.03
	(-54.70)	(-69.60)	(-43.54)	(9.44)
R^2	0.40	0.28	0.51	0.08
	(35.00)	(35.00)	(35.00)	(35.00)
N	3000	3000	3000	0