An Equilibrium Existence Theorem in Continuous-Time Finance

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Abstract

We state a theorem on the existence of equilibrium in a continuous-time finance model with multiple agents, including the case of dynamically incomplete markets as well as dynamically complete markets. In the case of dynamically incomplete markets, equilibrium prices may be discontinuous, even if all economically relevant primitives of the model (including the flow of information) are continuous. We provide a substantial part of the proof, along with a detailed outline of the remainder of the proof, clearly indicating the places where the details remain incomplete.

1 Introduction

Virtually all of the work in continuous-time finance takes as given that the prices of securities follow an exogenously specified stochastic process.[1] But prices of securities are in fact determined day by day and minute by minute by the balancing of supply and demand. A complete model of continuous-time trading requires the derivation of the pricing process as an *equilibrium* determined by more primitive data of the economy, in particular the agents' information, utility functions, and endowments, and the securities' dividend processes.

To date, the existence of equilibrium in continuous-time finance models has only been established in the case of a single agent (Bick [9], He and Leland [27], Cox, Ingersoll and Ross [11], Duffie and Skiadis [24], Raimondo [51][2]), or of multiple agents with a complete set of Arrow-Debreu contingent claims (Mas-Colell and Richard [45] and Bank and Riedel [8]). The assumption that there is a complete set of Arrow-Debreu contingent claims is clearly unrealistic as a description of actual security markets. It may be possible to extend the results that assume a complete set of Arrow-Debreu contingent claims to the more general situation in which markets are dynamically complete, i.e. a complete set of Arrow-Debreu contingent claims can be *constructed* from the set of primitive traded securities. However, it is not straightforward to do this. Suppose we are given a model, with securities specified by their dividend processes. If we can find Arrow-Debreu equilibrium prices, these prices can be used to price the given securities. If the resulting securities price process is dynamically complete, then the securities prices will be equilibrium prices in the specified model. However, there is no theorem in the literature specifying conditions on the primitives in the model that ensure that the securities prices induced by the Arrow-Debreu prices will be dynamically complete, and there is no clear intuition as to what kinds of conditions would help. Indeed, it is conceivable that a given model might have two Arrow-Debreu equilibrium prices: one which induces dynamically complete prices on the securities, and another which induces dynamically

[1]The most widely studied price process is geometric Brownian motion, but other Itô Processes and Lévy Processes have also been extensively studied.

[2]Of these papers, Raimondo [51] is the only one which provides sufficient conditions for existence of equilibrium expressed solely in terms of the primitives of the model—the dividends of the securities and the representative agent's endowment and preferences.

incomplete prices. In summary, there is no theorem in the literature which provides realistic assumptions about securities (defined by their dividends) and the endowments and preferences of the agents that suffice to ensure the existence of a financial market equilibrium.

This paper provides sufficient conditions for the existence of an equilibrium in a multi-agent securities model. The results do not depend on dynamic completeness in any way.

Generality is not our goal in this paper, and accordingly the model is special in some respects: agents have additively separable, time-independent utility functions, and the securities pay off only in the terminal period. The endowment in the terminal period is an function of the terminal value of a K-dimensional Brownian motion β. There are $J \leq K$ stocks (with net supply one); stock j pays off only in the terminal period T, when its payoff is $e^{\sigma_j \beta(T)}$. There is also a "bond" (with zero net supply) which pays one unit of consumption at date T. [3] Our existence theorem and pricing formula apply equally well to the case in which markets are potentially dynamically complete (i.e., $J = K$) and the case in which markets are necessarily dynamically incomplete ($J < K$).

We cannot at this point rule out the possibility that the equilibrium prices may be discontinuous, even if all the primitives of the economy (including the information flow) are continuous. Our method considers hyperfinite economies, a particular construction within nonstandard analysis which simultaneously exhibits the properties of discrete and continuous economies. At any given time t, an infinitesimal piece of information, which is visible in the hyperfinite model but not visible in the filtration generated by the underlying Brownian motions, might be used to coordinate which of two possible equilibria are followed for the remaining times in $(t, T]$. The terminal distribution of wealth might be quite different in these two equilibria; if so, since the current security prices depend on the terminal distribution of wealth, it follows that security prices will exhibit a discontinuity in this situation. The situation is somewhat analogous to the addition of extrinsic uncertainty to an economy, where it may lead to the existence of sunspot equilibria; or to a game, where it may allow players to coordinate on a correlated equilibrium.

[3]This is best thought of as a long-term zero coupon bond. While the bond pays off with certainty one unit of consumption at the terminal date, its price at dates before t will fluctuate in the equilibrium pricing process. It is thus not risk-free when viewed as a vehicle for transferring consumption across time.

We suspect that a more careful analysis of the hyperfinite economy (equivalently, of sequences of discrete economies) will allow us to establish the existence of an equilibrium with continuous prices in the continuous-time economy. Since we do not yet have such an analysis, our theorem asserts the existence of a continuous-time equilibrium with possibly discontinuous equilibrium prices. If it should turn out that equilibrium prices are in fact discontinuous in these models, we would not regard that as necessarily bad. Actual securities prices do exhibit very rapid price movements that are not consistent with Itô Processes, and there is a significant literature examining derivative pricing and other issues in situations in which prices are assumed to follow an exogenously specified model with discontinuitites. To the best of our knowledge, there is no paper in the literature in which discontinuous prices emerge endogenously, rather than being assumed by the exogenous imposition of a price process with jumps.

At this point, it may be helpful to discuss the relationship of our results to the martingale method of rationalizing arbitrage-free pricing systems. This method was initiated by Harrison and Kreps [26] in the complete markets case, and was subsequently extended to incomplete markets by Back [7], Duffie and Skiadis [24] and others. The essential idea is well-known in finance: if a system of prices does not admit arbitrage, it is a vector martingale (after time-discounting) with respect to an equivalent probability measure, i.e. a probability which is mutually absolutely continuous with respect to the true objective probability measure. Given a martingale pricing process, one can define a state-dependent felicity function which makes the pricing process an equilibrium for an economy with a representative agent who maximizes expected felicity. This is often understood as saying that "any arbitrage-free pricing system is an equilibrium."

However, in this paper (and the others cited below), we assert that equilibrium imposes more structure on finance models than that implied by the absence of arbitrage alone. Suppose, for example, we require that the single agent's utility function be the expected utility generated by some state-*independent* felicity function. In that case, the argument just cited that any arbitrage-free pricing system can be justified as an equilibrium will not hold; state-dependence is essential to the proof. Of course, one might object that state-dependence is commonly observed in practice. It is difficult to argue against this objection, but the implication of this objection is not that one should consider a pricing process justified if it can be supported as equilibrium with respect to a felicity function whose state-dependence is carefully chosen to match the peculiarities of the pricing process. The implication is that the state-dependence should be specified as part of the model, and the pricing process should be required to be an equilibrium with respect to the exogenously given state-dependent utility function. Allowing arbitrary state-dependence is a virtue in a result of the form "for all state-dependent felicity functions, ... ;" is is not a virtue in a result of the form "there exists a state-dependent felicity function ... ."

One of the reasons that state-independence appears natural in some models is that the models are partial equilibrium. If a significant portion of household wealth is held in housing, a model that includes stocks but not housing is a partial equilibrium model. Since changes in the value of housing induce wealth effects that alter individuals' willingness to hold stocks, changes in housing values seem, in a stock-only model, to be instances of state-dependent felicity. But in a general equilibrium model which includes both stocks and housing, the state-dependence disappears. In particular, we argue that the relationship between stock pricing and housing can only be properly studied in a general equilibrium model which includes both. More generally, in this and the companion papers, we take the position that all assets and securities should be included in the model, and that felicity functions (and in particular any state-dependence of felicity functions) should be taken as exogenously specified.

This approach has real economic and financial consequences. If one's sole criterion for validating a price process is the absence of arbitrage, then neither the number of agents, nor their endowments, nor their utility functions can play any role in determining prices: absence of arbitrage is a property of the pricing process alone, not of the underlying economic data. Thus, if one wishes to study the effect of demographics (in particular, the effect on asset prices of the sizes of various demographic cohorts), one simply cannot do it if absence of arbitrage is one's sole criterion for validating price processes. However, the large size of the American baby boom generation has had profound implications on the pricing of financial assets: see Geanakoplos, Magill and Quinzii [25].

This paper is the second of a series of papers providing foundations of financial markets from a GEI perspective.

1. As noted above, Raimondo [51] proves an existence theorem for repre-

sentative agent economies with dynamically incomplete markets. That paper introduces the methods that we utilize in this paper. He explicitly calculates the GEI equilibrium price process in terms of the form of the agent's utility function, and in particular her attitude toward risk; her endowment in the terminal period; and the realizations of the other sources of uncertainty in the economy. With Constant Relative Risk Aversion (CRRA) utility and no endowment in the terminal period, and exactly one source of uncertainty and an associated stock (i.e. $J = K = 1$), equilibrium prices follow a geometric Brownian motion. However, if any of these conditions fail, the stock price process is *not* geometric Brownian motion, nor is it a Lévy Process. For example, with Constant Absolute Risk Aversion (CARA) utility, the distribution of prices at any fixed time $t \in (0, T)$ is *not* log normal. With more than one security, the performance of one security induces wealth effects in the pricing process of other securities; as a consequence, the security price processes are not Lévy Processes.

2. We intend to generalize the results presented in this paper by considering more general utility functions and security payoffs.

3. We are developing (Anderson and Raimondo [5]) parametric and non-parametric tests for the pricing processes that arise as GEI equilibria.

4. If the pricing process is not geometric Brownian motion, then the Black-Scholes option pricing formula needs to be modified. We have computed generalized Black-Scholes formulas for certain of our pricing processes, and will address this question in detail (Anderson and Raimondo [6]).

5. As demonstrated in Raimondo [51], assets cannot be priced in isolation. The returns on one asset generate wealth effects that necessarily affect the risk tolerance of agents, and thus alter the price of every other asset. Thus, a model of stock pricing which does not take into account the bond, mortage and housing markets is necessarily incomplete; see Deng, Quigley and Van Order [20]. Similarly, assets cannot be priced without taking into account demographic factors, particularly the size of different age cohorts; see Geanakoplos, Magill and Quinzii [25]. Because it is a representative agent model, Raimondo [51] cannot address

questions concerning the effect of demographics on asset pricing. However, these questions *can* be addressed in the multi-agent model of this paper. We intend to pursue this question in a separate paper.

Most of the work in continuous-time finance has been done in the context of dynamically complete markets, in which the Bellman equation can be used to characterize a trader's optimal portfolio strategy, the agent's demand. There is a well-developed theory of existence of equilibrium in *finite* General Equilibrium Incomplete Markets (GEI) models. Our approach avoids the need to compute the agent's demand (and thus the need for dynamically complete markets) or to generalize the Duffie-Shafer [22, 23] fixed point argument to infinite-dimensional economies. Nonstandard analysis provides powerful tools to move from discrete to continuous time, and from discrete distributions like the binomial to continuous distributions like the normal; in particular, it provides the ability to transfer computations back and forth between the discrete and continuous settings. Thus, we invoke the Duffie-Shafer existence result for the discrete case and use nonstandard analysis to extend it to the continuous setting.

Anderson [1] provided a construction for Brownian motion and Brownian stochastic integration using nonstandard analysis. In nonstandard analysis, hyperfinite objects are infinite objects which nonetheless possess all the formal properties of finite objects. Anderson's Brownian motion is a hyperfinite random walk which, using a measure-theoretic construction called Loeb measure, can simultaneously be viewed as being a standard Brownian motion in the usual sense of probability theory. While the standard stochastic integral is motivated by the idea of a Stieltjes integral, the actual standard definition of the stochastic integral is of necessity rather indirect because almost every path of Brownian motion is of unbounded variation, and Stieltjes integrals are only defined with respect to paths of bounded variation. However, a hyperfinite random walk is of hyperfinite variation, and hence a Stieltjes integral with respect to it makes perfect sense. Anderson showed that the standard stochastic integral can be obtained readily from this hyperfinite Stieltjes integral.

This construction of Brownian motion has been used to answer a number of questions in stochastic processes. For the present paper, the most important generalizations of it are the work of Keisler [35] on stochastic differential equations with respect to Brownian motion, and work by Hoover and Perkins

[31] and Lindström [38, 39, 40, 41] on stochastic integration with respect to more general martingales. The nonstandard theory of stochastic integration has previously been applied to option pricing in Cutland, Kopp and Willinger [13, 14, 15, 16, 17, 18, 19]. Those papers primarily concern convergence of discrete versions of options to continuous-time versions, and their methods can likely be used to establish convergence results for the option pricing formulas developed in Anderson and Raimondo [6]. Nonstandard analysis has also previously been applied to finance in Khan and Sun [36, 37] to relate the Capital Asset Pricing Model and Arbitrage Pricing Theory in a Single-Period Setting.

Our starting point is a continuous-time model. We use a hyperfinite discretization procedure to construct a model with a hyperfinite number of trading dates, a simple binary tree with a hyperfinite number of nodes. In this setting, the generic existence of equilibrium follows immediately from Robinson's Transfer Principle and the Duffie-Shafer [22, 23] results on existence of equilibrium in GEI models with a finite number of dates and states;[4] without loss of generality, we may choose our discretization to lie in the generic set. We show that equilibrium consumptions are nonzero at all times and states. Consequently, as in Magill and Quinzii, we can use the first order conditions to characterize the equilibrium prices. The Central Limit Theorem then allows us to explicitly describe the prices as integrals with respect to a normal distribution; however, with more than one agent, the prices depend on the terminal distributions of wealth, which are not described in closed form. Then, as in the work of Anderson, Keisler, Hoover and Perkins, and Lindström cited above, we use the Loeb measure construction to produce an equilibrium of the original continuous-time model; the pricing process generated by that continuous-time equilibrium can be readily determined from the pricing formula in the hyperfinite economy.

In order to invoke Lindström's result linking hyperfinite Stieltjes integrals to standard stochastic integrals, we need to show that the pricing process is an SL^2 martingale; this condition is essentially the same as requiring the equilibrium prices in a sequence of finer and finer finite discretizations to be uniformly square integrable martingales. It is obvious that the price of the stocks (p_A in the formal model) is an internal martingale. It will be an SL^2 martingale provided that the prices at the terminal period $p_A(T, \cdot)$

[4] Magill and Quinzii [44] is an excellent reference on GEI models.

are SL^2. In order to guarantee that this condition is satisfied, we need to form our hyperfinite discretization in a particular way. Specifically, we restrict trading a particular infinitesimal amount of time before the terminal period T. At earlier nodes, trading can be conditioned on all movements of the hyperfinite random walks; however, in this last infinitesimal portion of the time, trading can be conditioned only on movements of the hyperfinite random walks corresponding to the Brownian motion components spanned by the securities' terminal dividends. We then consider the equilibria of the economy that results when the trading restriction is imposed. The trading restriction has two effects:

- If all traders' terminal endowments are spanned by the terminal dividends of the traded securities, we have essentially complete markets; Pareto optimality will then guarantee that the SL^2 condition holds. However, if at least one agent's terminal endowment is not spanned by the terminal dividends, then we show that the prices obtained from that agent's marginal utilities satisfy the SL^2 condition. The trading restriciton prevents the trader from selling forward essentially all of his/her endowment in the terminal period T. The trader might get very unlucky, and see the value of his/her holdings eroded to an infinitesimal at time T, but the probability of this happening is constrained by the distributions of the securities, and is too small to upset the SL^2 condition. However, since significant uncertainty remains to be resolved after trading is restricted, the trader cannot systematically sell forward his/her terminal consumption in such a way that consumption at time T is infinitesimal with noninfinitesimal (or large infinitesimal) probability without violating the nonnegavity restriction on consumption inherent in the definition of equilibrium.[5]

[5] Without the trading restriction, it is *feasible* for an agent to achieve infinitesimal consumption at time T with probablity one. Of course, this does not mean a trader would choose to have infinitesimal consumption at time T with noninfinitesimal probability. Indeed, with complete markets, equilibria are Pareto optimal, which suffices to rule out infinitesimal consumptions with noninfinitesimal probability. Even with incomplete markets, it appears to us unlikely a trader would deliberately choose to sell forward his/her endowment to leave infinitesimal consumption on a set of noninitesimal. However, we have not been able to prove that s/he does not make that choice. Moreover, based on their work computing equilibrium in GEI models (Judd, Kubler and Schmedders [33, 34]) have expressed (in a private conversation with us) doubts that such a result holds. Their work

- It aligns the strategic options available to the traders in the hyperfinite model with those in the standard continuous-time model. Zame [52] has demonstrated that a naive discretization of the continuous-time model may present traders with very different trading opportunities from those they face in the continuous-time model. Here is an example of this in the context of our model. Suppose $t < T$. The uncertainty is described by a scalar Brownian Motion β. The conditional distribution of $\beta(T)$ is normal with mean $\beta(t)$ and variance $T - t$; if t is very close to T, the conditional variance is very small. However, the probability of a very large change in β (for example, $|\beta(T) - \beta(t)| > 10^6$) is *not* zero. In particular, since the terminal endowments and terminal dividends of the securities are defined as functions of $\beta(T)$, the probability of a large change in the terminal endowments or terminal dividends is not zero. Consequently, at time $t < T$, the trader cannot have sold forward terminal consumption in a way that would reduce the agent's terminal consumption below zero under any possible realization of $\beta(T) \in (-\infty, \infty)$. For example, if agent 1's endoment is $e_1(s) = 0$ for $s \in [t, T)$, $e_1(T) = 1 + e^{\beta}(T)$, there is a bond which pays off one unit of consumption, and there is no stock in the model, then no matter how large $\beta(t)$ is, agent 1 cannot be short more than one unit of the bond at time t; if s/he were short more than one unit of the bond, then s/he will have negative consumption with positive probability at time T, and this is impossible at equilibrium. If one did not impose the trading restriction, the set of available trading strategies in the hyperfinite model would be considerably larger than the set of available trading strategies in the continuous-time model, and it would be very difficult to extract a continuous-time equilibrium from a hyperfinite equilibrium.

In the discussion that follows, we will distinguish between the hyperfinite and continuous time models by placing "hats" on the symbols associated with the hyperfinite model. Having shown that the equilibrium prices $\hat{p}$ in the hyperfinite model (with trading halt) are SL^2 martingales, we know by the results of Hoover and Perkins [31] and Lindström [38, 39, 40, 41] that the equilibrium prices do not chatter, hence we can extract a continuous-time

is in an infinite-horizon setting with heterogenous impatient agents; since our model is finite-horizon, it is not clear whether the conclusions of their model should hold in ours.

pricing process, denoted p, which will be our candidate equilibrium price in the continuous-time model. p is a square integrable martingale, and the stochastic integrals (capital gains) of nonchattering trading strategies with respect to the hyperfinite and continuous-time price processes agree up to an infinitesimal.

Let $\hat{z}$ be the equilibrium trading strategy in the hyperfinite model. We need to extract a standard continuous-time trading strategy z to serve as a candidate equilibrium trading strategy. If we knew that $\hat{z}$ were SL^2 in the quadratic variation measure induced by p, and that $\hat{z}$ does not chatter, we could simply use the results of Hoover and Perkins [31] and Lindstrom [38, 39, 40, 41].

One first needs to determine the correct form of the SL^2 condition on $\hat{z}$. The most obvious definition would be that each component of $\hat{z}$ is SL^2 with respect to the corresponding component of $\hat{p}$. However, just as in the standard continuous-time theory, this is too restrictive. In the standard continuous-time theory, it is well known that the set of trading strategies Itô integrable with respect to an Itô process p may be strictly larger than the set of trading strategies such that each component of the trading strategy is Itô integrable with respect to the corresponding component of p (see Example 2.8, page 61 of Nielsen [48]). There are two ways to explain this, the first technical and the second based on hedging:

- First, the Itô Integral in the case of a one-dimensional integrator is defined using the Itô Isometry, which says the L^2 norm of the integral is the integral of the L^2 norm of the integrand with respect to the quadratic variation measure of the integrator. With respect to a higher-dimensional integrand, the Itô Isometry fails. The change in the integral is the *inner product* of the integrand and the change in the integrator, and positive contributions from one component may cancel negative contributions from another component when the inner product is computed. The definition of the Itô Integral can be made using a variation of the Itô Isometry: the L^2 norm of the integral is the integral of the square of the inner product of the integrand and the change in the integrator, with respect to the quadratic variation measure of the integrator. Because of the cancellation in the inner product, there are processes that can be Itô integrated, even though some of their components cannot be Itô integrated with respect to the corresponding components of the integrand.

- Second, the changes in the components of p might be instantaneously perfectly correlated at some points. In this situation, a trader can take large positions in each of a number of securities, with the gains and losses in a given security being nearly perfectly hedged by losses and gains in other securities. For example, if the change in two securities is instantaneously perfectly negatively correlated at a point, a trading strategy that goes very long in one and very short in the other is instantaneously perfectly hedged at that point. Such a trading strategy is nearly perfectly hedged on a small interval around the point, and may be used to move a finite amount of consumption from certain states to other states over a finite time interval. However, at such a point, the capital gain typically cannot be comuted one security at a time because the integral of the square of the trading strategy over time does not converge. There might be an an infinite loss from holding one security offsetting an infinite gain from holding another security, leading to a finite gain or loss. Or we might find that the capital gain in a single security is an improper integral consisting of an infinite gain and an infinite loss.

The hyperfinite model behaves just like a discrete finite GEI model. At certain prices, Hart points, the rank of the matrix of price changes falls, and this may prevent the existence of equilibrium; the Duffie-Shafer existence result works on a generic set of economies, because on a generic set of economies, the candidate equilibrium prices are not Hart points. Because we take our hyperfinite model from the generic set, we know that the equilibrium prices are not Hart points. However, our hyperfinite equilibrium prices might conceivably be infinitesimal Hart points, in the sense that the matrix of price changes is only infinitesimally nonsingular at some nodes. At such nodes, the equilibrium trading strategy $\hat{z}$ might well take an infinite (i.e. a well-defined nonstandard number which is larger than any standard real number) long position in one security and an infinite short position in another. We believe that the strong (componentwise) form of the SL^2 condition is essentially equivalent to the statement that in the continuous time model, almost every path has the property that the dispersion of the price process if of full rank for all $t \in [0, T]$; in turn, this is essentially equivalent to the statement that, in the hyperfinite model, almost every (in the Loeb measure) path ω

contains no time $t \in \hat{\mathcal{T}}$ such that (t, ω) is an infinitesimal Hart point.[6] It may be possible to show that the componentwise SL^2 condition holds, perhaps on a generic set of standard continuous-time models, through a more careful analysis of finite GEI models.

We now turn to the weaker form of the SL^2 condition on $\hat{z}$. If we let $\hat{q}_{\hat{p}}$ denote the quadratic variation measure generated by $\hat{p}$, and let

$$\hat{z}_{\hat{p}}(t, \omega) = \frac{\hat{z}(t, \omega) \cdot \Delta \hat{p}(t, \omega)}{\sqrt{\hat{q}_{\hat{p}}(t, \omega)}}$$

then $\hat{z}_{\hat{p}}$ *is* SL^2 in the quadratic variation measure of $\hat{p}$. If not, we may modify $\hat{z}$ to obtain a trading strategy $\hat{z}'$ which is SL^2 componentwise with respect to the quadratic variation measure of $\hat{p}$; it follows that $\hat{z}' \cdot \Delta \hat{p}$ is SL^2 with respect to the quadratic variation measure. The cumulative gains process of $\hat{z}$ is a mean-preserving spread of the cumulative gains process of $\hat{z}'$, and this spread is independent of the filtration generated by $\hat{p}$, so it is independent of the marginal utilities of consumption. Consequently, the consumptions generated by $\hat{z}'$ give strictly higher utility than those generated by $\hat{z}$, which contradicts the fact that $\hat{z}$ is in the demand set. Hence, $\hat{z}_{\hat{p}}$ is SL^2 with respect to the quadratic variation measure of $\hat{p}$. The body of the paper gives a more detailed sketch of the argument. A full proof will be given in the next draft of the paper.

We also need to show that we can extract a standard continuous-time trading strategy z from $\hat{z}$. More precisely, we show there exists z such that for all $t \in [0, T]$,

$$^\circ\left(\int_0^t \hat{z}\, d\hat{p}\right) = \int_0^t z\, dp$$

This is saying, essentially, that $\hat{z}$ doesn't chatter, but we have to be careful. At infinitesimal Hart points, $\hat{z}$ itself could chatter; the rank reduction in the security prices allows for a one-dimensional family of approximately optimal trading strategies, and the optimal strategy could bounce around. Thus,

[6] If (t, ω) is not an infinitesimal Hart point then

$$\frac{\|\hat{z}(t, \omega)\|_2}{\|\hat{z}_{\hat{p}}(t, \omega)\|_2}$$

is finite, and the weak and strong SL^2 conditions are the same at (t, ω).

the condition is really saying that $\hat{z}_{\hat{p}}$ cannot chatter. Once again, this is an argument about mean-preserving spreads. The body of the paper gives a more detailed sketch of the argument. A full proof will be given in the next draft of the paper.

The attentive reader will have surmised that we do not consider the proofs given in this paper for the SL^2 and nonchattering conditions on $\hat{z}_{\hat{p}}$ to be air-tight. We are confident the arguments given can be turned into full proofs, and anticipate doing this shortly. However, if we are for some reason unable to do this, we would feel comfortable imposing these conditions as endogenous assumptions on the model. In our view, they are no more objectionable than endogenous assumptions that have been imposed in other papers on existence of continuous-time financial equilibrium that have obtained significantly weaker results.

We now return to the discussion of the discontinuity of equilibrium prices. We let $\hat{q}$ denote the quadratic variation of the $(K+J)$-dimensional process $\left(\hat{\beta}, \hat{p}\right)$. As noted above, we cannot rule out discontinuous equilibrium prices. In particular, it seems that the direction of a single random walk step can determine which of two quite different continuation equilibria occur starting at a given time. That single random walk step is a measurable event in the Loeb filtration, but not in the smaller filtration generated by the Brownian motions themselves (two random walk paths that differ at only a single step are infinitely close for all time, hence correspond to the same continuous path of the random walk in Anderson's construction). As a consequence, the conditional distribution of terminal wealths can jump in a single random walk step; since the security prices depend on the conditional distribution of terminal wealth, they will jump also. The fact that these jumps may not be visible in the Brownian filtration gives one reason why existence of equilibrium in the continuous-time, dynamically incomplete markets context, has been so resistant to previous analysis. The fact that the internal equilibrium prices are SL^2 implies that almost every path has at most countably many discontinuities.

A second complication occurs if the hyperfinite equilibrium exhibits discontinuities at two time s, t which are infinitely close together. There is no problem extracting a well-behaved standard pricing process in this case. However, the complication comes in specifying the set of trading strategies and the associated capital gains process. When two or more jumps occur within an infinitesimal length of time, agents in the hyperfinite model can make different choices of security holdings just before each jump, and can condition their choice of security holdings at later jumps on the outcomes of earlier jumps. When the pricing process is projected into a standard stochastic process, these two or more jumps are compressed to a single, more complex jump occurring at a single standard time. If one defines a standard trading strategy to require a single security holding at this complex jump, then the set of standard trading strategies will allow the agents less freedom to trade than they have in the hyperfinite economy. Thus, one needs to define the trading strategies in a way which takes into account the origin of these complex jumps as two or more simple jumps. This may be a second reason why existence in the continuous-time, dynamically incomplete context has been resistant to previous analysis. We address this problem by using the quadratic variation of the pair (β, p) as a time change in the model, to produce a notion of generalized equilibrium. With the time change, the strategic options in the continuous-time and hyperfinite models are closely aligned, and we can extract a generalized equilibrium of the continuous-time model from the equilibrium of the hyperfinite model. If the equilibrium prices are continuous almost surely, we can remove the time change and convert a generalized equilibrium to an equilibrium.

While nonstandard analysis plays a central role in the proof, we emphasize that the statement of the theorem is expressed entirely in terms of standard continuous-time model.

2 The Model

In this Section we define the continuous-time model.

1. Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure λ which agrees with Lebesgue measure on $[0, T)$ and such that $\lambda(\{T\}) = 1$.

2. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F}, \mu)$. The σ-algebra of predictable sets, denoted $\mathcal{P}$, is the σ-algebra generated by the sets $\{0\} \times F_0$ and $(s, t] \times F_s$ where $s < t \in \mathbf{R}_+$, $F_0 \in \mathcal{F}_0$, $F_s \in \mathcal{F}_s$ (see Metivier [47]). A stochastic process is said to be predictable if it is measurable with respect to $\mathcal{P}$.

3. There is a standard K-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)$ such that β_k is independent of $\beta_{k'}$ if $k \neq k'$ and such that the variance of $\beta_k(t, \cdot)$ is t and $\beta_k(t, \cdot) = E(\beta_k(T, \cdot)|\mathcal{F}_t)$. Notice that we do *not* assume that $\{\mathcal{F}_t : t \in [0, T]\}$ is the filtration generated by β; in general, $\mathcal{F}_t$ will contain more information than the history of β up to time t.

4. There are I agents $i = 1, \ldots, I$. The endowment of the agent i is a process

$$e_i(t, \omega) = \begin{cases} f_i(\beta(t, \omega)) & \text{if } t \in [0, T) \\ \rho_i + e^{\iota_i \cdot \beta(T, \omega)} & \text{if } t = T \end{cases}$$

where $f_i : \mathbf{R}^K \to \mathbf{R}_{++}$ is a continuous function which is uniformly bounded away from zero, $\rho_i \in \mathbf{R}_+$, and ι is an $I \times K$ matrix whose i^{th} row is ι_i. We make this assumption on the form of the endowment for simplicity in this draft, and we believe it can be significantly weakened. Note that since $\lambda([0, T)) = T$ and $\lambda(\{T\}) = 1$, the endowment in period $t \in [0, T)$ is interpreted as a rate of flow of endowment, while the endowment in period T is interpreted as a stock or lump.

5. Given a measurable consumption function $c : [0, T] \times \Omega \to \mathbf{R}_+$, the utility function of the agent is

$$U_i(c) = E_\mu \left[\int_0^T \varphi_{i1}(c_t) dt + \varphi_{i2}(c_T) \right]$$

where the functions $\varphi_{ik} : \mathbf{R}_+ \to \mathbf{R}$ $(k = 1, 2)$ are continuous on $\mathbf{R}_+$ and C^2 on $\mathbf{R}_{++}$ and satisfy

$$\begin{cases} \varphi_{ik}(0) & = & 0 \\ \lim_{c \to 0_+} \varphi'_{ik}(c) & = & \infty \\ \lim_{c \to \infty} \varphi'_{ik}(c) & = & 0 \\ \varphi'_{ik}(c) & > & 0 \text{ for } c \in \mathbf{R}_{++} \\ \varphi''_{ik}(c) & < & 0 \text{ for } c \in \mathbf{R}_{++} \end{cases}$$

6. There are $J + 1$ tradable securities, with $0 \leq J \leq K$. The payoffs of these securities are described by a $(J + 1) \times K$ matrix σ, whose j^{th} row

is denoted by σ_j. Security j pays off[7]

$$A_j(t,\omega) = \begin{cases} 0 & \text{if } t \neq T \\ e^{\sigma_j \beta(T,\omega)} & \text{if } t = T \end{cases}$$

We assume that $\sigma_0 = 0$, hence A_0 is a long-term zero-coupon bond which pays off

$$A_0(t,\omega) = \begin{cases} 0 & \text{if } t \neq T \\ 1 & \text{if } t = T. \end{cases}$$

We assume also that $\operatorname{rank} \sigma = J$, so there are no redundant securities. Agent i is initially endowed with deterministic security holdings $e_{iA} = (e_{iA_0}, \ldots, e_{iA_J}) \in \mathbf{R}^{J+1}_+$ satisfying

$$\begin{aligned} \sum_{i=1}^{I} e_{iA_0} &= 0 \\ \sum_{i=1}^{I} e_{iA_j} &= 1\,(j = 1, \ldots, J) \end{aligned}$$

thus, each stock is in net supply one, and the bond is in zero net supply, and the initial holdings are independent of the state ω.

7. In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time t not depend on information which has not been revealed by time t, and the trading strategy times the variation in the price yields a finite integral. Specifically,

   (a) A securities price process is a stochastic process $p_A = (p_{A_0}, \ldots, p_{A_J}) : \Omega \times [0,T] \to \mathbf{R}^{J+1}$, where each p_{A_j} is a right-continuous square-integrable martingale with respect to $\{\mathcal{F}_t\}$; and such that, for

[7] We believe this functional form for $A_j(T,\omega)$ can be generalized to $A_j(T,\cdot) = e^{\int_0^T \sigma_j \, d\beta}$, where σ is a $(J+1)\times K$ Itô integrable process, σ_j denotes the j^{th} row of σ, σ_0 is identically zero and $\operatorname{rank} \sigma = J$ almost surely.

almost all ω, $p_A(\cdot, \omega)$ is continuous except at a countable set of times (the set of times may be different for different ω). Securities are priced *cum dividend* at time T. Since p_A is a martingale, the Martingale Convergence Theorem implies that for all t,

$$p_A(t_-, \omega) = \lim_{s \nearrow t} p_A(s, \omega)$$

exists almost surely. A consumption price process is a stochastic process $p_C(t, \omega)$.

(b) In order to deal with price processes with compound jumps, as discussed in the introduction, we introduce the notion of a generalized securities price process. A generalized securities price processes is a pair $(\tilde{p}_A, \tau)$, where

i. $\tau : [0, T] \times \Omega \to \mathbf{R}_+$ is progressively measurable, i.e. $\tau|_{[0,t]\times\Omega}$ is measurable in the product of Lebesgue measure on $[0, t]$ and $(\Omega, \mathcal{F}_t, \mu)$.

ii. for all ω, $\tau(t, \omega)$ is strictly increasing in t

iii. we define the filtration $\{\tilde{\mathcal{F}}_\tau : \tau \in \mathbf{R}_+\}$ by $B \in \tilde{\mathcal{F}}_\tau$ if and only if there exists $C \in \tilde{\mathcal{F}}_t$ such that $\tau(\omega, t) \leq \tau$ for all $\omega \in C$ and either $\mu(B \Delta C) = 0$ or $\mu(B \Delta (\Omega \setminus C)) = 0$.

iv. $\tilde{p}_A$ is a right-continuous filtration $\{\tilde{\mathcal{F}}_\tau : \tau \in \mathbf{R}_+\}$.

v. $\tilde{p}_A(\tau, \omega) = \tilde{p}_A(\tau(T, \omega), \omega)$ for $\tau \geq \tau(T, \omega)$.

vi. $p_A(t, \omega) = \tilde{p}_A(\tau(t, \omega), \omega))$ is a securities price process.

(c) Given a generalized securities price process $(\tilde{p}_A, \tau)$, there are unique measures $\tilde{q}_{A_j}$ on the σ-algebra of $\{\tilde{\mathcal{F}}_\tau : \tau \in \mathbf{R}_+\}$-predictable sets, which measure the quadratic variation of the components of $\tilde{p}$; they are generated by

$$\tilde{q}_{A_j}((\sigma, \tau] \times F_\sigma) = \int_{F_\sigma} (\tilde{p}_{A_j}(\tau, \omega))^2 - (p_{A_j}(\sigma, \omega))^2 d\mu$$

for $\sigma < \tau$ and $F_\sigma \in \mathcal{F}_\sigma$ and $\tilde{q}_{A_j}(\{0\} \times \Omega) = 0$.

(d) Given a generalized securities price process $\tilde{p}_A$, a generalized trading strategy for agent i is a stochastic process $\tilde{z}_i$ where

i. $\tilde{z}_i : \mathbf{R}_+ \times \Omega \to \mathbf{R}^{J+1}$

ii. $\tilde{z}_i$ is an $\{\tilde{\mathcal{F}}_\tau : \tau \in \mathbf{R}_+\}$-predictable process

iii. $\tilde{z}_{iA_j} \in L^2([0,T] \times \Omega, \mathcal{F}, \tilde{q}_{A_j})$.

8. Given a generalized securities price process $(\tilde{p}_A, \tau)$ and a consumption price process p_C, let $t(\tau,\omega) = \inf\{t \in [0,T] : \tau(t,\omega) \geq \tau\}$. $t(\tau,\omega)$ is defined provided that $\tau \in [0, \tau(T,\omega)]$. The budget set for agent i is the set of all consumption plans c_i which satisfy the budget constraint

$$\begin{aligned}
&\tilde{e}_{iA}(\omega) \cdot \tilde{p}_A(0,\omega) + \int_0^\tau \tilde{z}_i \, d\tilde{p}_A + \int_0^{t(\tau,\omega)} p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds \\
&= \tilde{p}(\tau,\omega) \cdot \tilde{z}_i(\tau,\omega) \\
&\qquad \text{for almost all } \omega \text{ and all } \tau \in [0, \tau(T,\omega)) \\
&e_{iA}(0,\omega) \cdot \tilde{p}_A(0,\omega) + \int_0^{\tau(T,\omega)} \tilde{z}_i d\tilde{p}_A + \int_0^T p_C(s,\omega)(e_i(s,\omega) - c_i(s,\omega))ds \\
&\qquad + (e_i(T,\omega) + \tilde{z}_i(\tau(T,\omega),\omega)e^{\beta(T,\omega)} - c_i(T,\omega))p_C(T,\omega) \\
&= \tilde{p}(\tau(T,\omega),\omega) \cdot \tilde{z}_i(\tau(T,\omega),\omega) \text{ for almost all } \omega
\end{aligned}$$

for some generalized trading strategy $\tilde{z}_i$. We follow standard notation in writing

$$\int \tilde{z}_i d\tilde{p}_A = \sum_{i=0}^{J} \int \tilde{z}_{iA_j} d\tilde{p}_{A_j}$$

It is implicit in the definition that $p_C(\cdot,\omega)(e_i(\cdot,\omega) - c_i(\cdot,\omega)) \in L^1([0,T],\lambda)$

9. Given a price process p, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

10. A generalized equilibrium for the economy is a generalized securities price process $\tilde{p}_A$, a consumption price process p_C, a generalized trading strategy $\tilde{z}$ and a consumption plan c which lies in the demand set so that the securities and goods markets clear, i.e. for almost all ω

$$\begin{aligned}
\sum_{i=1}^{I} \tilde{z}_{iA_j}(\tau,\omega) &= 1 \text{ for } j = 1,\ldots,J \text{ and for all } \tau \in [0,\tau(T,\omega)] \\
\sum_{i=1}^{I} \tilde{z}_{iA_0}(\tau,\omega) &= 0 \text{ for all } \tau \in [0,\tau(T,\omega)]
\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{I} c_i(t,\omega) &= \sum_{i=1}^{I} e_i(t,\omega) \text{ for all } t \in [0,T) \\ \sum_{i=1}^{I} c_i(T,\omega) &= \sum_{i=1}^{I} e_i(T,\omega) + \sum_{i=1}^{I} z_i(0,\omega) \cdot A(T,\omega)\end{aligned}$$

11. An equilibrium for the economy is a generalized equilibrium in which for almost every $\omega \in \Omega$, $\tau(t,\omega) = t$ for all $t \in [0,T]$. Note that this is simply a roundabout version of the usual definition of equilibrium in a continuous-time finance model.

The *distribution* of the continuous time model is determined by the functions f_i and constants ρ_i, the matrices ι and σ, the initial security holdings e_{iA_j}, and the felicity functions φ_{i1} and φ_{i2}. The specification of the continuous time *model* also includes a specific probability space $(\Omega, \mathcal{F}, \mu)$, a specific filtration $\{\mathcal{F}_t : t \in [0,T]\}$, and a specific vector Brownian motion β. The distribution provides a complete description of the economic primitives of the economy. However, two models with the same distribution may behave quite differently technically. Indeed, changing the probability space and the filtration will change the set of trading strategies available to the agents, and it may well be that, of two models with the same distribution, one will have an equilibrium and the other will not. This is a well-known phenomenon in the study of stochastic differential equations (SDEs), in which a well-behaved SDE may have a solution with one filtration and not with another. Following the literature on SDEs, we take the point of view that the model should be chosen to minimize the technical difficulties, in particular to make the set of trading strategies well-behaved.

Theorem 2.1 *Fix K. There exists a standard probability space $(\Omega, \mathcal{F}, \mu)$, a filtration $\{\mathcal{F}_t\}$, and a K-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)$ adapted to $\{\mathcal{F}_t\}$ such that for any functions f_i, constants ρ_i, matrices ι and σ, initial security holdings e_{iA_j}, and felicity functions φ_{i1} and φ_{i2} satisfying the assumptions given above, the following statements hold:*

1. *The continuous-time model has a generalized equilibrium. For each i, if $\tilde{z}_i$ is the equilibrium trading strategy for agent i, the pricing process*

is given by

$$\begin{array}{rcl} p_{A_j}(\tau,\cdot) & = & e^{\sigma_j\beta(t(\tau,\omega),\omega)}E\left(\varphi'_{i2}\left(F_i(T,\omega)\right)e^{\sigma_j(\beta(T,\omega)-\beta(t(\tau,\omega),\omega))}\Big|\,\tilde{\mathcal{F}}_\tau\right) \\ p_C(\tau,\omega) & = & \varphi'_{i1}(c_i(\tau,\omega)) \text{ for } t<T \\ p_C(T,\omega) & = & \varphi'_{i2}\left(F_i(T,\omega)\right) \end{array} \quad (1)$$

where

$$F_i(T,\omega)=\rho_i\left(\beta(T,\omega)\right)+\tilde{z}_i(T,\omega)\cdot A(T,\omega) \quad (2)$$

2. *If the continuous-time model has a generalized equilibrium in which, for almost all* ω, $\tau(\cdot,\omega)$ *is continuous, the model has an equilibrium. For each* i, *if* $\tilde{z}_i$ *is the equilibrium trading strategy for agent* i, *the pricing process is given by*

$$\begin{array}{rcl} p_{A_j}(t,\cdot) & = & e^{\sigma_j\beta(t,\omega)}E\left(\varphi'_{i2}\left(F_i(T,\omega)\right)e^{\sigma_j(\beta(T,\omega)-\beta(t,\omega))}\Big|\,\mathcal{F}_t\right) \\ p_C(t,\omega) & = & \varphi'_{i1}(c_i(t,\omega)) \text{ for } t<T \\ p_C(T,\omega) & = & \varphi'_{i2}\left(F_i(T,\omega)\right) \end{array} \quad (3)$$

where F_i *is given by Equation (2).*

Remark 2.2 Because the pricing formula depends on the equilibrium trading strategies, the pricing formula is significantly less concrete than the explicit formula for the single agent case given in Raimondo [51]. Nonetheless, the formula allows us to derive many qualitative properties of the pricing process. For example, if there is more than one stock, the rank of σ is at least two, and φ_{i2} is strictly concave, then the pricing process of the j^{th} stock p_{A_j} cannot be measurable in the filtration generated by $\sigma_j\beta$ alone; changes in the value of each stock induce wealth effects that affect the willingness of agents to hold, and hence the price, of every other stock. We are exploring a range of specific predictions, which will be detailed in a later version of this paper; to get a sense of the form of these predictions, the reader may wish to consult the predictions in the representative agent model of Raimondo [51].

Remark 2.3 As discussed in the introduction, we have not been able to show that the equilibrium prices are continuous, despite the fact that all the primitives of the economy (including the information flow) are continuous. This discontinuity could arise because the conditional expectation of the terminal wealth distribution $E(F_i(T,\omega)|(t,\omega))$ might not be continuous.

Note also that the conditional distribution is taken with respect to the filtration $\mathcal{F}_t$, but this filtration is larger than the filtration generated by the K-dimensional Brownian motion β.

Remark 2.4 As noted in the introduction, the definition of generalized equilibrium is needed to handle situations in which the nonstandard equilibrium pricing process has two or more discontinuities at times that differ by an infinitesimal amount. The standard analogue would be that, in a sequence of discretizations approximating the continuous time economy, the sequence of equilibria exhibit discontinuities at times s_n and t_n with $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = t$. In this situation, the multiple discontinuities in the discrete model are combined into a single, complex discontinuity at t in the continuous time model, and the set of trading strategies in the discrete (standard or nonstandard) model is richer than in the continuous time model. In particular, in the discrete model, agents can choose different portfolios at the two discontinuities, and can condition their portfolio choice at the second discontinuity on the outcome at the first discontinuity. In a generalized securities price process, we can keep these discontinuities distinct; a generalized trading strategy allows traders to hold different portfolios at the second discontinuity than they held at the first discontinuity, just as in the discrete framwork.

Remark 2.5 It is well known that stochastic differential equations need not have solutions on the probability space and filtration on which they are defined. Roughly speaking, there may not be enough measurable sets to define the solution. As a consequence, existence theorems for solutions of stochastic differential equations take one of two forms: either they assume that the probability space and filtration have certain nice properties to start with and demonstrate the existence of so-called strong solutions, or they work with an arbitrary probability space and filtration and prove the existence of so-called "weak solutions" which can be defined on a richer probability space and filtration. We have chosen to take the former route and assume our probability space and filtration have nice properties to begin with, but our methods will also show that if we begin with an arbitrary probability space and filtration, one can obtain a "weak equilibrium," an equilibrium on a different probability space and filtration. Note that any two K-dimensional Brownian motions are essentially the same from the standpoint of probability theory; for example, they have the same finite-dimensional distributions, and they induce the same measure on $C([0,1],\mathbf{R}^K)$. Similarly, every weak equilibrium of the model just described will satisfy the pricing formula given in Equation (3).

3 Proof of Theorem 2.1

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson's construction of Brownian Motion and the Itô Integral ([1]) and Lindström's extension of that construction to stochastic integrals with respect to L^2 martingales [38, 39, 40, 41]. It is beyond the scope of this paper to develop these methods; Anderson [3] and Hurd and Loeb [32] are references to nonstandard analysis.

We construct our probability space, filtration and Brownian Motion following Anderson's construction [1]. Specifically, we construct a hyperfinite economy as follows:

1. By the Gram-Schmidt method, we may assume without loss of generality that the j^{th} security terminal dividend depends only on the first through j^{th} components of the Brownian motion, and that $\beta_{J+1},\ldots,\beta_K$ are independent of the terminal security dividends. The Gram-Schmidt process doesn't change the filtration generated by the Brownian motion.

2. Choose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$; for notational convenience, we will assume that $n = m^4$ for some infinite natural number m. For $t \in [0,T]$, define $\hat{t} = \frac{\lfloor nt \rfloor}{n}$, where $\lfloor x \rfloor$ denotes the greatest (nonstandard) integer less than or equal to x; in particular, $\hat{T} = \frac{\lfloor nT \rfloor}{n}$. Define $\Delta T = \frac{1}{n}$, $\mathcal{T} = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$, and

$$\hat{\Omega} = \left\{\omega : \mathcal{T} \setminus \{0\} \to \{-1,1\}^K\right\}$$

Intuitively, we have K independent random walks. At each time $t \in \hat{\mathcal{T}}$, each random walk moves either up or down. If $s \in \mathcal{T} \setminus \{0\}$, we write

$$\omega_s = (\omega_{s1},\ldots,\omega_{sK}) = \omega(s)$$

The internal hyperfinite measure $\hat{\mu}$ on $\hat{\Omega}$ is given by

$$\hat{\mu}(A) = \frac{|A|}{|\hat{\Omega}|}$$

for every $A \in \hat{\mathcal{F}}$, the algebra of all internal subsets of $\hat{\Omega}$. For $t \in \mathcal{T}$, $\hat{\mathcal{F}}_t$ is the algebra of all internal subsets of $\hat{\Omega}$ that respect the equivalence relation $\omega \sim_t \omega' \Leftrightarrow \omega_s = \omega'_s$ for all $s \leq t$. The internal measure $\hat{\lambda}$ on $\mathcal{T}$ is given by $\hat{\lambda}(\{t\}) = \Delta T$ if $t < \hat{T}$ and $\hat{\lambda}(\{\hat{T}\}) = 1$.

3. For $k = 1, \ldots, K$, define $\hat{\beta}_k : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}$ by

$$\hat{\beta}_k(t, \omega) = \sum_{0 < s \leq t,\, s \in \mathcal{T}} \frac{\omega_{sj}}{\sqrt{n}} \text{ and } \hat{\beta}(t, \omega) = (\hat{\beta}_1(t, \omega), \ldots, \hat{\beta}_K(t, \omega))$$

   Thus, $\hat{\beta}$ is a K-dimensional hyperfinite random walk.

4. Let $\Omega = \hat{\Omega}$ and let $(\Omega, \mathcal{F}, \mu)$ be the (complete) Loeb measure generated by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ (Loeb [42]). Although $(\Omega, \mathcal{F}, \mu)$ is generated by a nonstandard construction, Loeb showed that it is a probability space in the usual standard sense. Let

$$\begin{aligned} \mathcal{F}_t = \{B \in \mathcal{F} : \mu(B \Delta C) = 0 &\text{ for some } C \text{ which respects the} \\ &\text{equivalence relation } \omega \sim \omega' \Leftrightarrow \omega \sim_s \omega' \text{ for all } s \simeq t\} \end{aligned}$$

   Let $(\mathcal{T}, L(\hat{\lambda}))$ denote the complete Loeb measure generated by $\hat{\lambda}$ on $\mathcal{T}$.

5. Let $\beta : [0, T] \times \Omega \to \mathbf{R}^K$ be defined by $\beta(t, \omega) = {}^\circ\left(\hat{\beta}(\hat{t}, \omega)\right)$. Anderson [1] showed that β is a d-dimensional Brownian motion in the usual standard sense, and that $\beta(t, \cdot) = E(\beta(T, \cdot)|\mathcal{F}_t)$.[8]

6. Given an internal consumption plan $\hat{c} : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}_+$, the agent's utility is

$$\hat{U}(\hat{c}) = E_{\hat{\mu}}\left(\left(\Delta T \sum_{s \in \mathcal{T}, s < \hat{T}} {}^*\varphi_1(\hat{c}(t, \omega))\right) + {}^*\varphi_2(\hat{c}(\hat{T}, \omega))\right)$$

[8] Anderson used a slightly different filtration; Keisler [35] showed the result for the exact filtration used here.

7. We will use the Duffie-Shafer Theorem [22, 23] to perturb the endowments and security payoffs to ensure the existence of an equilibrium. Let $\hat{e}_i(t,\omega) \geq 0$, $\left|\hat{e}_i(t,\omega) - {}^*f_i(\hat{\beta}(t,\omega))\right| \leq \frac{1}{n^2}$ $(t < \hat{T})$ and $\hat{e}_i(\hat{T},\omega) \geq 0$, $\left|\hat{e}_i(\hat{T},\omega) - \rho_i + e^{\iota_i \cdot \hat{\beta}(\hat{T},\omega)}\right| \leq \frac{1}{n^2}$ denote the perturbed endowments. For all $\omega \in \hat{\Omega}$, let $\hat{A}(t,\omega)$ denote the perturbed security payoffs $\hat{A}(t,\omega) \geq 0$, $\left|\hat{A}(t,\omega) - A(t,\omega)\right| \leq \frac{1}{n^2}$ for all $t < \hat{T}$, and $\hat{A}(\hat{T},\omega) \geq 0$, $\left|\hat{A}(\hat{T},\omega) - e^{\sigma\hat{\beta}(\hat{T},\omega)}\right| \leq \frac{1}{n^2}$ (i.e. $\left|\hat{A}_j(\hat{T},\omega) - e^{\sigma_j \cdot \hat{\beta}(\hat{T},\omega)}\right| \leq \frac{1}{n^2}$, $j = 0,\ldots J$), $A(T,\omega) = e^{\beta(T,\omega)}$ (i.e. $A_j(T,\omega) = e^{\sigma_j \cdot \beta(T,\omega)}$). Note that $A(T,\omega) = {}^\circ\hat{A}(\hat{T},\omega)$ for μ-almost all ω.

8. A security price is an internal function $\hat{p}_A : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}^{J+1}$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t\in\hat{\mathcal{T}}}$. A consumption price is an internal function $\hat{p}_C : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}_+$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t\in\hat{\mathcal{T}}}$.

9. Let
$$\hat{T}_0 = \hat{T} - n^{-1/4}$$
Recall that $n = m^4$ for some $m \in {}^*\mathbf{N} \setminus \mathbf{N}$, so $\hat{T}_0 = \hat{T} - n^{-1/4} = \hat{T} - \frac{1}{m} = \hat{T} - \frac{m^3}{n} \in \mathcal{T}$. A trading strategy for agent i is $\hat{z}_i : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}^{J+1}$ which is adapted with respect to $\{\hat{\mathcal{F}}_t\}_{t\in\hat{\mathcal{T}}}$ with the additional property that if $t \in \mathcal{T}$ and $t \in \left[\hat{T}_0, \hat{T}\right]$, then $\hat{z}(t,\omega)$ does not depend on components ω_{js} for $j > J$ and $s > T_0$. In other words, we allow traders to freely trade the securities at each time $t \in \mathcal{T}$ such that $t \leq \hat{T}_0$, but at times $t > \hat{T}_0$, we do not allow traders to condition their trades on the post-T_0 random walk components that do not correspond to traded securities. As discussed in the introduction, this restriction on trading is imposed in order to eliminate certain strategies in the hyperfinite model which have no analogue in the continuous-time model. The restriction is critical to the proof that the hyperfinite equilibrium prices are SL^2.

10. A consumption plan for agent i is an internal function $\hat{c}_i : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}_+$. The budget set is the set of all consumption plans which satisfy the budget constraint

$$\hat{e}_{iA}(\omega) \cdot \hat{p}_A(0,\omega) \quad + \sum_{s\in\mathcal{T},s<t} \left(\hat{z}_i(s,\omega) \cdot (\hat{p}(s+\Delta T,\omega) - \hat{p}(s,\omega))\right)$$

$$
\begin{aligned}
& + \sum_{s\in\mathcal{T},s<t} \Delta T\left(\hat{p}_C(s,\omega)(\hat{e}_i(s,\omega)-\hat{c}_i(s,\omega))\right) \\
& = \hat{z}_i(t,\omega)\cdot\hat{p}(t,\omega) \\
& \quad \text{for all } t\in\mathcal{T} \text{ and all } \omega\in\hat{\Omega} \\
\hat{e}_{iA}(\omega)\cdot\hat{p}_A(0,\omega) & + \sum_{s\in\mathcal{T},s<\hat{T}} \left(\hat{z}_i(s,\omega)\cdot(\hat{p}(s+\Delta T,\omega)-\hat{p}(s,\omega))\right) \\
& + \sum_{s\in\mathcal{T},s<\hat{T}} \Delta T\left(\hat{p}_C(s,\omega)(\hat{e}_i(s,\omega)-\hat{c}_i(s,\omega))\right) \\
& + \hat{p}_C(\hat{T},\omega)\left(\hat{e}_i(\hat{T},\omega)+\mathbf{1}_J\cdot\left(z_{iA}(T,\omega)e^{\hat{\beta}(\hat{T},\omega)}\right)+z_{iB}(\hat{T},\omega)-\hat{c}_i(\hat{T},\omega)\right) \\
& = \hat{z}_i(T,\omega)\cdot\hat{p}(T,\omega) \\
& \quad \text{for all } \omega\in\hat{\Omega}
\end{aligned}
$$

for some trading strategy $\hat{z}_i$. Note that since $\hat{z}$ is required to be adapted to $\{\hat{\mathcal{F}}_t\}_{t\in\hat{\mathcal{T}}}$, it follows that $\hat{c}_i$ is adapted to $\{\hat{\mathcal{F}}_t\}_{t\in\hat{\mathcal{T}}}$.

11. Given a security price $\hat{p}$ and a consumption price $\hat{p}_C$, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

12. An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, a trading strategy $\hat{z}$ and a consumption plan $\hat{c}$ which lies in the demand set so that the securities and goods markets clear, i.e. for all ω

$$
\begin{aligned}
\sum_{i=1}^{I}\hat{z}_{iA_j}(t,\omega) &= 1 \text{ for } j=1,\ldots,J \text{ and for all } t<\hat{T} \\
\sum_{i=1}^{I}\hat{z}_{iA_0}(t,\omega) &= 0 \text{ for all } t<\hat{T} \\
\sum_{i=1}^{I}\hat{c}_i(t,\omega) &= \sum_{i=1}^{I}\hat{e}_i(t,\omega) \text{ for all } t<\hat{T} \\
\sum_{i=1}^{I}\hat{c}_i(\hat{T},\omega) &= \sum_{i=1}^{I}\left(\hat{e}_i(\hat{T},\omega)+\hat{z}_i(\hat{T},\omega)\cdot\hat{A}(T,\omega)\right)
\end{aligned}
$$

13. Now, we define the quadratic variation measure. Given $(t,\omega)\in\hat{\mathcal{T}}\times\hat{\Omega}$,

$\hat{q}_{\hat{p}}$, the quadratic variation measure of $\hat{p}$, assigns mass

$$E\left(\left|\hat{p}\left(t+\frac{1}{n}\right)-\hat{p}(t)\right|^2\middle|(t,\omega)\right)$$

to the node (t,ω); in other words, the mass assigned to (t,ω) is the average squared change in $\hat{p}$ between the node (t,ω) and its immediate successor nodes. $\hat{q}_{\hat{\beta}}$, the quadratic variation measure of the random walk $\hat{\beta}$, assigns mass $\frac{K}{n}$ to each node. We let $\hat{q}$ denote the quadratic variation of the $(K+J)$-dimensional process $\left(\hat{\beta},\hat{p}\right)$:

$$\hat{q}(t,\omega)=\hat{q}_{\hat{\beta}}(t,\omega)+\hat{q}_{\hat{p}}(t,\omega)=\frac{K}{n}+E\left(\left|\hat{p}\left(t+\frac{1}{n}\right)-\hat{p}(t)\right|^2\middle|(t,\omega)\right)$$

Theorem 3.1 *We may find perturbed endowments $\hat{e}_i$ and security payoffs $\hat{A}$ (each perturbed by at most $\frac{1}{n^2}$ at each (t,ω)) such that the hyperfinite economy just described has an equilibrium. The equilibrium pricing process is a scalar multiple of*

$$\begin{array}{rcl}\hat{p}_{A_j}(t,\omega) &=& E({}^*\varphi'_{i2}(\hat{c}_i(\hat{T},\cdot))\hat{A}_j(\hat{T},\cdot)|(t,\omega))\\ \hat{p}_C(t,\omega) &=& {}^*\varphi'_{i1}(\hat{c}_i(t,\omega)) \text{ for } t<\hat{T}\\ \hat{p}_C(\hat{T},\omega) &=& {}^*\varphi'_{i2}(\hat{c}_i(\hat{T},\omega))\end{array} \tag{4}$$

for each i. With the normalization given in Equation (4), the pricing processes $\hat{p}_{A_j}$ are internal SL^2 martingales. The equilibrium security holdings $\hat{z}$ are SL^2 with respect to the quadratic variation measure $\hat{q}_{\hat{p}}$, in the sense that

$$\hat{z}_{\hat{p}}(t,\omega)=\hat{z}\,(t,\omega)\cdot\frac{\left(\hat{p}\left(t+\frac{1}{n},\omega\right)-\hat{p}(t,\omega)\right)}{\sqrt{\hat{q}_{\hat{p}}(t,\omega)}}$$

is an SL^2 function with respect to $\hat{q}_{\hat{p}}$. There are standard right-continuous square-integrable martingales p_{A_j} with respect to the filtration $\{\mathcal{F}_t\}$ such that $\hat{p}_A$ lifts p_A in the sense of Lindström [38, 39, 40, 41]. For μ-almost all ω, each $p_{A_j}(\cdot,\omega)$ is continuous except at a countable number of times.

Remark 3.2 The statement of Theorem 3.1 and the proof are written as if the dividends were exactly zero at times $t<\hat{T}$. Magill and Quinzii Volume I

[44] asserts that it is sufficient to perturb the endowments, and that it is not necessary to perturb the dividends, and promises a proof in Volume II; however, Volume II has not appeared and we have been unable to obtain their proof of this claim. If the claim is in fact correct, we can omit the perturbations of the dividends, and the statement and proof of Theorem 3.1 can be used verbatim as given here. If not, one needs to perturb the dividends. The small $\frac{1}{n^2}$ perturbations we have used here do not have any effect visible in the standard continuous-time model, so Theorem 2.1 follows without any change. The statement and proof of Theorem 3.1 need only cosmetic changes. With nonzero dividends, the security prices are not martingales; however, security prices remain the expectation of future dividends times the marginal utility of consumption; the sum of the security price and the sum of realized past dividends is a martingale.

Proof: If we replace "hyperfinite" by "finite" everywhere in the definition of the hyperfinite economy, this is just a finite GEI economy.

If ι_i lies in the span of the rows of σ for each i, then $\hat{T}_0 = \hat{T}$ and trading occurs freely at all nodes. Moreover, we can take the perturbations to all be zero. Markets are in effect complete because the securities only pay off in the terminal period $\hat{T}$, and they span all the utility-relevant period $\hat{T}$ uncertainty. Hence, there is an equilibrium for the unperturbed economy, and the consumptions in the terminal period $\hat{T}$ are Pareto optimal for the economy restricted to the period $\hat{T}$.

Now, suppose that ι_i does not lie in the span of the rows of σ. We freeze trading between $\hat{T}_0$ and $\hat{T}$. By Duffie and Shafer [22, 23], there are arbitrarily small perturbations of the endowments and dividends such that a finite GEI economy has an equilibrium; by Robinson's Transfer Principle, there are perturbations of size at most $\frac{1}{n^2}$ such that the hyperfinite economy has an equilibrium.

Thus, whether or not the rows of σ span the rows of ι, we get an equilibrium for a (possibly perturbed) economy. Since the security payoffs are nonnegative for all (t, ω) and strictly positive for $(\hat{T}, \omega)$ for all ω, the absence of arbitrage guarantees that $\hat{p}_A(t, \omega) \gg 0$ for all (t, ω). Since utility is strictly increasing in consumption, $p_C(t, \omega) > 0$ for all (t, ω).

We claim that $\hat{c}_i(t, \omega) > 0$ for all (t, ω) and all i. If not, let $\hat{z}_i$ be the trading strategy that finances the consumption $\hat{c}_i$ at the prices $\hat{p}$, and let $\hat{\hat{z}}_i$ denote the no-trade trading strategy $\hat{\hat{z}}_i(t, \omega) = \hat{e}_{iA}(\omega)$ for all (t, ω) and $\hat{\hat{c}}_i$ the consumption it finances. Since $\hat{e}_i(t,\omega) > 0$ for all (t,ω), $\hat{\hat{c}}_i(t,\omega) > 0$ for all (t,ω). For $\alpha \in {}^*[0,1]$, let $\hat{z}_{i\alpha} = (1-\alpha)\hat{z}_i + \alpha\hat{\hat{z}}_1$. $\hat{z}_{i\alpha}$ is a trading strategy, which finances the consumption $\hat{c}_{i\alpha} = \alpha\hat{c}_i + (1-\alpha)\hat{\hat{c}}_i$. Since $\varphi'_{ik}(0) = \infty$ and $\hat{c}_i(t,\omega) = 0$ for some (t,ω), $\hat{U}_i(\hat{c}_{i\alpha}) > \hat{U}_i(\hat{c}_i)$ for a sufficiently small infinitesimal α. This contradicts the statement that $\hat{c}_i(t,\omega)$ is in the demand set at the prices $\hat{p}$.

Since $c_i(t,\omega) > 0$ for all t, individual i can adjust consumption at time t_0 and state ω_0 by a sufficiently small infinitesimal without violating the nonnegativity constraint at any time t. Specifically, agent i can do either of the following without violating the nonnegativity constraint:

1. Agent i can reduce consumption at (t_0,ω_0) by a sufficiently small infinitesimal α, buy $\frac{\alpha\hat{p}_C(t_0,\omega_0)}{\hat{p}_{Aj}(t_0,\omega_0)}$ units of stock A_j (or reduce her short position in the stock by that same number of units) and hold these units in addition to the holdings prescribed by $\hat{z}_i$, so that consumption is unchanged in periods t with $t_0 < t < \hat{T}$, and consumption in period $\hat{T}$ is increased by $\alpha e^{\sigma_j\beta(\hat{T},\omega)}$ for all $\omega \sim_{t_0} \omega_0$.

2. Agent i can increase consumption at (t_0,ω_0) by a sufficiently small infinitesimal α, sell $\frac{\alpha\hat{p}_C(t_0,\omega_0)}{\hat{p}_{Aj}(t_0,\omega_0)}$ units of stock A_j (or increase her short position in the stock by that same number of units) and hold this number of units fewer than the holdings prescribed by $\hat{z}_i$, so that consumption is unchanged in periods t with $t_0 < t < \hat{T}$, and consumption in period $\hat{T}$ is decreased by $\alpha e^{\beta_j(\hat{T},\omega)}$ for all $\omega \sim_{t_0} \omega_0$.

Therefore, the first order conditions imply that

$$\begin{aligned}
\frac{\hat{p}_{A_j}(t,\omega)}{\hat{p}_C(t,\omega)} &= \frac{E(\varphi'_{i2}(\hat{c}_i(\hat{T},\cdot))\hat{A}_j(\hat{T},\cdot)|(t,\omega))}{\varphi'_{i1}(\hat{c}_i(t,\omega))} \text{ for } t < \hat{T} \\
\frac{\hat{p}_{A_j}(\hat{T},\omega)}{\hat{p}_C(\hat{T},\omega)} &= \frac{E(\varphi'_{i2}(\hat{c}_i(\hat{T},\cdot))\hat{A}(\hat{T},\cdot)|(\hat{T},\omega))}{\varphi'_{i2}(\hat{c}_i(\hat{T},\omega))} = \hat{A}(\hat{T},\omega)
\end{aligned}$$

so normalizing prices by setting $\hat{p}_C(t,\omega) = \varphi'_{i1}(\hat{c}_i(t,\omega))$ for $t < \hat{T}$ and $\hat{p}_C(\hat{T},\omega) = \varphi'_{i2}(\hat{c}_i(\hat{T},\omega))$, we have

$$\hat{p}_{A_j}(t,\omega) = E(\varphi'_{i2}(\hat{c}_i(\hat{T},\cdot))\hat{A}_j(\hat{T},\cdot)|(t,\omega))$$

which shows that the $\hat{p}_{A_j}$ are internal martingales.

We now show that each $\hat{p}_{A_j}$ is an SL^2 martingale. Because it is a martingale, it is enough to show that $\hat{p}_{A_j}(\hat{T},\cdot)$ is SL^2 (Anderson [2]). In the case where the rows of σ span the rows of ι, the period $\hat{T}$ consumptions are Pareto optimal for the period $\hat{T}$ subeconomy, and optimality implies that ${}^\circ\left(\hat{c}_i(\hat{T},\omega)\right) \not\simeq 0$, so marginal utility is uniformly bounded above by a finite number, hence $\hat{p}_{A_j}(\hat{T},\cdot)$ is uniformly bounded above by a finite number, so *a fortiori* it is SL^2.

In the case where the rows of σ do not span the rows of ι, fix i such that ι_i is not in the span of the rows of σ. We use the fact that trader i is forced to use a buy-and-hold strategy at the hyperfinite times $\hat{T}_0,\ldots,\hat{T}$. Recall that $n = m^4$ for some hyperfinite integer m, so there are $\frac{n}{n^{-1/4}} = n^{3/4}$ times during which trade is frozen. With probability infinitely close to one, $\hat{\beta}(\hat{T},\omega) \simeq \hat{\beta}(\hat{T}_0,\omega)$. However, each component of ${}^\circ\left(\hat{\beta}(\hat{T},\cdot)\right)$ will take on every value in $[-\infty,\infty]$ with nonzero (but infinitesimal) probability; moreover, the components of $\hat{\beta}$ are independent. Since ι_i is not in the span of the rows of σ, and σ has rank J, the buy-and-hold strategy cannot short any security except possibly the bond A_0; if it did, consumption would be negative with nonzero (but possibly infinitesimal) probability, violating the feasibility condition. Thus, up to the $\frac{1}{n^2}$ pertubations, the conditional distribution of $\hat{c}_i(\hat{T},\omega)$ is a nonnegative-coefficient linear combination of logbinomials (the endowment $\hat{e}_i$ and the stocks A_j) with infinitesimal variance, plus a constant (which may be negative, because the strategy could short the bond $\hat{A}_0$). The feasibility constraint $\hat{c}_i(\hat{T},\omega) \geq 0$ for *all* ω guarantees that the short position on the bond is a finite amount smaller than the value that the logbinomials take on with probability infinitely close to one. This, plus the assumptions on the utility function φ_{i2}, guarantee that $\hat{p}_{Aj}(\hat{T},\cdot)$ is SL^2.

By Lindström [38, 39, 40, 41]. there is a price process p in the standard continuous time model such that $\hat{p}$ is a lifting of p; p is a square-integrable martingale, it is right continuous, and for almost all ω, $p(\cdot,\omega)$ is continuous except at a countable set of times. ■

As noted in the introduction, we do not regard the proof of the following proposition to be airtight. However, we believe the argument given is correct, and we anticipate circulating a full proof shortly.

Slightly Iffy Proposition 3.3 *The equilibrium trading strategy is SL^2 in the weak sense (i.e. $\hat{z}_{\hat{p}}$ is SL^2). $\hat{z}_{\hat{p}}$ does not chatter, i.e. there is a standard* *continuous-time process z such that for all $t \in [0,T]$,*

$$^\circ\left(\int_0^t \hat{z}\, d\hat{p}\right) = \int_0^t z\, dp \tag{5}$$

Proof: Suppose $\hat{z}_{\hat{p}}$ is not SL^2. $\hat{z}_{\hat{p}} \in {}^*L^2$; more to the point, feasibility of the equilibrium consumptions implies that the two-norm $\|\hat{z}_{\hat{p}}\|_2$ is finite. Thus, there is a set $\hat{C} \subset \mathcal{T} \times \hat{\Omega}$ such that $\hat{q}_{\hat{p}}(\hat{C}) \simeq 0$ and $\hat{z}_{\hat{p}}$ is SL^2 on the complement of $\hat{C}$. We let $\hat{z}'(t,\omega) = \hat{z}(t,\omega)$ for $(t,\omega) \notin \hat{C}$. For $(t,\omega) \in \hat{C}$, we would like to define $\hat{z}'(t,\omega)$ such that the wealth of the agent is the same at each of the immediate successor nodes to (t,ω).[9] If we could do that, then we could simply define $\hat{z}'$ to be the conditional expectation of $\hat{z}$ with respect to the filtration which glues together all the immediate successor nodes of every element of $\hat{C}$. It would be easy to see that $\hat{z}'$ would be in the budget set, and the consumptions produced by $\hat{z}$ are a noninfinitesimal mean-preserving spread of those that would be produced by $\hat{z}'$. More to the point, the mean-preserving spread would be (up to an infinitesimal) perpendicular to the prices $\hat{p}$, and these prices represent marginal utilities, so $\hat{z}'$ generates higher utility than $\hat{z}$. Since we cannot in general choose $\hat{z}'$ in this way, we need to allow a moderate variation in income at the immediate successor nodes (but much less variation than provided by $\hat{z}$). Then, we need to choose $\hat{z}'$ at successor nodes (not just immediate successor nodes) as convex combinations, near the conditional expectation, of the portfolios prescribed by $\hat{z}$. This will yield a trading strategy $\hat{z}'$ which lies in the budget set and generates consumptions yielding a higher utility than $\hat{z}$. The contradiction shows that $\hat{z}_{\hat{p}}$ is SL^2.

Let $\{\mathcal{F}_t\}$ denote the filtration generated by p; this filtration lives on $[0,T] \times \Omega$. Let $\{\mathcal{G}_t\}$ denote the filtration on $\mathcal{T} \times \hat{\Omega}$ obtained by lifting $\{\mathcal{F}_t\}$ by the inverse image of the map $(t,\omega) \to ({}^\circ t, \omega)$. Let z be the conditional expectation of ${}^\circ(\hat{z})$ with respect to $\{\mathcal{G}_t\}$, and let $\hat{w}$ be a lifting of z to a trading strategy in the hyperfinite model. If $\hat{w}_{\hat{p}} \simeq \hat{z}_{\hat{p}}$ almost everywhere, then Equation 5 follows. If not, the consumptions generated by $\hat{z}$ are a noninfinitesimal mean-preserving spread of an infinitesimal perturbation of

[9] If the model had a money-market account (a security which pays off the same amount in each successor node of every given node), this would be trivial. Hence, a simple way to resolve the issue would be to introduce a money-market acount in the continuous-time model.

the consumptions generated by $\hat{w}$. Moreover, this mean-preserving spread is uncorrelated with the securities prices $\hat{p}$, which equal marginal utilities of consumption, so the consumptions generated by $\hat{z}$ yield lower utility than the consumptions generated by $\hat{w}$, which contradicts the fact that $\hat{z}$ is an equilibrium trading strategy. Thus, Equation 5 must be satisfied. ■

Sketch of Proof of Theorem 2.1 from Theorem 3.1 and Slightly Iffy Proposition 3.3: The argument goes in essentially the same way as the proof in the one-agent case (Raimondo [51]). We have a candidate equilibrium price p and trading strategy z. Suppose there is a trading strategy w' whose associated consumptions make agent i strictly better off, given the prices p. Since w' is a trading strategy, the integral $\int w'\, dp$ must be defined. Since the integral is defined, we can approximate w' arbitrarily closely by trading strategies in $L^2([0,T] \times \Omega)$. Thus, we can find a trading strategy $w \in L^2([0,T] \times \Omega)$ whose associated consumptions give strictly higher utility than does z. Lift w componentwise to a hyperfinite trading strategy $\hat{w}$ which is SL^2 componentwise with respect to the quadratic variation measure $\hat{q}_{\hat{p}}$ and whose associated consumptions are a lifting of the consumptions generated by w. Then the utility generated by the consumptions of $\hat{w}$ is infinitely close to the utility generated by the consumptions of w, which is noninfinitesimally greater than the utility yielded by the consumptions generated by z, which is infinitely close to the utility yielded by the consumptions generated by $\hat{z}$. Therefore, $\hat{w}$ yields higher utility than $\hat{z}$, contradicting the fact that $\hat{z}$ is an equilibrium trading strategy, given the prices $\hat{p}$. We conclude that the price process p and the trading strategy z form an equilibrium for the continuous-time economy. The pricing formulae in Theorem 2.1 come from taking standard parts of the pricing formulae in Theorem 3.1. ■

References

[1] Anderson, Robert M., "A Nonstandard Representation of Brownian Motion and Itô Integration," *Israel Journal of Mathematics* **25**(1976), 15-46.

[2] Anderson, Robert M., "Star-finite Representations of Measure Spaces," *Transactions of the American Mathematical Society* **271**(1982), 667-687.

[3] Anderson, Robert M., *Infinitesimal Methods in Mathematical Economics,* Preprint (2000).

[4] Anderson, Robert M. and Salim Rashid, "A Nonstandard Characterization of Weak Convergence," *Proceedings of the American Mathematical Society* **69**(1978), 327-332.

[5] Anderson, Robert M. and Roberto C. Raimondo, "Empirical Tests of Securities Price Processes," (in preparation).

[6] Anderson, Robert M. and Roberto C. Raimondo, "Market Clearing and Option Pricing," preprint, 2001.

[7] Back, Kerry, "Asset Pricing for General Processes," *Journal of Mathematical Economics* **20**(1991), 371-395.

[8] Bank, Peter and Frank Riedel, "Existence and Structure of Stochastic Equilibria with Intemporal Substitution," *Finance and Stochastics* **5**(2001), 487-509.

[9] Bick, Avi, "On Viable Diffusion Price Processes of the Market Portfolio," *Journal of Finance* **15**(1990), 673-689.

[10] Campbell, John Y.and Lo, Andrew W. and MacKinley, A. Craig, *The Econometrics of Financial Markets,* Princeton University Press, Princeton, New Jersey, 1997.

[11] Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica* **53**(1985), 363-384.

[12] Cutland, Nigel J., "Infinitesimal Methods in Control Theory: Deterministic and Stochastic," *Acta Applicandae Mathematicae* **5**(1986), 105-135.

[13] Cutland, Nigel J., P. Ekkehard Kopp, and Walter Willinger, "A Nonstandard Approach to Option Pricing," *Mathematical Finance* **1**(1991a), 1-38.

[14] Cutland, Nigel J., P. Ekkehard Kopp, and Walter Willinger, "Nonstandard Methods in Option Pricing," *Proceedings of the 30th I.E.E.E. Conference in Decision and Control* (1991b), 1293-1298.

[15] Cutland, Nigel J., P. Ekkehard Kopp, and Walter Willinger, "Convergence of Cox-Ross-Rubinstein to Black-Scholes," manuscript (1991c).

[16] Cutland, Nigel J., P. Ekkehard Kopp, and Walter Willinger, "From Discrete to Continuous Financial Models: New Convergence Results for Option Pricing," *Mathematical Finance* **3**(1993), 101-123.

[17] Cutland, Nigel J., P. Ekkehard Kopp, and Walter Willinger, "Stock Price Returns and the Joseph Effect: A Fractional Version of the Black-Scholes Model," *Progress in Probability* **36**(1995a), 327-351.

[18] Cutland, Nigel J., P. Ekkehard. Kopp, and Walter Willinger, "From Discrete to Continuous Stochastic Calculus," *Stochastics and Stochastics Reports* **52**(1995b), 173-192.

[19] Cutland, Nigel J., P. Ekkehard Kopp, Walter Willinger and M. C. Wyman, "Convergence of Snell Envelopes and Critical Prices in the American Put," in M. H. A. Demmpster and S. R. Pliska, eds., *Mathematics for Derivative Securities*, Isaac Newton Institute Proceedings Volume, Cambridge University Press, Cambridge (forthcoming).

[20] Deng, Yongheng, John M. Quigley, and Robert Van Order, "Mortgage Terminations, Heterogeneity and Exercise of Mortgage Options," *Econometrica* **68**(2000).

[21] Duffie, Darrell, *Security Markets, Stochastic Models*, Academic Press, Boston, MA, 1995.

[22] Duffie, Darrell and Wayne Shafer, "Equilibrium in Incomplete Markets I: Basic Model of Generic Existence," *Journal of Mathematical Economics* **14**(1985), 285-300.

[23] Duffie, Darrell and Wayne Shafer, "Equilibrium in Incomplete Markets II: Generic Existence in Stochastic Economies," *Journal of Mathematical Economics* **15**(1986), 199-216.

[24] Duffie, Darrell and Costis Skiadas, "Continuous-time security pricing: A Utility Gradient Approach," *Journal of Mathematical Economics* **23**(1994), 107-131.

[25] John Geanakoplos, Michael Magill and Martine Quinzii, "Demography and the Predictability of the Stock Market," preprint, 2001.

[26] Harrison, J. Michael and David M. Kreps, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* **20** (1979), 381-408.

[27] He, Hua and Hayne Leland, "On Equilibrium Asset Price Processes," *Review of Financial Studies* **6**(1995), 593-617.

[28] Hindy, Ayman and Chi-Fu Huang, "On Intertemporal Preferences for Uncertain Consumption: A Continuous Time Approach," *Econometrica* **60**(1992), 781-801.

[29] Hindy, Ayman and Chi-Fu Huang, "Optimal Consumption and Portfolio Rules with Durability and Local Substitution," *Econometrica* **61**(1993), 85-121.

[30] Hindy, Ayman, Chi-Fu Huang and David M. Kreps, "On Intertemporal Preferences with a Continuous Time Dimension: The Case of Certainty," *Journal of Mathematical Economics* **21**(1992), 401-420.

[31] Hoover, Douglas N. and Edwin A. Perkins, "Nonstandard Construction of the Stochastic Integral and Applications to Stochastic Differential Equations," preprint, 1980.

[32] Hurd, Albert E. and Peter A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, Orlando, 1985.

[33] , "A Solution Method for Incomplete Asset Markets with Heterogeneous Agents," preprint, November 1999.

[34] , "Computing Equilibria in Infinite-Horizon Finance Economies: the Case of One Agent," *Journal of Economic Dynamics and Control* **24**(2000), 1047-1078.

[35] Keisler, H. Jerome, *An Infinitesimal Approach to Stochastic Analysis*, *Memoirs of the American Mathematical Society* **48**(1984), Number 297.

[36] Khan, M. Ali and Yeneng Sun, "The capital-asset-pricing model and arbitrage pricing theory: A unification," *Proceedings of the National Academy of Sciences of the United States of America* **94**(1997), 4229-4232.

[37] Khan, M. Ali and Yeneng Sun, "Asymptotic arbitrage and the APT with or without measure-theoretic structures," *Journal of Economic Theory* forthcoming.

[38] Lindström, T. L., "Hyperfinite Stochastic Integration I: the Nonstandard Theory," *Mathematica Scandinavica* **46**(1980), 265-292.

[39] Lindström, T. L., "Hyperfinite Stochastic Integration II: Comparison with the Standard Theory," *Mathematica Scandinavica* **46**(1980), 293-314.

[40] Lindström, T. L., "Hyperfinite Stochastic Integration III: Hyperfinite Representations of Standard Martingales," *Mathematica Scandinavica* **46**(1980), 315-331.

[41] Lindström, T. L., "Addendum to Hyperfinite Stochastic Integration III," *Mathematica Scandinavica* **46**(1980), 332-333.

[42] Loeb, Peter A., "Conversion from Nonstandard to Standard Measure Spaces and Applications in Potential Theory", *Transactions of the American Mathematical Society*, **211**(1975), 113-122.

[43] Lucas, Robert E. Jr., "Asset Prices in an Exchange Economy," *Econometrica* **46**(1978), 1426-1446.

[44] Magill, Michael and Martine Quinzii, *Theory of Incomplete Markets, Volume 1*, MIT Press, Cambridge, MA 1996.

[45] Mas-Colell, Andreu and Scott Richard, "A New Approach to the Existence of Equilibrium in Vector Lattices," *Journal of Economic Theory* **53**(1991), 1-11.

[46] Merton, Robert C. (1973), "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* **4**, 141-183.

[47] Métivier, M., *Reelle und Vektorwertige Quasimartingele und die Theorie der Stochastischen Integration*, *Lecture Notes in Mathematics* **607**, Springer-Verlag, Berlin-Heidelberg-New York, 1977

[48] Nielsen, Lars Tyge, *Pricing and Hedging of Derivative Securities*,Oxford University Press, 1999.

[49] Radner, Roy, "Existence of Equilibrium of Planes, Prices and Price Expectations in a Sequence of Markets," *Econometrica* **40**(1972), 289-303.

[50] Raimondo, Roberto C., "Incomplete Markets with a Continuum of States," preprint, 2001.

[51] Raimondo, Roberto C., "Market Clearing, Utility Functions, and Securities Prices," preprint, 2001.

[52] Zame, William R., "Continuous Trading: A Cautionary Tale," preprint, December 2001.