Leverage and Risk Taking

Santiago Moreno-Bromberg* and Guillaume Roger†

June 2016

Abstract

We study a dynamic contracting problem in which size is relevant. To boost performance, the agent may take on excessive risk that enhances current gains but exposes the principal to large, infrequent losses. To preserve incentive compatibility, the optimal contract uses size as an instrument: there is downsizing along the equilibrium path. The need to control size is interpreted as leverage management. The contract may be implemented using the full array of financial securities or as a regulatory one with a leverage ratio. Firms for which risk taking is less attractive can afford a higher leverage.

Keywords: asymmetric information; dynamic contracts; moral hazard; risk taking.

JEL Classification: D82, D86, G28, L43.

1 Introduction

It is generally accepted that the Global Financial Crisis (henceforth GFC) was precipitated by excessive risk taking on the part of financial institutions that were too highly levered (see for example Duffie (2010); Geithner (2014)). In this paper we show that leverage needs to be controlled to deter excessive risk taking, for the simple reason that (firm) size governs the returns on risk taking. This result provides foundations for equity requirements on firms, especially on financial institutions, and departs from the commonly-accepted wisdom that equity capital is required to sustain adverse shocks, such as economic downturns, or to

*Center for Finance and Insurance (DBF), University of Zurich: santiago.moreno@bf.uzh.ch

†University of Sydney: guillaume.roger72@gmail.com

absorb losses (see, e.g. Bonaccorsi di Patti et al. (2015)).[1] Instead, equity requirements, or a limit on financial leverage, are necessary to present the the potential speculator with the appropriate incentives and to curb risk taking. In other words, they are prevention measures rather than crisis-management tools. This fact was first observed by Jensen and Meckling (1976) (page 41); we offer a comprehensive analysis and propose a solution to the issue. In light of the GFC and of the many isolated incidents since (Société Générale, 2008, €4.9 Bn; UBS, 2011, US$ 2Bn or the London Whale at JP Morgan, 2012, US$ 7Bn) one ought to better understand the role of leverage in *fostering* crises – rather than just in amplifying them.[2]

To make our point we study a stylized, continuous-time contracting model with agency frictions, in which an agent controls the profitability of a project through a scalable arithmetic Brownian motion. The agent must be provided incentives to not divert funds (equivalently, to exert effort); she can also engage in excessively risky activities that increase the drift of the cashflows process but expose the firm to a Poisson process of very large losses. Put differently, she may take on tail risk on the asset side of the firm's balance sheet. Thanks to the aforementioned scale effect, the model exhibits rich dynamics.

Firm size emerges as a necessary control to satisfy incentive compatibility because the returns on risk taking are increasing in the size of the project and the agent may appropriate a fraction of these returns. Risk taking is deterred when the agent's continuation utility (her stake in the project) is sufficiently large compared to the size of the firm; the agent must have enough to lose so as to not speculate.[3] The incentive compatible, optimal contract includes downsizing along the equilibrium path in order to preserve incentive compatibility. If downsizing has to be so severe that the firm becomes inefficiently small, the principal prefers liquidating it. We characterize the downsizing process.

The downsizing option is only relevant at the lower boundary of the domain of the differential equation that characterizes the principal's value function; hence, all the firm-

[1] Diamond and Rajan (2000) suggest equity is not necessary in order to cover credit losses in a bank; any claim subordinated to deposits is sufficient. Therefore, long-term debt is sufficient to guarantee the depositors' confidence and to solve the issue of bank runs.

[2] We do not study the unraveling of leverage but its effect on incentives to take risks; for the former point, see Krishnamurthy (2010), Krishnamurthy and He (2012) or Brunnermeier and Sannikov (2014).

[3] In discrete time models firm size may also be used as a control, however to satisfy the more standard effort constraint (see for example, Fishman and DeMarzo (2007) or Biais et al. (2007)). That effect disappears in continuous time; in our model it pertains to tail risk instead.

size dynamics are centered around that boundary. Whenever the principal downsizes the project, he decreases that lower boundary as well: it is a *floating barrier*. The value function is homogeneous of degree one in the size variable, which has two benefits. First, in size-adjusted terms the floating barrier becomes a more standard reflecting one. Reaching it triggers downsizing, which in size-adjusted terms results in reflection. Second, away from the lower boundary, the differential equation that characterizes the (scale-adjusted) value function is the same as in DeMarzo and Sannikov (2006), the behavior of which is well understood. Our comparative statics indicate that the value to the principal uniformly decreases when risk taking becomes more attractive.

We suggest two implementations to connect our mathematical results to practical questions of economics and finance. In both instances, the risk-taking incentives affect both the value of the firm and its debt capacity. Quite intuitively, the milder the risk taking problem, the more valuable the cashflows and the more debt a firm can carry to finance itself. That is, in equilibrium less-risky firms are more highly levered: they face less stringent equity requirements. These empirical predictions are borne out by the data (Brown et al. (2008); Brown et al. (2012)). Brown and his co-authors study governance problems in hedge funds and find that a fund that is perceived by investors to have good governance has a larger debt capacity. Our optimal contract is first implemented as a capital structure using standard securities, but it requires a richer set of instruments than the implementation of DeMarzo and Sannikov (2006). In particular, it is essential that the agent holds enough equity to deter her from risk taking – and not just to allocate her cash-flow rights. This is achieved by adding covenants to the debt contract. In this interpretation, the (inverse) leverage measures the normalized value of inside equity. Second, we also implement the contract as a banking regulation, which mandates a minimum book value of equity as an intervention threshold. This is well in line with current practice.

Finally, we explore two extensions. The first one studies the opportunity to invest in the firm to increase its size. The principal only invests when the agent's continuation value is sufficiently high so as to avoid premature downsizing following costly investment. In the second extension, we assume that the project cannot be liquidated; for example, a large bank should not stop operating. This fits large financial institutions such as Global Systemically Important Banks (there are 29 worldwide) and Global Systemically Important Insurers (there are 9). The contractual stopping time is then a restructuring threshold at which the agent is terminated, yet the firm is no liquidated.

The papers closest to this work are Biais et al. (2010) (henceforth BMRV), He (2009), as well as DeMarzo et al. (2013) and Rochet and Roger (2015). BMRV study the problem of a firm exposed to large Poisson risks (accidents or losses), whose arrival probability is controlled through the agent's effort; there is no Brownian component. Incentive compatibility dictates that the agent's continuation value be decreased following a loss; it occurs along the equilibrium path because the probability of a loss is assumed to never be zero. Sometimes restoring incentive compatibility requires the firm to be downsized too. Our works departs from BMRV in at times subtle but important ways. First, absent the Brownian process, the optimal contract is trivial in our model – even with the Poisson risk: it is sufficient to offer the agent a constant wage. Without the Brownian dynamics to confound diversion, the latter becomes impossible. Moreover, the change in the drift becomes both perfectly observable by the principal and useless to the agent; hence she has no incentives to engage in risk taking. Second, BMRV allow for large losses even if the agent undertakes the good action . If one assumes in BMRV that the good action results in no large losses, as we do, there is *never* any downsizing. In contrast, even with no jump risk under the good action, we generate downsizing along the equilibrium path, thanks to the Brownian driver that introduces randomness in the agent's continuation value. Furthermore, downsizing occurs *exactly* at the lower boundary, which lends itself neatly to a policy interpretation – here as a leverage limit. Finally, we stress that the diversion and the risk-taking problems interact in this model, even though they are independent actions. This interaction arises because diversion requires the agent to be exposed to the cash flow process, which then generates the incentives to increase it.

DeMarzo et al. (2013) and Rochet and Roger (2015) (independently) study the double moral-hazard problem of diversion (or effort) and risk taking, however without any size effect. Incentive compatibility requires the agent's continuation value to remain sufficiently high but it is size independent. Adding a size effect allows for downsizing, which is used to preserve incentive compatibility – instead of termination in those models. Most notably, (i) the scaled (arithmetic) Brownian motion implies that the dynamics of the value function follow a partial differential equation reflecting the role of size dynamics, rather than an ordinary differential equation in DeMarzo et al. (2013); and (ii) in spite of homogeneity our solution is not stationary thanks the size dynamics at the boundary, unlike DeMarzo and

Sannikov (2006) or DeMarzo et al. (2013).

He (2009) extends the work of DeMarzo and Sannikov (2006) (henceforth DS) to a model where the cashflows follow a geometric Brownian motion. If writing our model with a geometric process rather than a scaled ABM, the same incentive constraints would emerge. However, the geometric assumption precludes using size as a control. It is precisely the ability to vary size that generates our rich dynamics at the lower boundary. Moreover, He (2009)'s contract cannot be implemented using standard securities. We suggest multiple, practical implementations for which we can derive comparative statics; all conform to intuition and are borne out by the data (Brown et al. (2008); Brown et al. (2012)).

This paper also connects to a large literature on leverage and risk taking, especially in financial institutions. In a series of papers, Anat Admati (and at times her coauthors) makes the argument for less leverage on the grounds of less fragility for individual firms, less subsidies from society and greater systemic resilience (see, e.g. Admati (2014)). From that perspective, leverage amplifies the transmission of shocks and the severity of crises. This paper adds a simple but salient point: with less levered institutions, there is also less risk-taking, which reinforces financial stability.

2 Model

We consider a principal-agent problem set in a continuous-time framework. The principal (shareholders or a regulator) must rely on the expertise of an agent (manager or firm) to operate their business. All parties are risk-neutral; the principal discounts future payments at rate $r > 0$ and the agent is (weakly) more impatient, as her discount rate $\rho \geq r$. In order to formally describe the principal-agent interaction, let us introduce the probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$. At each date $t \geq 0$, the agent chooses an *action* $a_t \in \{0, 1\}$. If the agent chooses the action 0 we say that she has *speculated*, whereas if her choice is 1 we say she has been *prudent*. Let $\boldsymbol{\mu}, \Delta\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ be strictly positive and define the functions

$$\mu(a) := \begin{cases} \boldsymbol{\mu} + \Delta\boldsymbol{\mu}, \text{ if } a = 0; \\ \boldsymbol{\mu}, \qquad\quad \text{if } a = 1, \end{cases} \qquad \text{and} \qquad \lambda(a) := \begin{cases} \boldsymbol{\lambda}, \text{ if } a = 0; \\ 0, \text{ if } a = 1. \end{cases} \tag{2.1}$$

For a given strategy $a = (a_t, 0 \leq t)$ we define $N(a) := (N_t(a_t), 0 \leq t)$ as the Poisson process with intensity $\lambda(a) = (\lambda(a_t), 0 \leq t)$. We shall also make use of the standard Brownian motion $Z = (Z_t, 0 < t)$ and, from this point on, we assume that $\mathbb{F}^a := (\mathcal{F}^a_t, t \geq 0) = \sigma\big(Z, N(a)\big)$ is the natural filtration induced by the agent's strategic choices. In the sequel we denote by $\mathcal{A}$ the set of all admissible strategies for the agent. For $a \in \mathcal{A}$ the firm's *operating cashflows* follow the process

$$dS^a_t = X_t\big(\mu(a_t)dt + \sigma dZ_t - LdN_t(a_t)\big), \quad S_0 = 0, \tag{2.2}$$

where X_t is the firm's size at date t (for a bank one could think of its balance sheet). Risk taking increases the drift of the cashflows but introduces exposure to large losses with small probabilities. We assume these losses are sufficient to wipe out the firm; thus, they are never desired by the principal. We provide a sufficient condition in Remark 3.

The $\mathbb{F}^a$-predictable *cumulative downsizing process* $X^d = \big(X^d_t, 0 \le t\big)$, which is one of the principal's controls, is non-decreasing. In other words, we allow for downsizing but not for growth; we relax this assumption in Section 6. Downsizing is undesirable for the principal, as it reduces the cash flows, but it may be necessary for incentive purposes. Downsizing implies partial liquidation and, for simplicity, we assume it generates no proceeds. The firm's initial size is denoted by X_0, its scale at date t is $X_t = X_0 - X^d_t$ and we take as given a minimal project scale $\underline{X} > 0$ below which the project cannot operate. At the first date such that $X_t = \underline{X}$, the agent is fired and the firm is liquidated (or restructured).[4] When the firm's scale is $X_t \ge \underline{X}$, liquidation earns the principal ΠX_t; we analyze restructuring as an extension.[5] The agent's outside option is zero.

There are two sources of frictions. First, the agent might divert some of the cashflows: she may hand over $d\widehat{S}_t < dS^a_t$ and appropriate the difference. A dollar diverted brings the agent $\eta \le 1$ dollars, i.e. misreporting results in the instantaneous profits $\eta|dS^a_t - d\widehat{S}_t|$. Second, the agent can secretly engage in excessively risky ("speculative") activities that generate the additional cashflows $X_t\Delta\mu dt$ but expose the firm to the catastrophic losses X_tL. For example, the firm sells (but does not buy) CDS (like AIG or Morgan Stanley during the GFC) or issues options. It could also speculate on electricity contracts (like Enron in the late 1990's). Since losses corresponding to a realization of the Poisson risk are catastrophic, there is no loss of generality in assuming that $\widehat{S}$ is a continuous process, which is also $\mathbb{F}^a$-adapted.[6]

[4] $\underline{X}$ may be conceived as a minimum efficient scale and $\underline{X}\Pi$ as the resale value of a fixed investment.

[5] By "restructuring" we mean that the firm cannot be liquidated – for example, a very large bank. Instead the agent is fired and replaced, and a new contract is initialized.

[6] We rule out private savings by the agent without loss of generality (see DS for details). Hence, we

3 The Contract

The principal seeks to maximizes the ex-ante value of the firm. It is optimal for the principal to always deter risk taking, given that it results in catastrophic losses. The contract between the principal and the agent is designed at date $t = 0$ and we assume that all parties can commit to it. A contract $\Xi = (X^d, I, \tau)$ stipulates, contingent on the history of observed cashflows, a cumulative downsizing process X^d, a non-decreasing process I of cumulative payments to the agent and a (random) termination time τ. The fact that I is non-decreasing reflects the agents *limited liability.* For a given contract Ξ, the agent chooses her strategy by solving

$$\sup_{a\in\mathcal{A}} U_0^a(\Xi) := \mathbb{E}\left[\int_0^\tau e^{-\rho s}\big(dI_s + \eta|dS_s^a - d\widehat{S}_s|\big)\right] \tag{3.1}$$

The contract is incentive compatible if it is designed in such a way that the agent never finds it optimal to divert cash nor to engage in speculative activities.

Following Spear and Srivastava (1987), who introduced the recursive approach to contracting, any contract can be characterized by the stochastic process W describing the continuation payoff to the agent when the contract Ξ is executed. If the agent chooses strategy $a \in \mathcal{A}$ then

$$W_t^a(\Xi) = \mathbb{E}\left[\int_t^\tau e^{-\rho(s-t)} d\widehat{C}_s^a + e^{-\rho(\tau-t)} W_\tau^a \middle| \mathcal{F}_t^a\right], \tag{3.2}$$

where $d\widehat{C}_s^a := dI_s + \eta|dS_s^a - d\widehat{S}_s|$ is the consumption process of the agent. In line with Sannikov (2008), we want to show $W^a(\Xi)$ can be represented as the solution to a stochastic differential equation in order to characterize its dynamics. Making use of the Martingale Representation Theorem for jump-diffusion processes (the details can be found in the proof of Proposition 1 in the Appendix), we may write Expression (3.2) as

$$dW_t^a = \rho W_t^a dt + \beta_t^a X_t dZ_t - d\widehat{C}_t^a - P_t^a\big[dN_t - \lambda(a_t)dt\big], \tag{3.3}$$

which is a jump-diffusion process. Here β_t^a and P_t^a represent the sensitivity of the agent's continuation payoffs to the volatility of cash flows and to large losses, respectively. Incentive compatibility can be characterized by simple conditions on these sensitivity parameters.

do not require $S - \widehat{S}$ to be of bounded variation. The difference here is that, since the agent is devoid of the possibility to over-report, we do not split the instantaneous effects of diversion/over-reporting into their positive and negative components (the term $[Y - \widehat{Y}]^\lambda$ in DS). This is only well defined under the bounded-variation hypothesis, but does not play a role here.

However, because we must account for both diversion and risk taking, our results depart from DS, Sannikov (2008) and He (2009), for instance.

Proposition 1 *For any strategy $a \in \mathcal{A}$ there is no cash diversion as long as*

$$\beta_t^a \geq \eta\sigma =: \beta, \tag{3.4}$$

and there is no risk taking, i.e. $a_t \equiv 1$, if and only if

$$P_t^a = P_t \geq \eta \frac{\Delta\mu}{\lambda} X_t \tag{3.5}$$

Combining Expressions (3.4) and (3.5), we have that a contract deters both risk taking and cash diversion only if

$$P_t \geq \frac{\beta_t}{\sigma} \frac{\Delta\mu}{\lambda} X_t,$$

and by limited liability $W_t \geq P_t$, so[7]

$$W_t \geq \frac{\beta_t}{\sigma} \frac{\Delta\mu}{\lambda} X_t =: W_m(X_t) =: w_m X_t. \tag{3.6}$$

Size does not enter Constraint (3.4). To gain some intuition, observe that the term

$$\left[\frac{\beta_t^a}{\sigma} - \eta\right]\left[dS_t^a - d\widehat{S}_t^a\right]$$

showcases the compromise the agent makes when deciding whether to divert funds: she enjoys the instantaneous benefit $\eta[dS_t^a - d\widehat{S}_t^a]$ from immediate consumption but forgoes $[\beta_t^a/\sigma][dS_t^a - d\widehat{S}_t^a]$ – the decrease in her continuation value. This is true for any strategy $a \in \mathcal{A}$, as we show formally in the Appendix. Due to the fact that the current scale appears on both sides of this compromise it is neutral on the agent's incentives to divert funds.

Turning to risk taking, consider two strategies as follows: $a_t \equiv 1$ (always prudent) and $\tilde{a} = \{a_s = 0, s < t; a_s = 1, s \geq t\}$ (take excessive risk until date t, then be prudent), for some $t > 0$. Since s, t are arbitrary and Bellman's Optimality Principle applies, what follows holds true for any strategy $a \in \mathcal{A}$. To deter risk taking, for any processes C_t and β_t^a, the principal needs to make the expected penalty

$$\int_0^t e^{-\rho s} \lambda P_s^{\tilde{a}} ds$$

larger than the corresponding expected gains

$$\int_0^t e^{-\rho s} \left(d\widetilde{C}_s^{\tilde{a}} - d\widetilde{C}_s^1\right) ds,$$

[7] The reason for relabeling $\beta_t \Delta\mu/(\sigma\lambda)$ as w_m will become apparent in Section 4.2.

representing an increase in consumption generated by following strategy $\tilde{a}$ until time t. This must hold regardless of whether $d\widetilde{C}_s = d\widehat{C}_s$ (diversion) or $d\widetilde{C}_s = dC_s$ (not). When there is diversion, we require $P_t \geq \eta\frac{\Delta\mu}{\lambda}X_t$ and when diversion is deterred $P_t \geq W_m(X_t)$ must hold. Unlike the diversion problem, X_t is only present in the right-hand side of the inequalities; therefore, size *does* matter for risk taking. This result is novel: Engaging in risk taking generates an additional gain $\Delta\mu X_t$, of which the agent always appropriates a fraction η or β_t/σ. The incentives are strongest when $P_t \equiv W_t$: the agent must be wiped out after a large adverse event. Any further penalty would violate limited liability, thus $W_t \geq W_m(X_t)$ so as to preserve incentive compatibility.

Remark 1 *Condition (3.6) is noteworthy: it requires the continuation value to grow linearly with the size of the firm X_t. This condition arises naturally here and may contribute to explain the increase in executive compensation at large firms (see Gabaix and Landier (2008)).*

Remark 2 *Absent diversion the risk taking problem has not object: it is sufficient to offer the agent a constant wage to induce participation. The the agent has no incentive to increase the drift $\mu(a)$.*

4 The value function and the optimal contract

The principal maximizes the discounted, expected cashflows net of payments to the agent over all incentive-compatible contracts, which we denote by IC. In other words, the principal's *value function* is given by

$$V(X,W) := \sup_{\Xi \in IC} \mathbb{E}\left[\int_0^\tau e^{-rs}(dS_s^a - dI_s)\Big| X_0 = X, W_0 = W\right]. \tag{4.1}$$

We conjecture (and later verify) that the optimal contract is such that

$$\beta_t^a = \beta \text{ and } P_t = W_m(X_t).$$

Given that the principal never wants to allow risk taking, along the optimal path W_t evolves according to the dynamics

$$dW_t = \rho W_t dt - dI_t + \beta X_t dZ_t. \tag{4.2}$$

Under an incentive compatible contract (which determines the dynamics of W), the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rV(X_t, W_t)dt = \mu X_t dt + \sup_{dX_t^d, dI_t} \Big\{ -dI_t + (\rho W_t dt - dI_t)V_W(X_t, W_t) - V_X(X_t, W_t)dX_t^d + \frac{\beta^2 X_t^2}{2} V_{WW}(X_t, W_t)dt \Big\}, \tag{4.3}$$

subject to the incentive compatibility Constraints (3.4) and (3.6).[8] In this problem the firm's current size matters for both incentives and continuation values. X_t enters the equations that determine the dynamics of V and W, but its impact does not stop there. Indeed (i) the agent is subject to downsizing at any point in time to preserve incentive compatibility – Condition (3.6) – and (ii) if downsizing leads the firm to become too small, it may be terminated:

$$V\big(\underline{X}, W_m(\underline{X})\big) = \Pi \underline{X}. \tag{4.4}$$

These conditions define an intervention threshold at $W_m(X_t)$. Equation (4.3) has a free upper boundary that may be determined through an optimality condition. Let $\widetilde{W}(X)$ be characterized by the condition $V_W(X, \widetilde{W}(X_t)) = -1$. As is now commonly known, $\widetilde{W}(X)$ is a *payment barrier*: transfers to the agent are postponed until the time when increasing her continuation utility becomes too expensive for the principal.

We postpone a more precise analysis of $\widetilde{W}(X)$ and instead stress that payments to the agent only take place at $\widetilde{W}(X)$ when $W_m(X) < \widetilde{W}(X)$, i.e. when the reflecting barrier is not in conflict with the incentive constraint (3.6). Whether this applies here depends on the risk-taking problem too. The complementary case described by $W_m(X) \geq \widetilde{W}(X)$ pits the reflecting barrier $\widetilde{W}(X)$ with the no risk-taking constraint (3.6). Then, of course, there can be no payment at $\widetilde{W}(X)$, the contract is not even incentive compatible at that point. In fact the principal must even reduce X_t to restore incentive compatibility.[9] The lower boundary induced by the termination condition (4.4) and the payment barrier may be confounded.

For now assume that $W_m(X) < \widetilde{W}(X)$; then, as long as $W_t \in \big(W_m(X_t), \widetilde{W}(X_t)\big)$, there are neither downsizing nor monetary transfers to the agent. In other words, both dI_t and

[8] Since X is a decreasing process it is of bounded variation. Moreover, as we show below, X has continuous paths, therefore the cross-variation term $\langle X, W\rangle_t$ and the quadratic variation $\langle X, X\rangle_t$ are both zero. As a consequence there are no V_{XW} nor V_{XX} terms in Equation (4.3). Observe that $dX_t = -dX_t^d$.

[9] In DS, He (2009) or BMRV, for example, the unique reflecting barrier is $\widetilde{W}$ independent of size X.

dX_t^d are zero on that open interval; then the continuation values for the principal and the agent are characterized by the equations

$$rV(X,W) = \mu X + \rho W V_W(X,W) + \frac{\beta^2 X^2}{2} V_{WW}(X,W) \tag{4.5}$$

and

$$dW_t = \rho W_t dt + \beta X_t dZ_t, \tag{4.6}$$

respectively. As long as the no risk-taking constraint remains slack, no changes occur to the payment barrier $\widetilde{W}(X_t)$ and the cumulative payments to the agent are such that W_t is *reflected* downwards. As in DS this reflecting barrier induces a boundary condition for the HJB equation (4.3) that is complemented by the super-contact condition $V_{WW}(X, \widetilde{W}(X)) = 0$. Together, these pasting conditions yield

$$rV(X, \widetilde{W}) + \rho \widetilde{W} = \mu X, \tag{4.7}$$

where the scale X also figures, and we have

$$dI_t \begin{cases} = 0, & \text{if } W_t < \widetilde{W}(X); \\ > 0, & \text{if } W_t \geq \widetilde{W}(X). \end{cases}$$

Intuitively speaking dI_t "equals" $\max\{0, W_t - \widetilde{W}(X_t)\}$, i.e. all the value in excess of the payment barrier is immediately paid out to the agent.[10] At the point $\widetilde{W}$, the continuation value W can no longer grow as it, together with $V(X,W)$, exhausts the cashflows μX. We stress that, as long as no downsizing occurs, X is simply a parameter in Equations (4.5) and (4.6). In the sequel our strategy is to simplify Equation (4.3) through the identification of the maximizing strategies and to characterize its solution.

4.1 Incentive compatibility and downsizing

A key feature of our model is the principal's option to downsize. This is necessary to prevent the no risk-taking constraint $W_t \geq W_m(X_t)$ from being violated, for then the agent would have incentives to speculate. Downsizing is then an alternative to termination–see for example, DeMarzo et al. (2013) or Rochet and Roger (2015). However, since $W_m(X_t)$ is a function of X_t, the incentive constraint (3.6) induces a *floating* (lower) boundary at

[10] In technical terms, the process I_t is the *local time* of W at the level $\widetilde{W}(X_t)$. We refer the reader to Revouz and Yor (1999) for a thorough exposition of Brownian local times.

$W_m(X_t)$. The floating boundary tracks the downward component of the agent's continuation utility (the negative increments of W_t) once the constraint becomes binding. Furthermore, when this occurs the dynamics of W are also impacted by the firm's downsizing, since the corresponding volatility is linear in X.

Recall that $w_m = \beta\Delta\mu/(\sigma\lambda)$ and define $R_t := \inf\{W_s, 0 \leq s \leq t\}$, the *running infimum* of W up to time t, i.e. the smallest value that the agent's continuation value has attained during her tenure. As long as $R_t > w_m X_0$ no downsizing has been required to preserve incentive compatibility. If that barrier is reached, the cumulative downsizing process is active as long as W pushes downwards. When this is no longer the case, the incentive constraint becomes slack again and downsizing stops. This is formalized by defining the cumulative downsizing process X^d via the relation

$$X_t^d := X_0 - \min\{R_t/w_m, X_0\}. \tag{4.8}$$

Loosely speaking, X^d counts the time that the agent's continuation value spends at the boundary $W_m(X)$, i.e. the length of each downsizing period. Put differently, as long as the incentive constraint remains slack, X_t is constant and $W_t \in \big(W_m(X_t), \widetilde{W}(X_t)\big)$. Whenever $W_t = \widetilde{W}(X_t)$, $dI_t > 0$: at this point if W "pushes upwards" it stays pegged at $\widetilde{W}(X_t)$ and as soon as W stops increasing, it is reflected downwards by its own dynamics. Reflection follows the fact that the process I prevents W from exceeding $\widetilde{W}(X_t)$. With downsizing the dynamics are quite similar, and it starts as soon as $W_t = w_m X_t$ if W "pushes downwards". Indeed,

Lemma 1 *Downsizing is necessary as soon as $W_t = W_m(X)$; more precisely, the distributions of the first-visitation and the first-crossing times*

$$\tau_v := \inf\{t \geq 0 | W_t = W_m(X)\} \quad \textit{and} \quad \tau_c := \inf\{t \geq 0 | W_t < W_m(X)\},$$

respectively, are identical.

We note that under the assumption that $W_m(X) < \widetilde{W}(X)$, it is clear that there should be no downsizing as long as $dI_t > 0$. Instead of downsizing the principal could simply set $dI_t = 0$, which would increase W_t one-to-one. In Figure 1 we depict the value function corresponding to the scales $X_0 = 1$ and $\underline{X} = 0.4$. and the parameters $\mu = 0.6$. $\beta = 0.8$, $r = 0.25$, $\rho = 0.3, w_m = 0.7$, and $\Pi = 0.16$. The scale-dependent payout and downsizing barriers, $\widetilde{W}(X)$ and $W_m(X)$ respectively, are highlighted.

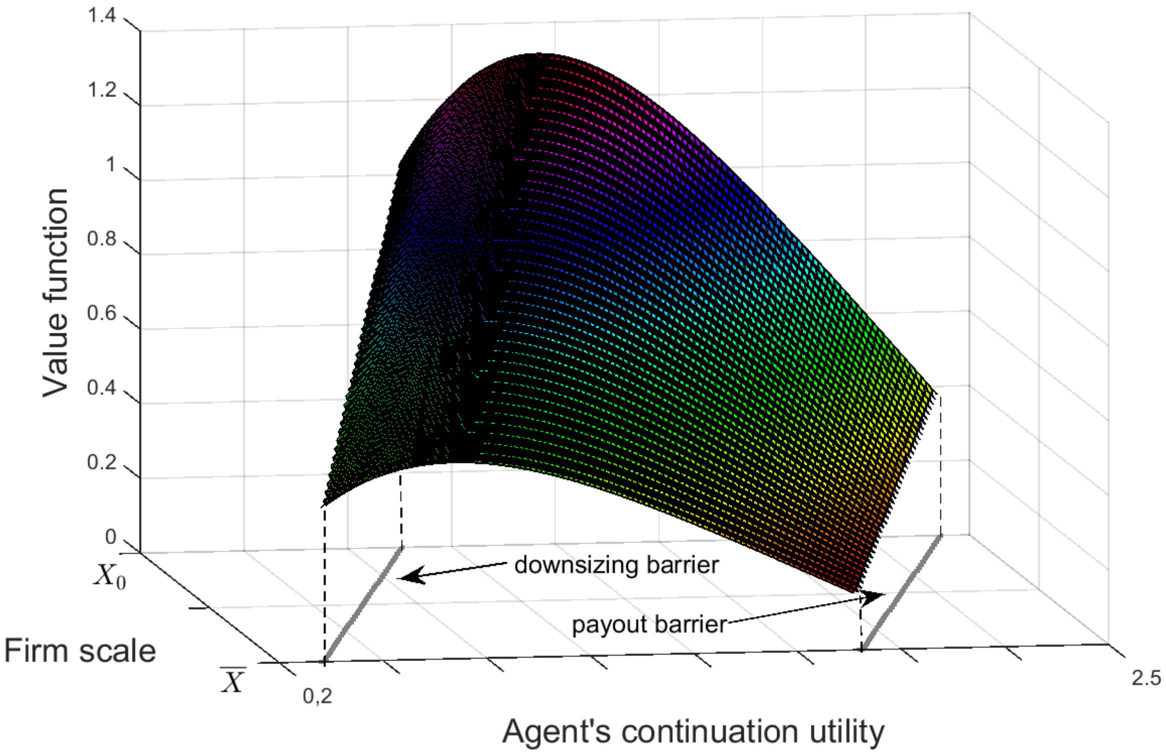


Figure 1: Value function and the payout and downsizing barriers

4.2 Homogeneity of $V(X,W)$

To study the solution to Equation (4.3) we make use of the fact that, by virtue of the cashflows being linear in X, the value function V is homogeneous in X. In other words, there exists a function $v : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$V(X,W) = Xv\Big(\frac{W}{X}\Big). \tag{4.9}$$

If we let $w_t := W_t/X_t$ we may write the size-adjusted version of Equation (4.3) as

$$rv(w)dt = \mu dt + \sup_{di_t, dx_t^d} \big\{ -di_t + v'(w)(\rho w_t dt - di_t) - (v(w) - w\,v'(w))dx_t^d + \frac{\beta^2}{2}v''(w)dt \big\}, \tag{4.10}$$

where $di_t := dI_t/X_t$, and $dx_t^d := dX_t^d/X_t$, subject to the no-diversion constraint (3.4) as before, together with the size-adjusted, no risk-taking constraint

$$w_t \geq w_m. \tag{4.11}$$

The boundary condition at $w = w_m$ follows directly from Equation (4.4): since

$$\Pi\underline{X} = \underline{X}v\left(\frac{W_m(\underline{X})}{\underline{X}}\right) = \underline{X}v(w_m), \tag{4.12}$$

we have $v(w_m) = \Pi$.[11]

In size-adjusted terms, the floating barrier $W_m(X_t)$ becomes a reflective one at w_m for the size-adjusted continuation value w_t of the agent. Using Itô's formula

$$dw_t = \frac{dW_t}{X_t} + W_t\left(\frac{dX_t^d}{X_t^2} + \frac{1}{X_t^3}d\langle X, X\rangle_t\right) + d\langle W, 1/X\rangle_t$$

We know that X is continuous and, since it is bounded away from zero, $1/X$ is continuous as well. Therefore, $\langle X, X\rangle_t = \langle W, 1/X\rangle_t = 0$, which yields

$$dw_t = \frac{dW_t}{X_t} + W_t\frac{dX_t^d}{X_t^2} = \rho w_t dt - di_t + \beta dZ_t + w_t dx_t^d.$$

Moreover, dx_t^d only carries mass on the set $\{w_t = w_m\}$, hence

$$dw_t = \rho w_t dt - di_t + \beta dZ_t + w_m dx_t^d. \tag{4.13}$$

The (size-adjusted) term $w_m dx_t^d$ in Equation (4.13) introduces an instantaneous reflection of the process w at the level $w = w_m$. Intuitively, as long as $w > w_m$, it holds that $dx_t^d = 0$. Whenever $w_t = w_m$, the process x is active if w "pushes downwards" (like W in the preceding section) and its role it precisely to maintain the inequality $w_t \geq w_m$. As soon as w "pushes upwards", we have $dx^d = 0$ and w is reflected upwards by its own dynamics. Formally

Lemma 2 *The term $w_m dx_t^d$ in the dynamics of the process $w = \big(w_t, 0 \leq t\big)$ induces an instantaneous reflection of the latter at the level $w = w_m$.*

The behavior at the boundary $w = w_m$ differs from both DS and Zhu (2013). In DS the lower boundary necessarily induces termination (there is no scale to adjust). In Zhu's model the boundaries are sticky for some classes of contracts. The continuation value w may remain at the lower bound w_m for a positive measure of time because the principal suspends the contract. Then the stochastic component dZ_t is neutralized and w is allowed to remain at w until the contract is reactivated. There is no suspension of the process dZ_t

[11] A comment regarding the term $(v(w) - w\,v'(w))dx_t^d$ is in order. Without $-dx_t^d$, $v(w) - w\,v'(w)$ corresponds to the value of the intersection of the tangent to the graph of v at $(w, v(w))$ and the vertical axis. It holds that $V_X(\underline{X}, w_m\underline{X}) = v(w_m) - w_m\,v'(w_m) = 0$, because at the downsizing boundary there is no gain/cost of downsizing. From the concavity of v we then have that for $w > w_m$ it holds that $v(w) - w\,v'(w) > 0$. Since x is a non-decreasing process, it is clear that it is optimal to have $-dx^d = 0$ whenever $w > w_m$. This complementarity condition is akin to the one concerning payouts to the entrepreneur, where the condition is $-(1 + v'(w))di_t$, which implies that payouts only take place when $v'(w) = -1$.

here; hence, the reflection is instantaneous (that is not to say downsizing is not costly to the agent; indeed, her scaled continuation value $W = wX$ is clearly lower after downsizing).

In analogous fashion to the scaled case, there exists $\widetilde{w}$ such that $di_t = 0$ if $w_t < \widetilde{w}$. Hence, the processes i and x are inactive on the open interval $(w_m, \widetilde{w})$. The reflection at level $w = \widetilde{w}$ follows from the condition $v'(\widetilde{w}) = -1$, with $\widetilde{w}$ characterized by the super-contact condition $v''(\widetilde{w}) = 0$, and Equation (4.10) becomes

$$rv(w) = \mu + \rho w v'(w) + \frac{\beta^2}{2} v''(w), \tag{4.14}$$

which is a well-known problem (see DS). In summary,

Proposition 2 *The function v is the unique solution on $(w_m, \tilde{w})$ to the differential equation*

$$rv(w) = \mu + \rho w v'(w) + \frac{\beta^2}{2} v''(w)$$

subject to the boundary conditions $v(w_m) = \Pi$ and $v'(\tilde{w}) = -1$. The mapping $w \mapsto v(w)$ is strictly concave on $[w_m, \tilde{w}]$ and the size-adjusted payment barrier is characterized by the super-contact condition $v''(\tilde{w}) = 0$.

Furthermore, directly from this proposition, we can show that the odd configuration in which termination (w_m) and payment ($\widetilde{w}$) are in conflict cannot arise in equilibrium.

Corollary 1 *The relation $w_m < \widetilde{w}$ holds in equilibrium.*

In Figure 2 we show the scale-adjusted value function for parameter values $\mu = 0.6$. $\beta = 0.8$, $r = 0.25$, $\rho = 0.3$, $w_m = 0.7$, $\Pi = 0.16$ and $\underline{X} = 0.4$. The resulting continuation value at the upper boundary $\tilde{w}$ equals 2.349.

If we define $\underline{x}_t := \underline{X}/X_t$, the stopping time τ becomes

$$\tau := \inf\left\{t \geq 0 | w_t = w_m, \underline{x}_t = 1\right\}. \tag{4.15}$$

For any two dates $s > t$, the continuation value w reaches w_m with positive probability: $\mathbb{P}\big\{w_s = w_m | w_t\big\} > 0$, which is simply a consequence of the Brownian stochastic driver in the dynamics of w. Furthermore the contract is terminated (x reaches $\underline{x}$) in finite time with probability one.

Proposition 3 *The stopping time defined in Expression* (4.15) *satisfies* $\mathbb{P}\big\{\tau < \infty\big\} = 1$.

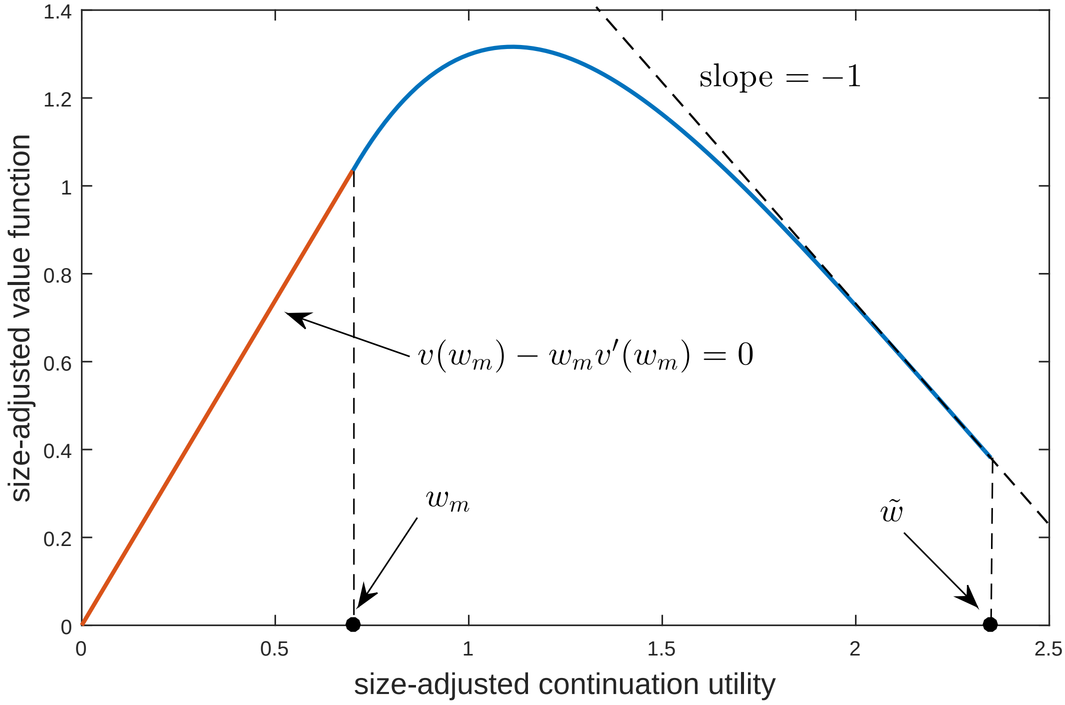


Figure 2: Size-adjusted value function and the reflecting barriers

4.3 Solution and optimal contract

Once we have found v, we can recover V using the homogeneity property (4.9). In particular, the payment barrier is given by

$$\widetilde{W}(X_t) = \widetilde{w}X_t \tag{4.16}$$

and, along the boundary $W = w_m X$,we have

$$V(X, w_m X) = \Pi X.$$

With enough information about the function $V(W, X)$ we can prove our conjecture regarding the optimal values for the agent's sensitivities to the volatility of cash flows and to large losses.

Proposition 4 *For any size X_t the sensitivity of the agent's continuation value and the penalty should be set as low as possible: $\beta_t \equiv \beta$ and $P_t = w_m X_t$.*

The first part of this claim is well known and follows directly by inspection of Equation (4.14), where the mapping $w \mapsto v(w)$ is concave – so β should be as small as possible.

Exposing the agent to risk is costly to the principal, who does it only just enough to generate the right incentives. The second part is new. Setting the penalty P_t any higher than $W_m(X_t)$ does not affect the incentive constraint (3.5). Furthermore, $W_m(X_t)$ is an intervention threshold: setting P_t higher than $W_m(X_t)$ only increases the probability of intervention. Whether corresponding to downsizing or termination, intervention is costly to the principal; its probability of occurrence should be minimized (subject to incentive compatibility). Last, downsizing is preferable to termination for size X_t larger than $\underline{X}$. From the homogeneity property (4.9) and Proposition 2 we derive the structure of the value function V.

Theorem 1 *When $\widetilde{W}(X) < W_m(X)$, there exists a unique solution V to Equation* (4.3), *together with the boundary conditions*

$$V_W(X, \widetilde{W}(X)) = -1 \text{ and } V(X, W_m(X)) = \Pi X,$$

which coincides with the value function defined in Expression (4.1). *For any $X \in [\underline{X}, X_0]$ the mapping $W \mapsto V(X,W)$ is concave and for any $W \in [w_m X, \widetilde{W}(X)]$ the mapping $X \mapsto V(X,W)$ is increasing. The contract starts at the initial value (X_0, W_0), where W_0 is the maximizer of $V(X_0, W)$. The payment barrier $\widetilde{W}(X)$ equals $\widetilde{w}X$ and the downsizing one corresponds to the line $W = w_m X$. The cumulative downsizing process X^d is defined as*

$$X_t^d = X_0 - \min\big\{R_t/w_m, X_0\big\},$$

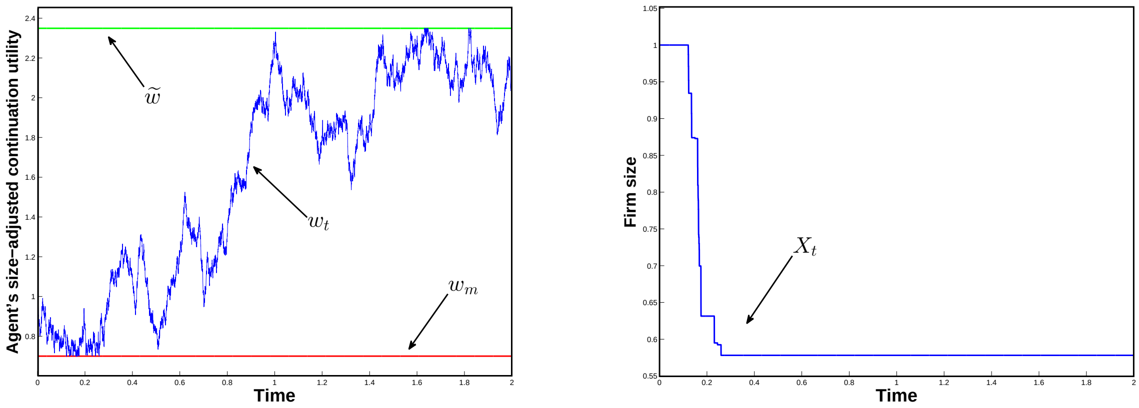
where $R_t := \inf\big\{W_s, 0 \leq s \leq t\big\}$. Finally the cumulative transfers to the agent are computed as

$$I_t = \int_0^t X_s di_s,$$

where the process $= i\big(i_t, t \geq 0\big)$ is the local time of the size-adjusted, continuation-value process w at the level $\tilde{w}$.

We plot in Figure 3(a) a path of the mapping $t \mapsto w_t$ for $t \leq 2$ using the same parameters as in Figure 2. The corresponding firm size, assuming $X_0 = 1$ is presented in Figure 3(b). At first the agent has a bad run and the firm is downsized considerably, but then he recovers and even receives payments.

To conclude our analysis we note that termination at $\underline{X}$ implies that $W_m(\underline{X}) = w_m \underline{X} > 0$ while the agent's reservation value is 0. This difference may be interpreted as a rent that the agent receives because of the risk-taking problem, or it could be extracted by imposing



(a) A path of w_t and the thresholds $\tilde{w}$ and w_m

(b) The corresponding firm size

Figure 3: A path of w_t and the corresponding downsizing

an "entry fee" in case the agent owns some assets (cash). In terms of implementation this entry fee is simply the purchase price of the equity she is made to hold.

Remark 3 *Theorem 1 is complete when the principal never wishes to induce risk taking; formally, whenever $\Delta\mu + r\Pi \leq \lambda L$. In the alternative, risk taking is also attractive to the principal and the no risk-taking problem is moot.*

4.4 Comparative statics

The simplicity of the differential Equation (4.14) allows for the derivation of accessible comparative statics. We do not present all the possible comparative statics results so as to not repeat prior work, but rather emphasize on the newer ones. In particular we focus on results pertaining the incentive constraint $w \geq \frac{\beta}{\sigma}\frac{\Delta\mu}{\lambda}$.

Because the function v and the upper boundary $\widetilde{w}$ are jointly determined and parameterized by the lower boundary w_m, changing a parameter in the latter affects both the solution v and the upper boundary $\widetilde{w}$. Heuristically, from the differential Equation (4.14) we can write $v(w)$ along the equilibrium path as

$$v(w) = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\mu + wv'(w) + \frac{\beta^2}{2}v''(w)\right) ds + e^{-r\tau}\Pi|w\right]$$

and then differentiate with respect to any parameter. For example, a marginal increase in the drift yields

$$\frac{\partial v}{\partial \mu} = \mathbb{E}\left[\int_0^\tau e^{-rs}\right].$$

That information can then be used to totally differentiate the boundary condition

$$rv(\widetilde{w}) + \rho\widetilde{w} = \mu$$

so as to obtain

$$\frac{\partial\widetilde{w}}{\partial\mu} = \frac{1}{\rho - r} - \frac{r}{\rho - r}\frac{\partial v}{\partial\mu},$$

and straightforward computations show that $\frac{\partial\widetilde{w}}{\partial\mu} > 0$.

Proposition 5 *For any $w \in [w_m, \tilde{w}]$, the principal's size-adjusted value $v(w)$ is (*i*) decreasing in the volatility σ, (*ii*) decreasing in the diversion efficiency η, (*iii*) decreasing in the return on risk taking $\Delta\mu$, (*iv*) increasing in the jump intensity λ.*

*The value of the barrier $\widetilde{w}$ (*i*) increases with the volatility σ, (*ii*) increases with the diversion parameter η, (*iii*) increases with the return to risk taking $\Delta\mu$ (*iv*) decreases with the jump intensity λ.*

Some of these results deserve comments. When returns are more volatile the agent must be offered more of the cash flow: $\beta = \eta\sigma$, so the value $v(w)$ to the principal decreases. Likewise with the diversion efficiency η: when it is easier to divert cash, more must be handed over to the agent so as to prevent her from doing so. In fact, $v(w)$ decreases in η on two accounts; first, more must be given to the agent to deter diversion. Second, increasing η directly increases

$$w_m = \eta\frac{\Delta\mu}{\lambda}.$$

In other words, it shifts the lower boundary and, therefore, it affects the value of $v(w)$ for any w. The intervention threshold w_m is likewise increasing in $\Delta\mu$ and decreasing in λ. Hence, inefficient termination is also triggered more frequently when $\Delta\mu$ increases, but less so when λ does so. This latter claim may be counterintuitive at first; however, one should bear in mind that losses only arise off the equilibrium path. A higher intensity λ induces a higher expected penalty λP_t for the agent (for a given P_t); thus, when λ increases, the said penalty may be lowered, which then reduces the likelihood of termination.

Parameters increasing (decreasing) the termination threshold w_m also increase (decrease) the barrier $\widetilde{w}$. The effect of changes in the parameter η are clear, as it acts like σ. For the remaining ones, increasing w_m decreases $v(w)$ for all w, so that the point $\widetilde{w}$ characterized by $v'(\widetilde{w}) = 1$ and $v'(\widetilde{w}) = 0$ lies further out. The intuition is that when preserving the risk taking condition is more difficult, the principal is more reluctant to disburse cash and prefers increasing the agent's continuation value w.

5 Implementation

We suggest two practical implementations of the optimal contract. First, we implement it using standard securities as in DS, but with more instruments, to show departures from this benchmark. Our optimal contract maps remarkably well into real-life financial arrangements. We also compute security prices. The second implementation is one of financial regulation, which is also faithful to what is observed in practice. In both cases, downsizing may be interpreted as the disposal of assets with a concurrent reduction in debt. Given the simplifying assumption that downsizing generates no proceeds, reducing debt means writing it off. This write-off is internalized by the value function $v(w)$ and, therefore, in the firm's debt capacity. If there were positive proceeds from downsizing, the value of the firm $v(w)$ would be larger and it could also borrow more.[12]

5.1 Capital structure

If we assume that the principal represents the financiers of a firm who contract with a CEO (the agent), the contract may be implemented, as in DS, with a mix of equity, debt, a credit line and a dividend policy to which we must add some critical covenants: the agent must accept the downsizing process X^d. Given that

$$W_t = \int_t^\tau e^{-\rho(s-t)} dIs + e^{-\rho\tau} W_m,$$

the market value of equity is

$$Y_t := \int_t^\tau e^{-\rho(s-t)} \frac{dIs}{\eta} + e^{-\rho\tau} \frac{W_m}{\eta} \tag{5.1}$$

The contract then bestow a fraction ηY_t of the firm's equity to the agent, while the balance rests with diffuse shareholders.

When W reaches $\widetilde{W}$ dividends are paid out to equity holders. On the other hand, when W reaches W_m the *principal* retains the right to downsize. As a consequence, all other (debt) contracts are also contingent on size. We interpret this as a contractual covenant.[13]

[12] Downsizing proceeds ι would enter the value function (4.1) as ιdX_s, and differ from zero only when the control dX_s is activated.

[13] Alternatively, debt may be converted into an equity claim; some of that equity may be granted to the agent to restore incentive compatibility. This is especially appealing if considering re-investing in the firm later – see our discussion below. We thank an anonymous referee for this suggestion.

The credit line keeps track of the agent's size-adjusted continuation value, as in DS; therefore it is the device that activates downsizing. In this sense, it matches quite well the characteristics of a revolving credit. Debt amounts to

$$rD_t = X_t \left(\mu - \rho \frac{w_m}{\eta} - \rho c \right),$$

which also has to be flexible. The point of decreasing size is to increase the agent's relative stake ($w_t = W_t/X_t$); thus, one must be able to manage the size of the debt. The first term in the brackets corresponds to the gross cash flows, the second one is the liquidation value to the agent (when incentive compatibility can no longer be guaranteed) and the last one is the limit on the credit line. The latter is characterized by

$$\begin{aligned} M_t &= (X_t - \underline{X}) \left(c - \frac{w_t - w_m}{\eta} \right) \quad \text{and} \\ c &= \frac{\widetilde{w} - w_m}{\eta}. \end{aligned}$$

The first line is the value of the balance M_t as a function of the current size X_t and the continuation value w_t. At w_m the balance is $M_t = c\,(X_t - \underline{X})$: the credit line is fully drawn, which triggers downsizing and results in a lower limit, so a lower balance. When X_t reaches $\underline{X}$ the balance on the credit line is zero because the firm is no longer allowed to draw credit; it is terminated. The second line determines the limit on the credit line – in size-adjusted terms. Hence, the credit line enforces covenants in the form of

1. a (inverse) leverage ratio: $w_t \geq w_m$ and

2. a downsizing process X^d

that correspond to the risk taking constraint and define a resolution mechanism when it is breached.

Our next corollary illustrates the usefulness of the analysis carried out in Sections 3 and 4. Only some of the the results of Proposition 5 can be directly exploited to analyze the behavior of the securities used to implement the contract. This implies that the observation of securities data may be insufficient to draw conclusions regarding what drives their behavior.

Corollary 2 *The debt level D_t is increasing in the intensity λ and decreasing in the gain $\Delta\mu$; the credit line c is ambiguous. Comparative statics of D_t and c with respect to η are ambiguous.*

The ambiguous result on the credit limit c stems from the fact that both $\widetilde{w}$ and w_m enter its definition, and that both are affected by the parameters of interest. The debt level increases when the penalty for risk taking is more effective (when λ increases). In such a case the agent may hold less equity and the same project may be financed with more debt instead. The converse holds true when risk taking is more attractive ($\Delta\mu$ decreases). These empirical predictions are borne out by the data (Brown et al, 2008; Brown et al, 2012); these authors find that hedge funds that display a higher leverage (lower w_t) are associated with lower operational risk (a proxy for better governance) and a better history of performance.

The face value of debt D_t and the credit line $C(X_t) = c(X_t - \underline{X})$ may also be computed. Substituting for c in D_t one has

$$\begin{aligned} rD_t &= X_t \left(\mu - \rho \frac{w_m}{\eta} - \rho c \right) \\ &= X_t \left[\mu - \rho \left(\frac{\Delta\mu}{\lambda} + \frac{\widetilde{w} - w_m}{\eta} \right) \right] \\ &= \frac{X_t}{\eta} \left[\mu(\eta - 1) + rv(\widetilde{w}) \right], \end{aligned}$$

where the last line uses the boundary condition $rv(\widetilde{w}) + \rho\widetilde{w} = \mu$. Therefore,

$$D_t = \frac{X_t}{\eta} \left[\frac{\mu}{r}(\eta - 1) + v(\widetilde{w}) \right]$$

and in the special case where $\eta = 1$, we have $D_t = X_t v(\widetilde{w})$. The face value of debt increases with the size X_t.[14] Likewise for the credit line, we find

$$\begin{aligned} c &= \frac{\widetilde{w} - w_m}{\eta} \\ &= \frac{1}{\eta} \left(\frac{\mu - rv(\widetilde{w})}{\rho} \right) - \frac{\Delta\mu}{\lambda} \end{aligned}$$

and c also depends on the value of the firm: a valuable firm (say, with a very profitable project) needs little revolving credit but may carry a large debt. Finally we see that w_m has a natural interpretation as an equity requirement: the extent of any debt holding D_t or $C(X)$ is decreased by w_m.

5.2 Regulation contract

Suppose now the principal is a regulator and the agent a regulated financial institution. Let Y_t denote the market value of the firm's equity, of which a fraction η is awarded to the agent.

[14] And indeed, since $v(w)$ reflects the absence of downsizing proceeds, it curtails the firm's debt capacity.

Define $y_t := Y_t/X_t$; therefore $w_t = \eta y_t$, so in particular $w_m = \eta y_m$ – this is an equity ratio. The fraction $1-\eta$ may represent outside equity held by a diffuse investor base. Bank debt is made of demand-deposit accounts, hence it has to be kept flexible

$$rD_t = X_t\left(\mu - \rho\frac{w_m}{\eta}\right) = X_t\left(\mu - \rho y_m\right),$$

which asserts that cash flows remunerate the debt, net of the flow value of the agent's termination payoff.

Here there is no credit line that may be used as a substitute for observing the continuation value of the agent. However, since the firm's equity is linear in said continuation value, one may keep track of the value of the firm instead. The accounting identity of the balance sheet reads

$$D_0 + Y_0 = A_0 + G_0,$$

where A_0 denotes assets, i.e. loans, and G_0 is *goodwill.* The book-to-market ratio is defined as

$$B/M := \frac{Y_0}{Y_t} = \frac{A_0 + G_0 - D_0}{Y_t};$$

hence, a leverage ratio implementing the constraint $w_t \geq w_m$ amounts to

$$y_t \geq y_m, \quad \text{that is,} \quad y_0 \geq y_m M/B.$$

Using mark-to-market accounting we have that $B/M = 1$, which yields:

Proposition 6 *The regulatory contract may be implemented with a equity requirement*

$$y_0 \geq \frac{w_m}{\eta} = \frac{\Delta\mu}{\lambda}$$

based on the book-value of equity Y_0 defined in Expression (5.1)*, provided mark-to-market accounting is used, together with the downsizing process X_t^d and the dividend payout policy characterized by the process I.*

The value y_0 of the equity requirement (equity/ratio) based on the book-value of equity is an intervention threshold, together with a downsizing policy. Here the equity requirement is not designed as a buffer against losses, which are too large anyway. Rather, it is a preemptive threshold. The regulatory contract goes as far as prescribing

- that the agent (say, the CEO) holds enough equity;

- that mark-to-market accounting is used;
- a dividend policy; and
- that the agent does not control the size of the firm.

We note that the use of mark-to-market accounting and restrictions on dividend policies accord well with current practice. In Australia or the UK, for instance, the local regulators go as far as regulating some aspects of executive compensation in banks. More generally, the last point suggests that the allocation of control on the size of a firm is an important aspect or corporate governance.

6 Extensions

6.1 Upsizing: costly investment

In our main analysis we only consider downsizing, i.e. X is a non-increasing process. Clearly, however, firms also grow and, given that dS_t is linear in X_t, the principal may sometimes find it profitable to invest in the firm. In this section we look precisely into that option. To do so, we restrict the size X_t of the firm to be at most the initial size X_0. That level is not essential – it may be interpreted as some optimal size, as we also show below. For technical reasons, though, it is important that X remains bounded.

6.1.1 Proportional investment costs

We introduce the process $g = (g_t, t \geq 0)$ of investment rate, which is naturally (and strictly) bounded by the principal's discount rate: there exists $\overline{g} \in [0, r)$ such that $g_t \in [0, \overline{g}]$ for all $t \geq 0$. Intuitively speaking, if $g_s \equiv g > 0$ for $s \in (t, t+dt)$ then $X_{t+dt} = (1+g)X_t dt$. Investing carries the unitary cost k. With a positive investment cost, there is never simultaneous downsizing and investment; therefore, the process X remains of bounded variation and, in fact, continuous. Hence, the arguments regarding the absence of cross-variation terms in Section 4 remain valid. As a consequence, the HJB equation becomes

$$rV(X_t, W_t)dt = \mu X_t dt + \sup_{dX_t^d, dI_t, g_t} \Big\{ - dI_t + V_W(X_t, W_t)(\rho W_t dt - dI_t) - V_X(X_t, W_t)dX_t^d + \frac{\beta^2 X^2}{2} V_{WW} dt + g_t X_t \left(V_X(X_t, W_t) - k\right) dt \Big\}$$

where the last term represents the instantaneous benefit of increasing the venture's size. It is precisely through the factor $V_X(X_t, W_t) - k$ that we determine the investment region. Notice that investing has no bearing on the agent's incentives. The HJB equation becomes

$$\begin{aligned} rv(w_t)dt = \mu dt + \sup_{di_t, dx_t^d, g_t} \Big\{ &- di_t + v'(w_t)(\rho w_t dt - di_t) - (v(w_t) - w_t v'(w_t))dx_t^d \\ &+ \frac{\beta^2}{2} v''(w_t)dt + g_t \left(v(w_t) - w_t v'(w_t) - k\right) dt \Big\}, \end{aligned} \tag{6.1}$$

still subject to Constraints (3.4) and (4.11). As noted in BMRV, a necessary condition for investment to ever take place is

$$v(\widetilde{w}) + \widetilde{w} > k.^{15}$$

That is, the maximum of the (unit) social value of the firm must exceed the unit cost of investment. We suppose this holds. From the HJB Equation (6.1) investment only takes place if

$$v(w) - wv'(w) > k,$$

at which point the linearity of the investment return implies that g_t is as high as possible. Since $v(w_m) - w_m v'(w_m) = 0$ we have the following result:

Proposition 7 *Suppose the principal may invest in the firm. Then investment takes place only when the agent's continuation reaches the threshold w_i characterized by the condition*

$$v(w_i) - w_i v'(w_i) = k; \tag{6.2}$$

at that point investment takes place at the maximum rate. In other words, whenever $w_t \geq w_i$ we have $g_t = \overline{g}$. Furthermore, $w_m < w_i < \widetilde{w}$ and the value function is twice continuously differentiable at w_i.

The investment threshold w_i is strictly interior. It clearly cannot coincide with the downsizing barrier w_m as investing is costly. When the principal invests, he delays payments to the agent ($\widetilde{W} = X_t \widetilde{w}$ so increasing the size pushes out the payment barrier $\widetilde{W}$); it is as if everyone became more patient, as one observes in Equation (6.1). Hence, albeit costly, investment reduces the cost of delaying payments. However, it also moves the boundary W_m (also linearly) and so it opens the door to future downsizing. Thanks to the concavity of v, though, the investment threshold w_i is strictly bounded away from w_m. Investing requires a history of good news (high enough a continuation value) after any downsizing to ensure there will be no downsizing for some time to come. We can complement this result with

[15] The equality may be ignored.

Corollary 3 *The investment threshold w_i*

1. *increases with* (i) *the volatility σ,* (ii) *the diversion parameter η,* (iii) *the risk taking benefit $\Delta\mu$;*

2. *decreases with the intensity λ.*

The proof is immediate (and therefore omitted) as soon as one considers that increasing the threshold w_m is accompanied by a uniform decrease in v (see Equation (A25) in the proof of Proposition 5). Since the condition for investment (6.2) is relevant on the increasing range of v, it can only be satisfied at a higher level w_i. Worsening the moral hazard problem in the sense of higher diversion benefits η or risk taking benefit $\Delta\mu$ is deleterious for investment. That is, the principal "delays" investment for longer because the intervention threshold w_m is more demanding when the risk-taking incentives are worse and future downsizing is more likely.

6.1.2 Lumpy investment: fixed cost

Investing may entail frictions such as costly fund raising, which is documented to feature fixed costs, like the installation of fixed-size assets (machinery, office space...) in addition to the marginal cost k. In the sequel we denote the fixed investment cost by F and by ΔX the change in firm size following an investment decision.[16] For (X, W) given, an investment decision is made so as to maximize

$$\underbrace{V(X+\Delta X, W) - V(X, W)}_{\text{increment in principal's value}} - \underbrace{(k\Delta X + F)}_{\text{invest. costs}}.$$

Provided the mapping $\Delta X \mapsto V(X+\Delta X, W) - k\Delta X$ is quasi-concave, the first-order conditions for optimality is

$$V_X(X+\Delta X, W) = k,$$

which is akin to Expression (6.2). Assuming that $V_{XX} \neq 0$ always holds[17] we may invoke the Implicit Function Theorem (IFT). Then, there exists a function $\xi : \mathbb{R} \to \mathbb{R}$ such that

[16]Given the presence of the time-independent quantities F and ΔX, homogeneity of the value function $V(X, W)$ in X does not carry over. A thorough study of the principal's value function would require the use of new methods beyond the scope of the current paper; hence the presentation below is mathematically not as rigorous as the rest of the paper.

[17]In the prequel we had $V_{XX}(X, W) = -wv''(w) > 0$, so this is by no means far-fetched.

$V_X(\xi(W), W) = k$, and ΔX satisfies

$$\Delta X = \xi(W) - X. \tag{6.3}$$

The magnitude of any upsizing is contingent on the agent's current continuation utility, as before, and the firm's current size. As a consequence of Expression (6.3), upsizing is lumpy – precisely because of the fixed investment costs. Fixing $X \in (\underline{X}, X_0)$, the principal chooses to increase the firm's size if W satisfies

$$\nu(X, W) := V(\xi(W), W) - V(X, W) - k(\xi(W) - X) - F = 0. \tag{6.4}$$

Observe that the total derivative

$$\frac{d}{dW}\nu(X, W) = \underbrace{\Big[V_X(\xi(W), W) - k\Big]\xi'(W)}_{=0 \text{ from the def. of } \xi(W)} + \underbrace{\frac{\partial}{\partial W}\Big[V(\xi(W), W) - V(X, W)\Big]}_{V_X \xi'(W) > 0}.$$

Using the again the IFT, there exists $\omega : \mathbb{R} \to \mathbb{R}$ such that

$$\nu(X, \omega(X)) = 0.$$

Increasing the size X discretely may violate incentive compatibility. Hence we must require that, at the time of upsizing, W_t satisfies $W_t \geq w_m (X_t + \Delta X)$; in other words

$$\omega(X) \geq w_m \xi(W).$$

Finally, notice that the principal is still subject to the usual incentive Constraints (3.4) and (4.11). In summary,

Proposition 8 *Consider the fixed- and proportional-cost investment problem. The principal's value function V is the solution to the differential equation*

$$rV(X, W) = \mu X + \rho W V_W(X, W) + \frac{\beta^2 X^2}{2} V_{WW}(X, W)$$

on the domain delimited by the graph of $W = \omega(X)$ and the inequalities $W \leq \widetilde{W}(X)$ and $X \in [\underline{X}, X_0]$ with boundary conditions

$$V_W(X, \widetilde{W}(X)) = -1,\ V(\underline{X}, w_m \underline{X}) = \Pi \quad \text{and} \quad \nu(X, \omega(X)) = 0.$$

For each $X \in (\underline{X}, X_0)$ upsizing occurs whenever $W_t = \omega(X)$ and it is of size $\Delta X = \xi(W) - X$.

6.1.3 Endogenous bound on size

Hitherto we have worked under the assumption that, at any date $t \geq 0$, the scale of the firm is bounded above by its initial size X_0. When the investment cost is linear, as in Section 6.1.1 this bound is necessary to guarantee that the process X is of bounded variation, but its exogeneity is not particularly restrictive.

Suppose instead that the investment cost is a convex function C of the investment $g_t X_t dt$ with $C', C'' > 0$. Following the logic of Section 6.1.1, the HJB equation dictates that investment should take place whenever

$$g_t X_t V_X \geq C(g_t X_t), \text{ or } V_X \geq \frac{C(g_t X_t)}{g_t X_t},$$

where $V_X = v(w) - wv'(w)$ is independent of X_t. By the convexity of C, there exists a size X_t such that

$$\frac{C(g_t X_t)}{g_t X_t} > c,$$

for any $c < \infty$. Therefore, the investment condition

$$v(w_i) - w_i v'(w_i) = \frac{C(g_t X_t)}{g_t X_t}$$

eventually pins down a finite maximal size.

6.2 No liquidation at the boundary

Thus far we simplified the analysis by imposing a termination condition with an exogenous liquidation value at the lower boundary. In many cases, however, liquidation may not be desirable or even possible. For example, liquidating Lehman Brothers proved to be very disruptive and socially costly. Likewise, a large utility company (e.g. electricity transmission) or a clearinghouse can hardly be liquidated. In such cases, continuation of service prevents liquidation. Instead, the contract must be terminated, the agent replaced and a new contract with a new agent must be initialized. This is a costly endeavor that we model by introducing the fixed restructuring cost $K > 0$. Assuming that the principal would aim at restructuring the firm so as to have its scale back to X_0, the total benefit of restructuring reads

$$V(X_0, W_0^*) - V(\underline{X}, W_m) - (X_0 - \underline{X}) - K,$$

where W_0^* maximizes the mapping $W \mapsto V(X_0, W)$. This has to be contrasted with Π, the benefits of liquidation. In this case we would have that the boundary condition at

$(X, W_m(X))$ is

$$V(X, W_m(X)) = \max\Big\{\Pi X, V(X_0, W_0^*) - (X_0 - X) - K\Big\}. \tag{6.5}$$

The above condition introduces a fixed-point type problem, since V appears on both sides of Expression (6.5). If K is not too large then the principal opts for the restructuring option, whereas too-high restructuring costs would lead to termination. Since Equation (6.5) is a boundary condition, the HJB equation that describes the principal's value function remains, modulo the new condition at $\big\{(X, W_m(X)) | X \in [\underline{X}, X_0]\big\}$, unaltered. However, now the solution is parametrized by K and the corresponding payout threshold becomes $\widetilde{W}(X; K)$.

7 Discussion

Leverage and scale. The term $w_t = W_t/X_t$ may be interpreted in several ways. In the context of executive compensation, it may be understood as the stake(s) of the executive(s) in the firm. This is the "inside equity". If so, then this model suggests that said inside equity holdings must be significant enough compared to the wealth of the agent and it must increase in the size of the firm.[18]

An alternative interpretation, which we suggest when implementing the contract, is that of leverage. In such case W_t maps to the equity market-value of the firm and the model (*i*) provides foundations for leverage regulation and roots its origin squarely in the opportunity to speculate and (*ii*) it shows that a firm has incentives to gamble for resurrection when the market value of its equity is too low. In line with Jensen and Meckling (1976) we show that equity is *necessary* to prevent excessive risk-taking. We go beyond Jensen and Meckling (1976) in that we give provide micro-foundations to the agency problem and show that it can be solved with the appropriate contract.

The model also predicts that intervention is necessary along the equilibrium path, where it takes the form of downsizing. This maps cleanly into divestment by firms and was widely observed in the recent GFC, e.g. AIG selling assets like Hartford Steam Boiler, 21st Century Insurance, Transatlantic Re or ALICO to pay debt off, i.e. to reduce its leverage. Likewise, early in the GFC, Merrill Lynch sought to sell its commercial finance business to GE.

[18] In this model the agent's continuation value is her only wealth.

Proceeds from downsizing. For simplicity we suppose that downsize generates no proceeds; the implication is that debt is written off upon downsizing, which is bound to happen on the equilibrium path. Given that the value of the firm accounts for there being no proceeds and the face value of the debt D depends on the value of the firm $v(w)$, all is completely consistent. The assumption of zero proceeds then simply limits the borrowing capacity of the firm.

Liquidation proceeds are independent of the agent's incentives; therefore Equation (4.2) describing the evolution of the agent's continuation utility is unaltered. Let us now assume there are proportional liquidation proceeds of magnitude J that satisfy $J < \Pi$ (downsizing remains undesirable). Since now the proceeds of partial liquidation are JdX_t^d then $V_X(X_t, W_m(X_t)) = J$ and, from the assumption $J < \Pi$, $V_X(X_t, W_t) > J$ for $W_t > W_m(X_t)$. As a consequence, the process dX_t^d remains only active at $W = W_m(X_t)$ and the value function becomes

$$V(X, W) = \sup_{\Xi \in IC} \mathbb{E}\left[\int_0^\tau e^{-rs}(dS_s^a - dI_s + JdX_s^d)\Big|X_0 = X, W_0 = W\right].$$

The derivation of the size-adjusted case does not significantly deviate from that is Section 4.2, but now the HJB equation (4.10) becomes

$$rv(w)dt = \mu dt + \sup_{di_t, dx_t^d}\big\{-di_t + v'(w)(\rho w_t dt - di_t) - (v(w) - wv'(w) - J)dx_t^d + \frac{\beta^2}{2}v''(w)dt\big\}.$$

Given that dx_t^d only carries mass on $\{w = w_m\}$, the differential equation (4.14) still holds on the open interval $(w_m, \widetilde{w})$. However, the boundary condition at $w = w_m$ now has to account for the downsizing proceeds. From the relation $v(w_m) - w_m v'(w_m) = J$ we have that

$$v'(w_m) = (\Pi - J)/w_m. \tag{7.1}$$

Proposition 2 remains valid as long as we substitute the Neumann boundary condition (7.1) for the previous one ($v(w_m) = \Pi$).

Monitoring. Monitoring is a standard remedy to moral hazard. In a dynamic model, Piskorski and Westerfield (2015) allow for stochastic monitoring of the diversion decision, where diversion is not socially wasteful. Here diversion should always be avoided as it is socially costly; hence monitoring is only relevant to the risk-taking decision. Then one has to be careful in regards to *what* should be monitored. Monitoring that somehow results in reducing the change $\Delta\mu$ in the drift is uniformly positive: it reduces w_m by curtailing the incentives to engage in risky activities. However, monitoring to reduce the probability λ of catastrophic losses is uniformly bad(!); it increases w_m and decreases both v and $\widetilde{w}$. It is a license to speculate: a large loss is even less likely. Hence, under the lens of system-wide risk management, it is better to reduce the magnitude of losses L than their frequency λ.

Rents. In this model we require $w_m > 0$ for incentive reasons even though the agent's outside option is 0, i.e. the agent earns rents. Observationally, this corresponds to providing executives with seemingly too generous incentive packages, especially if some cash payouts are necessary for subsistence. Likewise, it may appear that some banks are made out hold too much equity; here it is justified as the only means to deter risk taking. As we have noted earlier, this ex-post rent may be extracted ex-ante through a participation fee that may take the form of a concession fee or a buy-in cost as in a partnership.

8 Conclusion

This paper proposes a contracting model in continuous time with a scalable arithmetic Brownian process and the option to speculate. Speculation, or excessive risk taking, improves the drift of the cashflows but also introduces the risk of large losses governed by a Poisson process.

Incentive compatibility requires that the agent has a large-enough continuation value at all times. This justifies equity holdings as a contractual or regulatory instrument. Due to the scale effect, the aforementioned continuation value must exceed a threshold that is linear in the size of the project. This implies that size becomes an instrument for incentive compatibility. Satisfying the risk-taking condition requires downsizing along the equilibrium path, which induces a floating barrier. This justifies leverage regulation as in banking, for instance.

We implement the contract using a broad array of instruments, especially covenants that enable the use of the downsizing process. This suggests, importantly for corporate governance, that *who* controls size in organizations is an important question that seems to have been somewhat neglected so far.

Acknowledgments. We thank Jean-Charles Rochet and Quynh Anh Vo for helpful comments and suggestions, as well as participants at the 2015 Stanford Institute for Theoretical

Economics (Segment IV) for stimulating discussion and seminar participants at Melbourne, UNSW, Sydney, UTS, Monash, the FIRN Corporate Finance conference and the FIRN Financial Stability conference. The research leading to these results has received funding from the ERC (grant agreement 249415-RMAC) from NCCR FinRisk (project "Banking and Regulation") and from the Swiss Finance Institute (project "Systemic Risk and Dynamic Contract Theory"), and it is gratefully acknowledged.

Appendix

A Proofs

Proof of Proposition 1: In a nutshell the Martingale Representation Theorem (MRT) for jump-diffusion processes states that any process Y that is a martingale with respect to the filtration $\mathbb{A}$ generated by a Wiener process $B = (B_t, 0 < t)$ and a Poisson process $M = (M_t, 0 < t)$ with intensity θ can be written as

$$Y_t = y_0 - \int_0^t h_s \big[dM_s - \theta_s ds\big] + \int_0^t g_s dB_s,$$

where $h = \big(h_t, t \geq 0\big)$ and $g = \big(g_t, t \geq 0\big)$ are unique, $\mathbb{A}$-adapted processes in $\mathcal{L}^*$.[19] A thorough exposition of the MRT at hand can be found, for instance, in Applebaum (2009).

Given a contract Ξ and an action $a \in \mathcal{A}$, we define the agent's total utility at date t as

$$\widehat{\psi}_t^a(\Xi) := \mathbb{E}\left[\int_0^\tau e^{-\rho s} d\widehat{C}_s^a \Big| \mathcal{F}_t^a\right],$$

which is clearly a $\mathbb{F}^a$-martingale. From this point on we omit Ξ as an argument. Applying the MRT, there exist $\mathbb{F}^a$-adapted processes $P^a = \big(P_t^a, t \geq 0\big)$ and $\beta^a = \big(\beta_t^a, t \geq 0\big)$ such $\widehat{\psi}_t^a$ may be written as

$$\widehat{\psi}_t^a = \widehat{\psi}_0^a - \int_0^t e^{-\rho s} P_s^a \big[dN_s - \lambda(a_s)ds\big] + \int_0^t e^{-\rho s} \beta_s^a X_s dZ_s, \tag{A1}$$

where $e^{-\rho s} > 0$ is a scaling factor. Moreover, since $\int_0^t e^{-\rho s} d\widehat{C}_s^a$ is $\mathcal{F}_t^a$-measurable, $\widehat{\psi}_t^a$ can be rewritten as

$$\widehat{\psi}_t^a = \int_0^t e^{-\rho s} d\widehat{C}_s^a + e^{-\rho t} W_t^a. \tag{A2}$$

Equations (A1) and (A2) then imply

$$\widehat{\psi}_0^a - \int_0^t e^{-\rho s} P_s^a \big[dN_s - \lambda(a_s)ds\big] + \int_0^t e^{-\rho s} \beta_s^a X_s dZ_s = e^{-\rho t} W_t^a + \int_0^t e^{-\rho s} d\widehat{C}_s^a. \tag{A3}$$

With these preliminaries we can proceed with the proof.

We deal first with diversion, in the spirit of DS, Sannikov (2008) and He (2009). Using Expression (2.2), we may rewrite the dynamics of W^a as

$$dW_t^a = \rho W_t^a dt + \frac{\beta_t^a}{\sigma}\Big(dX_t^a - \mu(a_t) X_t^a dt\Big) - d\widehat{C}_t - P_t^a \big[dN_t - \lambda dt\big]. \tag{A4}$$

[19]Processes in the $\mathcal{L}^*$ space are square-integrable with finite expected value: $\mathbb{E}\left[\int_0^t h_s^2 ds\right] < \infty$.

Furthermore, we have that

$$\frac{\beta_t^a}{\sigma}\Big(dS_t^a - \mu(a_t)X_t dt\Big) - d\widehat{C}_t^a = \left[\frac{\beta_t^a}{\sigma} - \eta\right]\left[dS_t^a - d\widehat{S}_t^a\right] + \frac{\beta_t^a}{\sigma}\big[d\widehat{S}_t^a - \mu(a_t)X_t dt\big]. \tag{A5}$$

On the one hand,

$$\mathbb{E}\left[P_t^a\big[dN_t - \lambda dt\big]\right] = 0.$$

On the other one, it holds that

$$\mathbb{E}\big[d\widehat{S}_t^a - \mu(a_t)X_t dt\big] \leq 0,$$

since the agent cannot over-report and will not report a jump. This implies that

$$\mathbb{E}\left[dW_t^a - \rho W_t^a dt\right] \geq 0 \Leftrightarrow \left[\frac{\beta_t^a}{\sigma} - \eta\right]\left[dS_t^a - d\widehat{S}_t^a\right] \geq 0 \tag{A6}$$

and because $dS_t^a - d\widehat{S}_t^a \geq 0$, the latter inequality requires $\frac{\beta_t^a}{\sigma} - \eta \geq 0$, which is our first constraint.

We now turn our attention to risk taking. Recall the definition of the strategies $a \equiv 1$ and $\tilde{a} := \big(a_s = 0, s < t; a_s = 1, s \geq t\big)$. The agent's total utility under strategy $\tilde{a}$ satisfies

$$\begin{aligned} \widehat{\psi}_t^{\tilde{a}} &= \widehat{\psi}_t^1 + \int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big) \\ &= \widehat{\psi}_0^1 - \int_0^t e^{-\rho s} P_s^a\big[dN_s(a_s) - \lambda(a_s)ds\big] + \int_0^t e^{-\rho s}\beta_s^a dZ_s + \int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big). \end{aligned} \tag{A7}$$

where the second line exploits the MRT and follows from the fact that $\widehat{\psi}_t^{\tilde{a}}$ is a martingale. Let $\mathbb{P}^{\tilde{a}}$ be the probability distribution induced by the strategy $\tilde{a}$ on the paths of $N(\tilde{a})$. Under $\mathbb{P}^{\tilde{a}}$, Equation (A7) becomes

$$\begin{aligned} \widehat{\psi}_t^{\tilde{a}} &= \widehat{\psi}_0^1 - \int_0^t e^{-\rho s} P_s^{\tilde{a}}\big[dN_s(\tilde{a}_s) - \lambda(\tilde{a}_s)ds\big] - \int_0^t e^{-\rho s}\lambda P_s^{\tilde{a}} ds \\ &+ \int_0^t e^{-\rho s}\beta_s^{\tilde{a}} dZ_s + \int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big). \end{aligned}$$

In order to guarantee that the strategy $a \equiv 1$ is preferable to $\tilde{a}$, it suffices to to make sure that the drift of the semimartingale $\widehat{\psi}^{\tilde{a}}$ is negative, which holds if and only if

$$\int_0^\tau e^{-\rho s}\big(d\tilde{C}_s^{\tilde{a}} - d\tilde{C}_s^1\big) \leq \int_0^t e^{-\rho s}\lambda P_s^{\tilde{a}} ds. \tag{A8}$$

for an arbitrary time t. If the agent chooses to speculate he may divert $\eta\Delta\boldsymbol{\mu}X_t$ or he may report truthfully to earn a higher payment. In the first case Expression (A8) implies that

$$\eta\Delta\boldsymbol{\mu}X_t \leq \lambda P_t,$$

is sufficient, whereas in the second one we have the sufficient condition

$$\frac{\beta_t}{\sigma}\Delta\boldsymbol{\mu}X_t \leq \lambda P_t,$$

which is our second (risk taking) constraint. ■

Proof of Lemma 1: We have said that downsizing takes place when Constraint 3.5 is violated. One could think that, in principle, the level of the agent's continuation utility could hit $W_m(X)$ and immediately "bounce back up"; thus, there would be no need to downsize the firm at that point. This is in fact not the case. Indeed, away from $\widetilde{W}(X)$, the agent's continuation utility evolves like

$$dW_t = \rho W_t dt + \beta X_t dZ_t. \tag{A9}$$

Let us assume that for some date $\bar{t}$ it holds that $W_{\bar{t}} = w_m X_{\bar{t}}$ and that at that point there is no downsizing. Then, instantaneously the dynamics of W are

$$dW_t = w_m X(\rho dt + \beta dZ_t), \quad W_{\bar{t}} = w_m X \tag{A10}$$

Let us consider the auxiliary process defined via the equation

$$db_t = \rho dt + \beta dZ_t, \quad b_{\bar{t}} = 1 \tag{A11}$$

and define, for $\epsilon > 0$,

$$B(\epsilon) := \inf_{s\in[\bar{t},\epsilon]} \big\{ b_s \mid W_{\bar{t}} = 1 \big\}. \tag{A12}$$

Using the Cameron-Martin theorem we know there is an equivalent measure $\mathbb{Q}$ under which b is a standard Brownian Motion. Furthermore the infimum of a standard Brownian motion From Chesney et al. (2009), page 147, we have that for any $a \leq 1$,

$$\mathbb{Q}\big\{B(\epsilon) > a\big\} = \Phi\Big(\frac{-(a-1)+\rho\epsilon}{\beta\sqrt{\epsilon}}\Big) - e^{2(a-1)\rho}\Phi\Big(\frac{(a-1)+\rho\epsilon}{\beta\sqrt{\epsilon}}\Big), \tag{A13}$$

where Φ is the standard normal cumulative distribution function. Letting $a \to 1$ we obtain that for all $\epsilon > 0$

$$\mathbb{Q}\big\{B(\epsilon) > 1\big\} = 0,$$

which, since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, implies $\mathbb{P}\big\{B(\epsilon) > 1\big\} = 0$. Therefore, for all $\epsilon > 0$ it holds that

$$\mathbb{P}\Big\{\inf_{[\bar{t},\bar{t}+\epsilon]} \big\{W_s | W_{\bar{t}} = w_m X\big\} > w_m X\Big\} = 0,$$

which concludes the proof. ■

Proof of Lemma 2: Observe that, analogously to Eq. (10) in DS, the increments of the scale-process X are given by

$$dX_t^d = \frac{1}{w_m} \max\big\{w_m X_t - W_t, 0\big\}.$$

This means that

$$w_m dx_t^d = \max\big\{w_t - w_m, 0\big\},$$

which in turn implies that

$$w_m x_t^d = \max\Big\{0, \max_{0 \le s \le t}\big\{w_t - w_m\big\}\Big\}.$$

From Lemma 6.14 in Karatzas and Shreve (1991) we have that $w_m x^d = \big(w_m x_t^d, 0 \le t\big)$ is the unique continuous process such that, added to the process $\widetilde{w}$ whose dynamics are given by the equation

$$d\widetilde{w}_t = \rho \widetilde{w}_t dt + \sigma dZ_t$$

and defining w via $dw_t = d\widetilde{w}_t + w_m dx_t^d$, results in:

1. $w_t \ge w_m$ almost surely and
2. the measure $w_m dx^d$ only carries mass on the set $\big\{w = w_m\big\}$:

   $$\int_0^\tau \mathbb{I}_{\{w_s > w_m\}}(w_m dx_s^d) = 0,$$

   for any w-stopping time τ, i.e the non-decreasing process $-w_m x$ only increases on the set $\big\{w = w_m\big\}$.

In other words, if we consider the following problem on $[w_m, \tilde{w}]$: find three processes (w, H^*, G^*) that satisfy

$$\begin{aligned} & w_t = w_0 + \rho \int_0^t w_s ds + \beta \int_0^t dZ_s - H_t^* - G_t^*, \\ & w_m \le w_t \le \tilde{w},\ t \ge 0, \\ & \int_0^\infty \mathbf{1}_{\{w_t < \tilde{w}\}} dH_t^* = \int_0^\infty \mathbf{1}_{\{w_t > w_m\}} dG_t^* = 0, \end{aligned} \tag{A14}$$

then for all $t \ge 0$ we have $i_t \equiv H_t^*$ and $w_m x_t \equiv G_t^*$. The solution to the so-called Skorokhod Problem (A14) is discussed, for instance, in Karatzas and Shreve (1991).

We conclude that $-w_m x$ is the local time at level w_m of $\widetilde{w}$; thus, adding $w_m dx_t^d$ to its dynamics yields a the process that, by construction, will exhibit an instantaneous reflection at the level w_m, which concludes the proof. ■

Proof of Proposition 2: Any solution to the differential Equation (4.14) may be written as the sum of the particular solution $v \equiv \mu/r$ and one particular solution to the homogeneous equation

$$rh(w) = \rho w h'(w) + \frac{\beta^2}{2} h''(w) \tag{A15}$$

Let us denote by h_0 and h_1 the particular solutions to Equation (A15) that satisfy $h_0(w_m) = 1, h_1(w_m) = 0, h_0'(w_m) = 0$ and $h_1'(w_m) = 1$. Using these *basis functions* we may write

$$v(w) = \frac{\mu}{r} + a_0 h_0(w) + a_1 h_1(w), \;\; w \in [w_m, \widetilde{w}]$$

for some $\widetilde{w} \geq w_m$. In order to determine a_0 and a_1 we use the boundary conditions $v(w_m) = \pi := \frac{\Pi}{\underline{X}}$ and $v'(\widetilde{w}) = -1$:

$$\frac{\mu}{r} + a_0 h_0(w_m) + a_1 h_1(w_m) = \pi \Rightarrow a_0 = \pi - \frac{\mu}{r}$$

and

$$a_0 h_0'(\widetilde{w}) + a_1 h_1'(\widetilde{w}) = -1, \Rightarrow a_1 = -\frac{1}{h_1'(\widetilde{w})}\left[1 + \left(\pi - \frac{\mu}{r}\right) h_0'(\widetilde{w})\right].$$

Therefore

$$v(w) = v(w; \widetilde{w}) = \frac{\mu}{r} + \left(\pi - \frac{\mu}{r}\right) h_0(w) - \left[1 + \left(\pi - \frac{\mu}{r}\right) h_0'(\widetilde{w})\right] \frac{h_1(w)}{h_1'(\widetilde{w})}, \tag{A16}$$

which is indexed by $\widetilde{w}$, and satisfies the boundary $v(w_m) = \pi$. Next we need to show that for $w_m \geq 0$ given, there exists a unique $\widetilde{w} \geq w_m$ and a unique corresponding function $v(\cdot; \widetilde{w})$ such that $v''(\widetilde{w}; \widetilde{w}) = 0$. To this end we show that the function $h_1(\cdot)$ is strictly increasing for all $w \geq w_m$. Indeed, if this were not the case, there would exist some $\hat{w}$ such that $h_1'(\hat{w}) = 0$ and $h_1'(w) \leq 0, \;\; w \in (\hat{w}, \hat{w} + \epsilon)$ for some $\epsilon > 0$. In other words, $\widetilde{w}$ would be a local maximum of $h_1(\cdot)$; thus, $h_1''(\hat{w}) \leq 0$. From the latter and Equation (A15) we obtain that $h_1(\hat{w}) \leq 0$. However, by construction $h_1(\cdot)$ is strictly increasing on $[w_m, \hat{w})$, so that $h_1(\hat{w}) > h_1(w_m) = 0$. This is a contradiction so we must have $h_1'(w) > 0$ for all $w \geq w_m$.

Next we show that the $\widetilde{w}$ that satisfies $v''(\widetilde{w}; \widetilde{w}) = 0$ and the corresponding function $v(\cdot; \widetilde{w})$ are jointly unique. Let us define $\phi(w) := h_0(w) h_1'(w) - h_1(w) h_0'(w)$ and observe that $\phi(w_m) = 1$. Using the boundary condition $v'(\widetilde{w}) = -1$,

$$
\begin{aligned}
\frac{\beta^2}{2} v''(\widetilde{w}) &= rv(\widetilde{w}) + \rho\widetilde{w} - \mu \\
&= \rho\widetilde{w} + (r\pi - \mu)\left(\frac{h_0(\widetilde{w})h_1(\widetilde{w}) - h_1(\widetilde{w})h_0'(\widetilde{w})}{h_1'(\widetilde{w})}\right) - r\frac{h_1(\widetilde{w})}{h_1'(\widetilde{w})} \\
&= \rho\widetilde{w} + (r\pi - \mu)\frac{\phi(\widetilde{w})}{h_1'(\widetilde{w})} - r\frac{h_1(\widetilde{w})}{h_1'(\widetilde{w})} \qquad \text{(A17)}
\end{aligned}
$$

where the second line follows from substituting Equation (A16) and the third one from a simple rearrangement of terms. Now, the boundary-value problem

$$
\begin{aligned}
\phi'(w) &= h_0(w)h_1''(w) - h_1(w)h_0''(w) \\
&= \frac{2\rho w}{\beta^2}\left[h_1(w)h_0'(w) - h_1'(w)h_0(w)\right] \\
&= -\frac{2\rho w}{\beta^2}\phi(w) \qquad \text{(A18)}
\end{aligned}
$$

where the second line uses Equation (A15), together with the boundary condition $\phi(w_m) = 1$ can be solved in closed form:

$$
\phi(w) = \exp\left\{-\frac{\rho}{\beta^2}(w^2 - w_m^2)\right\}.
$$

Now multiply both sides of Equation (A17) times $h_1'(w)/\phi(w)$ and re-arrange to obtain

$$
\frac{\beta^2}{2} v''(w)\frac{h_1'(w)}{\phi(w)} = \rho w\frac{h_1'(w)}{\phi(w)} - r\frac{h_1(w)}{\phi(w)} + r\pi - \mu.
$$

We want to show that $v''(\widetilde{w}) = 0$, for which we need that the function

$$
\varphi(w) := \left[\rho w h_1'(w) - r h_1(w)\right]\frac{1}{\phi(w)}
$$

satisfies $\varphi(\widetilde{w}) = r\pi - \mu$. Observe that it must hold that $r\pi - \mu < 0$, otherwise there is no object in hiring the agent, and $\varphi(w_m) = \rho w_m > 0$. Hence, it is enough to show that $\varphi(\cdot)$ is strictly decreasing on $[w_m, \infty)$. Differentiating and using Equation (A15) we have

$$
\begin{aligned}
\varphi'(w) &= e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)}\left[\rho w h_1'' - r h_1' + \frac{2\rho w}{\beta^2}(\rho w h_1' - r h_1)\right] \\
&= -e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)} r h_1'(w) < 0,
\end{aligned}
$$

where the last inequality follows from the fact that $h_1' > 0$. Similarly

$$
\begin{aligned}
\varphi''(w) &= e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)}\left[-r h_1'' - \frac{2\rho w}{\beta^2} r h_1'\right] \\
&= -e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)} r\left[h_1'' + \frac{2\rho w}{\beta^2} h_1'\right] \\
&= -e^{\frac{\rho}{\beta^2}(w^2 - w_m^2)} r\frac{2}{\beta^2} h_1(w) < 0
\end{aligned}
$$

where the last line uses Equation (A15) again. Hence $\varphi(\cdot)$ is decreasing and strictly concave on $[w_m, \infty)$, so $\widetilde{w}$ is unique.

The fact that, when $w_m < \tilde{w}$, the mapping $w \mapsto v(w)$ is strictly concave is a result of DS. The condition $v''(\widetilde{w}) = 0$, which guarantees that v and be extended in $\mathcal{C}^2$-fashion to $[w_m, \infty)$ corresponds to the (optimality) super-contact condition in Dumas (1991). ■

Proof of Corollary 1: The result follows directly from the proof of Proposition 2, where it is shown there exists a unique $\widetilde{w} \geq w_m$ and a unique corresponding function $w(\cdot, \widetilde{w})$ that is a solution. ■

Proof of Proposition 3: Observe that from Lemma 1 we have that the liquidation event $\{w_t = w_m, \underline{x}_t = 1\}$ coincides with the event $\{W_t = w_m \underline{X}\}$. Observe that for all $t > 0$ it holds that $W_t < \widetilde{W}(X_0) = \tilde{w}$ and $X_t \geq \underline{X}$. Let define a family of processes via the equations

$$dW_t^{a,b} = adt + bdZ_t - dI_t, \quad W_0^{a,b} = \widetilde{W}_0$$

for $a, b > 0$ $\widetilde{W}_0 > w_m \underline{X}$ independent of a and b and I inducing reflection at the level $W^{a,b} = \tilde{w}$. The first visitation time of $W^{a,b}$ to the level $w_m \underline{X}$ is increasing in a decreasing in b. Therefore, we can bound τ above with the first visitation time

$$\tilde{\tau} = \inf \Big\{ t > 0 \Big| W_t^{\rho \tilde{w}, \beta \underline{X}} = w_m \underline{X} \Big\}.$$

In particular

$$\mathbb{P}\{\tau < t\} \leq \mathbb{P}\{\tilde{\tau} < t\}. \tag{A19}$$

From Proposition 3.1 in Hu et al. (2012) we know there exist sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{c_n\}_{n \in \mathbb{N}}$ of positive numbers such that

$$\mathbb{P}\{\tilde{\tau} < t\} = 1 - \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}.$$

Letting $t \to \infty$ yields, together with Equation (A19), the desired result. ■

Proof of Proposition 4: The function characterized by Equation (4.14) and the boundary conditions $v(w_m) = \Pi / \underline{X}$ and $v'(\tilde{w}) = -1$ is strictly concave on $[w_m, \tilde{w}]$ (see DS) so that $v''(w) < 0$. Therefore, the smaller the coefficient in from of the second-order term is, the larger (in the $\mathbb{L}^\infty$ sense) the solution to Equation (4.14) becomes. Hence β_t should be set as small as possible without violating the no-diversion constraint (3.4), i.e. $\beta_t \equiv \beta$.

The second claim follows from the fact that downsizing is costly to the principal and it is only done for incentive reasons. This results in the fact that it is optimal to set $P_t = w_m X_t$

in Expression (3.6), since

$$\mathbb{P}\{w_s \leq w_m\}$$

is decreasing in w_m. ■

Proof of Theorem 1: The existence and uniqueness of a solution Equation (4.3) together with the boundary conditions $V_W(X, \widetilde{W}(X)) = -1$ and $V(\underline{X}, W(\underline{X})) = \Pi\underline{X}$ follow from the homogeneity Property (4.9) and Proposition 2, as do the values from the payment and downsizing barriers.

We obtain that for all $X \in [\underline{X}, 1]$ the mapping $W \mapsto V(X, W)$ is concave from the simple observation that $V(X, W) = Xv(W/X)$ and that the mapping $W \mapsto Xv(X/W)$ is concave, since $v(\cdot)$ is a concave function.

To show that, for any $W \in [w_m\underline{X}, \widetilde{W}(X)]$, the mapping $X \mapsto V(X, W)$ is increasing we compute the total derivative

$$\frac{d}{dX}V(X, W) = \frac{d}{dX}Xv(W/X) = v(W/X) - \frac{W}{X}v'(W/X) = v(w) - wv'(w),$$

where $w = W/X$. It follows from the concavity of $v(\cdot)$ that the mapping $w \mapsto v(w) - wv'(w)$ is increasing for all $w \geq 0$. Given that $v(w_m) - w_m v'(w_m) = \Pi - w_m v'(w_m) = 0$, it follows that the mapping $X \mapsto V(X, W)$ is increasing for all $W \in [w_m\underline{X}, \widetilde{W}(X)]$.

To finalize the proof we require a verification result that guarantees that the aforementioned contract is indeed an argmax of the principal's constrained optimization Problem (4.1). To this end, consider an arbitrary, incentive-compatible contract Ξ and initial size X and agent's continuation value W. Recall that the corresponding continuation-utility process W evolves according to the SDE

$$dW_t^{\Xi} = \rho W_t dt - dI_t + \beta X_t dZ_t - P_t^a[dN_t^a - \lambda(a_t)dt], \quad W_0 = W.$$

Recall that, under an incentive compatible contract (which determines the dynamics of W), the value function satisfies the following HJB equation:

$$\begin{aligned} rV(X_t, W_t)dt = \mu X_t dt + \sup_{dX_t, dI_t} \Big\{ &- dI_t + (\rho W_t dt - dI_t)V_W(X_t, W_t) \\ &+ V_X(X_t, W_t)dX_t + \frac{(\beta_t^a)^2 X_t^2}{2} V_{WW}(X_t, W_t)dt \Big\}, \end{aligned} \tag{A20}$$

Let us apply the generalized Itô's formula to $f(t, X, W) = e^{-\rho t}V(X, W)$, from which we obtain,

$$
\begin{aligned}
e^{-rt}V(X_t,W_t) = {} & V(X,W) \\
& + \int_0^t e^{-rs}\Big(- rV(X_s,W_s)ds + V_X(X_s,W_s)dX_s + V_{WW}(X_s,W_s)\frac{(\beta^a_s)^2X_s^2}{2}ds\Big) \\
& + \int_0^t e^{-rs}V_W(X_s,W_s)\Big(\rho W_s ds - dI_s + \beta X_s dZ_s - P^a_s[dN^a_s - \lambda(a_s)ds]\Big) \\
& + \sum_{s\in\Gamma_1\cup\Gamma_2}^{t} e^{-rs}\Big(V(X_s,W_s) - V(X_{s-},W_{s-}) - V_X(X_{s-},W_s)\beta^a_s dX_s\mathbf{1}_{\{s\in\Gamma_1\}} \\
& - V_X(X_s,W_{s-})dI_s\mathbf{1}_{\{s\in\Gamma_2\}}\Big) \\
& + \sum_{s\in\Gamma_3}^{t} e^{-rs}\Big(V(X_s,W_s) - V(X_s,W_{s-}) - V_X(X_s,W_{s-})P^a_s\Big),
\end{aligned}
\tag{A21}
$$

where $\Gamma_1,\ \Gamma_2$ and Γ_3 are the sets of discontinuities of the processes X, I and N^a until date t, respectively. We shall denote the "jump terms" corresponding to the summations in the last three lines of Expression (A21) by $J_t(X,I,N^a)$. Observe that we have assumed, without loss of generality, that X and W never jump simultaneously. Next, we define the auxiliary process

$$G_t := \int_0^t e^{-rs}(dS_s - dIs) + e^{-rt}V(X_t,W_t), \tag{A22}$$

which, making use of Expression (A21), can be rewritten as

$$
\begin{aligned}
G_t - V(X,W) = {} & \int_0^t e^{-rs}\Big(\mu X_s ds - dI_s + (\rho W_t ds - dI_s)V_W(X_s,W_s) + V_X(X_s,W_s)dX_s \\
& + \frac{(\beta^a_s)^2X_s^2}{2}V_{WW}(X_s,W_s)ds - rV(X_s,W_s)ds\Big) \\
& + \int_0^t e^{-rs}(\beta^a_s+\sigma)X_s dZ_s - \int_0^t e^{-rs}P^a_s[dN^a_s - \lambda(a_s)ds] \\
& - \int_0^t e^{-rs}X_s L dN^a_s + J_t(X,I,N^a).
\end{aligned}
$$

Analyzing the right-hand sides of the above equation, we have from Expression (A20) that the first two lines are non-positive, the third line is a martingale and the fourth line is also non-positive. Since, by construction $V(X,W) = G_0$, we have that the process $G = \big(G_t, t\geq 0\big)$ is a supermartingale. When the payout and downsizing processes correspond to the proposed optimal ones and $\beta^a\equiv\beta$, then the first two lines equal zero. Furthermore, from Proposition 1 we know that setting $P^a = \frac{\beta}{\sigma\lambda}\Delta\boldsymbol{\mu}X_t$ result in $\lambda(a)\equiv 0$. This, together with the fact that the proposed optimal downsizing and payout policies are continuous, implies that the corresponding $J(X,I,N^a)\equiv 0$. In other words, for the proposed optimal contract, the process G is a martingale. To finalize, we compute the principal's total value from an arbitrary, incentive-compatible contract:

$$\mathbb{E}\Big[\int_0^\tau e^{-rs}(dS_s - dI_s) + e^{-t\tau}\Pi\Big] =$$

$$= \mathbb{E}\Big[G_{t\wedge\tau} + \mathbf{1}_{\{t<\tau\}}\Big(\int_t^\tau e^{-rs}(dS_s - dI_s) + e^{-r\tau}\Pi - e^{-rt}V(X_t, W_t)\Big)\Big]$$
$$= \mathbb{E}\big[G_{t\wedge\tau}\big] + e^{-rt}\mathbb{E}\Big[\mathbf{1}_{\{t<\tau\}}\Big(\int_t^\tau e^{-r(t-s)}(dS_s - dI_s) + e^{-r(\tau-t)} - V(X_t, W_t)\Big)\Big]$$

From the supermartingale property of G we have $\mathbb{E}\big[G_{t\wedge\tau}\big] \leq G_0 = V(X, W)$. Furthermore, from Proposition 3 we know that $\tau < \infty$. This implies that, by letting $t \to \infty$, we have

$$\mathbb{E}\Big[\int_0^\tau e^{-rs}(dS_s - dI_s) + e^{-t\tau}\Pi\Big] \leq V(X, W). \tag{A23}$$

With the proposed optimal contract the above expression holds with equality. This concludes the theorem's proof. ∎

Proof of Proposition 5: Proceeding analogously as in Lemma 6 in DS, we have that, using θ as a dummy for one of the model's parameters, the following Feynman-Kac-type representation for $\partial v/\partial\theta$ holds:

$$\frac{\partial v_\theta(w)}{\partial\theta} = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\frac{\partial\mu}{\partial\theta} + \frac{\partial\rho}{\partial\theta}w_s v'(w_s) + \frac{1}{2}\frac{\partial\beta^2}{\partial\theta}v''(w_s)\right)ds + e^{-r\tau}\frac{\partial}{\partial\theta}\frac{\Pi}{\underline{X}}\Big|w_0 = w\right].$$

Recall that $\beta = \eta\sigma$. Then we have

$$\frac{\partial v(w)}{\partial\sigma} = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\eta^2\sigma v''(w)\right)ds\Big|w_0 = w\right] < 0,$$

by concavity of $v(\cdot)$. Similarly,

$$\frac{\partial v(w)}{\partial\eta} = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\eta\sigma^2 v''(w)\right)ds\Big|w_0 = w\right] < 0.$$

.

For the last set of results we use the first and second-order boundary conditions at $\tilde{w}$ to obtain the expression

$$rv(\widetilde{w}) + \rho\widetilde{w} = \mu$$

and compute its partial derivatives with respect to the generic parameter θ :

$$r\left(\frac{\partial v(\widetilde{w})}{\partial\theta} + v'(\widetilde{w})\frac{\partial\widetilde{w}}{\partial\theta}\right) + \rho\frac{\partial\widetilde{w}}{\partial\theta} = 0 \Rightarrow \frac{\partial\widetilde{w}}{\partial\theta} = -\frac{r}{\rho - r}\frac{\partial v(\widetilde{w})}{\partial\theta} \tag{A24}$$

The task is then to compute $\partial v(\widetilde{w})/\partial\theta$. To this end, we denote, as in DS

$$g_\tau(w) := \mathbb{E}\left[e^{-r\tau}\big|w_0 = w\right].$$

Given that the principal's profit remains the same if the agent's continuation value at liquidation increases by dw_m and liquidation value increases by $v'(w_m)dw_m$, the effect of a change in w_m on the principal's profit is

$$\frac{\partial}{\partial w_m} v(w) = -v'(w_m) g_\tau(w). \tag{A25}$$

From the risk taking constraint

$$w_m = \frac{\beta}{\sigma}\frac{\Delta\mu}{\lambda} = \eta\frac{\Delta\mu}{\lambda},$$

it is immediate that $dw_m/d\eta > 0$, $dw_m/d\Delta\mu > 0$ and $dw_m/d\lambda < 0$. Therefore

$$\frac{\partial v(\widetilde{w})}{\partial\lambda} = \frac{\partial}{\partial w_m} v(\widetilde{w})\frac{dw_m}{d\lambda} = -v'(w_m) g_\tau(\widetilde{w})\frac{dw_m}{d\lambda} < 0,$$

$$\frac{\partial v(\widetilde{w})}{\partial\eta} = \frac{\partial}{\partial w_m} v(\widetilde{w})\frac{dw_m}{d\eta} = -v'(w_m) g_\tau(\widetilde{w})\frac{dw_m}{d\eta} > 0$$

and analogously for $\partial v(\widetilde{w})/\partial\Delta\mu$. Inserting these expressions into Equation (A24) and recalling that $\rho > r$ we obtain

$$\frac{\partial\widetilde{w}}{\partial\lambda} = -\frac{r}{\rho - r}\frac{\partial v(\widetilde{w})}{\partial\lambda} > 0,$$

$$\frac{\partial\widetilde{w}}{\partial\eta} = -\frac{r}{\rho - r}\frac{\partial v(\widetilde{w})}{\partial\eta} < 0$$

and analogously for $\partial\widetilde{w}/\partial\Delta\mu$, which concludes the proof. ■

Proof of Corollary 2: Using the definition of the limit on the credit line c, rewrite

$$rD = \underline{X}\left(\mu - \rho\frac{\widetilde{w}}{\eta}\right)$$

then

$$\begin{aligned}
\frac{\partial rD_t}{\partial\Delta\mu} &= -\underline{X}\frac{\rho}{\eta}\frac{r}{\rho - r}\frac{\partial v}{\partial\Delta\mu} < 0\\
\frac{\partial rD_t}{\partial\lambda} &= -\underline{X}\frac{\rho}{\eta}\frac{r}{\rho - r}\frac{\partial v}{\partial\lambda} > 0\\
\frac{\partial D_t}{\partial\eta} &= \underbrace{-\underline{X}\frac{\rho}{\eta}\frac{\partial\widetilde{w}}{\partial\eta}}_{<0} + \underbrace{\underline{X}\frac{\rho}{\eta^2}\widetilde{w}}_{>0}
\end{aligned}$$

making use of the results of Proposition 5. Similar computations can be carried out for the credit limit c; they yield ambiguous results. ■

Proof of Proposition 7: We need to understand what value w_i of the continuation utility w is conducive of investment. Clearly there cannot be any investment at the boundary w_m: at that point the principal is forced to downsize to preserve incentive compatibility. So for any $k \geq 0$, $w_i > w_m$ – even when $k = 0$ because of the risk taking problem at w_m. To characterize w_i more precisely, note that the form of the HJB equation of the value function depends on whether there is investment; that is, on whether $w_t \geq w_i$. On the no-investment range (w_m, w_i) the function v_n satisfies

$$\begin{aligned} rv_n(w) = \mu dt + \sup_{di_t, dx_t, g_t} \big\{ &- di_t + v_n'(w)(\rho w_t dt - di_t) \\ &+ (v_n(w) - wv_n'(w))dx_t + \frac{\beta^2}{2} v_n''(w)dt \big\}, \end{aligned} \tag{A26}$$

while above w_i the function v_i follows

$$\begin{aligned} (r - g_t)v_i(w) = \mu dt - g_t k + \sup_{di_t, dx_t, g_t} \big\{ &- di_t + v_1'(w)[(\rho - g_t)w_t dt - di_t] \\ &+ (v_i(w) - wv_i'(w))dx_t + \frac{\beta^2}{2} v_i''(w)dt \big\}. \end{aligned} \tag{A27}$$

We need to show there exists a solution to the differential Equation (A27). As before, any such solution may be written as the sum of the particular solution $v \equiv \frac{\mu - gc}{r - g}$ and one particular solution to the homogeneous equation

$$rh(w) = \rho wh'(w) + \frac{\beta^2}{2} h''(w) \tag{A28}$$

as in the proof of Proposition 2. Let h_0 and h_1 denote the particular solutions to Equation A28 that satisfy $h_0(w_m) = 1, h_1(w_m) = 0, h_0'(w_m) = 0$ and $h_1'(w_m) = 1$. With these basis functions,

$$v_i(w) = \frac{\mu - gc}{r - g} + b_0 h_0(w) + b_1 h_1(w), \;\; w \in [w_m, \widetilde{w}]$$

for some $\widetilde{w} \geq w_m$. The boundary conditions $v_i(w_m) = \pi := \frac{\Pi}{X}$ implies

$$\frac{\mu - gc}{r - g} + b_0 h_0(w_m) + b_1 h_1(w_m) = \pi \Rightarrow b_0 = \pi - \frac{\mu - gc}{r - g}$$

and $v_i'(\widetilde{w}) = -1$ yields

$$b_0 h_0'(\widetilde{w}) + b_1 h_1'(\widetilde{w}) = -1, \Rightarrow b_1 = -\frac{1}{h_1'(\widetilde{w})} \left[1 + \big(\pi - \frac{\mu - gc}{r - g}\big) h_0'(\widetilde{w}) \right].$$

Therefore

$$v_i(w) = v_i(w; \widetilde{w}) = \frac{\mu - gc}{r - g} + \left(\pi - \frac{\mu - gc}{r - g} \right) h_0(w) - \left[1 + \big(\pi - \frac{\mu - gc}{r - g}\big) h_0'(\widetilde{w}) \right] \frac{h_1(w)}{h_1'(\widetilde{w})}, \tag{A29}$$

which is also indexed by $\widetilde{w}$, and satisfies the boundary $v_i(w_m) = \pi$. The rest proceeds as in the proof of Proposition 2.

Second we combine these solution at the investment threshold w_i. That is,

$$v(w) = \begin{cases} v_n(w), & \text{for } w_m < w \leq w_i\text{; and} \\ v_{ni}(w), & \text{for } w_i \leq w < \widetilde{w}. \end{cases}$$

with equality at w_i: the principal must be indifferent between investing and not, that is, $v_n(w_i) = v_i(w_i)$ and the function $v(w)$ must be $\mathcal{C}^2$. Re-arranging:

$$v(w_i) - w_i v'(w_i) = k \tag{A30}$$

More precisely, the function $v(w)$ must obey the termination condition at w_m, so $a_0 = \pi - \mu/r$ as in Proposition 2, and the boundary condition $v'(\widetilde{w}) = -1$, now given by $v_n(w)$. That is,

$$b_0 h_0'(\widetilde{w}) + b_1 h_1'(\widetilde{w}) = -1 \Rightarrow b_0 = -\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})}$$

We still have to determine b_1, a_1 and w_i, for which we may exploit continuity, smooth-pasting and super contact at w_i.

$$\begin{aligned} \frac{\mu}{r} + a_0 h_0(w_i) + a_1 h_1(w_i) &= \frac{\mu - gc}{r - g} + b_0 h_0(w_i) + b_1 h_1(w_i) \\ a_0 h_0'(w_i) + a_1 h_1'(w_i) &= b_0 h_0'(w_i) + b_1 h_1'(w_i) \\ a_0 h_0''(w_i) + a_1 h_1''(w_i) &= b_0 h_0''(w_i) + b_1 h_1''(w_i) \end{aligned}$$

$b_0 - a_0 = -\left[\frac{1+b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})} + \pi - \frac{\mu}{r}\right]$ and from the last two equations,

$$b_1 - a_1 = -(b_0 - a_0)\frac{h_0''(w_i)}{h_1''(w_i)} = \frac{h_0''(w_i)}{h_1''(w_i)}\left[\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})} + \pi - \frac{\mu}{r}\right]$$

Hence w_i is identified by

$$\frac{g(rc - \mu)}{r(r - g)} = -\left[\frac{1 + b_1 h_1'(\widetilde{w})}{h_0'(\widetilde{w})} + \pi - \frac{\mu}{r}\right]\left(h_0(w_i) - \frac{h_0''(w_i)}{h_1''(w_i)} h_1(w_i)\right)$$

To bound w_i from above we make use of the boundary condition at $\widetilde{w}$ and of the necessary condition for investment $v(\widetilde{w}) + \widetilde{w} > k$. Together they imply

$$w_i < \widetilde{w},$$

which is intuitive if we recall that an important benefit of investment is to delay payments to the agent. For this it must take place before that payment barrier is reached. ■

References

Admati, A. (2014), The compelling case for stronger and more effective leverage regulation in banking, Working Paper 159, Rock Center for Corporate Governance at Stanford University.

Applebaum, D. (2009), *Levy Processes and Stochastic Calculus (2nd Edition)*, number 116 *in* 'Studies in Advanced Mathematics', Cambridge University Press.

Biais, B., T. Mariotti, G. Plantin and J.-C. Rochet (2007), 'Dynamic security design: Convergence to continuous time and asset pricing implications', *Review of Economic Studies* **74**(2), 345–390.

Biais, B., T. Mariotti, J.-C. Rochet and S. Villeneuve (2010), 'Large risks, limited liability, and dynamic moral hazard', *Econometrica* **78**(1), 73–118.

Bonaccorsi di Patti, E., A. DIgnazio, M. Gallo and G. Micucci (2015), 'The role of leverage in firm solvency: Evidence from bank loans', *Italian Economic Journal* **1**(2), 253–286.

Brown, S., W. Goetzmann, B. L. and C. Schwarz (2008), 'References mandatory disclosure and operational risk: Evidence from hedge fund registration', *Journal of Finance* **63**(6), 2785–2815.

Brown, S., W. Goetzmann, B. L. and C. Schwarz (2012), 'Trust and delegation', *Journal of Financial Economics* **103**(2), 221–234.

Brunnermeier, M. and Y. Sannikov (2014), 'A macroeconomic model with a financial sector', *American Economic Review* **104**(2), 379–421.

Chesney, M., M. Jeanblanc and M. Yor (2009), *Mathematical Methods for Financial Markets*, Springer.

DeMarzo, P., D. Livdan and A. Tchistyi (2013), Risking other people's money: Gambling, limited liability and optimal incentives, mimeo, Stanford GSB.

DeMarzo, P. M. and Y. Sannikov (2006), 'Optimal security design and dynamic capital structure in a continuous–time agency model', *The Journal of Finance* **61**(6), 2681–2724.

Diamond, D. W. and R. G. Rajan (2000), 'A theory of bank capital', *The Journal of Finance* **55**(6), 2431–2465.

Duffie, D. (2010), *How Big Banks Fail.*, Princeton University Press, PUP - United States of America.

Dumas, B. (1991), 'Super contact and related optimality conditions', *Journal of Economic Dynamics and Control* **15**, 675–685.

Fishman, M. and P. DeMarzo (2007), 'Optimal long-term financial contracting', *Review of Financial Studies* **20**(6), 2079–2128.

Gabaix, X. and A. Landier (2008), 'Why has ceo pay increased so much?', *The Quarterly Journal of Economics* **123**(1), 49–100.

Geithner, T. (2014), *Stress Test*, Crown Publishing Group (NY).

He, Z. (2009), 'Optimal executive compensation when firm size follows geometric brownian motion', *Review of Financial Studies* **22**(2), 859–892.

Hu, Q., Y. Wang and X. Yang (2012), 'The hitting time density for a reflected brownian motion', *Computational Economics* **40**(1), 1–18.

Jensen, M. and W. Meckling (1976), 'Theory of the firm: Managerial behavior, agency cost and ownership structure', *Journal of Financial Economics* **3**(4), 305–360.

Karatzas, I. and S. Shreve (1991), *Brownian Motion and Stochastic Calculus*, Springer Verlag, New York.

Krishnamurthy, A. (2010), 'Ammplification mechanisms in liquidity crises', *American Economic Journal: Macroeconomics* **2**(3), 1–30.

Krishnamurthy, A. and Z. He (2012), 'A model of capital and crises', *Review of Economic Studies* **79**(2), 735–777.

Piskorski, T. and M. Westerfield (2015), Optimal dynamic contracts with moral hazard and costly monitoring, mimeo, Columbia University.

Revouz, D. and M. Yor (1999), *Continuous Martingales and Brownian Motion.*, Springer, Germany.

Rochet, J.-C. and G. Roger (2015), 'Risky utilities', *Economic Theory (forthcoming)* pp. 1–22.

Sannikov, Y. (2008), 'A continuous- time version of the principal: Agent problem', *The Review of Economic Studies* **75**(3), 957–984.

Spear, S. E. and S. Srivastava (1987), 'On repeated moral hazard with discounting', *Review of Economic Studies* **54**(4), 599–617.

Zhu, J. Y. (2013), 'Optimal contracts with shirking', *The Review of Economic Studies* **80**(2), 812–839.