Bargaining with Deadlines and Private Information

VERY PRELIMINARY AND INCOMPLETE

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Abstract

1 Introduction

Many negotiations have a preset deadline by which an agreement must be reached. For example, with a known trial date looming ahead, parties engage in pretrial negotiations. Before international summits, countries bargain over the terms of the accords to be signed at the summit. Broadcasters selling advertising space for some live event have until the event takes place to reach an agreement with the advertisers. Negotiations to renew labor contracts have until the expiration date of the current contract or the pre-set strike date if conflicts are to be avoided.

Financial considerations might also act as an effective deadline. Countries that have large debt repayments ahead of them bargain with international agencies such as the IMF for financing that would help them avoid default.[1] Private companies also face refinancing deadlines or deadlines to obtain financing in order to be able to invest in a given venture.

Finally, negotiations can be affected by regulatory deadlines. For example, to take advantage of the home buyer credit program, buyers and sellers of homes had to close their transactions by a given deadline to qualify for the subsidy.

Both the experimental literature and the empirical evidence have documented that a large fraction of the agreements are reached in the "eleventh hour". Cramton and Tracy (1992) study a sample of 5002 labor contract negotiations involving large bargaining units they claim a "clear 'deadline effect' exists in the data" since 31% of agreements are reached on the deadline.[2] So why is it that parties only reach an agreement on the morning of the trial date (see Spier 1992) or in the wee hours of the night before the labor contract expires?

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[1] For example, the current negotiations between Greece the EU and IMF are carried under the looming refinancing needs due to loans maturing. " Greece must refinance 54 billion euros in debt in 2010, with a crunch in the second quarter as more than 20 billion euros becomes due." http://www.reuters.com/article/idUSTRE63F2ZR20100416

[2] They interpret agreements reached even a day after the contract expiration as reached "on" the deadline.

If we had common knowledge that we are going to agree in period T and delay in reaching an agreement is costly, then we could surely reach an agreement at $T-1$ and save some of the costs of delay. Similarly, if we knew we were going to agree at $T-1$ we could reach an agreement at $T-2$ and save even more. Therefore, trade should be immediate. Indeed, this would be the correct logic with no asymmetric information.

When there is asymmetric information and we are simply bargaining over a pot of money (wages, price or settlement amounts) for trade to take place immediately, it must be that all of the informed types get the same terms. There is no way to tell them apart if they all must agree at the same time. This implies that the uniformed party is getting very little surplus since the terms she offers must be acceptable to the "lowest" type of the informed player. If the uniformed party could commit to a time interval between offers, then there is room to offer different terms to different informed types.[3] When delay is costly and more so for higher types, the willingness to bear the cost of waiting for the next offer allows the informed party to signal that his type is not too high.[4,5] Believing (correctly in equilibrium) that she is facing lower types, the uninformed party is then willing to make a lower offer and so on until the offer is accepted. When the uninformed party has no commitment power (i.e she can revise her offers down continuously), there are no deadlines and the (opportunity) cost of reaching an agreement is independent of the type she is dealing with, then despite the asymmetric information, trade would also take place in the first instant. Therefore, the terms of the agreement need to be acceptable to the lowest type of the informed party. Hence, leaving little or no surplus to the uninformed party. These results are known as the Coase conjecture and have been first formalized by Nancy Stokey (1981), Jeremy Bulow (1982), Drew Fudenberg, David K. Levine and Jean Tirole (1985) (henceforth FLT) and Faruk Gul, Hugo Sonnenschein and Robert Wilson (1986) (henceforth GSW).[6]

When there is a deadline, in the last instant the uninformed party gets the opportunity to effectively make a take it or leave it offer. If the horizon is not too long then, rather than settling immediately, the uninformed party could wait until the end (by making unacceptable offers) to make her take it or leave it offer. This would not be an equilibrium because high types would then be willing to trade at a slightly higher price than the last offer in earlier periods and, the seller would benefit by making such offers. As the uninformed party starts making offers that would be acceptable to some types and they get rejected, her belief about the type she is facing decreases. As these beliefs decrease the seller is willing to further decrease her offers, since what she expects to get with her last take it or leave it offer must also be decreasing. We show that in equilibrium the uninformed party sets the terms in a way that she is indifferent between reaching an agreement at the current terms or waiting until she gets to make her last take it or leave it offer. Furthermore, the prices must decrease at a rate such that the informed types that are supposed to trade in every particular instant cannot benefit from trading sooner or later.

The properties described above imply that in equilibrium there is smooth trade until the deadline. Once the deadline is reached and the last take it or leave it offer is made, there is a large probability it is accepted but there is also the possibility that the offer is not accepted. In that case, the players get their disagreement payoffs. Cramton and Tracy (1992) report that 57 percent of the labor negotiations in their sample end in disputes (strikes (10%), holdouts (47%) and lockouts(0.4%)) This seems very surprising since the failure to

[3] See Admati and Perry (1987) for a model in which the players can choose the length for which they commit to theirs offers.

[4] This is known as the skimming property.

[5] If the informed pary is allowed to make offers that too might lead to the updating of beliefs but we will consider a model where only the uninformed party is allowed to make offers.

[6] See Lawrence M. Ausubel, Peter Cramton, and Raymond J. Deneckere (2001) for a general survey of the literature.

reach an agreement could be very inefficient. For example, when the existing contract between the NHL and its players expired on September 15, 2004 without having reached an agreement, the entire season was cancelled. Indeed, we show that the larger the efficiency loss from the failure to reach an agreement, the smaller the likelihood that parties will fail to reach some agreement. This might be a possible explanation of why strikes in professional sports leagues are very infrequent relative to other activities.

Each of the bargaining environments provided as examples in this introduction have idiosyncratic and potentially important details that would affect the way negotiations are carried forward. We present a model that will abstract from many of those details, yet it is rich enough to capture the effect of deadlines and the consequence of not reaching an agreement on the bargaining outcome. The model is essentially a finite horizon version of Fuchs and Skrzypacz (2010) where if by time T the agents have not reach an agreement, they receive some disagreement payoff. This payoff could also be thought as the expected payoffs the agents would get from the continuation game that would start at $T+1$. For example, if the private information is revealed at $T+1$ the players then start bargaining with full information.

We solve for the equilibria by using backwards induction arguments. We show that when offers can be frequently revised, there is a unique equilibrium. Since working in the continuous time limit allows us to get an intuitive and tractable characterization, we focus our analysis and comparative static results on the limiting case when offers can be revised continuously.

Additional Literature review and further discussion of results to be added.

2 The Model

There is a seller and a buyer. The seller has an indivisible good (or asset) to sell. The buyer has a privately known type $v \in [0,1]$ that represents his value of the asset. Types are distributed according to a *c.d.f.* $F(v)$ which is assumed to be an analytic function and have full support. We denote its density by $f(v)$. The seller's value of the asset is zero.[7]

There is a total amount of time $T < \infty$ for the parties to try to reach an agreement. Time is divided in discrete periods of length $\Delta > 0$.[8] The timing within periods is as follows. In the beginning of the period an event arrives with probability $1 - e^{-\Delta\lambda}$ that ends the game (λ represents a Poisson arrival rate). If the event does not arrive, the seller makes a price offer p. The buyer then decides whether to accept this price or to reject it. If he accepts, the game ends. If he rejects, the game moves to the next period. If time T is reached the game ends.

A strategy for the seller for a given period $P\left(p^{t-1}, T-t; \Delta\right)$ is a mapping from the histories of rejected prices p^{t-1} and the remaining time $(T-t)$ to a current period price offer p_t. Similarly, in each period a strategy of the buyer of type v, which we denote $A_v\left(p^t, T-t; \Delta\right)$ is a mapping from the history of prices (rejected plus current) and the remaining time to a choice wether to accept or reject the current offer.

The payoffs are as follows. If the game ends in disagreement the buyer gets a disagreement payoff of

[7] The only non-trivial assumption about the range of v and the seller's value is that the seller's value is no lower than the lowest buyer's value - i.e. the "no-gap case". The rest is a normalization.

[8] For notational simplicity we will only consider values for Δ such that $\frac{T}{\Delta} \in \mathbb{N}$.

$e^{-rT}\beta v$ and seller gets $e^{-rT}\alpha v$ where r is a common discount rate. We assume $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.[9] For example, the case $\alpha = \beta = 0$ represents that the opportunity to trade disappears after the deadline.

If the game ends with the buyer accepting price p at time t, then the seller's payoff is $e^{-rt}p$ and the buyer's payoff is $e^{-rt}(v - p)$.[10] Finally, if the game ends with the event arriving at time t, then the payoffs are:

$$e^{-rt}W(v) \text{ for the buyer,}$$
$$e^{-rt}\Pi(v) \text{ for the seller.}$$

Let $V_A(k) = \int_0^k \Pi(v)\frac{f(v)}{F(k)}dv = E[\Pi(v)|v \leq k]$ denote the seller's expected payoff conditional on the arrival of the event and buyer type being distributed according to a truncated $F(v)$ over $v \in [0, k]$.

To justify the reduced-form payoffs $W(v)$ and $\Pi(v)$, consider the following examples. Let the arrival represent a second buyer arriving and suppose the seller runs an English auction upon arrival. If the buyers' valuations are i.i.d. then $\Pi(v) = \int_0^1 \min\{x, v\} dF(x)$ and $W(v) = \int_0^v F(x) dx$. The arrival could also represent the buyer's information becoming public and the beginning of a bargaining game with complete information.[11]

We assume:

Assumption 1

i) $\frac{e^{-\Delta r}(1-e^{-\Delta\lambda})}{1-e^{-\Delta(r+\lambda)}}(\Pi(v) + W(v)) < v$ for all $v > 0$.

ii) $W(v)$ is increasing and analytic, with $v - W(v)$ strictly increasing.

iii) $\Pi(v)$ is strictly increasing and analytic.

iv) $\Pi(0) = W(0) = 0$.

v) $v - \frac{1-F(v)}{f(v)}\frac{1-\beta}{1-\beta-\alpha}$ is strictly increasing

These assumptions are not too restrictive and are satisfied in many environments (including the examples above).

Conditions (i) to (iv) are only relevant if we want to consider the model with arrivals. The main results all go through without arrivals. Condition (i) is assumed so that from the point of view of the two parties involved in the negotiation delay is inefficient and if it were not for the information frictions there would be no delay in equilibrium. If it was violated delay would be a natural consequence of waiting for the total surplus to grow. (ii) simply states that higher types are more eager to trade immediately. This guarantees that the *skimming property* holds (see below). The properties of $\Pi(v)$ in (iii), in particular $\Pi'(v) > 0$, only play an important role in the equilibrium dynamics as $T \to \infty$- they are necessary in the limit for slow screening over types in equilibrium. (iv) is assumed to simplify the analysis since it saves us from solving for a fix point problem to find the relevant lowest type that trades. Condition (v) is important even in the absence of arrivals. We impose this condition in order to guarantee uniqueness of the optimal price to be offered by the seller in the last period. If $\alpha = 0$ this assumption simply says that $F(v)$ satisfies the downward-sloping marginal revenue condition.[12]

[9] If $\alpha + \beta = 1$ there would be no trade in the last period since there are no gains from trade. We could then think of the game upto $T - 1$ with $\alpha + \beta = e^{-r\Delta} < 1$.

[10] We focus on the case $\Delta \to 0$, i.e. no commitment power, so it is more convenient to count time in absolute terms rather than in periods. Period n corresponds to real time $t = n\Delta$.

[11] See Fuchs and Skrzypacz (2009) for additional examples.

[12] Myerson (1981) calls this condition increasing virtual valuation, or the regular case.

2.1 Equilibrium Definition

A complete strategy for the seller $\mathbf{P} = \left\{P\left(p^{t-1}, T-t; \Delta\right)\right\}_{t=0}^{t=T}$ determines the prices to be offered in every period after any possible price history.[13] As usual (in dynamic bargaining games), in any equilibrium the buyer types remaining after any history are a truncated sample of the original distribution (even if the seller deviates from the equilibrium prices). This is due to the *skimming property* which states that in any sequential equilibrium after any history of offered prices p^{t-1} and for any current offer p_t, there exists a cutoff type $\kappa(p_t, p^{t-1}, T-t; \Delta)$ such that buyers with valuations exceeding $\kappa(p_t, p^{t-1}, T-t; \Delta)$ accept the offer p_t and buyers with valuations less than $\kappa(p_t, p^{t-1}, T-t; \Delta)$ reject it. Best responses satisfy the skimming property because it is more costly for the high types to delay trade than it is for the low types. We can hence summarize the buyers' strategy by $\boldsymbol{\kappa} = \left\{\kappa(p_t, p^{t-1}, T-t; \Delta)\right\}_{t=0}^{t=T}$.

Definition 1 *A pair of strategies $(\mathbf{P}, \boldsymbol{\kappa})$ constitute a subgame perfect Nash equilibrium of the game if they are mutual best responses after every history of the game. That requires:*
1)That given the buyers' acceptance strategy every period (after every history) the seller chooses his current offer as to maximize his current value.
2) For every history, given the seller's future offers which would follow on the continuation equilibrium path induced by $(\mathbf{P}, \boldsymbol{\kappa})$, the buyers' acceptance choice of the current offer is optimal.

Proposition 1 *There exist a $\bar{\Delta} > 0$ such that for all games with period lengths Δ such that $\Delta < \bar{\Delta}$ there is a unique subgame perfect equilibrium.*

The proof can be found in the Appendix. It is based on a backwards induction argument. We show that, conditional on reaching T, the last offer the seller makes can only depend on his belief about the current cutoff type k. This offer is unique given Assumption $(1.v)$ which implies that the seller faces an increasing marginal revenue. Hence, the seller's last period price choice $P\left(p^{T-1}, 0; \Delta\right)$ as a function of the types remaining can be simply denoted by $p_T(k)$ which is unique, increasing, continuous and in fact analytic.[14] The induction step then establishes that if in the next period the equilibrium strategies in any equilibrium depend only on the time remaining and on k, are unique and $p_{t+1}(k)$ is increasing and analytic then:

a) $\kappa(p_t, p^{t-1}, T-t; \Delta)$ is continuous and strictly increasing and analytic in p_t. Furthermore, it is independent of p^{t-1} beyond its role in determining k.

b) The seller problem is concave and analytic, hence has a unique solution for a given k furthermore as a function of k this solution which we label $p_t(k)$ is analytic and increasing.[15]

Hence, in any equilibrium, if we arrive at time t with belief k, the continuation equilibrium is unique and depends only on t and k. Hence, by backwards induction, at the start of the game $t = 0$ and $k = 1$ there is a unique equilibrium. Moreover, this equilibrium is Markov on the current cutoff k and the time left $(T-t)$. These two variables define the payoff-relevant state of the game and equilibrium strategies are only conditioned on this state. The past history of price offers is only relevant to the extent that it induces the current state.

[13] At $t = 0$ there is no history of prior prices so $P(\emptyset, T; \Delta)$ should be simply interpretes as $P(T; \Delta)$. The same applies to the buyer's strategy.

[14] Establishing that $p_T(k)$ is analytic and that in the induction step this property is maintained is important given our proof strategy. This is so because at each induction step pricing choices depend on the derivative of the pricing best response from the previous step but then to be able to do the full argument by induction we need differentiability at every step.

[15] The induction requires that Δ is small enough to show that $p_t(k)$ is analytic.

Corollary 1 *The unique equilibrium for $\Delta < \bar{\Delta}$ is Markovian on the current cutoff type k and on the time remaining to reach an agreement $(T-t)$.*

The classic papers in dynamic bargaining (FLT, GSW, Lawrence Ausubel and Raymond Deneckere (1989), henceforth AD) which study the case with $T=\infty$ and $\lambda=0$ have shown existence of stationary equilibria (equilibria which are Markov in k) and that these equilibria are unique in the gap case. As shown by AD, in the no gap case there can also exist non-stationary equilibria that exhibit delay and a positive seller's payoff even as $\Delta \to 0$.[16] The fact that in the limit of our game, as we take $T \to \infty$ we would converge to a stationary equilibrium can provide some additional justification for focusing on the stationary equilibria rather than on the non-stationary ones.

For the rest of our analysis we will use the Markovian nature of the equilibrium to simplify our notation. Instead of $\kappa(p_t, p^{t-1}, T-t;\Delta)$ and $P\left(p^{t-1}, T-t;\Delta\right)$ we will use $\kappa(p,k,T-t;\Delta)$ and $P\left(k,T-t;\Delta\right)$ where k is the cutoff type induced by the history p^{t-1}.

Note that the pair $(\mathbf{P},\boldsymbol{\kappa})$ determines the future sequence of prices starting at any history described by k *and* $(T-t)$. The current equilibrium price is $p = P(k,T-t;\Delta)$, the next period price is $P(\kappa\left(p,k,T-t;\Delta\right), T-(t+\Delta);\Delta)$ and so on. They induce a decreasing step function $K\left(t;\Delta\right)$ which specifies the highest remaining type in equilibrium as a function of time (with $K(0;\Delta)=1$) and a decreasing step function $\Upsilon\left(v;\Delta\right)$ (with $\Upsilon\left(1;\Delta\right)=0$) which specifies the time at which each type v trades conditional on no arrival. For notational purposes, we let $k_+ = \kappa\left(P\left(k,T-t;\Delta\right), k, T-t;\Delta\right)$ denote highest remaining type at the beginning of the next period given current cutoff k and the strategies $(\mathbf{P},\boldsymbol{\kappa})$.

Let $V\left(k,T-t;\Delta\right)$ be the expected continuation payoff of the seller given a cutoff k with $(T-t)$ time left and the strategy pair $(\boldsymbol{\kappa},\mathbf{P})$. For $t<T$ we can express $V\left(k,T-t;\Delta\right)$ recursively as:

$$V\left(k,T-t;\Delta\right) = \left(1-e^{-\Delta\lambda}\right)V_A\left(k\right) + e^{-\Delta\lambda}\left[\begin{array}{c}\left(\frac{F(k)-F(k_+)}{F(k)}\right)P\left(k,T-t;\Delta\right)\\ +\frac{F(k_+)}{F(k)}e^{-\Delta r}V\left(k_+, T-(t+\Delta);\Delta\right)\end{array}\right] \tag{1}$$

and for $t=T$ we have:

$$V\left(k,0;\Delta\right) = \left(1-e^{-\Delta\lambda}\right)V_A\left(k\right) + e^{-\Delta\lambda}\left[\begin{array}{c}\left(\frac{F(k)-F(k_+)}{F(k)}\right)P\left(k,0;\Delta\right)\\ +\frac{F(k_+)}{F(k)}\int_0^{k_+}\alpha v\frac{f(v)}{F(k_+)}dv\end{array}\right] \tag{2}$$

For $t<T$, the seller's strategy is a best response to the buyer's strategy $\kappa(p,t;\Delta)$ if:

$$P\left(k,T-t;\Delta\right) \in \arg\max_p\left[\left(\frac{F\left(k\right)-F\left(\kappa\left(p,t;\Delta\right)\right)}{F\left(k\right)}\right)p + \frac{F\left(\kappa\left(p,t;\Delta\right)\right)}{F\left(k\right)}e^{-\Delta r}V\left(\kappa\left(p,t;\Delta\right), t+\Delta;\Delta\right)\right] \tag{3}$$

At $t=T$, the seller's strategy is a best response to the buyer's strategy $\kappa(p,T;\Delta)$ if:

$$P\left(k,0;\Delta\right) \in \arg\max_p\left[\left(\frac{F\left(k\right)-F\left(\kappa\left(p,T;\Delta\right)\right)}{F\left(k\right)}\right)p + \frac{F\left(\kappa\left(p,T;\Delta\right)\right)}{F\left(k\right)}\int_0^{\kappa(p,T;\Delta)}\alpha v\frac{f\left(v\right)}{F\left(\kappa\left(p,T;\Delta\right)\right)}dv\right] \tag{4}$$

These best response problems capture the seller's lack of commitment: in every period he chooses the price to maximize his current value (instead of committing to a whole sequence of prices at time 0).

[16] In contrast, the stationary equilibria all satisfy the Coase conjecture: as $\Delta \to 0$ the expected time to trade converges to zero and the profit of the seller converges to zero (and prices converge to seller's cost)

A necessary condition for the buyer's strategy κ to be a best response is that given the expected path of prices the cutoffs satisfy:

For $t < T$:

$$\underbrace{k_+ - P(k, T-t; \Delta)}_{\text{trade now}} = e^{-\Delta r}\left(1-e^{-\Delta\lambda}\right)\underbrace{W(k_+)}_{\text{arrival}} + e^{-\Delta(r+\lambda)}\underbrace{(k_+ - P(k_+, T-(t+\Delta); \Delta))}_{\text{trade tomorrow}} \tag{5}$$

For $t = T$:

$$\underbrace{k_+ - P(k, 0; \Delta)}_{\text{trade now}} = \underbrace{\beta k_+}_{\text{disagreement payoff}} \tag{6}$$

Given the buyer's best response at $t = T$ we can show that the seller's last offer $P(k, 0; \Delta)$ solves:

$$F\left(\frac{P(k,0;\Delta)}{1-\beta}\right) + \frac{P(k,0;\Delta)}{1-\beta}\frac{1-(\alpha+\beta)}{1-\beta}f\left(\frac{P(k,0;\Delta)}{1-\beta}\right) = F(k) \tag{7}$$

As shown in Proposition 1 there is a unique $P(k, 0; \Delta)$ that solves the equation above but a full characterization of the equilibrium is quite intractable. Looking at the limit of equilibria as we take $\Delta \to 0$ helps in this respect.

3 Limit of Equilibria as $\Delta \to 0$.

As shown in Fuchs and Skrzypacz (2010) even though the equilibrium strategies are hard to characterize for $\Delta >> 0$, in the limit as $\Delta \to 0$ we are able to provide a more tractable characterization of the limit of equilibria.

Proposition 2 *As* $\Delta \to 0$:
1)$V(k, T-t; \Delta) \to V(k, T-t) = \frac{\lambda}{r+\lambda}V_A(k) + e^{-(\lambda+r)(T-t)}\left(V(k,0) - \frac{\lambda}{r+\lambda}V_A(k)\right)$
2)$P(k, T-t; \Delta) \to P(k, T-t) = \frac{\lambda}{r+\lambda}\Pi(k) + e^{-(\lambda+r)(T-t)}\left(P(k,0) - \frac{\lambda}{r+\lambda}\Pi(k)\right)$
3)$K(t, T-t; \Delta) \to K(t, T-t)$ *which is defined for* $t < T$ *by the differential equation:*[17]

$$\begin{pmatrix} r\left(K(t,T-t) - P(K(t,T-t), T-t)\right) \\ +\lambda\left((K(t,T-t) - P(K(t,T-t), T-t)) - W(K(t,T-t))\right) \end{pmatrix} = \begin{pmatrix} -\frac{\partial P(K(t,T-t),T-t)}{\partial t} \\ -\frac{\partial P(K(t,T-t),T-t)}{\partial k}\dot{K}(t,T-t) \end{pmatrix}$$

with boundary condition:

$$K(0,T) = 1$$

and an atom of trade at time $t = T$.

$$m(T) = K(T,0) - \kappa\left(P(K(T,0),0), K(T,0), 0\right)$$

Beyond being more tractable, the limiting expressions are also quite intuitive. Consider first the value for the seller $V(k, T-t)$, his value is simply what he would get from just waiting and either collecting the expected discounted value he gets upon an arrival or from the final payoff from making the last offer. If

[17] Where $\dot{K} = \lim_{\Delta\to 0}\frac{k_+ - k}{\Delta}$.

the horizon were infinite this value would be $\frac{\lambda}{r+\lambda}V_A(k)$. Now, since there is finite horizon T with probability $e^{-(\lambda)(T-t)}$ there is no arrival and hence instead of getting $e^{-(r)(T-t)}\frac{\lambda}{r+\lambda}V_A(k)$ he gets the payoff $e^{-(r)(T-t)}V(k,T)$ of getting to make the last offer. This implies that the seller's value is driven down to his outside option of simply waiting to make his last offer. This is similar to the result in Fuchs and Skrzypacz (2010) in which we show that although there is delay, part of the Coase Conjecture holds in the sense that the seller cannot capture more than his reservation value.

The equilibrium prices display a no-regret property, the seller is indifferent between collecting $P(k,T-t)$ from type k today or getting what this type would contribute to his value either via an arrival or upon reaching the end of the game. The differential equation that defines $K(t,T-t)$ is more convoluted but it is still quite intuitive. It follows from taking the continuous time limit of the buyers best response condition given in (5). The RHS represents the change in price that results from the horizon getting closer and the seller updating his beliefs downwards. The LHS in turn captures the costs of delay for the buyer. The first source is simply from the delay in realizing the value from trade and the second source is that if there is an arrival he gets $W(v)$ rather than his value from trading.

In order to better understand the equilibrium properties it is useful to look both at the limit when $T\to\infty$ and the setting with no arrivals $\lambda=0$.

Corollary 2 *A) Letting first $\Delta\to 0$ and then taking $T\to\infty$ we get:*
1)$V(k,T-t)\to V(k)=\frac{\lambda}{r+\lambda}V_A(k)$
2)$P(k,T-t)\to P(k)=\frac{\lambda}{r+\lambda}\Pi(k)$
3)$K(t,T-t)\to K(t)$ *which is defined by the differential equation:*

$$-\dot{K}(t)=\frac{\lambda(K(t)-W(K(t))-\Pi(K(t,)))+rK(t)}{\frac{\lambda}{r+\lambda}\Pi'(k)}$$

with boundary condition:

$$K(0)=1$$

B) Letting $\lambda=0$ and $\Delta\to 0$ we get:
1)$V(k,T-t)=e^{-r(T-t)}V(k,0)$
2)$P(k,T-t)=e^{-r(T-t)}P(k,0)$
3)$K(t,T-t)$ *is defined by the differential equation:*

$$r(k-P(k,T-t))=-P_k(k,T-t)\dot{K}+P_t(k,T-t)$$

with boundary condition:

$$K(0,T)=1$$

and an atom of trade at time $t=T$.

$$m(T)=K(T,0)-\kappa(P(K(T,0),0),K(T,0),0)$$

Part A delivers the same characterization that we obtained in Fuchs and Skrzypacz (2010) where we directly work with $T=\infty$. Hence, as the model converges we also have convergence of the limit of equilibria.

In this case the seller's value is simply the expected discounted value he gets upon an arrival and the speed of trade is depends on how sensitive the seller arrival payoff is to the current buyer's type.[18] In contrast, when the horizon is finite and there are no arrivals, the seller's payoff is given entirely by what he can obtain from getting to make the last take it or leave at $t = T$. Prices at any given period can be calculated by figuring our how much the seller would charge in the last period if we were to fix the current cutoff $P(k, 0)$ and then simply discounting it by the time remaining. In this way the seller is indifferent in trading with a given type at time t for price $P(k, T - t)$ or waiting to trade with this type at time T.

Proposition 3
(i) (Delay): For all $0 < T < \infty$ the expected time to trade is strictly positive.
(ii) (Coase conjecture): When $\lambda = 0$ as $T \to \infty$, the expected time to trade and transaction prices converge to 0 for all types (i.e. $\Upsilon(v) \to 0$ and $P(k, T) \to 0$).

(i) Follows directly from our characterization, but the intuition is as follows: for there to be no delay in equilibrium the transaction prices for all types have to be close to zero, implying a seller's payoff close to zero, in particular, less than $e^{-r(T-t)}V(k, T) > 0$. But that leads to a contradiction since the seller can guarantee himself that by just waiting for the last period.

Part (ii) shows that our limit-equilibrium with $\lambda = 0$ converges to the equilibria in GSW and FLT: as we make the horizon very long (convergence of the model) trade takes place immediately and the buyer captures all the surplus (convergence of equilibrium).

It is worth noting that it is very different if we face a fixed deadline date versus a stochastic deadline that arrives at a Poisson rate after which the opportunity to trade disappears, even if from time zero perspective the expected time available to reach an agreement is the same. This difference arises because knowing that the game ends at T allows the seller to make a credible last take it or leave it offer at T. This possibility allows him to extract a positive amount of surplus. Instead, a stochastic loss of the opportunity of trade that arrives as a surprise is equivalent to having a higher discount rate $\hat{r} = r + \lambda$. If we instead allowed the seller to make a last take it or leave it offer when the stochastic deadline materialized the outcome would be much closer (but not the same) to what we would obtain with a deterministic deadline. The seller would prefer the stochastic deadline since his value would be $\frac{\lambda}{\lambda+r}V(1, 0)$ instead of $e^{-\frac{r}{\lambda}}V(1, 0)$ and $\frac{\lambda}{\lambda+r} > e^{-\frac{r}{\lambda}}$. This follows simply because the present value of a dollar is a convex function of the time at which it is generated. Therefore, a mean preserving spread increases value.

3.1 Applications and Comparative Statics

In this section we study the effects of the disagreement payoffs on the trade dynamics and division of surplus. We decompose the analysis in two parts. First we look at the effect of different degrees of inefficiency upon not reaching an agreement by changing $\alpha + \beta$ (lower values => more inefficiency) and keeping $\frac{\alpha}{\beta}$ constant. Then we consider the converse, for a certain amount of inefficiency how does the relative disagreement payoffs affect the equilibrium of the game.

For tractability we will abstract from the possibility of arrivals ($\lambda = 0$) and focus on the uniform case $F(v) = v$. This allows us to obtain simpler expression for all the objects of interest.

We start by using the fact that $\lambda = 0$ and Proposition 2 to obtain:

[18] See Fuchs and Skrzypacz (2010) for a detailed analysis and comparative statics results.

$$V(k,T-t) = e^{-r(T-t)}V(k,0)$$

$$P(k,T-t) = e^{-r(T-t)}P(k,0)$$

The buyer's best response problem for $t<T$ is then given by:

$$r(k-P(k,T-t)) = -P_k(k,T-t)\dot{K} + P_t(k,T-t) \tag{8}$$

the LHS is the cost of delaying trade (due to discounting) and the RHS is the benefit of waiting consisting of the reduction of price . The benefits and costs are evaluated at the current cutoff type.

Using $P(k,T-t)$ we found above in the buyer's indifference condition allows us to find $K(t)$.

$$-\dot{K} = \frac{P_t(k,T-t) + r(k-P(k,T-t))}{P_k(k,T-t)} = \frac{rk}{e^{-r(T-t)}P_k(k,0)} \tag{9}$$

The seller's last offer solves:

$$V(k,0) = \max_p p\frac{k-\frac{p}{1-\beta}}{k} + \frac{\alpha}{2k}\left(\frac{p}{1-\beta}\right)^2.$$

The solution for p is:

$$P(k,0) = \left(\frac{(1-\beta)^2}{[1-(\alpha+\beta)]+(1-\beta)}k\right) \tag{10}$$

and hence:

$$V(k,0) = \left(\frac{(1-\beta)^2}{[1-(\alpha+\beta)]+(1-\beta)}\right)\frac{k}{2}$$

Furthermore, taking the derivative with respect to k of (10) we can further simplify the expression for $\dot{K}$:

$$-\dot{K} = e^{r(T-t)}rk\frac{[1-(\alpha+\beta)]+(1-\beta)}{(1-\beta)^2}$$

Which together with the boundary condition $K(0,T-t)=1$ allows us to explicitly solve the differential equation:

$$K(t,T-t) = \exp\left(\frac{[1-(\alpha+\beta)]+(1-\beta)}{(1-\beta)^2}e^{rT}\left(e^{-rt}-1\right)\right)$$

In particular we have:

$$K(T,0) = \exp\left(\frac{[1-(\alpha+\beta)]+(1-\beta)}{(1-\beta)^2}\left(1-e^{Tr}\right)\right)$$

This allows us to figure out the last period price:

$$P(K(T,0),0) = \frac{(1-\beta)^2\exp\left(\frac{[1-(\alpha+\beta)]+(1-\beta)}{(1-\beta)^2}\left(1-e^{Tr}\right)\right)}{[1-(\alpha+\beta)]+(1-\beta)}$$

The buyer's indifference condition the last period implies:

$$\kappa(p,T) = \frac{P(K(T,0),0)}{1-\beta}$$

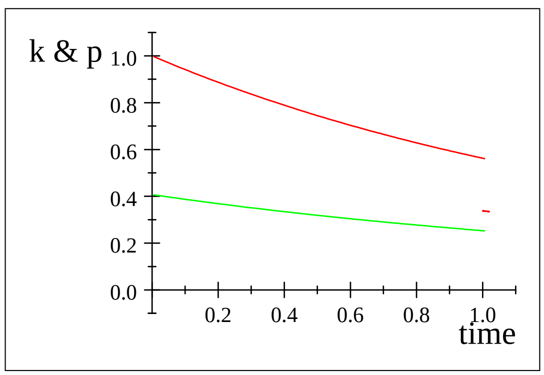
Which we can use to calculate the mass of buyers that trade just before the deadline:

$$m(T) = K(T,0) - \kappa(p,T)$$
$$m(T) = \frac{(1-(\alpha+\beta))}{[1-(\alpha+\beta)]+(1-\beta)} \exp\left(\frac{[1-(\alpha+\beta)]+(1-\beta)}{(1-\beta)^2}\left(1-e^{Tr}\right)\right)$$

Finally, the probability a agreement is reached either in the last offer or before is:

$$\Pr(agreement) = 1 - \kappa(p,T)$$
$$= 1 - \frac{(1-\beta)\exp\left(\frac{[1-(\alpha+\beta)]+(1-\beta)}{(1-\beta)^2}\left(1-e^{Tr}\right)\right)}{[1-(\alpha+\beta)]+(1-\beta)}$$

Below we plot an example of the path of cutoff types and the prices over time.[19] Note the atom of trade at T that aproximately includes types between 0.6 and 0.4.



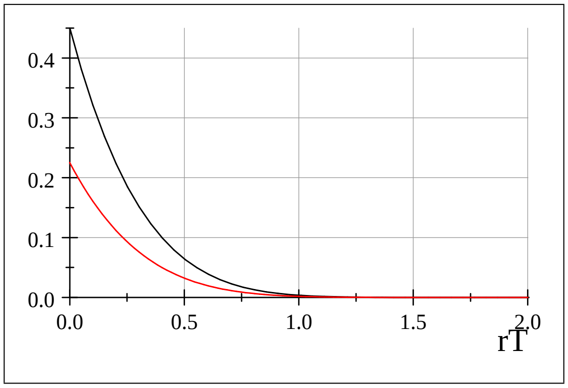
3.1.1 Comparative Statics:

Time to the Deadline T.

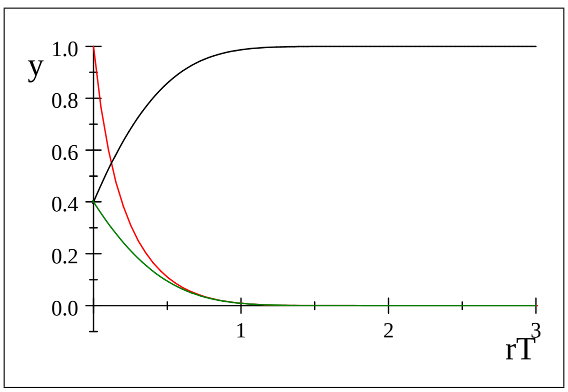
Proposition 4 *$V(k,T-t)$ and $P(k,T-t)$ are decreasing in T and go to 0 as $T \to \infty$. The probability of agreement is increasing in T and goes to 1 as $T \to \infty$. The last atom is $m(T)$ is decreasing in T and goes to 0 as $T \to \infty$. $K(t,T-t)$ is decreasing in T and goes to 0 as $T \to \infty$.*

Below we graph the seller's value at time zero (black) and the initial price demanded (red) for $\alpha = \beta = .25$. As we can see from the plot, we quickly converge to the Coasian results, initial prices and seller's value are very close to 0 for $rT > 1$.

[19] The parameters used are $T = 1$, $r = 10\%$, $\lambda = 0$, $\alpha = \beta = \frac{1}{4}$.



Below we graph the probability of agreement (black) and atom size (green) and percentage of trades that take place at the deadline (red) for $\alpha = \beta = .25$. The plot shows how although for short horizons most of the trade takes place in the last offer as the horizon increases this quickly changes. Furthermore, when rT is greater than one trade takes place with very high probability and it mostly takes place before the deadline.



Constant efficiency with varying disagreement positions.

Proposition 5 *Keeping $\alpha + \beta$ constant, $V(k, T-t)$ and $P(k, T-t)$ are increasing in α/β and for every $t > 0$ $K(t, T-t)$ will be higher with higher α/β. Additionally, the size of the atom at time T, $m(T)$, and the probability trade takes place on or before T is decreasing in α/β.*

Proof. First note that keeping $\alpha + \beta$ constant while increasing α/β leads to a higher value of $P(k, 0)$ for all k. Letting $x = [1 - (\alpha + \beta)]$

$$\frac{\partial P(k,0)}{\partial \beta} = \frac{\partial \left[\frac{(1-\beta)^2 \exp\left(\frac{[x]+(1-\beta)}{(1-\beta)^2} \left(1 - e^{Tr}\right) \right)}{[x]+(1-\beta)} \right]}{\partial \beta} < 0$$

That in turn implies $P(k, T-t)$ is higher as well. Similarly, a higher value of $P(k, 0)$ implies a higher value for $V(k, T-t)$. Regarding the size of the last atom of trade note that

$$\frac{\partial m(T)}{\partial \beta} > 0$$

therefore, the size of the atom is decreasing in α/β

Finally,

$$\frac{\partial \Pr(agreement)}{\partial \beta} > 0$$

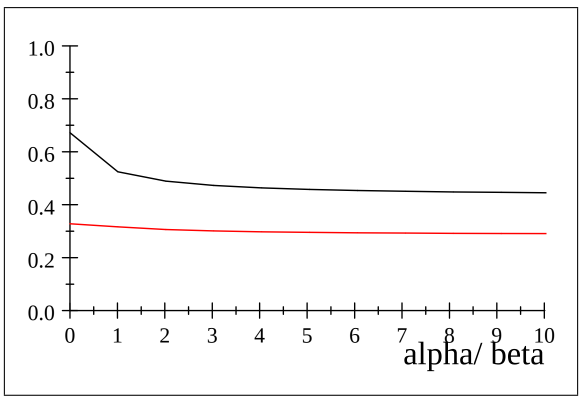
Hence, the probability of agreement is decreasing in α/β

$$\frac{\partial K(t, T-t)}{\partial \beta} < 0$$

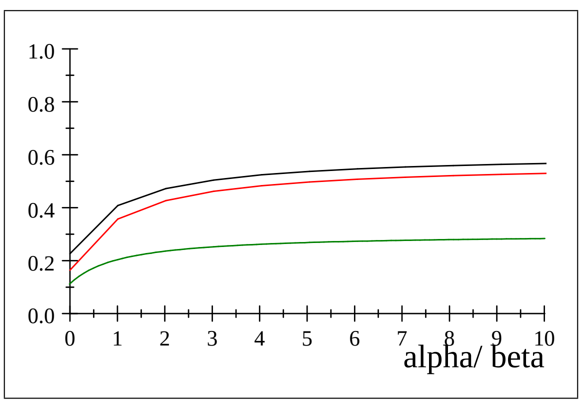
Hence, as we increase α/β trade takes place at a slower pace. That is for all t, $K(t, T-t)$ will be higher with higher α/β. ■

Summarizing, for a given amount of disagreement surplus. As the fraction of that surplus that goes to the buyer increases (higher β) the lower the prices the seller charges. There is also more trade before the deadline is reached. When the deadline is reached the atom of trade is also larger since the seller is decreasing the price both because of the decrease in her disagreement payoff and because of the increase in the seller's reservation value. Finally, more and faster trade implies that the surplus of the trading relationship will be higher. The decrease in the sellers value is (on average) more than offset by the increase in the buyer's value.

Below we graph the probability of agreement (black) and atom size (red) for $\alpha + \beta = .5$ $r = 10\%$ and $T = 1$.



In the figure below first period price is in black, last offer is in red and seller's value at time 0 in green.

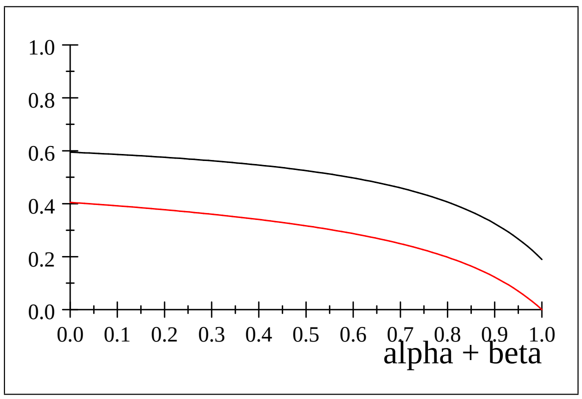


Equal disagreement payoffs with varying efficiency:

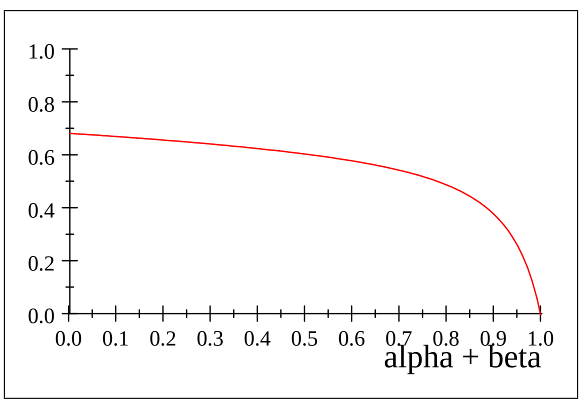
Proposition 6 *Letting $\alpha = \beta \leq 1/2$ we have that for $rT \leq 1$ the overall probability of trade and the amount of trade at T are decreasing in the disagreement payoffs. Furthermore, condition on that an agreement is reached the likelihood it will happen at T is also decreasing in $\alpha + \beta$. For $\alpha + \beta < 2/3$ $K(T, 0)$ is decreasing in the surplus and for $\alpha + \beta > 2/3$ it is increasing. $P(k, 0)$ is increasing in $\alpha + \beta$ if $\alpha + \beta > 2/3$ and decreasing otherwise.*

The bleaker the prospects if they do not reach an agreement the more likely they will reach one. The last atom of trade larger with worse diagreement payoffs. Furthermore, proportionally, more of the trade now takes place in the last instant when it is clear that not agreeing will be bad for both. The buyer will be more eager to accept a given offer. Also, if the seller were facing the same demand function he would ask for less. Now, since the seller is more eager to trade, he might even raise his demands but never to an extent that would reduce the likelihood of trade.

Below we graph the probability of agreement (black) and atom size (red) for $\alpha = \beta$ $r = 10\%$ and $T = 1$.



Percentage of Agreements at T.



4 Conclusions

To be added.

5 Appendix

Proof of Proposition 1. We will use a backward induction argument to establish this result. Consider first the last period. In period $t = T$, a buyer with type k will accept price p if and only if

$$k(1-\beta) \geq p$$

So we can think of the seller as choosing the optimal cutoff $k_T^*(k)$ rather than the optimal price $p_T^*(k)$ since $k_T^{*-1}(k_T) \equiv p_T^*(k)$.

Suppose the seller arrives to period T with a belief that the types are truncated below k.Then the seller's maximization problem at T is

$$\max_{k_T} [F(k) - F(k_T)] k_T (1-\beta) + \alpha \int_0^{k_T} v f(v) \, dv$$

The FOC can be written as:

$$k_T - \frac{F(k) - F(k_T)}{f(k_T)} \frac{1-\beta}{1-\beta-\alpha} = 0$$

Let $k_T^*(k)$ be the solution to the maximization problem. ■

Lemma 1 *Suppose* $v - \frac{1-F(v)}{f(v)} \frac{1-\beta}{1-\beta-\alpha}$ *is strictly increasing. Then* $k_T^*(k)$ *is unique, continuous and strictly increasing for all* k.

Proof. This is analogous to the proof of Lemma (1) in Section IV of Fuchs and Skrzypacz (2009) with the difference that now we have the extra term $\frac{1-\beta}{1-\beta-\alpha}$. In the case $\alpha = \beta = 0$ it is exactly the same.

The inverse function is:

$$k_T^{*-1}(k_T) = F^{-1}\left(F(k_T) + f(k_T) k_T \frac{1-\beta}{1-\beta-\alpha}\right)$$

Since we know that $k_T^*(k)$ is strictly increasing, so is the inverse function. Since F is analytic and strictly increasing, so are $k_T^*(k)$ and $p_T^*(k)$.

The Induction Step.

We show that if $p_{t+\Delta}^*(k)$ is a strictly increasing analytic function then $p_t^*(k)$ is a strictly increasing analytic function and so is $k_+(k,t)$, at least for Δ small enough. Where, $k_+ = \kappa(P(k, T-t; \Delta), k, T-t; \Delta)$ denotes the highest remaining type at the beginning of the next period given current cutoff k and the strategies $(\mathbf{P}, \boldsymbol{\kappa})$.

At time t given a current cutoff k the seller solves the following problem:

$$U(k, T-t) = \max_{k^* \in [0,k]} \left(\begin{array}{c} \left(1 - e^{-\lambda\Delta}\right) U_A(k) + e^{-\lambda\Delta} (F(k) - F(k^*)) \hat{p}_t(k) \\ + e^{-(r+\lambda)\Delta} U(k^*, T - (t+\Delta)) \end{array} \right) \quad (11)$$

where:

$$\hat{p}_t(k) = k\left(1 - e^{-(r+\lambda)\Delta}\right) - e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right) W(k) + e^{-(r+\lambda)\Delta} p_{t+\Delta}^*(k) \quad (12)$$

Since $p_{t+\Delta}^*(k), W(k)$ are analytic functions $\hat{p}_t(k)$ is an analytic function.

By the envelope theorem we have

$$\frac{\partial}{\partial k} U(k, T-t) = \left(1 - e^{-\lambda\Delta}\right) f(k) \Pi(k) + e^{-\lambda\Delta} f(k) p_t^*(k) > 0 \quad (13)$$

From (12) we can compute

$$\frac{\partial}{\partial k}\hat{p}_t(k) = \left(1 - e^{-(r+\lambda)\Delta}\right) - e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right)W'(k) + e^{-(r+\lambda)\Delta}\frac{\partial}{\partial k}p^*_{t+\Delta}(k)$$

Note that $p^*_t(k) \equiv \hat{p}_t(k_+(k,t))$ where $k_+(k,t)$ is the maximizer in (11) (which we later show to be unique for Δ small enough). Therefore, if $k_+(k,t)$ is analytic, then $p^*_t(k)$ is analytic as well.
To see this, the FOC of (11) is

$$-f(k^*)\hat{p}_t(k^*) + (F(k) - F(k^*))\frac{\partial}{\partial k}\hat{p}_t(k^*) + e^{-r\Delta}\frac{\partial}{\partial k}U(k^*, T - (t+\Delta)) = 0$$

Using the envelope theorem we have

$$e^{-r\Delta}\frac{\partial}{\partial k}U(k^*, T-(t+\Delta)) = e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right)f(k^*)\Pi(k^*) + e^{-(\lambda+r)\Delta}f(k^*)p^*_t(k^*)$$

and we have:

$$\hat{p}_t(k) = k\left(1 - e^{-(r+\lambda)\Delta}\right) - e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right)W(k) + e^{-(r+\lambda)\Delta}p^*_{t+\Delta}(k)$$

So the FOC becomes

$$\begin{aligned}&(F(k) - F(k^*))\begin{pmatrix} 1 - e^{-(r+\lambda)\Delta} - e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right)W'(k^*) \\ +e^{-(r+\lambda)\Delta}\frac{\partial}{\partial k}p^*_{t+\Delta}(k^*)\end{pmatrix} - \left(1 - e^{-(r+\lambda)\Delta}\right)f(k^*)G(k^*;\Delta) \\ = \;& 0\end{aligned}$$

where we denoted $G(k;\Delta) = k - \frac{e^{-r\Delta}\left(1-e^{-\lambda\Delta}\right)}{\left(1-e^{-(r+\lambda)\Delta}\right)}(W(k) + \Pi(k))$, which is is strictly positive and analytic.
Now, note that $H(k,t;\Delta) \equiv 1 - e^{-(r+\lambda)\Delta} - e^{-r\Delta}\left(1 - e^{-\lambda\Delta}\right)W'(k) + e^{-(r+\lambda)\Delta}\frac{\partial}{\partial k}p^*_{t+\Delta}(k) > 0$ uniformly for all Δ and k. As $\Delta \to 0$ it converges to

$$\frac{\partial}{\partial k}p^*_{t+\Delta}(k) > 0$$

Define $J(k,t;\Delta) = \frac{f(k)G(k;\Delta)}{H(k,t;\Delta)}$. For any Δ this is an analytic function, it is strictly positive and it has a bounded derivative, bounded in absolute value uniformly for all Δ and k (albeit it can be positive or negative).
As $\Delta \to 0$, $J(k,t;\Delta) \to \frac{f(k)\left(k - \frac{\lambda}{\lambda+r}(W(k)+\Pi(k))\right)}{\frac{\partial}{\partial k}p^*_{t+\Delta}(k)} > 0$.
The first order condition is then

$$(F(k) - F(k^*)) - \left(1 - e^{-(r+\lambda)\Delta}\right)J(k^*,t;\Delta) = 0 \tag{14}$$

For Δ small enough, the LHS is decreasing in k^* and hence it has a unique solution. Moreover, the solution is strictly increasing in k and the derivative is

$$\frac{\partial}{\partial k}k_+(k,t) = \frac{f(k)}{f(k^*) + \left(1 - e^{-(r+\lambda)\Delta}\right)\frac{\partial}{\partial k}J(k^*,t;\Delta)} \tag{15}$$

which is strictly positive for small enough Δ (uniformly in all Δ).
Since $k_+(k,t)$ is the inverse of

$$k_+^{-1}(k^*,t) = F^{-1}\left(F(k^*) + \left(1 - e^{-(r+\lambda)\Delta}\right)J(k^*,t;\Delta)\right)$$

it is also an analytic function. So

$$p_t^*(k) = \hat{p}_t(k_+(k,t))$$

is also an analytic function. It remains to prove that $\frac{\partial}{\partial k}p_t^*(k) > 0$ uniformly for all k, (for small enough Δ).

$$\begin{aligned}
\frac{\partial}{\partial k}p_t^*(k) &= \underbrace{\frac{\partial k_+(k,t)}{\partial k}}_{>0}\left(\frac{\partial}{\partial k}\hat{p}_t(k)\right) \\
&= \underbrace{\frac{\partial k_+(k,t)}{\partial k}}_{>0}\left(\left(1-e^{-(r+\lambda)\Delta}\right) - e^{-r\Delta}\left(1-e^{-\lambda\Delta}\right)W'(k) + e^{-(r+\lambda)\Delta}\frac{\partial}{\partial k}p_{t+\Delta}^*(k)\right) \\
&\geq \frac{\partial k_+(k,t)}{\partial k}\left(e^{-(r+\lambda)\Delta}\frac{\partial}{\partial k}p_{t+\Delta}^*(k)\right) \\
&\geq \frac{\partial k_+(k_t,t)}{\partial k} * \frac{\partial k_+(k_{t+\Delta},t+\Delta)}{\partial k} * .... * \frac{\partial k_+(k_{T-\Delta},T-\Delta)}{\partial k}e^{-(r+\lambda)(T-t)}\frac{\partial}{\partial k}p_T^*(k) \\
&\geq \left(\begin{array}{c} \frac{f(k_t)}{f(k_{t+\Delta})+\left(1-e^{-(r+\lambda)\Delta}\right)A} * \frac{f(k_{t+\Delta})}{f(k_{t+2\Delta})+\left(1-e^{-(r+\lambda)\Delta}\right)A} * \\ *... * \frac{f(k_{T-\Delta})}{f(k_T)+\left(1-e^{-(r+\lambda)\Delta}\right)A}e^{-(r+\lambda)(T-t)}\frac{\partial}{\partial k}p_T^*(k) \end{array}\right)
\end{aligned}$$

Now, the first term can be re-written as

$$\begin{aligned}
&\frac{f(k_t)}{f(k_{t+\Delta})+\left(1-e^{-(r+\lambda)\Delta}\right)A} * \frac{f(k_{t+\Delta})}{f(k_{t+2\Delta})+\left(1-e^{-(r+\lambda)\Delta}\right)A} * ... * \frac{f(k_{T-\Delta})}{f(k_T)+\left(1-e^{-(r+\lambda)\Delta}\right)A} \\
= &\frac{f(k_t)}{f(k_T)+\left(1-e^{-(r+\lambda)\Delta}\right)A} * \frac{f(k_{t+\Delta})}{f(k_{t+\Delta})+\left(1-e^{-(r+\lambda)\Delta}\right)A} * ... * \frac{f(k_{t+T-\Delta})}{f(k_{T-\Delta})+\left(1-e^{-(r+\lambda)\Delta}\right)A}
\end{aligned}$$

Now, since $f(v)$ is bounded from above and below, there exists $B > 0$ such that this term can be bounded from below by

$$\frac{f(k_t)}{f(k_T)+\left(1-e^{-(r+\lambda)\Delta}\right)A} * \left(\frac{B}{B+\left(1-e^{-(r+\lambda)\Delta}\right)A}\right)^{\frac{T-t}{\Delta}}$$

This in turn converges to

$$\lim_{\Delta\to 0^+}\left(\frac{B}{B+\left(1-e^{-(r+\lambda)\Delta}\right)A}\right)^{\frac{T-t}{\Delta}} = \exp\left(-A(r+\lambda)\frac{(T-t)}{B}\right) > 0$$

So indeed $\frac{\partial}{\partial k}p_t^*(k)$ is bounded uniformly away from zero for all k and Δ. Hence, establishing that $p_t^*(k)$ is a strictly increasing analytic function. This concludes our induction step since we had already shown $k_+(k,t)$ was also a strictly increasing analytic function.

Summarizing: we showed that $p_T^*(k)$ is strictly increasing and analytic and then showed the induction step that if $p_{t+\Delta}^*(k)$ is strictly increasing and analytic then for small enough Δ, $k_+(k,t)$ and $p_t^*(k)$ are strictly increasing and analytic. Hence, by induction, $k_+(k,t)$ and $p_t^*(k)$ are strictly increasing and analytic for all t. That implies that there is a unique equilibrium since in each period for every k the seller has a unique optimal price to offer and the buyers have an acceptance strategy that is consistent with this. ■

Proof of Proposition 2.

(1)We will work with $U(k,T-t) = V(k,T-t;\Delta)F(k)$ and $U_A(k) = V_A(k)F(k)$

$$U(k,T-t) = \max_{k^*\in[0,k]}\left(\begin{array}{c} e^{-\lambda\Delta}(F(k)-F(k^*))\hat{p}(k^*,T-t) \\ +e^{-(r+\lambda)\Delta}U(k^*,T-(t+\Delta)) + \left(1-e^{-\lambda\Delta}\right)U_A(k) \end{array}\right)$$

■

The speed-up constraint is

$$\begin{aligned} & \hat{p}\left(k_{t+\Delta},t\right)\left[F\left(k_t\right)-F\left(k_{t+\Delta}\right)\right]+e^{-\Delta r}U\left(k_{t+\Delta},t+\Delta\right) \\ \geq\ & \hat{p}\left(k_{t+2\Delta},T-t\right)\left[F\left(k_t\right)-F\left(k_{t+2\Delta}\right)\right]+e^{-\Delta r}U\left(k_{t+2\Delta},T-\left(t+\Delta\right)\right) \end{aligned}$$

Using definition of

$$e^{-\Delta r}U\left(k_{t+\Delta},T-\left(t+\Delta\right)\right)=\left(\begin{array}{c} e^{-(\lambda+r)\Delta}\left(F\left(k_{t+\Delta}\right)-F\left(k_{t+2\Delta}\right)\right)\hat{p}\left(k_{t+2\Delta},T-\left(t+\Delta\right)\right) \\ +e^{-\Delta r}e^{-(r+\lambda)\Delta}U\left(k_{t+2\Delta},T-\left(t+2\Delta\right)\right)+e^{-\Delta r}\left(1-e^{-\lambda\Delta}\right)U_A\left(k\right) \end{array}\right)$$

we have

$$\begin{aligned} & \left(\hat{p}\left(k_{t+\Delta},T-t\right)-\hat{p}\left(k_{t+2\Delta},T-t\right)\right)\left[F\left(k_t\right)-F\left(k_{t+\Delta}\right)\right] \\ & +\left(e^{-(\lambda+r)\Delta}\hat{p}\left(k_{t+2\Delta},T-\left(t+\Delta\right)\right)-\hat{p}\left(k_{t+2\Delta},T-t\right)\right)\left[F\left(k_{t+\Delta}\right)-F\left(k_{t+2\Delta}\right)\right]+e^{-\Delta r}\left(1-e^{-\lambda\Delta}\right)U_A\left(k\right) \\ \geq\ & e^{-r\Delta}\left(\left(1-e^{-(r+\lambda)\Delta}\right)U\left(k_{t+2\Delta},T-\left(t+\Delta\right)\right)+e^{-r\Delta}\left[U\left(k_{t+2\Delta},T-\left(t+\Delta\right)\right)-U\left(k_{t+2\Delta},T-\left(t+2\Delta\right)\right)\right]\right) \end{aligned}$$

Divide both sides by Δ and take the limit.

As we show below in Lemma 2 we have:

1) $\frac{\hat{p}(k_{t+\Delta},T-t)-\hat{p}(k_{t+2\Delta},T-t)}{\Delta}\rightarrow O\left(const\right)$

2) $\frac{(\hat{p}(k_{t+2\Delta},T-(t+\Delta))-\hat{p}(k_{t+2\Delta},T-t))}{\Delta}\rightarrow O\left(const\right)$

We use of these results to show the LHS converges to

$$\lambda U_A\left(k\right)$$

Similarly, the RHS converges to

$$\left(r+\lambda\right)U\left(k,T-t\right)-\frac{\partial U\left(k,T-t\right)}{\partial t}$$

So the inequality is

$$\lim_{\Delta\rightarrow 0}U\left(k,T-t\right)\leq\frac{\lambda}{\lambda+r}U_A\left(k\right)+\frac{1}{\lambda+r}\frac{\partial U\left(k,T-t\right)}{\partial t} \tag{16}$$

Since the seller has the option to just wait for the arrival or T we must have:

$$\lim_{\Delta\rightarrow 0}U\left(k,T-t\right)\geq\frac{\lambda}{r+\lambda}U_A\left(k\right)-e^{-(\lambda+r)(T-t)}\left(\frac{\lambda}{r+\lambda}U_A\left(k\right)-U\left(k,0\right)\right)\equiv U^*\left(k,T-t\right) \tag{17}$$

as $t\rightarrow T$, $U\left(k,T-t\right)\rightarrow U^*\left(k,T-t\right)$.

Suppose that there exists t' such that $U\left(k,T-t'\right)>U^*\left(k,T-t'\right)$. That implies that there exists $t''>t'$ such that

$$\frac{\partial U\left(k,T-t''\right)}{\partial t}<\frac{\partial U^*\left(k,T-t''\right)}{\partial t}=-\left(\lambda+r\right)e^{-(\lambda+r)\left(T-t''\right)}\left(\frac{\lambda}{r+\lambda}U_A\left(k\right)-U\left(k,0\right)\right)$$

Hence at that t'':

$$\begin{aligned} & \frac{\lambda}{\lambda+r}U_A\left(k\right)+\frac{1}{\lambda+r}\frac{\partial U\left(k,T-t''\right)}{\partial t} \\ <\ & \frac{\lambda}{\lambda+r}U_A\left(k\right)-e^{-(\lambda+r)\left(T-t''\right)}\left(\frac{\lambda}{r+\lambda}U_A\left(k\right)-U\left(k,0\right)\right)=U^*\left(k,T-t''\right) \end{aligned}$$

But that leads to a contradiction between (16) and (17). Hence, as $\Delta \to 0$, at all t and k,

$$U(k, T-t) \to U^*(k, T-t)$$

or equivalently:

$$V(k, T-t; \Delta) \to V(k, T-t) = \frac{\lambda}{r+\lambda} V_A(k) + e^{-(\lambda+r)(T-t)} \left(V(k, 0) - \frac{\lambda}{r+\lambda} V_A(k) \right)$$

(2) Next, we can see that prices converge. We have from the envelope theorem

$$\frac{\partial U(k, T-t)}{\partial k} \to f(k) p^*(k, T-t)$$

We also have from the above result that

$$\begin{aligned} \frac{\partial U(k, T-t)}{\partial k} \quad &\to \quad \frac{\partial U^*(k, T-t)}{\partial k} = \frac{\lambda}{r+\lambda} \frac{\partial U_A(k)}{\partial k} - e^{-(\lambda+r)(T-t)} \left(\frac{\lambda}{r+\lambda} \frac{\partial U_A(k)}{\partial k} - \frac{\partial U(k, 0)}{\partial k} \right) \\ &= \quad \left(1 - e^{-(\lambda+r)(T-t)}\right) \frac{\lambda}{r+\lambda} \Pi(k) f(k) + e^{-(\lambda+r)(T-t)} p^*(k, 0) f(k) \end{aligned}$$

Hence

$$p^*(k, T-t) \to \left(1 - e^{-(\lambda+r)(T-t)}\right) \frac{\lambda}{r+\lambda} \Pi(k) + e^{-(\lambda+r)(T-t)} p^*(k, 0)$$

This establishes the second part of the proposition:

$$P(k, T-t; \Delta) \to P(k, T-t) = \frac{\lambda}{r+\lambda} \Pi(k) + e^{-(\lambda+r)(T-t)} \left(P(k, 0) - \frac{\lambda}{r+\lambda} \Pi(k) \right)$$

It only remains to show (3) what $K(t, T-t; \Delta)$ converges to.

We get from the buyers' indifference condition

$$k_{t+\Delta} - p^*(k_t, T-t) = e^{-r\Delta} \left(\left(1 - e^{-\Delta\lambda}\right) W(k_{t+\Delta}) + e^{-\Delta\lambda} \left(k_{t+\Delta} - p^*(k_{t+\Delta}, T-(t+\Delta))\right) \right)$$

Subtract $e^{-\Delta(r+\lambda)} (k_{t+\Delta} - p^*(k, t))$ from both sides, divide by Δ and take the limit

$$\frac{\left(1 - e^{-\Delta(r+\lambda)}\right)}{\Delta} (k_{t+\Delta} - p^*(k_t, T-t)) = e^{-r\Delta} \left(\begin{array}{c} \frac{\left(1-e^{-\Delta\lambda}\right)}{\Delta} W(k_{t+\Delta}) \\ +e^{-\Delta\lambda} \left(\frac{p^*(k_t, T-t) - p^*(k_{t+\Delta}, T-(t+\Delta))}{\Delta} \right) \end{array} \right)$$

$$\begin{aligned} (r+\lambda)(K(t) - p^*(K(t), T-t)) \quad &= \quad \lambda W(K(t)) - \frac{d}{dt} p^*(K(t), T-t) \\ &= \quad \lambda W(K(t)) - \frac{\partial p^*(K(t), T-t)}{\partial t} - \frac{\partial p^*(K(t), T-t)}{\partial k} \dot{K}(t) \end{aligned}$$

$$\begin{aligned} p^*(k, T-t) \quad &= \quad \left(1 - e^{-(\lambda+r)(T-t)}\right) \frac{\lambda}{r+\lambda} \Pi(k) + e^{-(\lambda+r)(T-t)} p^*(k, 0) \\ \frac{\partial p^*(k, T-t)}{\partial k} \quad &= \quad \left(1 - e^{-(\lambda+r)(T-t)}\right) \frac{\lambda}{r+\lambda} \Pi'(k) + e^{-(\lambda+r)(T-t)} \frac{\partial p^*(k, 0)}{\partial k} \\ \frac{\partial p^*(k, T-t)}{\partial t} \quad &= \quad -e^{-(\lambda+r)(T-t)} \left[\lambda \Pi(k) - (\lambda+r) p^*(k, 0)\right] \end{aligned}$$

The differential equation simplifies to:

$$-\dot{K}(t, T-t) = \frac{(r+\lambda)(K(t, T-t) - p^*(K(t, T-t), T-t)) - \lambda W(K(t, T-t)) + \frac{\partial p^*(K(), T-t)}{\partial t}}{\frac{\partial p^*(K(t, T-t), T-t)}{\partial k}}$$

or as stated in the Proposition:

$$\left(\begin{array}{c} r\left(K\left(t,T-t\right)-P\left(K\left(t,T-t\right),T-t\right)\right) \\ +\lambda\left(\left(K\left(t,T-t\right)-P\left(K\left(t,T-t\right),T-t\right)\right)-W\left(K\left(t,T-t\right)\right)\right) \end{array}\right)=\left(\begin{array}{c} -\frac{\partial p^{*}\left(K\left(t,T-t\right),T-t\right)}{\partial t} \\ -\frac{\partial p^{*}\left(K\left(t,T-t\right),T-t\right)}{\partial k}\dot{K}\left(t,T-t\right) \end{array}\right)$$

with the boundary condition $K\left(0,T\right)=1$
So we have convergence of all: $V\left(k,T-t\right)$, $P\left(k,T-t\right)$,and $K\left(t,T-t\right)$.

In the proof above we make use of:

Lemma 2 *1)* $\frac{\hat{p}(k_{t+\Delta},t)-\hat{p}(k_{t+2\Delta},t)}{\Delta}\rightarrow O\left(const\right)$
2) $\frac{\left(\hat{p}(k_{t+2\Delta},t+\Delta)-\hat{p}(k_{t+2\Delta},t)\right)}{\Delta}\rightarrow O\left(const\right)$

Proof.

$$\frac{\hat{p}\left(k_{t+\Delta},T-t\right)-\hat{p}\left(k_{t+2\Delta},T-t\right)}{\Delta}=\frac{\hat{p}\left(k_{t+\Delta},T-t\right)-\hat{p}\left(k_{t+2\Delta},T-t\right)}{k_{t+\Delta}-k_{t+2\Delta}}\frac{k_{t+\Delta}-k_{t+2\Delta}}{\Delta}$$

$$k_{+}^{-1}\left(k^{*},t\right)=F^{-1}\left(F\left(k^{*}\right)+\left(1-e^{-(r+\lambda)\Delta}\right)J\left(k^{*},t;\Delta\right)\right)$$

$$\begin{aligned}\frac{k_{t+\Delta}-k_{t+2\Delta}}{\Delta} &= \frac{k_{+}^{-1}\left(k_{t+2\Delta},t+\Delta\right)-k_{t+2\Delta}}{\Delta} \\ &= \frac{F^{-1}\left(F\left(k_{t+2\Delta}\right)+\left(1-e^{-(r+\lambda)\Delta}\right)J\left(k_{t+2\Delta},t;\Delta\right)\right)-k_{t+2\Delta}}{\Delta}\end{aligned}$$

■

Since for any analytic g (and so is F^{-1})

$$g\left(x+O\left(\Delta\right)\right)=g\left(x\right)+O\left(\Delta\right)$$

we get that

$$\frac{k_{t+\Delta}-k_{t+2\Delta}}{\Delta}\rightarrow O\left(const\right)$$

We already have that $\frac{\hat{p}(k_{t+\Delta},T-t)-\hat{p}(k_{t+2\Delta},T-t)}{k_{t+\Delta}-k_{t+2\Delta}}\rightarrow O(const)$ so $\frac{\hat{p}(k_{t+\Delta},T-t)-\hat{p}(k_{t+2\Delta},T-t)}{\Delta}\rightarrow O\left(const\right)$
Second,

$$\frac{\left(\hat{p}\left(k_{t+2\Delta},T-(t+\Delta)\right)-\hat{p}\left(k_{t+2\Delta},T-t\right)\right)}{\Delta}=\left(\begin{array}{c} \frac{\left(\hat{p}(k_{t+2\Delta},T-(t+\Delta))-\hat{p}(k_{t+\Delta},T-t)\right)}{\Delta} \\ +\frac{\hat{p}(k_{t+\Delta},T-t)-\hat{p}(k_{t+2\Delta},T-t)}{k_{t+\Delta}-k_{t+2\Delta}}\frac{k_{t+\Delta}-k_{t+2\Delta}}{\Delta} \end{array}\right)$$

From the buyer problem, $\frac{\left(\hat{p}(k_{t+2\Delta},T-(t+\Delta))-\hat{p}(k_{t+\Delta},T-t)\right)}{\Delta}\rightarrow O(const)$. $\frac{\hat{p}(k_{t+\Delta},T-t)-\hat{p}(k_{t+2\Delta},T-t)}{k_{t+\Delta}-k_{t+2\Delta}}\rightarrow O\left(const\right)$ since this is just the derivative of $p^{*}\left(k,T-t\right)$. Finally, as we argued above, $\frac{k_{t+\Delta}-k_{t+2\Delta}}{\Delta}\rightarrow O\left(const\right)$, completing the proof.

6 References (to be completed)

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