

Systematic Risk*

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Abstract

The systematic risk of an asset represents the contribution of the asset to the total risk of a portfolio. In this paper we study systematic risk in a setting that allows risk to capture a variety of attributes such as high distribution moments and rare disasters. We offer two different approaches. First is an axiomatic approach in which we specify desirable properties of a systematic risk measure, resulting in a unique solution to the problem. Second is an equilibrium framework generalizing the Capital Asset Pricing Model (CAPM) to allow for a broad set of risk attributes. The equilibrium approach generalizes classic results including the two fund separation theorem, the efficiency of the market portfolio, and the security market line. Both approaches lead to the same result in which systematic risk is measured by a scaled version of the Aumann-Shapley (1974) diagonal formula. In the special case where the only risk aspect of interest is the variance, our systematic risk measure coincides with the traditional beta.

1 Introduction

Risk is a complex concept. The definition of risk and its implications have long been the subject of both academic and practical debate. This issue has gained even more prominence during the recent financial crisis, when markets and individual assets were hit by catastrophic events whose ex-ante probabilities were considered negligible. Indeed, these events demonstrate that “risk” accounts for much more than what is measured by the variance of the returns of an asset. High distribution moments, rare disasters, and downside risk are just some of the different aspects that may be of interest when measuring risk.

In this paper we allow “risk” to take a very general form. We then re-visit the classic question of how to “fairly” allocate total portfolio risk among different assets in the portfolio. Such a “fair” allocation should reflect the contribution of an asset to the risk of the portfolio, which is often termed “systematic risk.” Traditional measures of

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systematic risk focus on a narrow set of risk attributes. In particular, the most well-known and widely used measure of systematic risk is the “beta” of the asset, which is the slope from regressing the asset returns on portfolio returns (Sharpe (1964), Lintner (1965a,b), and Mossin (1966)). Beta is the contribution of an asset to the risk of the portfolio as measured by the variance of its return. It is a fundamental concept in financial economics which sets the foundations to all risk-return analysis as part of the Capital Asset Pricing Model (CAPM). However, the traditional beta ignores all aspects of risk other than the variance such as high distribution moments and rare disasters.

We offer two different approaches to the risk allocation problem. First is an axiomatic approach in which we specify desirable properties of a systematic risk measure which lead to a unique solution to the problem. Second, we study an equilibrium framework generalizing the traditional CAPM to allow for a broad set of risk attributes. The equilibrium approach allows us to generalize classic results such as the two fund separation theorem, the efficiency of the market portfolio, and the security market line. Both approaches lead to the same result in which systematic risk is measured by a scaled version of the Aumann-Shapley (1974) diagonal formula, which was developed as a tool to fairly allocate cost or surplus among players in cooperative games. In the special case where the only risk aspect of interest is the variance, our systematic risk measure coincides with the traditional CAPM beta.

We begin with a broad definition of what would constitute a measure of risk. We define a risk measure as any mapping from random variables to real numbers. That is, a risk measure is simply a summary statistic that encapsulates the randomness using just one number. The variance (or standard deviation) is obviously the most commonly used risk measure. However, many other risk measures have been proposed and used. For example, high distribution moments can account for skewness and tail risk, downside risk accounts for the variation in losses, and value-at-risk is a popular measure of disaster risk. Recently, Aumann and Serrano (2008) and Foster and Hart (2009) offered two appealing risk measures that account for all distribution moments and for disaster risk.¹ All of these measures fall under our wide umbrella of risk measures. Moreover, any linear combination of risk measures is itself a risk measure. Thus, one can easily create measures of risk that account for a number of dimensions of riskiness, assigning the required weight to each dimension.

Many cases of interest involve measuring the contribution of one asset to the risk of a portfolio of assets. For example, the asset pricing literature has tied the return of a financial asset to the return of the market through “beta,” which is a measure of the contribution of the asset to the variance of the market portfolio. Similarly, it may be desirable to estimate the contribution of banks and other financial institutions to the total market risk (known as systemic risk). Banks and other financial institutions may also find it useful to calculate the contribution of different assets on their balance sheet to the total risk of the institution, so that each asset or business unit could be “taxed” appropriately. All of these problems are essentially risk allocation problems

¹See Hart (2011) for a unified treatment of these two measures and Kadan and Liu (2013) for an analysis of the moment properties of these measures.

in which total risk should be allocated among the constituents of a portfolio.

We first tackle this problem from an axiomatic point of view. We state four economically plausible properties that risk allocation measures are expected to satisfy. These properties are:

- **Normalization.** The weighted average of the systematic risk across all assets in a portfolio is normalized to 1. This implies that assets with high contribution to total risk have systematic risk higher than 1, while assets with low contribution have systematic risk lower than 1.
- **Linearity.** If a risk measure is the sum of two other risk measures, then the systematic risk of each asset is a linear transformation of the systematic risks under the individual risk measures.
- **Proportionality.** We broaden the notion of “perfect correlation” to arbitrary risk measures. We then ask that if assets are perfectly correlated in the appropriate sense, then their systematic risks are proportionally related.
- **Monotonicity.** We broaden the notion of “positive correlation” to arbitrary risk measures. We then ask that if all assets in the portfolio are positively correlated in the appropriate sense, then their systematic risks are non-negative.

We show that there is a unique systematic risk measure that satisfies these four properties. This measure is given by a scaled version of the Aumann-Shapley (1974) diagonal formula. Essentially, this formula calculates for each asset the average of its marginal contributions to portfolios along a diagonal starting from the origin and ending at the portfolio of interest. In the common case where the risk measure is homogenous of some degree, our solution becomes very simple, assigning to each asset its marginal contribution to portfolio risk scaled by the weighted average of marginal contributions of all assets. The proof of the axiomatization result relies on a mapping between risk allocation problems to the cost allocation problems studied in Billera and Heath (1982).

Our systematic risk measure is easy to calculate for a variety of risk measures using simple calculus methods. Importantly, when risk is measured using “variance,” our measure boils down to the familiar beta. More generally, we show how our systematic risk measure can be calculated in closed form for various risk measures accounting for high distribution moments, rare disasters as well as other risk attributes.

Our next step is to show that our measure of systematic risk arises naturally in an equilibrium model generalizing the CAPM to a broad set of risk measures. The idea is simple. In the classic CAPM setting investors are assumed to have mean-variance preferences. That is, their utility is increasing in the expected payoff and decreasing in the variance of their payoffs. In our generalized setting we assume that investors have mean-risk preferences, where the term “risk” stands for a host of potential risk measures. We provide mild sufficient conditions under which these preferences are locally consistent with expected utility in the sense of Machina (1982).

We consider an exchange economy with a finite number of risky assets, one risk-free asset, and a finite number of investors with mean-risk preferences. As usual, in equilibrium each investor chooses a portfolio of assets from the set of efficient portfolios, minimizing risk for a given expected return. However, due to the generality of the risk measure, the geometry of this set is more complicated than in the case where risk is measured by the variance. Nevertheless, we establish sufficient conditions under which the solution to each investor’s problem satisfies Tobin’s (1958) two-fund money separation property. That is, each investor’s optimal portfolio of assets can be presented as a linear combination of the risk-free asset and a unique portfolio of risky assets. The sufficient conditions we propose consist of the following three properties of risk measures:

- Convexity. The risk of a portfolio of assets (with a positive weight assigned to each asset) is smaller than the corresponding weighted average risk of the constituent assets.
- Homogeneity. Scaling up a random investment by a factor $t > 0$ increases risk by a factor of t^k for some k .
- Risk-free property. Assets paying a constant amount are the least risky assets, and adding a risk-free asset to a risky asset does not affect total risk.

We demonstrate that many risk measures satisfy these sufficient conditions, where the variance is just one special case. A consequence of two-fund money separation is that the equilibrium market portfolio lies on the efficient frontier. Using this, and under a smoothness condition we establish a generalization of the classic security market line (SML) to a large class of risk measures. Specifically, in equilibrium, the expected return of each risky asset i satisfies

$$E(\tilde{z}_i) = r_f + \mathcal{B}_i (E(\tilde{z}^M) - r_f),$$

where \tilde{z}_i is the risky return of asset i , \tilde{z}^M is the risky return of the market portfolio, r_f is the risk-free rate, and \mathcal{B}_i is our measure of systematic risk. Thus, our measure of systematic risk emerges naturally in an equilibrium setting as a generalization of beta. In particular, in equilibrium, each asset’s expected return reflects its marginal contribution to the total risk (broadly defined) of the market portfolio.

Our paper contributes to several strands of the literature. First, the paper adds to the growing literature on risk measurement. This literature dates back to Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970) who extend the notion of riskiness beyond the “variance” framework by introducing stochastic dominance rules. Artzner, Delbaen, Eber, and Heath (1999) specify desirable properties of coherent risk measures. More recently Aumann and Serrano (2008), Foster and Hart (2009, 2013), and Hart (2011) came up with appealing risk measures that generalize conventional stochastic-dominance rules. Notably, all the risk measures discussed in this literature are idiosyncratic in nature. Our paper adds to this literature by specifying a method to calculate the systematic risk of an asset for

any given risk measure. This in turn allows us to study the fundamental risk-return trade-off associated with a risk measure.

Our paper also adds to the recent literature on the measurement of systemic risk, which is the risk that the entire economic system collapses. Adrian and Brunnermeier (2008) define the $\Delta CoVaR$ measure as the difference between the value-at-risk of the banking system conditional on the distress of a particular bank and the value-at-risk of the banking system given that the bank is solvent. Acharya, Pedersen, Philippon, and Richardson (2010) propose the Systemic Expected Shortfall measure (SES), which estimates the exposure of a particular bank in terms of under-capitalization to a systemic crisis. Our paper takes a general approach to the problem of estimating the contribution of one asset to the risk of a portfolio of assets. We rely on both an axiomatic approach and an equilibrium framework to provide an easy-to-calculate and intuitive measure that applies to a wide variety of risk measures, as well as in an array of contexts.

The paper also contributes to the growing literature on high distribution moments and disaster risk and their effect on prices. Kraus and Litzenberger (1976), Jean (1971), Kane (1982), and Harvey and Siddique (2000) argue that investors favor right-skewness of returns, and demonstrate the cross-sectional implications of this effect. In addition, Barro (2006, 2009), Gabaix (2008, 2012), Gourio (2012), Chen, Joslin, and Tran (2012), and Wachter (2013) study the aversion of investors to tail risk and rare disasters. Our paper adds to this literature by outlining a general measure of systematic risk that can capture the contribution of an asset to a range of market risk aspects such as high distribution moments, rare disasters, and downside risk. Our measure applies to homogeneous and non-homogeneous risk measures, and can be calculated easily when one needs to estimate the contribution of a particular asset to the risk of a portfolio. Our approach to equilibrium essentially follows a reduced form, where preferences are described through the aversion to broadly defined risk. It should be emphasized, however, that our approach is stylized and does not account for the dynamics of returns, cash flows and consumption as do modern consumption based asset pricing models (e.g., Bansal and Yaorn (2004) and Campbell and Cochrane (1999)). These models rely on the specification of a utility function (such as Epstein and Zin (1989) preferences or preferences reflecting past habits). One advantage of our approach is that it provides a very parsimonious and simple one factor model that can capture different aspects of risk in a manner that may lend itself naturally to empirical investigation.

Additionally, the paper adds to an extensive list of studies applying the Aumann-Shapley solution concept in different contexts. For example, Billera, Heath, and Raanan (1978) solve a telephone billing allocation problem, Samet, Tauman, and Zang (1984) solve a transportation costs allocation problem, and Powers (2007) studies the allocation of insurance risk. Billera, Heath, and Verrecchia (1981) use a related procedure to allocate production costs. Finally, Tarashev, Borio, and Tsatsaronis (2010) use the Shapley value (Shapley (1953), a discrete version of the Aumann-Shapley solution concept) to measure systemic risk. Our paper offers theoretical foundations for their practical approach.

The paper proceeds as follows. In Section 2 we define the notion of risk measures. Section 3 presents the axiomatic approach to the risk allocation problem. In Section 4 we study the equilibrium setup and offer a generalization of the CAPM. Section 5 concludes. Technical proofs are in Appendix I, and additional technical results are provided in Appendices II and III.

2 Risk Measures

Let (Ω, \mathcal{F}, P) be a probability space, where Ω is the state space, \mathcal{F} is the σ -algebra of events, and $P(\cdot)$ is a probability measure. As usual, a random variable is a measurable function from Ω to the reals. In the context of investments, we typically consider random variables representing the payoffs or the returns of financial assets. Thus, we often refer to random variables as “investments” or “random returns.” We generically denote random variables by \tilde{z} , which is a shorthand notation for $\tilde{z}(\omega) \forall \omega \in \Omega$. We restrict attention to random variables for which all moments exist. We denote the expected value of \tilde{z} by $E(\tilde{z})$ and its k^{th} central moment by $m_k(\tilde{z}) = E(\tilde{z} - E(\tilde{z}))^k$, where $k \geq 2$.

A risk measure is simply a function that assigns to each random variable a single number summarizing its riskiness. Formally,

Definition 1 *A risk measure is a function mapping random variables to the reals.*

We generically denote risk measures by $R(\cdot)$. The simplest and most commonly used risk measure is the variance ($R(\tilde{z}) = m_2(\tilde{z})$). However, many other risk measures have been proposed in the literature, capturing higher distribution moments and other risk attributes. A risk measure $R(\cdot)$ is *homogeneous of degree k* , if for any random return \tilde{z} and positive number $\lambda > 0$,

$$R(\lambda\tilde{z}) = \lambda^k R(\tilde{z}).$$

This condition guarantees, among other things, that the risk ranking between two investments does not depend on scaling. As we illustrate below, many common risk measures are homogenous. For the results in Section 3 we do not require that risk measures be homogenous. However, it will turn out that our systematic risk formula takes a particularly simple form when the risk measure is homogeneous of some degree. By contrast, homogeneity is required for all of the equilibrium results in Section 4.

Risk measures can be applied to individual random variables or to portfolios of random variables. Formally, assume there are n random variables represented by the vector $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$. A portfolio is represented by a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ where x_i is the dollar amount invested in \tilde{z}_i . Then, $\mathbf{x} \cdot \tilde{\mathbf{z}} = \sum_{i=1}^n x_i \tilde{z}_i$ is itself a random variable. We then say that the risk of portfolio \mathbf{x} is simply $R(\mathbf{x} \cdot \tilde{\mathbf{z}})$. When the vector of random variables is unambiguous, we often abuse notation and denote $R(\mathbf{x})$ as a shorthand for $R(\mathbf{x} \cdot \tilde{\mathbf{z}})$. We say that a risk measure is *smooth* if for any vector of random returns $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ and for all portfolios $\mathbf{x} = (x_1, \dots, x_n)$ we have that $R(\mathbf{x} \cdot \tilde{\mathbf{z}})$ is continuously differentiable in x_i for $i = 1, \dots, n$.

2.1 Examples of Risk Measures

Below we present some popular examples of risk measures.

Example 1 Central moments. For any integer $k \geq 2$, $R(\tilde{z}) = m_k(\tilde{z})$ is a risk measure which is homogeneous of degree k and smooth.

Example 2 Value at risk. A risk measure widely used in financial risk management is the value at risk (VaR), designed to capture the risk associated with rare disasters or downside risk. VaR measures the amount of loss not exceeded with a certain confidence level. Formally, given some confidence level $\alpha \in (0, 1)$, for any random return \tilde{z} , the VaR measure is defined as the negative of the α -quantile of \tilde{z} , i.e.,

$$\text{VaR}_\alpha(\tilde{z}) = -\inf \{z \in \mathbb{R} : F(z) \geq \alpha\}, \quad (1)$$

where $F(\cdot)$ is the cumulative distribution function of \tilde{z} . Notice that we include the minus sign to reflect the fact that a larger loss indicates higher risk. In particular, when \tilde{z} is continuously distributed with a density function $f(\cdot)$, (1) is implicitly determined by

$$\int_{-\infty}^{-\text{VaR}_\alpha(\tilde{z})} f(z) dz = \alpha. \quad (2)$$

This risk measure is homogenous of degree 1. Moreover, if the vector of random returns \tilde{z} follows a joint distribution with a continuously differentiable probability density function, then $R(\tilde{z}) = \text{VaR}_\alpha(\tilde{z})$ is also smooth.²

Example 3 Expected shortfall. This measure captures the average amount of loss from disastrous events, where a disastrous event is defined as one involving a loss larger than the VaR. Formally, given some confidence level $\alpha \in (0, 1)$, for any random return \tilde{z} the Expected Shortfall (ES) is defined as the negative of the conditional expected value of \tilde{z} below the α -quantile. That is,

$$\text{ES}_\alpha(\tilde{z}) = -\frac{1}{\alpha} \int_{-\infty}^{-\text{VaR}_\alpha(\tilde{z})} z dF(z), \quad (3)$$

where as before, $F(\cdot)$ is the cumulative distribution of \tilde{z} . Similar to VaR, ES is homogeneous of degree 1 and smooth.

Example 4 The Aumann-Serrano and Foster-Hart risk measures. Two measures of riskiness have recently been proposed by Aumann and Serrano (2008, hereafter AS) and Foster and Hart (2009, hereafter FH). These measures generalize the notion of second order stochastic dominance (SOSD). The AS measure $R^{AS}(\tilde{z})$ is given by the unique positive solution to the implicit equation

$$\mathbb{E} \left[\exp \left(-\frac{\tilde{z}}{R^{AS}(\tilde{z})} \right) \right] = 1. \quad (4)$$

²Smoothness follows from an application of the implicit function theorem to (2). The proof is available upon request.

The FH measure $R^{FH}(\tilde{z})$ is given by the unique positive solution to the implicit equation

$$\mathbb{E} \left[\log \left(1 + \frac{\tilde{z}}{R^{FH}(\tilde{z})} \right) \right] = 0. \quad (5)$$

Hart (2011) shows that R^{AS} and R^{FH} correspond, respectively, to the wealth-uniform dominance order and the utility-uniform dominance order, both of which are complete orders and coincide with SOSD whenever the latter applies. Both these measures are homogeneous of degree 1 and smooth.

3 Systematic Risk as a Solution to a Risk Allocation Problem

The systematic risk of an asset is typically conceived as a measure of the contribution of the asset to the risk of a diversified portfolio. The classical approach to this issue uses the variance as a risk measure, in which case systematic risk is measured by the “beta” of the asset. Specifically, the systematic risk of asset i relative to portfolio $\mathbf{x} = (x_1, \dots, x_n)$ of risky assets $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ is the slope-coefficient from a regression of individual returns on portfolio returns

$$\mathcal{B}_i = \frac{\text{Cov}(\tilde{z}_i, \mathbf{x} \cdot \tilde{\mathbf{z}})}{\text{Var}(\mathbf{x} \cdot \tilde{\mathbf{z}})}. \quad (6)$$

A priori, it is not clear how to generalize the notion of beta to arbitrary risk measures that account for high distribution moments and other risk aspects. In this section we develop a framework to measure systematic risk that generalizes the traditional beta. Our approach is to consider this issue as a risk allocation problem, where the total risk of the portfolio is “fairly” allocated among its components. To this end, we follow the literature on solutions to cost allocation problems. We offer four axioms that describe reasonable properties of solutions to risk allocation problems. We then show that these axioms define a unique formula for the systematic risk of an asset - the contribution of the asset to the risk of the portfolio. In the special case where risk is measured using variance, our systematic risk measure coincides with the traditional beta given by (6).

The solution to the risk allocation problem we will obtain is a scaled version of the Aumann-Shapley diagonal formula (Aumann and Shapley (1974)), which is a widely used solution concept from cooperative game theory. It is a continuous generalization of the Shapley value (Shapley (1953)).

3.1 Axiomatic Characterization of Systematic Risk

A *risk allocation problem* of order $n \geq 1$ is a pair (R, \mathbf{x}) , where R is a smooth risk measure with $R(0) = 0$, $\mathbf{x} \in \mathbb{R}_+^n$ is a portfolio specifying the dollar amount invested in each of n risky assets $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$, and $R(\mathbf{x}) \neq 0$. Denote the total dollar amount invested by $\bar{x} = \sum_{i=1}^n x_i$. Also, let $\boldsymbol{\alpha}$ be the vector of portfolio weights, i.e., $\alpha_i = \frac{x_i}{\bar{x}}$.

A *systematic risk measure* is a function mapping any risk allocation problem of order n to a vector $\mathbf{B}^R(\mathbf{x}) = (\mathcal{B}_1^R(\mathbf{x}), \dots, \mathcal{B}_n^R(\mathbf{x}))$ in \mathbb{R}^n . Intuitively, one can think of $\mathcal{B}_i^R(\mathbf{x})$ as the contribution of asset i to the total risk of portfolio \mathbf{x} , which is $R(\mathbf{x} \cdot \tilde{\mathbf{z}})$. Note that the notion of systematic risk is defined relative to a particular portfolio \mathbf{x} . In Section 4, when we study an equilibrium setup, the portfolio \mathbf{x} will arise endogenously as the market portfolio. But here we allow it to be any given portfolio. For example, one can think about \mathbf{x} as describing the different assets on a bank's balance sheet. Then, $\mathcal{B}_i^R(\mathbf{x})$ is the contribution of a particular asset to the total bank risk.

We now state four axioms specifying desirable economic properties of systematic risk. The first axiom postulates that the weighted average of systematic risk across all assets is 1. The intuition for this requirement comes from considering the traditional beta (see (6)). Assets that contribute strongly to the risk of the portfolio (aggressive assets) have a beta greater than 1, whereas assets that have little contribution to total risk (defensive assets) have a beta less than 1. The weighted average of all asset betas is 1. We ask that a generalized systematic risk measure have the same property.

Axiom 1 *Normalization:* $\sum_{i=1}^n \alpha_i \mathcal{B}_i^R(\mathbf{x}) = 1$.

The sum of any two risk measures is itself a risk measure. The next axiom requires that in such a case the systematic risk measure of the sum will be a risk-weighted average of systematic risk based on each of the two risk components.

Axiom 2 *Linearity:* If $R(\cdot) = R^1(\cdot) + R^2(\cdot)$, then

$$\mathcal{B}_i^R(\mathbf{x}) = \frac{R^1(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R^1}(\mathbf{x}) + \frac{R^2(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R^2}(\mathbf{x}) \text{ for all } i = 1, \dots, n.$$

When risk is measured using variance, the notion of systematic risk is closely tied to the concepts of correlation and covariance. It is not easy to generalize these concepts to arbitrary risk measures. However, two features can be easily generalized laying the foundations for the next two axioms.

First, while the concept of “correlation” is not easy to generalize, the idea of “perfect correlation” does lend itself to a natural generalization. We say that assets $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are *R-perfectly correlated* if there exists a function $g(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ and a non-zero vector $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{R}_+^n$, such that for any portfolio $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ we have $R(\boldsymbol{\eta} \cdot \tilde{\mathbf{z}}) = g(\boldsymbol{\eta} \cdot \mathbf{q})$. That is, the n assets are perfectly correlated if the risk of any portfolio of these assets only depends on some linear combination of their investment amounts. In essence, this means that the n assets can be aggregated into one “big” asset by assigning each asset a certain weight specified by the vector \mathbf{q} .³

³To see the correspondence to the standard notion of perfect correlation, consider the following example. Assume risk is measured using variance and let $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ with $\tilde{z}_2 = 2\tilde{z}_1$ and $\tilde{z}_3 = 5\tilde{z}_1$. Then, all three assets are perfectly correlated and for any portfolio (η_1, η_2, η_3) we have

$$\text{Var}(\eta_1 \tilde{z}_1 + \eta_2 \tilde{z}_2 + \eta_3 \tilde{z}_3) = (\eta_1 + 2\eta_2 + 5\eta_3)^2 \text{Var}(\tilde{z}_1).$$

Thus, we can set $g(t) = t^2$ and the vector of weights is $\mathbf{q} = \sqrt{\text{Var}(\tilde{z}_1)}(1, 2, 5)$. More generally, it is easy to verify that when risk is measured using variance, the concept of *R*-perfect correlation coincides with the standard definition of perfect correlation.

The next axiom then imposes the requirement that if the n assets are R -perfectly correlated, then their systematic risk measures are proportional to each other.

Axiom 3 *Proportionality: If $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are R -perfectly correlated with weights $\mathbf{q} = (q_1, \dots, q_n)$, then*

$$q_j \mathcal{B}_i^R(\mathbf{x}) = q_i \mathcal{B}_j^R(\mathbf{x}) \text{ for all } i, j = 1, \dots, n. \quad (7)$$

Next we turn to generalize the idea of “positive correlation.” Assume first that risk is measured using variance. Then, if two assets are positively correlated, adding additional units of each asset to the portfolio of the two always increases total variance. We can then use this feature to get a generalized notion of positive correlation. Specifically, we say that assets $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are R -positively correlated if $R_i(\boldsymbol{\eta} \cdot \tilde{\mathbf{z}}) \geq 0$ for all $\boldsymbol{\eta} \in \mathbb{R}_+^n$ and for all $i = 1, \dots, n$. Namely, the assets are R -positively correlated if adding one more unit of each asset to a portfolio with non-negative weights can never reduce total risk.⁴ The next axiom requires that when the assets are R -positively correlated, the systematic risk of all assets is non-negative.

Axiom 4 *Monotonicity: If $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are R -positively correlated, then $\mathcal{B}_i^R(\mathbf{x}) \geq 0$ for all $i = 1, \dots, n$.*

Our main result in this section follows. It states that Axioms 1-4 are sufficient to pin down a unique systematic risk measure, which takes on a very simple and intuitive form.

Theorem 1 *There exists a unique systematic risk measure satisfying Axioms 1-4. For each risk allocation problem (R, \mathbf{x}) of order n , it is given by*

$$\mathcal{B}_i^R(\mathbf{x}) = \frac{\bar{x} \int_0^1 R_i(tx_1, \dots, tx_n) dt}{R(x_1, \dots, x_n)} \text{ for } i = 1, \dots, n. \quad (8)$$

Furthermore, if R is homogeneous of some degree k , then (8) reduces to

$$\mathcal{B}_i^R(\mathbf{x}) = \frac{R_i(\boldsymbol{\alpha})}{\sum_{j=1}^n \alpha_j R_j(\boldsymbol{\alpha})} \quad (9)$$

$$= \frac{R_i(\boldsymbol{\alpha})}{kR(\boldsymbol{\alpha})}. \quad (10)$$

The intuition for the systematic risk measure we obtain is easy to see first from the expression in (9). It states that when R is homogenous (which is a common case), systematic risk of asset i is measured simply as the marginal contribution of asset i to the total risk of the portfolio, scaled by the weighted average of marginal contributions of all assets. When the risk measure is not homogeneous, the expression in (8) shows that systematic risk depends not only on marginal contributions at \mathbf{x} ,

⁴It is easy to check that when risk is measured using variance, the assets are R -positively correlated if and only if the correlation between any two assets is non-negative.

but rather on marginal contributions along a diagonal between $(0, \dots, 0)$ and \mathbf{x} . This is a variation on the well-known diagonal formula of Aumann and Shapley (1974). The integral can be interpreted as an average of marginal contributions of asset i to the risk of portfolios along the diagonal. Then, $\mathcal{B}_i^R(\mathbf{x})$ is simply a scaled version of the integral where the scaling ensures that Axiom 1 is satisfied.

Note that when the risk measure is homogenous, $\mathcal{B}_i^R(\mathbf{x})$ depends only depends on the portfolio weights $\alpha_i = \frac{x_i}{\bar{x}}$ (and not on the dollar amounts invested in each asset). By contrast, when the risk measure is not homogeneous, Theorem 1 shows that there is no systematic risk measure satisfying Axioms 1-4 that depends on portfolio \mathbf{x} only, and the entire diagonal between $\mathbf{0}$ and \mathbf{x} has to be considered. Indeed, in the homogenous case $R_i(tx_1, \dots, tx_n)$ is proportional to $R_i(x_1, \dots, x_n)$ for all $t \in [0, 1]$, yielding the simple expression in (9).

The proof of Theorem 1 is in Appendix I. It relies on the solution to cost allocation problems established in Billera and Heath (1982), which in turn relies on the tools developed in Aumann and Shapley (1974). What we do in the proof is draw a 1-1 mapping between risk allocation problems and cost allocation problems, and from systematic risk measures to solutions of cost allocation problems. Then, we show that given these mappings, our set of axioms is both necessary and sufficient for the set of Conditions specified in Billera and Heath (1982). This in turn allows us to apply their result and obtain Theorem 1.

To get some additional intuition, we now explain why (8) satisfies Axioms 1-4. Suppose that $\mathcal{B}_i^R(\mathbf{x})$ is given by (8). Then,

$$\begin{aligned} \sum_{i=1}^n \alpha_i \mathcal{B}_i^R(\mathbf{x}) &= \sum_{i=1}^n \frac{x_i}{\bar{x}} \frac{\int_0^1 R_i(tx_1, \dots, tx_n) dt}{R(x_1, \dots, x_n)} \\ &= \frac{\int_0^1 \sum_{i=1}^n x_i R_i(tx_1, \dots, tx_n) dt}{R(x_1, \dots, x_n)} \\ &= \frac{\int_0^1 \frac{dR(tx_1, \dots, tx_n)}{dt} dt}{R(x_1, \dots, x_n)} = 1, \end{aligned}$$

and so Axiom 1 holds. To see Axiom 2, suppose $R(\cdot) = R^1(\cdot) + R^2(\cdot)$. Then,

$$\begin{aligned} \mathcal{B}_i^R(\mathbf{x}) &= \frac{\bar{x} \int_0^1 R_i(tx_1, \dots, tx_n) dt}{R(x_1, \dots, x_n)} \\ &= \frac{\bar{x} \int_0^1 R_i^1(tx_1, \dots, tx_n) dt}{R^1(x_1, \dots, x_n)} R^1(x_1, \dots, x_n) + \frac{\bar{x} \int_0^1 R_i^2(tx_1, \dots, tx_n) dt}{R^2(x_1, \dots, x_n)} R^2(x_1, \dots, x_n)}{R(x_1, \dots, x_n)}, \end{aligned}$$

as required. Next, for Axiom 3, suppose that $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are R -perfectly correlated. Then, there exists $g(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ and a nonzero vector $\mathbf{q} \in \mathbb{R}_+^n$ such that for all $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ we have $R(\boldsymbol{\eta}) = g(\boldsymbol{\eta} \cdot \mathbf{q})$. It follows that

$$R_i(\boldsymbol{\eta}) = q_i g'(\boldsymbol{\eta} \cdot \mathbf{q}) \text{ for all } i = 1, \dots, n.$$

Hence, for all $i = 1, \dots, n$

$$\mathcal{B}_i^R(\mathbf{x}) = \frac{\bar{x}q_i \int_0^1 g'(x \cdot q) dt}{R(x_1, \dots, x_n)},$$

which implies (7). Finally, given the definition of R -positive correlation, it is immediate that (8) satisfies Axiom 4.

3.2 Applying the Result

All of the examples below relate to homogeneous risk measures. Thus, in light of Theorem 1, we dispense with dollar amounts and state portfolio weights only. When the risk measure is the variance, \mathcal{B}_i^R boils down to the traditional CAPM beta given in (6). To see this, assume $R(\cdot) = \text{Var}(\cdot)$ and consider portfolio weights $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Then,

$$R(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \text{Cov}(\tilde{z}_i, \tilde{z}_j).$$

Hence,

$$R_i(\alpha_1, \dots, \alpha_n) = 2\text{Cov}(\tilde{z}_i, \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}).$$

It follows that

$$\mathcal{B}_i^R = \frac{2\text{Cov}(\tilde{z}_i, \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}})}{2 \sum_{j=1}^n \alpha_j \text{Cov}(\tilde{z}_j, \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}})} = \frac{\text{Cov}(\tilde{z}_i, \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}})}{\text{Var}(\boldsymbol{\alpha})}. \quad (11)$$

We next illustrate how our general notion of systematic risk applies to other risk measures.

3.2.1 Central Moments

Let $R(\cdot) = m_k(\cdot)$ for $k \geq 2$ be the k^{th} central moment. That is, given any random return \tilde{z} ,

$$R(\tilde{z}) = \text{E} \left[(\tilde{z} - \text{E}(\tilde{z}))^k \right].$$

Consider portfolio weights $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Then

$$R(\alpha_1, \dots, \alpha_n) = \text{E} \left[(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} - \boldsymbol{\alpha} \cdot \text{E}(\tilde{\mathbf{z}}))^k \right] = \text{E} \left[\left(\sum_{i=1}^n \alpha_i (\tilde{z}_i - \text{E}(\tilde{z}_i)) \right)^k \right].$$

We then have,

Proposition 1 *Let $R(\cdot) = m_k(\cdot)$ for $k \geq 2$. Then, the systematic risk is given by*

$$\mathcal{B}_i^R = \frac{\text{Cov} \left(\tilde{z}_i, (\boldsymbol{\alpha} \cdot (\tilde{\mathbf{z}} - \text{E}(\tilde{\mathbf{z}})))^{k-1} \right)}{\text{E} \left[(\boldsymbol{\alpha} \cdot (\tilde{\mathbf{z}} - \text{E}(\tilde{\mathbf{z}})))^k \right]}.$$

That is, the systematic risk of asset i is proportional to the covariance of \tilde{z}_i with the $(k-1)^{th}$ power of the demeaned portfolio return. In the special case of $k=2$ (variance), this boils down to (11) as expected.

Obviously, we can apply the same procedure to linear combinations of different moments considered as risk measures. As an example, let

$$R(\cdot) = \theta_1 m_2(\cdot) + \theta_2 (m_3(\cdot))^{\frac{2}{3}}$$

for some real θ_1 and θ_2 . This risk measure captures both variance and skewness. Intuitively, it makes sense to have $\theta_1 > 0$ and $\theta_2 < 0$ to reflect aversion to both variance and negative skewness. Raising the skewness to the $\frac{2}{3}$ -power ensures that the resulting measure is homogeneous. Then it is straightforward to show that

$$\begin{aligned} \mathcal{B}_i^R &= \frac{\theta_1 \text{Cov}(\tilde{z}_i, \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}) + \theta_2 (m_3(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}))^{-\frac{1}{3}} \text{Cov}\left(\tilde{z}_i, (\boldsymbol{\alpha} \cdot (\tilde{\mathbf{z}} - \mathbf{E}(\tilde{\mathbf{z}})))^2\right)}{\theta_1 m_2(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}) + \theta_2 (m_3(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}))^{\frac{2}{3}}} \\ &= \frac{R_1(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R_1}(\mathbf{x}) + \frac{R_2(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R_2}(\mathbf{x}), \end{aligned}$$

where $R_1(\cdot) = \theta_1 m_2(\cdot)$ and $R_2(\cdot) = \theta_2 (m_3(\cdot))^{\frac{2}{3}}$ (in line with Axiom 2).

3.2.2 The AS and FH Measures

We now calculate the systematic risk of an asset based on the AS and FH measures. We have shown that both measures are smooth and homogeneous of degree 1. In addition, although $R^{AS}(0, \dots, 0) = R^{AS}(0)$ and $R^{FH}(0, \dots, 0) = R^{FH}(0)$ are not defined, they can be approximated using a limiting argument. Specifically, take any random return \tilde{z} satisfying $\mathbf{E}(\tilde{z}) > 0$ and $P(\{\tilde{z} < 0\}) > 0$, and for both $R(\cdot) = R^{AS}(\cdot)$ and $R(\cdot) = R^{FH}(\cdot)$, we can then define $R(0)$ by

$$R(0) = \lim_{t \rightarrow 0} R(t\tilde{z}) = \lim_{t \rightarrow 0} tR(\tilde{z}) = R(\tilde{z}) \lim_{t \rightarrow 0} t = 0,$$

where the second equality follows since both the AS and the FH measures are homogeneous of degree 1. Hence, we can apply Theorem 1 to obtain the systematic risk of individual assets associated with the AS and FH measures. The results are stated in the following propositions.

Proposition 2 *If $R(\cdot) = R^{AS}(\cdot)$, then, the systematic risk is given by*

$$\mathcal{B}_i^R = \frac{\mathbf{E}\left[\exp\left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})}\right) \tilde{z}_i\right]}{\mathbf{E}\left[\exp\left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})}\right) \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}\right]}.$$

If $R(\cdot) = R^{FH}(\cdot)$, then, the systematic risk is given by

$$\mathcal{B}_i^R = \frac{\mathbf{E}\left[\frac{\tilde{z}_i}{R(\boldsymbol{\alpha}) + \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}\right]}{\mathbf{E}\left[\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha}) + \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}\right]}.$$

4 Systematic Risk in an Equilibrium Setting

The approach to measuring systematic risk in the previous section did not rely on equilibrium considerations. Traditionally, systematic risk is derived from an equilibrium setting known as the Capital Asset Pricing Model (CAPM). We will now present such a generalized setting. We will show that for a wide variety of risk measures an equilibrium in this setting exists, and that any equilibrium exhibits two-fund separation. Furthermore, in equilibrium the standard CAPM formula holds where systematic risk coincides with the scaled Aumann-Shapley diagonal formula derived in Section 3.

An advantage of the approach in the previous section is that it asks very little of the risk measures allowed. We only required that $R(0) = 0$ and that R be smooth. To derive the equilibrium results below we need to impose more structure on the allowable risk measures. We begin by specifying the properties of risk measures that will turn out useful for our results. We then present the setup for the equilibrium model, study the geometry of solutions, and present a two-fund separation result implying the efficiency of the market portfolio. Finally, we derive a variant of the security market line, enabling us to obtain a generalization of beta.

4.1 Properties of Risk Measures

The first property of risk measures which will become useful in this section is convexity. Formally, we say that a risk measure R is *convex* if for any two random returns \tilde{z}_1 and \tilde{z}_2 , and for any $\lambda \in (0, 1)$, we have

$$R(\lambda\tilde{z}_1 + (1 - \lambda)\tilde{z}_2) \leq \lambda R(\tilde{z}_1) + (1 - \lambda)R(\tilde{z}_2),$$

with equality holding only when $\tilde{z}_1 = \tilde{z}_2$ with probability 1. Notice that $\lambda\tilde{z}_1 + (1 - \lambda)\tilde{z}_2$ can be considered as the return of a portfolio that assigns weights λ and $1 - \lambda$ to \tilde{z}_1 and \tilde{z}_2 , respectively. Then the convexity condition says that the risk of the portfolio should not be higher than the corresponding weighted average risk of the constituent investments. Thus, convexity of a risk measure captures the idea that diversifying among two investments lowers the total risk.

Next we would like to formalize a property dealing with the type of assets that are risk-free. If an asset \tilde{z} has $R(\tilde{z}) = 0$ we say that \tilde{z} is *R-risk-free*. We say that a risk measure $R(\cdot)$ has the *risk-free* property, if (i) $R(\tilde{z}) \geq 0$ for all \tilde{z} ; (ii) \tilde{z} is *R-risk-free* if and only if $P(\{\tilde{z} = c\}) = 1$ for some constant c ; and (iii) $R(\tilde{z}_1 + \tilde{z}_2) = R(\tilde{z}_1)$ whenever \tilde{z}_2 is *R-risk-free*. Namely, R has the risk-free property if the only risk-free assets are those that pay a constant amount with probability 1, if all other assets have a strictly positive risk, and if adding a risk-free asset does not change risk.

Finally, note that all of the properties we discussed (including homogeneity and smoothness) are maintained when taking convex combinations of different risk measures with the same degree of homogeneity. Thus, we can easily create new risk measures satisfying these properties from existing risk measures. That is, let s be a positive integer and let $R^1(\cdot), \dots, R^s(\cdot)$ be risk measures and choose $\theta = (\theta_1, \dots, \theta_s)$

such that $\sum_{i=1}^s \theta_i = 1$ and $\theta_i \geq 0 \forall i$. We can then define a new risk measure by

$$R^\theta(\tilde{z}) = \sum_{i=1}^s \theta_i R^i(\tilde{z}).$$

We then have the following trivial but useful lemma.

Lemma 1 *Fix k , assume that each R^i is homogeneous of degree k , smooth, convex, and satisfies the risk-free property. Then, R^θ also satisfies all of these properties.*

4.1.1 Examples

We now re-visit the examples in Section 2.1 to illustrate when the three properties defined in the previous section hold and when they fail.

Example 5 Even central moment. *For all k even, $R(\tilde{z}) = m_k(\tilde{z})$ is a risk measure which is convex and has the risk-free property. Indeed, the risk-free property is trivial in this case. Convexity, follows from the following stronger result, which shows that the k^{th} root of an even moment is a convex risk measure. Thus, $m_k(\tilde{z})$ is a fortiori convex.*

Proposition 3 *Let $w_k(\tilde{z}) = (m_k(\tilde{z}))^{\frac{1}{k}}$ be the normalized k^{th} moment. Then, for all k even, $R(\tilde{z}) = w_k(\tilde{z})$ is a convex risk measure.*

Example 6 Combinations of normalized even moment. *Proposition 3 implies that we can define $R(\tilde{z}) = (w_k(\tilde{z}))^t$ for any $t \geq 1$, and obtain a risk measure that satisfies all three properties above, as well as homogeneity and smoothness. Also, using Lemma 1 we can take convex combinations of different even moments by normalizing them to have the same level of homogeneity and relying on Proposition 3 to ensure that convexity is maintained. This creates new risk measures which account for several distribution moments. For example, suppose we would like a risk measure that accounts for both dispersion and tail risk. Such a risk measure should reflect both the second and the fourth central moments. In particular, for all $\theta \in [0, 1]$ let*

$$\begin{aligned} R^\theta(\tilde{z}) &= \theta m_2(\tilde{z}) + (1 - \theta) \sqrt{m_4(\tilde{z})} \\ &= \theta (w_2(\tilde{z}))^2 + (1 - \theta) (w_4(\tilde{z}))^2. \end{aligned}$$

Then, $R^\theta(\tilde{z})$ is a family of risk measures parametrized by θ , which incorporate both the dispersion (as reflected by the variance $m_2(\tilde{z})$) and the tail risk (as reflected by the fourth central moment $m_4(\tilde{z})$) of a random return. The parameter θ specifies the weight assigned to the two moments. A higher θ reflects a larger weight assigned to the variance relative to the fourth moment. By Lemma 1 these risk measures satisfy all of the three properties as well as homogeneity of degree 2 and smoothness.

Example 7 Odd central moments. *Assume $R(\tilde{z}) = m_k(\tilde{z})$ for odd $k > 2$. In contrast to the even central moments, neither convexity nor the risk-free property*

holds in this case. To see the former, consider the simple example of two random returns, \tilde{z}_1 and \tilde{z}_2 that are independent and have negative third central moments $m_3(\tilde{z})$. Then by independence and the homogeneity of central moments

$$m_3\left(\frac{1}{2}\tilde{z}_1 + \frac{1}{2}\tilde{z}_2\right) = \left(\frac{1}{2}\right)^3 m_3(\tilde{z}_1) + \left(\frac{1}{2}\right)^3 m_3(\tilde{z}_2) > \frac{1}{2}m_3(\tilde{z}_1) + \frac{1}{2}m_3(\tilde{z}_2),$$

since $m_3(\tilde{z}_1) + m_3(\tilde{z}_2) < 0$. To see the latter, note that the third moment can be negative, violating the risk-free property.

Example 8 Value at risk. For any risk-free return \tilde{z} with $P(\{\tilde{z} = c\}) = 1$, we have $\text{VaR}_\alpha(\tilde{z}) = -c$, implying that the VaR of risk-free assets depends on the risk-free return. Hence, the risk-free property is not satisfied. In addition, it is not hard to find examples where convexity is violated for the VaR measure.

Example 9 Expected short-fall. Similar to the VaR, ES does not satisfy the risk-free property. Unlike VaR, ES does satisfy the convexity property as shown in the next proposition.

Proposition 4 For any $\alpha \in (0, 1)$, $R(\tilde{z}) = \text{ES}_\alpha(\tilde{z})$ is convex.

Example 10 The AS and FH risk measures. These two risk measures satisfy the convexity property.⁵ By contrast, technically, these measures cannot apply to risk-free assets, and the risk-free property is irrelevant.

4.2 Model Setup

Investors, Assets, and Timing. Assume a market with $n + 1$ assets $\{0, \dots, n\}$. Assets $1, \dots, n$ are risky and pay a random amount denoted by $(\tilde{y}_1, \dots, \tilde{y}_n)$. Asset 0 is risk-free, paying an amount \tilde{y}_0 which is equal to some constant $y_0 \neq 0$ with probability 1. Denote $\tilde{\mathbf{y}} = (\tilde{y}_0, \dots, \tilde{y}_n)$. There are ℓ investors in the market, all of whom agree on the parameters of the model. The choice set of each investor is \mathbb{R}^{n+1} , where $\zeta^j \in \mathbb{R}^{n+1}$ represents the number of shares investor j chooses in each asset $i = 0, \dots, n$; i.e., ζ^j is a bundle of assets. Negative numbers represent short sales, and we impose no short-sale constraints. The initial endowment of investor j is $\mathbf{e}^j \in \mathbb{R}_+^{n+1}$. We assume that $\sum_{j=1}^{\ell} e_i^j > 0$ for $i = 1, \dots, n$ and $\sum_{j=1}^{\ell} e_0^j = 0$. That is, risky assets are in positive net supply and the risk-free asset is in zero net supply. An *allocation* is an ℓ -tuple $\mathcal{A} = (\zeta^1, \dots, \zeta^\ell)$ consisting of a bundle $\zeta^j \in \mathbb{R}^{n+1}$ for each investor. An allocation \mathcal{A} is *attainable* if $\sum_{j=1}^{\ell} \zeta^j = \sum_{j=1}^{\ell} \mathbf{e}^j$, that is, if it clears the market. A *price system* is a vector $\mathbf{p} = (p_0, \dots, p_n)$ specifying a price for each asset. The random return of asset i is then given by $\tilde{z}_i = \frac{\tilde{y}_i}{p_i}$, where the return from the risk-free asset \tilde{z}_0 is equal to some constant r_f with probability 1. Similar to the standard CAPM setting, there are two dates. At Date 0, investors trade with each other and prices are set. At Date 1, all random variables are realized.

⁵This follows since these risk measures are sub-additive and homogeneous of degree 1

Risk and Preferences. The traditional approach has investors having mean-variance preferences, i.e., they prefer higher mean and lower variance of investments. Instead, we will assume that investors have mean-risk preferences. Formally, fix a risk measure $R(\cdot)$. The utility that investor $j = 1, \dots, \ell$ assigns to a bundle $\zeta \in \mathbb{R}^{n+1}$ is given by

$$U^j(\zeta) = V^j(\mathbb{E}(\zeta \cdot \tilde{\mathbf{y}}), R(\zeta \cdot \tilde{\mathbf{y}})), \quad (12)$$

where V^j is continuous, strictly increasing in its first argument (expected return) and strictly decreasing in its second argument (risk of return), and quasi-concave.

Note that $U^j(\zeta)$ cannot be in general supported as a von Neumann-Morgenstern utility. Nevertheless in Appendix II we show that if V^j is differentiable and if the risk measure is a differentiable function of a finite number of moments, then $U^j(\zeta)$ is a local expected utility function in the sense of Machina (1982). Namely, comparisons of “close by” investments are well approximated by expected utility. These conditions apply to a wide range of risk measures representing high distribution moments.

An implication of quasi-concavity of V^j is that when plotted in mean-risk space, the upper contour of each indifference curve is convex. Similar to the standard mean-variance case, we will assume that a risk-free asset cannot be created synthetically from risky assets. That is, there is no redundant risky asset: for any $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n) \in \mathbb{R}^{n+1}$ we have $R(\zeta \cdot \tilde{\mathbf{y}}) \neq 0$ unless $(\zeta_1, \dots, \zeta_n) = (0, \dots, 0)$.⁶

Equilibrium. An *equilibrium* is a pair $(\mathbf{p}, \mathcal{A})$ where $\mathbf{p} \neq 0$ is a price system and $\mathcal{A} = (\zeta^1, \dots, \zeta^\ell)$ is an attainable allocation, such that for each $j \in \{1, \dots, \ell\}$, $\mathbf{p} \cdot \zeta^j = \mathbf{p} \cdot \mathbf{e}^j$, and if $\zeta \in \mathbb{R}^{n+1}$ and $U^j(\zeta) > U^j(\zeta^j)$ then $\mathbf{p} \cdot \zeta > \mathbf{p} \cdot \mathbf{e}^j$. In words, an equilibrium is a price system and an allocation that clears the market such that each investor optimizes subject to her budget constraint. The next lemma specifies conditions under which an equilibrium exists.

Lemma 2 *Suppose that $R(\cdot)$ is convex, smooth, and satisfies the risk-free property. Then, an equilibrium exists.*

It is well known that the CAPM setting can yield negative or zero prices (see for example Nielsen (1992)). The reason for this is that preferences are not necessarily monotone. Specifically, the expected return to an investor’s bundle increases as she holds more shares of a (risky) asset, but so does the risk. It may well be that at some point, the additional expected return gained from adding more shares to the bundle is not sufficient to compensate for the increase in risk. If the equilibrium happens to fall in such a region then the asset becomes undesirable, rendering a negative price. For our following results we will need that prices are positive for all assets. The literature has suggested several ways to guarantee such an outcome. In Appendix III we provide one sufficient condition which follows Nielsen (1992). Other (and possibly weaker) sufficient conditions may be obtained, but are beyond the scope of this paper.

From now on we will only consider equilibria with positive prices. Given positivity of prices, naturally, each equilibrium induces a vector of random returns $\tilde{z}_i = \frac{\tilde{y}_i}{p_i}$, and

⁶In the standard mean-variance case this condition corresponds to the variance-covariance matrix of risky assets being positive-definite.

so we can talk about the expected returns and the risk of the returns in equilibrium, as in the usual CAPM setting. We now study these returns.

4.3 A Generalized CAPM

4.3.1 Geometry of Efficient Portfolios

Let $(\mathbf{p}, \mathcal{A})$ be an equilibrium. The equilibrium allocation $(\zeta^1, \dots, \zeta^\ell)$ naturally induces a portfolio for each investor j given by $\mathbf{x}^j = (x_0^j, \dots, x_n^j)$ where $x_i^j = p_i \zeta_i^j$ is the amount invested in asset i , and where the vector of portfolio weights of investor j is denoted by α^j , and given by $\alpha_i^j = \frac{x_i^j}{\sum_{h=0}^n x_h^j}$. Let

$$\mu^j = \sum_{i=0}^n \alpha_i^j \mathbf{E}(\tilde{z}_i)$$

be the expected return obtained by investor j in equilibrium. The next lemma shows that the standard procedure of “minimizing risk for a given expected return” applies to the equilibrium setting.

Theorem 2 *Suppose that $R(\cdot)$ is homogeneous. Then, in an equilibrium with positive prices, for all investors $j \in \{1, \dots, \ell\}$, α^j is the unique solution to*

$$\begin{aligned} & \min_{\alpha \in \mathbb{R}^{n+1}} R(\alpha \cdot \tilde{\mathbf{z}}) & (13) \\ & \text{s.t.} \\ & \sum_{i=0}^n \alpha_i \mathbf{E}(\tilde{z}_i) = \mu^j. \\ & \sum_{i=0}^n \alpha_i = 1. \end{aligned}$$

Given this, we can now discuss the geometry of portfolios in the μ - R plane where the horizontal axis is the risk of the return of a portfolio (R) and the vertical axis is the expected return (μ). The locus of portfolios minimizing risk for any given expected return is the boundary of the portfolio opportunity set. This set is convex in the μ - R plane whenever $R(\cdot)$ is a convex risk measure. This follows simply because the expectation operator is linear, implying that the line connecting any two portfolios in the μ - R plane lies to the right of the set of portfolios representing convex combinations of these portfolios. Figure 1 illustrates two curves. The blue curve depicts the opportunity set of risky assets only. The red curve depicts portfolios minimizing risk for a given expected return, corresponding to Program (13). In general, both of these are defining convex sets, unlike in the special case of the standard deviation, where we have an extreme case of a straight line connecting the risk-free asset and risky portfolios. We say that a portfolio is *efficient* if it solves Program (13) for some μ^j . Thus, the red curve in Figure 1 corresponds to the set of efficient portfolios.

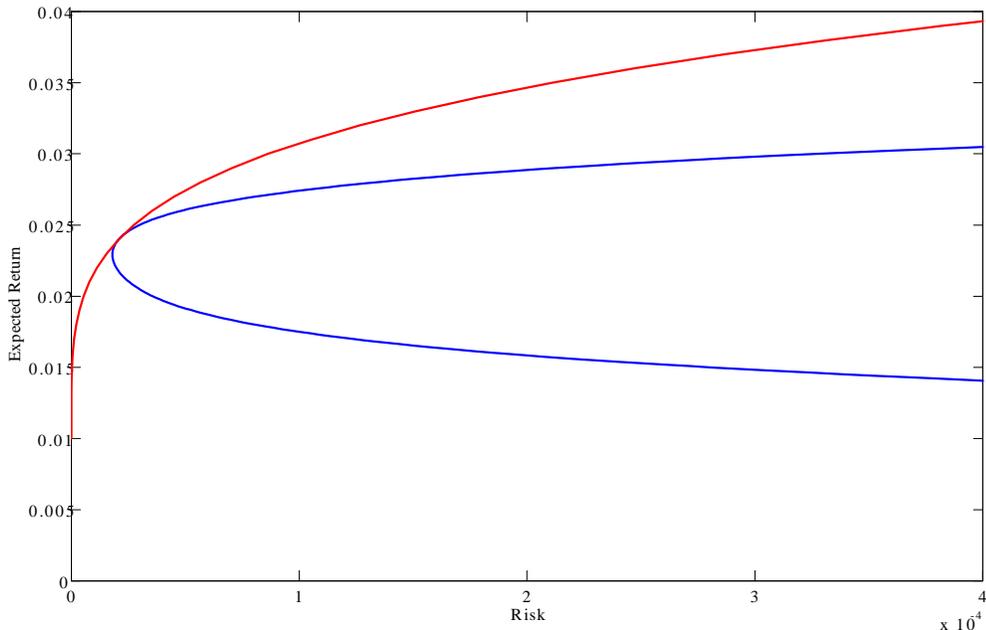


Figure 1: Portfolio Opportunity Set and Efficient Frontier

4.3.2 Two-Fund Separation

We say that two-fund money separation holds if the equilibrium optimal portfolios for all investors can be spanned by the risk-free asset and a unique portfolio of risky assets only. The idea of two-fund separation was introduced by Tobin (1958). Since then the literature discussed different sufficient conditions for two-fund separation (see Cass and Stiglitz (1970), Ross (1978), and more recently Dybvig and Liu (2012)). These papers provide sufficient conditions for two-fund separation based on either restrictions on the return distributions or on the utility functions. Here we take a different approach, as we specify sufficient conditions for two-fund money separation in terms of properties of the risk measure.

Theorem 3 *Consider an equilibrium with positive prices. Assume that $R(\cdot)$ is homogeneous of some degree k , convex, and satisfies the risk-free property. Then, two-fund money separation holds. That is, there exists a unique portfolio with weights α^P such that $\alpha_0^P = 0$, and for all investors $j \in \{1, \dots, \ell\}$, the solution to Problem (13) is a linear combination of α^P and the risk-free asset.*

The proof is very intuitive, and we show it here. Let α^1 and α^2 be solutions of Problem (13) for investors $j_1 \neq j_2$, respectively, and without loss of generality assume $j_1 = 1$ and $j_2 = 2$. Assume without loss of generality that both α^1 and α^2 have non-zero weights in some risky assets.⁷ By the risk-free property and by the

⁷If only one investor holds non-zero weights in risky assets then two fund separation is trivial.

non-redundancy assumption, $R(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}}) > 0$ for $j = 1, 2$. Hence, $\mu^j = E(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}}) > r_f$ for $j = 1, 2$, since otherwise $\boldsymbol{\alpha}^j$ would be mean-risk dominated by the risk-free asset, and thus would not be optimal.

Now, consider all the linear combinations of these two portfolios with the risk-free asset. Since $R(\cdot)$ is assumed convex, the resulting curves are concave in the μ - R plane as illustrated in Figure 2. Note that both $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ can be presented as a linear combination of the risk-free asset and some portfolios $\boldsymbol{\alpha}^{P_1}$ and $\boldsymbol{\alpha}^{P_2}$ of risky assets only (i.e., $\alpha_0^{P_1} = \alpha_0^{P_2} = 0$). To show two-fund separation we need to show that $\boldsymbol{\alpha}^{P_1} = \boldsymbol{\alpha}^{P_2}$. Assume this is not the case. Then let $\hat{\boldsymbol{\alpha}}^1$ be a portfolio of $\boldsymbol{\alpha}^{P_2}$ and the risk-free asset such that $E(\hat{\boldsymbol{\alpha}}^1 \cdot \tilde{\mathbf{z}}) = \mu^1$. Similarly, let $\hat{\boldsymbol{\alpha}}^2$ be a portfolio of $\boldsymbol{\alpha}^{P_1}$ and the risk-free asset such that $E(\hat{\boldsymbol{\alpha}}^2 \cdot \tilde{\mathbf{z}}) = \mu^2$. By convexity of $R(\cdot)$, $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ are the unique solutions for Program (13) for $j = 1, 2$. Hence,

$$R(\hat{\boldsymbol{\alpha}}^1 \cdot \tilde{\mathbf{z}}) > R(\boldsymbol{\alpha}^1 \cdot \tilde{\mathbf{z}}) \quad \text{and} \quad R(\hat{\boldsymbol{\alpha}}^2 \cdot \tilde{\mathbf{z}}) > R(\boldsymbol{\alpha}^2 \cdot \tilde{\mathbf{z}}). \quad (14)$$

Thus, as illustrated in Figure 2, the two curves must cross at least once. We will now show that such crossings are impossible.

By homogeneity, we have for any $\lambda > 0$,

$$R(\lambda \boldsymbol{\alpha}^1 \cdot \tilde{\mathbf{z}}) = \lambda^k R(\boldsymbol{\alpha}^1 \cdot \tilde{\mathbf{z}}) < \lambda^k R(\hat{\boldsymbol{\alpha}}^1 \cdot \tilde{\mathbf{z}}) = R(\lambda \hat{\boldsymbol{\alpha}}^1 \cdot \tilde{\mathbf{z}}),$$

which together with risk-free property implies

$$R(\lambda \boldsymbol{\alpha}^1 \cdot \tilde{\mathbf{z}} + (1 - \lambda) r_f) < R(\lambda \hat{\boldsymbol{\alpha}}^1 \cdot \tilde{\mathbf{z}} + (1 - \lambda) r_f).$$

This means that all linear combinations of $\boldsymbol{\alpha}^1$ with the risk-free asset (with positive λ) lie strictly to the left of all linear combinations of $\hat{\boldsymbol{\alpha}}^1$ with the risk-free asset. In particular, $\hat{\boldsymbol{\alpha}}^2$ can be obtained as a linear combination of $\boldsymbol{\alpha}^1$ with the risk-free asset by setting

$$\lambda = \frac{\mu^2 - r_f}{\mu^1 - r_f} > 0,$$

where the inequality follows since $\mu^j > r_f$ for $j = 1, 2$. But, using this λ we obtain

$$R(\hat{\boldsymbol{\alpha}}^2 \cdot \tilde{\mathbf{z}}) < R(\boldsymbol{\alpha}^2 \cdot \tilde{\mathbf{z}}),$$

contradicting (14). Thus, two-fund separation must hold.

A corollary is that the unique portfolio $\boldsymbol{\alpha}^P$ is efficient. Indeed, let $\mu^P = E(\boldsymbol{\alpha}^P \cdot \tilde{\mathbf{z}})$. Since in equilibrium all investors hold a linear combination of the risk-free asset and $\boldsymbol{\alpha}^P$, and since $\mu^j = E(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}}) \geq r_f$ for all j with strict inequality for some j , we have two cases:⁸ (i) all investors hold $\boldsymbol{\alpha}^P$ with a non-negative weight, and $\mu^P > r_f$; or (ii) all investors hold $\boldsymbol{\alpha}^P$ with a non-positive weight, and $\mu^P < r_f$. But, the second case is impossible since then the market cannot clear for at least one risky asset, which is held in positive weight in $\boldsymbol{\alpha}^P$. Thus, $\mu^P > r_f$.

Now, assume that $\boldsymbol{\alpha}' \neq \boldsymbol{\alpha}^P$ solves Problem (13) for $\mu^j = \mu^P$. Then, $R(\boldsymbol{\alpha}' \cdot \tilde{\mathbf{z}}) < R(\boldsymbol{\alpha}^P \cdot \tilde{\mathbf{z}})$, and so by the same argument as in the proof of Theorem 3, all linear

⁸If all investors choose the risk-free asset then the market for risky assets cannot clear.

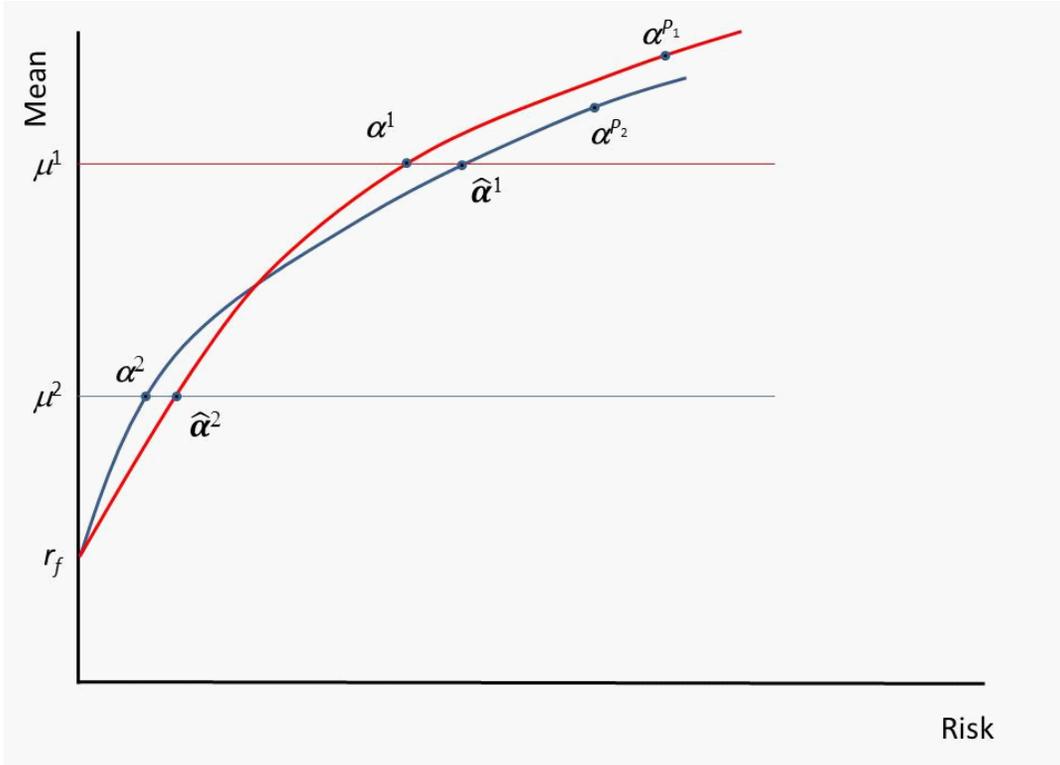


Figure 2: Graphical Illustration of the Proof of Theorem 3

combinations of α' with the risk-free asset would have strictly lower risk than the corresponding linear combinations of α^P with the risk-free asset. This contradicts that α^P and the risk-free asset span all efficient portfolios. We thus have:

Corollary 1 *Under the conditions of Theorem 3, the portfolio α^P solves Problem (13) for some $\mu^P > r_f$.*

Let $x_i^M = \sum_{j=1}^{\ell} x_i^j$ be the total amount invested in asset i . We call $\mathbf{x}^M = (x_0^M, \dots, x_n^M)$ the *market portfolio*. Note that since the risk-free asset is in zero net supply we have $x_0^M = 0$. Let α^M be the corresponding portfolio weights, where in particular $\alpha_0^M = 0$. By Theorem 3, in equilibrium, the market portfolio is equal to α^P the unique portfolio of risky assets that together with the risk-free asset spans all efficient portfolios. Moreover, by corollary 1, the market portfolio is efficient, and its expected return is strictly higher than r_f .

Corollary 2 *Under the conditions of Theorem 3, the market portfolio solves Problem (13) for some $\mu^M > r_f$.*

4.3.3 A Generalized Security Market Line

In the traditional CAPM framework, the security market line describes the equilibrium relation between the expected returns of individual assets and the market

expected return. In particular, the expected return of any asset in excess of the risk-free rate is proportional to the excess market expected return, with the proportion being equal to the traditional beta. The following theorem provides sufficient conditions under which a similar relation holds with respect to our Aumann-Shapley based systematic risk measure.

Theorem 4 *Consider an equilibrium with positive prices and let α^M be the market portfolio. Assume that $R(\cdot)$ is homogeneous of some degree k , smooth, convex, and satisfies the risk-free property. Then, for each asset $i = 1, \dots, n$,*

$$E(\tilde{z}_i) = r_f + \mathcal{B}_i^R (E(\alpha^M \cdot \tilde{\mathbf{z}}) - r_f), \quad (15)$$

where

$$\mathcal{B}_i^R = \frac{R_i(\alpha^M)}{\sum_{j=1}^n \alpha_j^M R_j(\alpha^M)} = \frac{R_i(\alpha^M)}{kR(\alpha^M)}$$

is the scaled Aumann-Shapley measure of systematic risk.

As an illustration, consider

$$R(\cdot) = \theta m_2(\cdot) + (1 - \theta) \sqrt{m_4(\cdot)} \quad (16)$$

for $\theta \in [0, 1]$. This risk measure captures both variance and tail risk, and it satisfies all of the conditions in Theorem 4. Moreover, the utility function V^j derived from this risk measure is a local expected utility in the sense of Machina (1982), see Appendix II. It is straightforward to verify that

$$\begin{aligned} \mathcal{B}_i^R &= \frac{\theta \text{Cov}(\tilde{z}_i, \alpha \cdot \tilde{\mathbf{z}}) + (1 - \theta) (m_4(\alpha \cdot \tilde{\mathbf{z}}))^{-\frac{1}{2}} \text{Cov}(\tilde{z}_i, (\alpha \cdot (\tilde{\mathbf{z}} - E(\tilde{\mathbf{z}})))^3)}{\theta m_2(\alpha \cdot \tilde{\mathbf{z}}) + (1 - \theta) (m_4(\alpha \cdot \tilde{\mathbf{z}}))^{\frac{1}{2}}} \quad (17) \\ &= \frac{R_1(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R_1}(\mathbf{x}) + \frac{R_2(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R_2}(\mathbf{x}), \end{aligned}$$

where $R_1(\cdot) = \theta m_2(\cdot)$ and $R_2(\cdot) = (1 - \theta) \sqrt{m_4(\cdot)}$.

4.3.4 Discussion

Before concluding this section we point out two additional issues. First, while we described sufficient conditions under which two-fund money separation holds and use this to argue that the market portfolio is efficient, these conditions are by no means necessary. Weaker conditions that guarantee two-fund money separation may exist. Further, even when two-fund money separation fails, it does not necessarily mean that market efficiency is rejected. The literature explores market efficiency from both theoretical (see, for example, Dybvig and Ross (1982)) and empirical (see, for example, Levy and Roll (2010)) views. Our generalized SML remains valid as long as we have evidence that the market portfolio is mean-risk efficient.

Second, the SML pricing formula (15) can indeed be applied not only to the market portfolio, but with respect to any efficient portfolio. Specifically, suppose

that α^* represents a portfolio that is mean-risk efficient. Then, it can be easily seen from the proof of Theorem (4) that in equilibrium, we have for each asset i ,

$$E(\tilde{z}_i) = r_f + \frac{R_i(\alpha^*)}{kR(\alpha^*)} (E(\alpha^* \cdot \tilde{\mathbf{z}}) - r_f).$$

5 Conclusion

In this paper we develop a general framework to measuring systematic risk, the contribution of an asset to the risk of a portfolio of assets. Our axiomatic approach specifies four economically meaningful conditions that pin down a unique measure of systematic risk. Our equilibrium approach shows that results attributed to the classic CAPM hold much more broadly. In particular, aspects of the geometry of efficient portfolios, two fund money separation, and the security market line are derived in a setting where risk can account for a variety of attributes. Both approaches lead to the conclusion that systematic risk should be measured as a scaled version of the Aumann-Shapley (1974) diagonal formula.

When risk is confined to measure the variance of a distribution, our systematic risk measure coincides with beta, the slope from regressing asset returns on portfolio returns. More generally, systematic risk is not a regression coefficient. Rather, it should be interpreted as the average of the marginal contributions of an asset to the risk of portfolios along a diagonal from the original to the portfolio of interest. This interpretation takes a particularly simple form when the risk measure is homogeneous. In these cases systematic risk is simply the marginal contribution of the asset to the risk of the portfolio of interest, scaled by the weighted average of all such marginal contributions.

Our axiomatic approach applies to a wide variety of risk measures, requiring of them only smoothness and zero risk for zero investment. The equilibrium framework imposes additional conditions in the form of homogeneity, convexity, and the risk-free property. Nevertheless, even in the equilibrium framework we are still left with an extensive class of risk measures. Indeed, this class is sufficiently broad to potentially account for high distribution moments, rare disasters, downside risk, as well as many other aspects of risk. Future research may direct at developing weaker conditions that further expand the set of applicable risk measures that can be studied in an equilibrium framework.

Finally, our framework is agnostic regarding the choice of a particular risk measure. Indeed, which risk measures better capture the risk preferences of investors is ultimately an empirical question. Our framework therefore provides foundations for testing the appropriateness of risk measures and consequently selecting those that are supported by the data.

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Appendix I: Proofs

Proof of Theorem 1: Consider a risk allocation problem (R, \mathbf{x}) of order n given some underlying risky assets $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$. Since R is smooth and satisfies $R(0) = 0$ we can view (R, \mathbf{x}) as a cost allocation problem as defined in Billera and Heath (1982, hereafter BH). Let c denote a cost allocation procedure as defined in BH. That is, for each cost allocation problem (R, \mathbf{x}) of order n , $c(R, \mathbf{x}) \in \mathbb{R}^n$, and should be interpreted as the cost allocated to each of the n goods or services. We can then consider a natural mapping between systematic risk measures and cost allocation procedures as follows. If $\mathbf{B}^R(\mathbf{x})$ is a systematic risk measure of the risk allocation problem (R, \mathbf{x}) , then

$$c(R, \mathbf{x}) = \frac{\mathbf{B}^R(\mathbf{x}) R(\mathbf{x} \cdot \tilde{\mathbf{z}})}{\bar{x}}, \quad (18)$$

is a cost allocation procedure for the corresponding cost allocation problem (R, \mathbf{x}) . Namely, risk allocation measures can be viewed as scaled versions of cost allocation procedures for the corresponding problems.

Claim 5 *A systematic risk measure $\mathbf{B}^R(\mathbf{x})$ satisfies Axioms 1-4 if and only if the corresponding cost allocation procedure $c(R, \mathbf{x})$ satisfies Conditions (2.1)-(2.4) in BH.*

It is important to note that we do not argue that Axioms 1-4 and Conditions (2.1)-(2.4) in BH are equivalent individually. Rather, our four axioms as a set are equivalent to their four conditions as a set.

The proof of this claim follows from the following four steps, which apply to any risk allocation problem and corresponding cost allocation problem of order n .

Step 1. Axiom 1 is satisfied if and only if Condition (2.1) holds. Indeed, $\sum_{i=1}^n \alpha_i \mathcal{B}_i^R(\mathbf{x}) = 1$ is equivalent to $\sum_{i=1}^n \frac{x_i R(\mathbf{x}) \mathcal{B}_i^R(\mathbf{x})}{\bar{x}} = R(\mathbf{x})$, which using (18) is equivalent to $\sum_{i=1}^n x_i c_i(R, \mathbf{x}) = R(\mathbf{x})$. This is Condition (2.1).

Step 2. Axiom 2 is satisfied if and only if Condition (2.2) holds. Indeed, suppose $R(\cdot) = R^1(\cdot) + R^2(\cdot)$ and

$$\mathcal{B}_i^R(\mathbf{x}) = \frac{R^1(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R^1}(\mathbf{x}) + \frac{R^2(\mathbf{x})}{R(\mathbf{x})} \mathcal{B}_i^{R^2}(\mathbf{x}).$$

Then

$$\frac{\mathcal{B}_i^R(\mathbf{x}) R(\mathbf{x})}{\bar{x}} = \frac{R^1(\mathbf{x}) \mathcal{B}_i^{R^1}(\mathbf{x})}{\bar{x}} + \frac{R^2(\mathbf{x}) \mathcal{B}_i^{R^2}(\mathbf{x})}{\bar{x}}.$$

That is,

$$c(R, \mathbf{x}) = c(R^1, \mathbf{x}) + c(R^2, \mathbf{x}),$$

which is Condition (2.2). The other direction is similar (recalling that $R(\mathbf{x}) \neq 0$).

Step 3. Condition (2.3) implies Axiom 3. Axioms 1 and 3 jointly imply Condition (2.3).

Assume first that Condition (2.3) in BH holds. Suppose that $\tilde{\mathbf{z}} = (\tilde{z}_1, \dots, \tilde{z}_n)$ are R -perfectly correlated. Then, for any $\boldsymbol{\eta} \in \mathbb{R}^n$, $R(\boldsymbol{\eta} \cdot \tilde{\mathbf{z}}) = g(\boldsymbol{\eta} \cdot \mathbf{q})$ for some function $g(\cdot)$ and a non-zero $\mathbf{q} \in \mathbb{R}^n$. Condition (2.3) implies that $c_i(R, \mathbf{x}) = c(g, \mathbf{x} \cdot \mathbf{q}) q_i$ for $i = 1, \dots, n$. It follows that for all $i, j = 1, \dots, n$

$$q_j c_i(R, \mathbf{x}) = q_i c_j(R, \mathbf{x}).$$

Applying (18) we obtain that Axiom 3 is satisfied.

Assume now that both Axioms 1 and 3 are satisfied and assume that for all $\boldsymbol{\eta} \in \mathbb{R}^n$,

$$R(\boldsymbol{\eta} \cdot \tilde{\mathbf{z}}) = g(\boldsymbol{\eta} \cdot \mathbf{q}) \quad (19)$$

for some function $g(\cdot)$ and a non-zero vector $\mathbf{q} \in \mathbb{R}^n$. Then, $(\tilde{z}_1, \dots, \tilde{z}_n)$ are R -perfectly correlated.

By Axiom 3 for all $i, j = 1, \dots, n$

$$q_j \mathcal{B}_i^R(\mathbf{x}) = q_i \mathcal{B}_j^R(\mathbf{x}). \quad (20)$$

By Axiom 1 we know that $\sum_{i=1}^k \alpha_i \mathcal{B}_i^R(\mathbf{x}) = 1$. Plugging (20) we have

$$q_j = \sum_{i=1}^n \alpha_i q_i \mathcal{B}_j^R(\mathbf{x}) = (\boldsymbol{\alpha} \cdot \mathbf{q}) \mathcal{B}_j^R(\mathbf{x}) \text{ for } j = 1, \dots, n.$$

By (18), and recalling that $R(\mathbf{x}) \neq 0$

$$q_j = (\boldsymbol{\alpha} \cdot \mathbf{q}) \frac{c_j(R, \mathbf{x}) \bar{x}}{R(\mathbf{x})} = (\mathbf{x} \cdot \mathbf{q}) \frac{c_j(R, \mathbf{x})}{R(\mathbf{x})} \text{ for } j = 1, \dots, n. \quad (21)$$

If $\mathbf{x} \cdot \mathbf{q} = \mathbf{0}$ this implies that $q_j = 0$ for all j , contradicting that \mathbf{q} is a non-zero vector. Hence, $\mathbf{x} \cdot \mathbf{q}$ is not zero. We then have

$$c_j(R, \mathbf{x}) = \frac{q_j R(\mathbf{x})}{(\mathbf{x} \cdot \mathbf{q})} \text{ for all } j = 1, \dots, n. \quad (22)$$

Consider an asset with return $\tilde{w} = \frac{\mathbf{x} \cdot \tilde{\mathbf{z}}}{\mathbf{x} \cdot \mathbf{q}}$. Namely, investing $\mathbf{x} \cdot \mathbf{q}$ dollars in this asset yields the same return as of the portfolio \mathbf{x} . Then,

$$R((\mathbf{x} \cdot \mathbf{q}) \tilde{w}) = R(\mathbf{x} \cdot \tilde{\mathbf{z}}) = g(\mathbf{x} \cdot \mathbf{q}).$$

Consider now the risk allocation problem of order 1 with the single asset \tilde{w} held at the amount $\mathbf{x} \cdot \mathbf{q}$. By Axiom 1 the systematic risk measure of this asset must satisfy

$$\mathcal{B}^R(\mathbf{x} \cdot \mathbf{q}) = 1,$$

or equivalently using (18),

$$c(g, \mathbf{x} \cdot \mathbf{q}) = \frac{R((\mathbf{x} \cdot \mathbf{q}) \tilde{w})}{\mathbf{x} \cdot \mathbf{q}} = \frac{g(\mathbf{x} \cdot \mathbf{q})}{\mathbf{x} \cdot \mathbf{q}}.$$

Plugging back into (22) and using that $R(\mathbf{x}) = g(\mathbf{x} \cdot \mathbf{q})$ we have

$$c_j(R, \mathbf{x}) = c(g, \mathbf{x} \cdot \mathbf{q}) q_j,$$

which is exactly what Condition (2.3) in BH requires.

Step 4. Axiom 4 is satisfied if and only if Condition (2.4) holds. This follows directly from (18) and the definition of R -positive correlation.

Having established this claim we can now rely on the main theorem in BH to conclude that the unique cost allocation procedure associated with any risk allocation problem (R, \mathbf{x}) (and its corresponding cost allocation problem) of order n satisfying Axioms 1-4 is given by.

$$c_i(R, \mathbf{x}) = \int_0^1 R_i(t\mathbf{x}) dt.$$

Applying (18) we obtain that the unique systematic risk measure satisfying Axioms 1-4 is given by (8).

Finally, to see that (8) and (9) are equivalent when R is homogenous of degree k , note first that in this case

$$\begin{aligned} \int_0^1 R_i(tx_1, \dots, tx_n) dt &= R_i(x_1, \dots, x_n) \int_0^1 t^{k-1} dt \\ &= \frac{R_i(x_1, \dots, x_n)}{k}, \end{aligned}$$

where the first equality follows since R_i is homogeneous of degree $k - 1$. It follows that

$$\begin{aligned} \mathcal{B}_i^R(\mathbf{x}) &= \frac{\bar{x} \int_0^1 R_i(tx_1, \dots, tx_n) dt}{R(x_1, \dots, x_n)} \\ &= \frac{\bar{x} R_i(x_1, \dots, x_n)}{k R(x_1, \dots, x_n)} \\ &= \frac{\bar{x} R_i(\bar{x}\alpha_1, \dots, \bar{x}\alpha_n)}{k R(\bar{x}\alpha_1, \dots, \bar{x}\alpha_n)} \\ &= \frac{R_i(\alpha_1, \dots, \alpha_n)}{k R(\alpha_1, \dots, \alpha_n)} \\ &= \frac{R_i(\alpha_1, \dots, \alpha_n)}{\sum_{j=1}^n \alpha_j R_j(\alpha_1, \dots, \alpha_n)}, \end{aligned}$$

where the penultimate equality follows from the homogeneity of degrees k and $k - 1$ of R and R_i respectively, and the last equality follows from Euler's homogeneous function theorem. This completes the proof of Theorem 1. ■

Proof of Proposition 1: The partial derivatives of $R(\cdot)$ with respect to the portfolio weights are given by $\forall i$,

$$\begin{aligned} R_i(\alpha_1, \dots, \alpha_n) &= k\mathbf{E} \left[(\tilde{z}_i - \mathbf{E}(\tilde{z}_i)) \left(\sum_{j=1}^n \alpha_j (\tilde{z}_j - \mathbf{E}(\tilde{z}_j)) \right)^{k-1} \right] \\ &= k\mathbf{E} \left[(\tilde{z}_i - \mathbf{E}(\tilde{z}_i)) (\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} - \boldsymbol{\alpha} \cdot \mathbf{E}(\tilde{\mathbf{z}}))^{k-1} \right] \\ &= k\text{Cov} \left(\tilde{z}_i, (\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} - \boldsymbol{\alpha} \cdot \mathbf{E}(\tilde{\mathbf{z}}))^{k-1} \right). \end{aligned}$$

Thus, using (10), the systematic risk of asset i becomes

$$\mathcal{B}_i^R = \frac{\text{Cov} \left(\tilde{z}_i, (\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} - \boldsymbol{\alpha} \cdot \mathbf{E}(\tilde{\mathbf{z}}))^{k-1} \right)}{\mathbf{E} \left[(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} - \boldsymbol{\alpha} \cdot \mathbf{E}(\tilde{\mathbf{z}}))^k \right]}.$$

■

Proof of Proposition 2: When $R(\cdot) = R^{AS}(\cdot)$, $R(\boldsymbol{\alpha})$ is implicitly determined by

$$\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \right] = 1.$$

By the implicit function theorem,

$$\begin{aligned} R_i(\boldsymbol{\alpha}) &= \frac{\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \frac{\tilde{z}_i}{R(\boldsymbol{\alpha})} \right]}{\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{(R(\boldsymbol{\alpha}))^2} \right]} \\ &= R(\boldsymbol{\alpha}) \frac{\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \tilde{z}_i \right]}{\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} \right]}. \end{aligned}$$

Since the AS measure is homogeneous of degree 1, (10) becomes

$$\mathcal{B}_i^R = \frac{\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \tilde{z}_i \right]}{\mathbf{E} \left[\exp \left(-\frac{\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}}}{R(\boldsymbol{\alpha})} \right) \boldsymbol{\alpha} \cdot \tilde{\mathbf{z}} \right]}.$$

The proof for the FH measure is similar. ■

Proof of Proposition 3: What we need to show is that for any random returns \tilde{z}_1 and \tilde{z}_2 , and any $0 \leq \lambda \leq 1$,

$$w_k(\lambda \tilde{z}_1 + (1 - \lambda) \tilde{z}_2) \leq \lambda w_k(\tilde{z}_1) + (1 - \lambda) w_k(\tilde{z}_2). \quad (23)$$

Letting $\hat{z}_1 = \tilde{z}_1 - \mathbf{E}(\tilde{z}_1)$ and $\hat{z}_2 = \tilde{z}_2 - \mathbf{E}(\tilde{z}_2)$, (23) can be rewritten as

$$\left(\mathbf{E} \left[(\lambda \hat{z}_1 + (1 - \lambda) \hat{z}_2)^k \right] \right)^{\frac{1}{k}} \leq \lambda \left(\mathbf{E} \left[\hat{z}_1^k \right] \right)^{\frac{1}{k}} + (1 - \lambda) \left(\mathbf{E} \left[\hat{z}_2^k \right] \right)^{\frac{1}{k}}. \quad (24)$$

From the binomial formula we know that for any two numbers p and q ,

$$(p + q)^k = \sum_{i=0}^k \binom{k}{i} p^{k-i} q^i.$$

Applying this to the LHS of (24) implies that we need to show

$$\left(\sum_{i=0}^k \binom{k}{i} \lambda^{k-i} (1-\lambda)^i \mathbb{E} \left(\hat{z}_1^{k-i} \hat{z}_2^i \right) \right)^{\frac{1}{k}} \leq \lambda \left(\mathbb{E} \left[\hat{z}_1^k \right] \right)^{\frac{1}{k}} + (1-\lambda) \left(\mathbb{E} \left[\hat{z}_2^k \right] \right)^{\frac{1}{k}}.$$

Since k is even, replacing each \hat{z}_1 and \hat{z}_2 with $|\hat{z}_1|$ and $|\hat{z}_2|$ will not affect the RHS, but it might increase the LHS. So, it is sufficient to show that

$$\left(\sum_{i=0}^k \binom{k}{i} \lambda^{k-i} (1-\lambda)^i \mathbb{E} \left(\left| \hat{z}_1^{k-i} \hat{z}_2^i \right| \right) \right)^{\frac{1}{k}} \leq \lambda \left(\mathbb{E} \left[|\hat{z}_1|^k \right] \right)^{\frac{1}{k}} + (1-\lambda) \left(\mathbb{E} \left[|\hat{z}_2|^k \right] \right)^{\frac{1}{k}}.$$

Since both sides are positive we can raise both sides to the k^{th} power, maintaining the inequality. Thus, it would be sufficient to show

$$\sum_{i=0}^k \binom{k}{i} \lambda^{k-i} (1-\lambda)^i \mathbb{E} \left(\left| \hat{z}_1^{k-i} \hat{z}_2^i \right| \right) \leq \left(\lambda \left(\mathbb{E} \left[|\hat{z}_1|^k \right] \right)^{\frac{1}{k}} + (1-\lambda) \left(\mathbb{E} \left[|\hat{z}_2|^k \right] \right)^{\frac{1}{k}} \right)^k.$$

Applying the binomial formula to the RHS implies that it would be sufficient to show

$$\sum_{i=0}^k \binom{k}{i} \lambda^{k-i} (1-\lambda)^i \mathbb{E} \left(\left| \hat{z}_1^{k-i} \hat{s}^i \right| \right) \leq \sum_{i=0}^k \binom{k}{i} \lambda^{k-i} (1-\lambda)^i \left(\mathbb{E} \left[|\hat{z}_1|^k \right] \right)^{\frac{k-i}{k}} \left(\mathbb{E} \left[|\hat{z}_2|^k \right] \right)^{\frac{i}{k}}.$$

To establish this inequality we will show that it actually holds term by term. That is, it is sufficient to show that for each $i = 0, \dots, k$,

$$\mathbb{E} \left(\left| \hat{z}_1^{k-i} \hat{z}_2^i \right| \right) \leq \left(\mathbb{E} \left[|\hat{z}_1|^k \right] \right)^{\frac{k-i}{k}} \left(\mathbb{E} \left[|\hat{z}_2|^k \right] \right)^{\frac{i}{k}}.$$

To see this, note that it is equivalent to show that

$$\mathbb{E} \left(\left| \hat{z}_1^{k-i} \hat{s}^i \right| \right) \leq \left(\mathbb{E} \left[\left| \hat{z}_1^{k-i} \right|^{\frac{k}{k-i}} \right] \right)^{\frac{k-i}{k}} \left(\mathbb{E} \left[\left| \hat{z}_2^i \right|^{\frac{k}{i}} \right] \right)^{\frac{i}{k}}.$$

Now, denote $Z_1 = \hat{z}_1^{k-i}$, $Z_2 = \hat{z}_2^i$, $p = \frac{k}{k-i}$ and $q = \frac{k}{i}$. Note that $\frac{1}{p} + \frac{1}{q} = 1$. Then, what we need to show is

$$\mathbb{E} (|Z_1 Z_2|) \leq \mathbb{E} (|Z_1|^p)^{\frac{1}{p}} \mathbb{E} (|Z_2|^q)^{\frac{1}{q}}.$$

But, this is immediate from Holder's inequality, and we are done. ■

Proof of Proposition 4: In the definition of Expected Short-fall we assumed the existence of a cumulative distribution function $F(\cdot)$ applied to realizations of random

variables. For the sake of this proof it will be more useful to work directly with the probability space Ω and with the underlying probability measure $P(\cdot)$. We first prove that $\text{ES}_\alpha(\tilde{z})$ is subadditive. That is, for any two random returns \tilde{z}_1 and \tilde{z}_2 ,

$$\text{ES}_\alpha(\tilde{z}_1 + \tilde{z}_2) \leq \text{ES}_\alpha(\tilde{z}_1) + \text{ES}_\alpha(\tilde{z}_2). \quad (25)$$

By definition, for any random return \tilde{z} , $\text{ES}_\alpha(\tilde{z})$ can be expressed as

$$\text{ES}_\alpha(\tilde{z}) = -\frac{1}{\alpha} \int_{\{\omega: \tilde{z} \leq -\text{VaR}_\alpha(\tilde{z})\}} \tilde{z} dP(\omega).$$

Define $\tilde{z}_3 = \tilde{z}_1 + \tilde{z}_2$. Let $\Omega_i = \{\omega \in \Omega : \tilde{z}_i \leq -\text{VaR}_\alpha(\tilde{z}_i)\}$ for $i = 1, 2, 3$. Then, (25) is equivalent to

$$\int_{\Omega_3} \tilde{z}_3 dP(\omega) \geq \int_{\Omega_1} \tilde{z}_1 dP(\omega) + \int_{\Omega_2} \tilde{z}_2 dP(\omega),$$

which can be rewritten as

$$\int_{\Omega_3} \tilde{z}_1 dP(\omega) + \int_{\Omega_3} \tilde{z}_2 dP(\omega) \geq \int_{\Omega_1} \tilde{z}_1 dP(\omega) + \int_{\Omega_2} \tilde{z}_2 dP(\omega).$$

This is true if we have

$$\int_{\Omega_3} \tilde{z}_1 dP(\omega) \geq \int_{\Omega_1} \tilde{z}_1 dP(\omega), \quad (26)$$

and

$$\int_{\Omega_3} \tilde{z}_2 dP(\omega) \geq \int_{\Omega_2} \tilde{z}_2 dP(\omega). \quad (27)$$

For brevity, we will only prove (26) below. The proof of (27) is parallel.

To prove (26), define $\Omega_4 = \{\omega \in \Omega : \tilde{z}_1 \leq -\text{VaR}_\alpha(\tilde{z}_1), \tilde{z}_3 \leq -\text{VaR}_\alpha(\tilde{z}_3)\}$, $\Omega_5 = \{\omega \in \Omega : \tilde{z}_1 \leq -\text{VaR}_\alpha(\tilde{z}_1), \tilde{z}_3 > -\text{VaR}_\alpha(\tilde{z}_3)\}$ and $\Omega_6 = \{\omega \in \Omega : \tilde{z}_1 > -\text{VaR}_\alpha(\tilde{z}_1), \tilde{z}_3 \leq -\text{VaR}_\alpha(\tilde{z}_3)\}$. Since $\Omega_4 \cap \Omega_5 = \emptyset$, $\Omega_4 \cup \Omega_5 = \Omega_1$, and $\Omega_4 \cap \Omega_6 = \emptyset$, $\Omega_4 \cup \Omega_6 = \Omega_3$, we have that

$$\int_{\Omega_1} dP(\omega) = \int_{\Omega_4} dP(\omega) + \int_{\Omega_5} dP(\omega),$$

and

$$\int_{\Omega_3} dP(\omega) = \int_{\Omega_4} dP(\omega) + \int_{\Omega_6} dP(\omega).$$

By the definition of VaR, we know

$$\int_{\Omega_1} dP(\omega) = \int_{\Omega_3} dP(\omega) = \alpha.$$

Thus, we obtain

$$\int_{\Omega_5} dP(\omega) = \int_{\Omega_6} dP(\omega). \quad (28)$$

Similarly, we know

$$\int_{\Omega_1} \tilde{z}_1 dP(\omega) = \int_{\Omega_4} \tilde{z}_1 dP(\omega) + \int_{\Omega_5} \tilde{z}_1 dP(\omega),$$

and

$$\int_{\Omega_3} \tilde{z}_1 dP(\omega) = \int_{\Omega_4} \tilde{z}_1 dP(\omega) + \int_{\Omega_6} \tilde{z}_1 dP(\omega).$$

Hence,

$$\begin{aligned} & \int_{\Omega_1} \tilde{z}_1 dP(\omega) - \int_{\Omega_3} \tilde{z}_1 dP(\omega) \\ = & \int_{\Omega_5} \tilde{z}_1 dP(\omega) - \int_{\Omega_6} \tilde{z}_1 dP(\omega) \\ \leq & \int_{\Omega_5} [-\text{VaR}_\alpha(\tilde{z}_1)] dP(\omega) - \int_{\Omega_6} [-\text{VaR}_\alpha(\tilde{z}_1)] dP(\omega) \\ = & -\text{VaR}_\alpha(\tilde{z}_1) \left[\int_{\Omega_5} dP(\omega) - \int_{\Omega_6} dP(\omega) \right], \end{aligned}$$

where the inequality follows from $\tilde{z}_1 \leq -\text{VaR}_\alpha(\tilde{z}_1)$ when $\omega \in \Omega_5$ and $\tilde{z}_1 > -\text{VaR}_\alpha(\tilde{z}_1)$ when $\omega \in \Omega_6$. By (28), we have

$$\int_{\Omega_5} dP(\omega) - \int_{\Omega_6} dP(\omega) = 0,$$

which implies

$$\int_{\Omega_1} \tilde{z}_1 dP(\omega) - \int_{\Omega_3} \tilde{z}_1 dP(\omega) \leq 0.$$

Therefore, (26) is obtained, and hence $\text{ES}_\alpha(\tilde{z})$ is subadditive.

Then, convexity follows immediately from homogeneity and subadditivity, since for any $\lambda \in [0, 1]$,

$$\begin{aligned} \text{ES}_\alpha(\lambda \tilde{z}_1 + (1 - \lambda) \tilde{z}_2) & \leq \text{ES}_\alpha(\lambda \tilde{z}_1) + \text{ES}_\alpha((1 - \lambda) \tilde{z}_2) \\ & = \lambda \text{ES}_\alpha(\tilde{z}_1) + (1 - \lambda) \text{ES}_\alpha(\tilde{z}_2). \end{aligned}$$

■

Proof of Lemma 2: Our setting is a special case of the setting in Nielsen (1989). To show the existence of equilibrium Nielsen requires that preferences satisfy the following three conditions: (i) each investor's choice set is closed and convex, and contains her initial endowment; (ii) The set of $\{\zeta \in \mathbb{R}^{n+1} : U^j(\zeta) \geq U^j(\zeta')\}$ is closed for all $\zeta' \in \mathbb{R}^{n+1}$ and for all $j = \{1, \dots, \ell\}$; (iii) If $\zeta, \zeta' \in \mathbb{R}^{n+1}$ and $U^j(\zeta') > U^j(\zeta)$, then $U^j(t\zeta' + (1-t)\zeta) > U^j(\zeta)$ for all t in $(0, 1)$.

Condition (i) is satisfied in our setting since the choice set of each investor is \mathbb{R}^{n+1} , which is closed and convex, and contains e^j for all j . Condition (ii) holds since V is assumed continuous and R is assumed smooth, and so their composition is continuous. Condition (iii) follows since $V(\cdot)$ is quasi-concave, strictly increasing in its first argument and strictly decreasing in its second argument, and $R(\cdot)$ is a convex risk measure.

Given these properties of the preferences, Nielsen (1989) establishes two conditions as sufficient for the existence of a quasi-equilibrium: (i) positive semi-independence

of directions of improvement, and (ii) non-satiation at Pareto attainable portfolios. Condition (i) follows in our setting as in Nielsen (1990, Proposition 1) since in our setting all investors agree on all parameters of the problem (in particular on the expected returns), and due to the non-redundancy of risky assets assumption. To see why condition (ii) holds in our setting note that we assume the existence of a risk-free asset paying a non-zero payoff with probability 1. Since $R(\cdot)$ satisfies the risk-free property, we have that $R(\tilde{z}_1 + \tilde{z}_2) \leq R(\tilde{z}_1)$ whenever \tilde{z}_2 is R -risk-free with $P(\{\tilde{z}_2 > 0\}) = 1$. Thus, adding a positive risk-free asset can only (weakly) reduce risk. It follows that we can always add this positive risk-free asset to any bundle ζ , strictly increasing the expected return while weakly decreasing risk. This implies that in our model there is no satiation globally. Thus, a quasi-equilibrium exists in our setting. Moreover, any quasi-equilibrium is, in fact, an equilibrium in our setting. This follows from the conditions in Nielsen (1989 p. 469). Indeed, in our setting each investor's choice set is convex and unbounded, and the set $\{\zeta \in \mathbb{R}^{n+1} : U^j(\zeta) > U^j(\zeta')\}$ is open for all j and $\zeta' \in \mathbb{R}^{n+1}$. ■

Proof of Theorem 2: Suppose that the equilibrium bundle of investor j is ζ^j . Let $\bar{x}^j = \sum_{i=0}^n x_i^j = \mathbf{p} \cdot \zeta^j$ be the total dollar amount of investment of investor j . Then,

$$\begin{aligned}
U^j(\zeta^j) &= V^j \left(\mathbb{E} \left(\sum_{i=0}^n \zeta_i^j \tilde{y}_i \right), R \left(\sum_{i=0}^n \zeta_i^j \tilde{y}_i \right) \right) \\
&= V^j \left(\bar{x}^j \mathbb{E} \left(\sum_{i=0}^n \frac{\zeta_i^j}{\bar{x}^j} \tilde{y}_i \right), (\bar{x}^j)^k R \left(\sum_{i=0}^n \frac{\zeta_i^j}{\bar{x}^j} \tilde{y}_i \right) \right) \\
&= V^j \left(\bar{x}^j \mathbb{E} \left(\sum_{i=0}^n \frac{\zeta_i^j p_i}{\bar{x}^j p_i} \tilde{y}_i \right), (\bar{x}^j)^k R \left(\sum_{i=0}^n \frac{\zeta_i^j p_i}{\bar{x}^j p_i} \tilde{y}_i \right) \right) \\
&= V^j \left(\bar{x}^j \mathbb{E} \left(\sum_{i=0}^n \frac{x_i^j}{\bar{x}^j} \tilde{z}_i \right), (\bar{x}^j)^k R \left(\sum_{i=0}^n \frac{x_i^j}{\bar{x}^j} \tilde{z}_i \right) \right) \\
&= V^j \left(\bar{x}^j \mathbb{E} \left(\sum_{i=0}^n \alpha_i^j \tilde{z}_i \right), (\bar{x}^j)^k R \left(\sum_{i=0}^n \alpha_i^j \tilde{z}_i \right) \right) \\
&= V^j \left(\bar{x}^j \mathbb{E}(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}}), (\bar{x}^j)^k R(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}}) \right),
\end{aligned} \tag{29}$$

where k is the degree of homogeneity of $R(\cdot)$. From the definition of equilibrium, each investor chooses ζ^j to maximize $U^j(\zeta^j)$ subject to $\bar{x}^j \leq \mathbf{p} \cdot \mathbf{e}^j$, where in equilibrium $\bar{x}^j = \mathbf{p} \cdot \mathbf{e}^j$ is given. From (29) and since V^j is strictly increasing in the first argument and strictly decreasing in the second argument, we have that for any positive \bar{x}^j , $U^j(\zeta^j)$ is strictly increasing in $\mathbb{E}(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}})$ and strictly decreasing in $R(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}})$. Therefore, in equilibrium, $\boldsymbol{\alpha}^j$ must minimize $R(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}})$ for a given level of expected return $\mathbb{E}(\boldsymbol{\alpha}^j \cdot \tilde{\mathbf{z}})$, and thus solve Problem (13). The solution is unique since we assumed that $R(\cdot)$ is a convex risk measure, and so $R(\boldsymbol{\alpha} \cdot \tilde{\mathbf{z}})$ is convex as a function of $\boldsymbol{\alpha}$. ■

Proof of Theorem 4: Since $R(\cdot)$ is smooth and by Lemma 2 the solution to Problem (13) for some $\mu^j = \mu$ is determined by the first order conditions. To solve this program, form the Lagrangian

$$\mathcal{L}(\boldsymbol{\alpha}) = R(\boldsymbol{\alpha}) - \lambda \left(\sum_{i=1}^n \alpha_i E(\tilde{z}_i) + \left(1 - \sum_{i=1}^n \alpha_i\right) r_f - \mu \right),$$

where λ is a Lagrange multiplier. Equivalently,

$$\mathcal{L}(\boldsymbol{\alpha}) = R\left(1 - \sum_{i=1}^n \alpha_i, \alpha_1, \dots, \alpha_n\right) - \lambda \left(\sum_{i=1}^n \alpha_i E(\tilde{z}_i) + \left(1 - \sum_{i=1}^n \alpha_i\right) r_f - \mu \right)$$

The first order condition states that $\forall i = 1, \dots, n$,

$$-R_0(\boldsymbol{\alpha}^*) + R_i(\boldsymbol{\alpha}^*) - \lambda(E(\tilde{z}_i) - r_f) = 0, \quad (30)$$

where $\boldsymbol{\alpha}^*$ represents the optimal portfolio. By the risk-free property, $R_0(\boldsymbol{\alpha}^*) = 0$. Hence,

$$R_i(\boldsymbol{\alpha}^*) - \lambda(E(\tilde{z}_i) - r_f) = 0. \quad (31)$$

Multiplying by α_i^* , we obtain

$$\alpha_i^* R_i(\boldsymbol{\alpha}^*) - \lambda \alpha_i^* (E(\tilde{z}_i) - r_f) = 0.$$

Summing up over $i = 1, \dots, n$ yields

$$\sum_{i=1}^n \alpha_i^* R_i(\boldsymbol{\alpha}^*) - \lambda \sum_{i=1}^n \alpha_i^* (E(\tilde{z}_i) - r_f) = 0.$$

Noting again that $R_0(\boldsymbol{\alpha}^*) = 0$, this is equivalent to

$$\sum_{i=0}^n \alpha_i^* R_i(\boldsymbol{\alpha}^*) - \lambda \sum_{i=1}^n \alpha_i^* (E(\tilde{z}_i) - r_f) = 0.$$

By Euler's homogeneous function theorem, we can rewrite this as

$$kR(\boldsymbol{\alpha}^*) - \lambda \sum_{i=1}^n \alpha_i^* (E(\tilde{z}_i) - r_f) = 0.$$

Adding and subtracting r_f inside the summation yields

$$kR(\boldsymbol{\alpha}^*) - \lambda \left(\sum_{i=1}^n \alpha_i^* E(\tilde{z}_i) + r_f \left(1 - \sum_{i=1}^n \alpha_i^*\right) - r_f \right) = 0,$$

which by the optimization constraint is equivalent to

$$kR(\boldsymbol{\alpha}^*) - \lambda(\mu - r_f) = 0.$$

Hence,

$$\lambda = \frac{kR(\boldsymbol{\alpha}^*)}{\mu - r_f}. \quad (32)$$

Plugging (32) into (31) gives

$$R_i(\boldsymbol{\alpha}^*) - \frac{kR(\boldsymbol{\alpha}^*)}{\mu - r_f} (\mathbb{E}(\tilde{z}_i) - r_f) = 0,$$

which can be rewritten as

$$\mathbb{E}(\tilde{z}_i) = r_f + \frac{R_i(\boldsymbol{\alpha}^*)}{kR(\boldsymbol{\alpha}^*)} (\mu - r_f). \quad (33)$$

This result applies to any efficient portfolio $\boldsymbol{\alpha}^*$. By Corollary 2, the market portfolio is efficient, and hence

$$\mathbb{E}(\tilde{z}_i) = r_f + \frac{R_i(\boldsymbol{\alpha}^M)}{kR(\boldsymbol{\alpha}^M)} (\mu^M - r_f).$$

as claimed. ■

Appendix II: Mean-Risk Preferences and Expected Utility

Background

One would wonder how the mean-risk preferences considered in Section 4 are related to the commonly assumed von Neumann-Morgenstern utility. It is widely known that a von Neumann-Morgenstern investor with a quadratic utility function only cares about the mean and the variance of his investments in the sense that he prefers a high expected wealth and a low variance. In this sense, the mean-risk preference is consistent with the von Neumann-Morgenstern utility when variance is used as the risk measure. Alternatively, when returns are distributed according to a two-parameter elliptical distribution (normal being a special case), mean-variance preferences can also be supported by expected utility. These instances, however, are quite restrictive. First, the quadratic utility is not very intuitive since it implies increasing absolute risk aversion. Second, elliptical distributions, being determined by the first two moments only, limit our ability to describe the dependence of risk on high distribution moments. Thus, in general, mean-variance preferences are not consistent with expected utility. The approach taken in this paper is much more general, allowing for a variety of risk measures. Whether a particular risk measure is consistent with expected utility depends on the actual choice of the risk measure. For example, risk measures that are simple linear combinations of raw moments up to the k^{th} degree can be represented by a k^{th} degree polynomial (Müller and Machina (1987)), generalizing the mean-variance result.

While in general the preferences defined in (12) cannot be supported by expected utility, they are often consistent with expected utility *locally*. The idea is based on

Machina’s (1982) “Local Utility Function.” To facilitate this approach we first restrict attention to risk measures that depend on the distribution of the random variables only. Thus, we consider risk measures that are functions from the distribution of realizations to the reals rather than functions from the random variables themselves. Practically, this does not present a binding restriction since all the examples in this paper and all standard risk measures only rely on the distribution of realizations anyway. In this case the preferences in (12) can be written as

$$U(\zeta) = V(E(F_{\zeta \cdot \tilde{y}}), R(F_{\zeta \cdot \tilde{y}})),$$

where $F_{\zeta \cdot \tilde{y}}$ is the cumulative distribution of the random variable $\zeta \cdot \tilde{y}$. When the random variable in question is clear, we will omit it from the notation and write the utility as $U(F) = V(E(F), R(F))$.

According to Machina (1982), if the realizations of all random variables are contained in some bounded and closed interval I and $U(F)$ is Fréchet differentiable with respect to the L^1 norm,⁹ then for any two distributions F_1, F_2 on I there exists $u(\cdot; F_1)$ differentiable almost everywhere on I such that

$$U(F_2) - U(F_1) = \int_I u(y; F_1) dF_2(y) - \int_I u(y; F_1) dF_1(y) + o(\|F_2 - F_1\|), \quad (34)$$

where $\|\cdot\|$ denotes the L^1 norm. That is, starting from a wealth distribution F_1 , if an investor moves to another “close” distribution F_2 , then he compares the utility from these two distributions as if he is maximizing his expected utility with a local utility function $u(\cdot; F_1)$.

The key to applying Machina’s result is to find sufficient conditions on the risk measure which guarantee that $U(F)$ is Fréchet differentiable. This can be done in many ways. Next we provide one simple but effective approach which is sufficient to validate many popular risk measures as consistent with local expected utility.

Risk Measures as Functions of Moments

Let $\mu_k^F = \int y^k dF(y)$ be the k^{th} raw moment given distribution F , and $m_k^F = \int (y - \mu_1^F)^k dF(y)$ be the k^{th} central moment given distribution F . Consider risk measures which are a function of a finite number of (raw or central) moments. We denote such risk measures by $R(\mu_{j_1}^F, \dots, \mu_{j_l}^F, m_{k_1}^F, \dots, m_{k_n}^F)$. We assume that R is differentiable in all arguments. The utility function in (12) then takes the form

$$U(F) = V(\mu_1^F, R(\mu_{j_1}^F, \dots, \mu_{j_l}^F, m_{k_1}^F, \dots, m_{k_n}^F)), \quad (35)$$

where V is differentiable in both mean and risk. This class of utility functions is quite general and it allows the risk measure to depend on a large number of high distribution moments. We then have the following proposition.

⁹Fréchet differentiability is an infinite dimensional version of differentiability. The idea here is that $U(F)$ changes smoothly with F , where changes in F are topologized using the L^1 norm. See Luenberger (1969 p. 171).

Proposition 5 *If $U(F)$ takes the form (35) then for any two distributions F_1, F_2 on I there exists $u(\cdot; F_1)$ differentiable almost everywhere on I such that (34) holds.*

Proof of Proposition 5: We need to show that $U(F)$ is Fréchet differentiable. By the chain rule for Fréchet differentiability (Luenberger (1969, p. 176), we know that if both μ_k^F and m_k^F are Fréchet differentiable for any k , then so is $U(\cdot)$. The Fréchet differentiability of μ_k^F is obvious, since

$$\mu_k^{F_2} - \mu_k^{F_1} = \int_I y^k dF_2(y) - \int_I y^k dF_1(y) = -k \int_I (F_2(y) - F_1(y)) y^{k-1} dy.$$

Now we show that m_k^F is Fréchet differentiable. We have

$$\begin{aligned} m_k^F &= \int (y - \mu_1^F)^k dF(y) \\ &= \int \sum_{i=0}^k \frac{k!}{i!(k-i)!} y^i (\mu_1^F)^{k-i} dF(y) \\ &= \sum_{i=0}^k \frac{k!}{i!(k-i)!} (\mu_1^F)^{k-i} \int y^i dF(y) \\ &= \sum_{i=0}^k \frac{k!}{i!(k-i)!} (\mu_1^F)^{k-i} \mu_i^F, \end{aligned}$$

which is a differentiable function of the μ_i^F 's. By the chain rule, it follows immediately that m_k^F is also Fréchet differentiable. This completes the proof. ■

Appendix III: Sufficient Conditions for Positive Prices

In this appendix we provide a sufficient condition for the positivity of equilibrium prices. Let $\zeta \in \mathbb{R}^{n+1}$ be a bundle. Denote the gradient of investor j 's utility function at ζ by $\nabla \mathbf{U}^j(\zeta) = (U_0^j(\zeta), \dots, U_n^j(\zeta))$, where a subscript designates a partial derivative in the direction of the i^{th} asset. Also, let $\gamma^j(\zeta) = -\frac{V_2^j(\mathbb{E}(\zeta \cdot \tilde{\mathbf{y}}), R(\zeta \cdot \tilde{\mathbf{y}}))}{V_1^j(\mathbb{E}(\zeta \cdot \tilde{\mathbf{y}}), R(\zeta \cdot \tilde{\mathbf{y}}))} > 0$ be the marginal rate of substitution of the expected payoff of the bundle for the risk of the bundle. This is the slope of investor j 's indifference curve in expected payoff-risk space. For brevity we often omit the arguments of this expression and use $\gamma^j(\zeta) = -\frac{V_2^j}{V_1^j}$.

Proposition 6 *Assume that for each asset i there is some investor j such that $\mathbb{E}(\tilde{y}_i) > \gamma^j(\zeta) R_i(\zeta \cdot \tilde{\mathbf{y}})$ for all ζ . Then, prices of all assets are positive in all equilibria.*

Proof: As Nielsen (1992) indicates, at an equilibrium, all investors' gradients point in the direction of the price vector. So the price of asset i must be positive in any equilibrium if there is some investor j such that $U_i^j(\zeta) > 0$ for all ζ . Recall that

$$U^j(\zeta) = V^j(\mathbf{E}(\zeta \cdot \tilde{\mathbf{y}}), R(\zeta \cdot \tilde{\mathbf{y}})).$$

Thus,

$$\begin{aligned} U_i^j(\zeta) &= V_1^j \mathbf{E}(\tilde{y}_i) + V_2^j R_i(\zeta \cdot \tilde{\mathbf{y}}) \\ &= V_1^j [\mathbf{E}(\tilde{y}_i) - \gamma^j(\zeta) R_i(\zeta \cdot \tilde{\mathbf{y}})], \end{aligned}$$

where $R_i(\zeta \cdot \tilde{\mathbf{y}})$ denotes the partial derivative of $R(\zeta \cdot \tilde{\mathbf{y}})$ with respect to ζ_i .

Since $V_1^j > 0$, $U_i^j(\zeta) > 0$ corresponds to

$$\mathbf{E}(\tilde{y}_i) - \gamma^j(\zeta) R_i(\zeta \cdot \tilde{\mathbf{y}}) > 0$$

as required. ■

Note that $\gamma^j(\cdot)$ can serve as a measure of risk aversion for investor j . We can thus interpret this proposition as follows. If each asset's expected return is sufficiently high relative to some investor's risk aversion and the marginal contribution of the asset to total risk, then this asset will always be desirable by some investor, and so, its price will be positive in any equilibrium.