

TECHNOLOGICAL IMPROVEMENT AND THE DECENTRALIZATION PENALTY IN A SIMPLE PRINCIPAL/AGENT MODEL

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May 2, 2018

We consider the organizer of a firm who compares a decentralized arrangement where divisions are granted total autonomy with an arrangement where perfect monitoring and policing guarantee that all divisions make the choices the organizer wants them to make. We ask: when does improvement in the divisions' technology strengthen the case for decentralization and when does it weaken it? The question is difficult and it is natural to start with a stripped-down model, where there is just one division. In the decentralized mode, the organizer appoints a Principal who rewards a single autonomous Agent (the division manager). The Agent freely chooses an effort x . The effort need not be hidden. The Agent bears its cost. The firm then achieves the surplus $R(x) - t \cdot C(x)$, where R is revenue, t is a positive technology parameter known to both parties, and $t \cdot C(x)$ is the cost of the Agent's effort. When technology improves, t drops. The Agent receives a share of the revenue, namely $r \cdot R(x)$, where $0 < r < 1$. The Principal receives the residual $(1 - r) \cdot R(x)$. In the *exogenous* case the share r is determined outside the model (perhaps by Principal/Agent bargaining). In the *endogenous* case the Principal, who knows how the Agent responds to every possible r given the current t , chooses the r which maximizes the residual revenue. The Decentralization Penalty for a given t equals the maximal possible surplus for that t — attained under perfect monitoring — minus the surplus achieved in the decentralized mode. It turns out that there are no simple conditions on C and R which imply that the Penalty grows (shrinks) when technology improves. Instead, we obtain a variety of results about relations between t , surplus, the Principal's "generosity" (the size of the Principal's chosen share in the endogenous case), and "effectiveness" (the effect of a small rise in the share on the Agent's effort).

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1. Introduction

Does the case for decentralizing a firm get stronger or weaker when the production technology used by one or more of its divisions improves? Consider the Organizer of the firm, who seeks a good balance between the cost of the divisions' efforts and the revenue which those efforts yield. One way to achieve a good balance may be intrusive but perfect monitoring and policing, which fully reveals the chosen efforts and guarantees that they are those the Organizer prefers.

Perfect monitoring/policing may be very costly. A better mode of organizing might be “decentralization”, where the divisions are totally autonomous, though their choices may be influenced by appropriate rewards and penalties. In the decentralized mode that we shall study there is a Principal who treats each division as an Agent. Each Agent freely chooses her effort and bears the effort's cost. The Principal observes the realized revenue and rewards the Agents. Each Agent's reward is a function of revenue, and her net earnings are her reward minus the cost of her chosen effort. The reward functions the Principal chooses are acceptable to the Agents and are preferred by the Principal to other possible reward functions that are also acceptable to the Agents. The Principal pockets the residual revenue which is left over after the rewards have been paid. When an Agent's technology improves, the cost of a given effort drops.

The Organizer compares the decentralized Principal/Agents mode with perfect monitoring/policing. Many production technologies rapidly improve, but at the same time the costs of perfect monitoring may rapidly drop as well, because of dramatic advances in monitoring techniques. So the relative merit of the two modes requires regular reassessment. We shall let the Organizer take a “welfare” point of view in comparing the two modes. The Organizer's focus is the firm's surplus: the revenue earned by the divisions' efforts minus the cost of those efforts. Perfect monitoring/policing guarantees maximal surplus. The *Decentralization Penalty* is the welfare loss due to decentralizing. It is the gap between maximal surplus and the surplus achieved in the decentralized Principal/Agents mode; so an alternative term for the Penalty is *surplus gap*.¹

Our central question is whether the Decentralization Penalty grows or shrinks when a technical advance lowers Agents' effort costs. If the Penalty substantially grows, then perfect monitoring may now be worth what it costs. (We will not explicitly model the cost of monitoring). If the Penalty shrinks, then perfect monitoring becomes less attractive even if monitoring techniques have advanced. Our central question is tricky for the following reason: when the Agents' technology improves, maximal surplus rises (under weak assumptions). Maximal surplus is a “moving target”. Decentralized surplus also rises, under reasonable assumptions. But that does NOT mean, in general, that as technology improves, the rising decentralized surplus gets *closer* to the moving surplus target. Our question appears to be very rarely asked in the abundant Principal/Agent literature. The cost of an Agent's effort appears in many papers and so does

¹If the firm is a regulated monopoly, for example, then high surplus might be the regulator's goal. The regulator compares decentralization with perfect monitoring/policing and favors the mode that achieves the higher surplus. Alternatively surplus may be viewed as the firm's profit. If the Organizer is also the firm's owner then profit may be what he seeks to maximize.

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2. The model

We shall study a highly simplified model. There is a single effort variable x , chosen from a set $\Sigma \subseteq \mathbb{R}^+$ of possible positive efforts. The set Σ may be finite or it may be a continuum. There is no uncertainty about the consequences of a given effort. The effort x generates a positive *revenue* $R(x)$, where R is strictly increasing. The effort x costs $t \cdot C(x)$, where C is positive and strictly increasing. A drop in t occurs when technology improves (or there is a fall in the price of the inputs which effort requires). For a given t , we consider the *surplus at the effort* x , denote $\tilde{W}(x, t)$. Thus

$$\tilde{W}(x, t) = R(x) - t \cdot C(x).$$

In the centralized mode perfect monitoring/policing guarantees that effort is "first-best": it maximizes surplus. In the decentralized mode there is no direct monitoring. Instead there is a self-interested Principal and a single self-interested Agent who freely chooses $x \in \Sigma$ and bears the cost $t \cdot C(x)$. The functions R and C , and the technology parameter t , are known to both parties. The Principal observes the revenue $R(x)$. Since R is strictly increasing, that observation also reveals the Agent's chosen x . The Principal rewards the Agent, using a reward which is a function of the observed revenue. We study an extremely simple reward scheme, namely linear revenue sharing. The Principal pays the Agent a *share* $r \in (0, 1]$ of the revenue. So if the Agent chooses the effort x , she earns $rR(x) - t \cdot C(x)$ and the net amount received by the Principal is the residual $(1 - r) \cdot R(x)$. We will assume that for every (r, t) there is an effort $x \in \Sigma$ such that the Agent's gain $rR(x) - tC(x)$ is nonnegative, and this is sufficient for the Agent to be willing to participate. The Agent chooses to exert the effort $\hat{x}(r, t)$, the smallest maximizer of $rR(x) - t \cdot C(x)$ on the set Σ . We denote *the surplus when the share is* r by $W(r, t)$ (the tilde is deleted). So

$$W(r, t) \equiv \tilde{W}(\hat{x}(r, t), t) = R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t)).$$

Note that if $r = 1$, then the Agent's effort choice $\hat{x}(1, t)$ is surplus-maximizing. Thus

$$W(1, t) = \tilde{W}(\hat{x}(1, t), t) \text{ is the largest possible surplus.}$$

In the centralized mode, perfect monitoring/policing insures that $W(1, t)$ is achieved.

In our study of the Decentralization Penalty we consider two cases. In the *exogenous* case, the reward share is determined outside the model. It might, for example, be the result of previous bargaining between Principal and Agent, or it might be prescribed by law. In the *endogenous case*, the Principal considers all the shares in the open interval $(0, 1)$ and chooses a share which maximizes $(1 - r) \cdot R(\hat{x}(r, t))$, the residual when the Agent uses the best-effort function \hat{x} in responding to a given share. We let $r^*(t)$ denote the maximizer which the Principal chooses. So in the endogenous case the Agent's effort is $\hat{x}(r^*(t), t)$ and surplus is $\tilde{W}(\hat{x}(r^*(t), t), t) = W(r^*(t), t)$.

The “moving target” remark that we made above suggests that the effect of a drop in t on the Decentralization Penalty is subtle. On the other hand, it is hard to imagine a model simpler than ours. So one might hope that in our simple model there are simple conditions on Σ, R, C under which the Penalty rises (falls) when t drops. It turns out, however, that even in our model there is a striking diversity of results. There are simple examples where the Penalty rises and simple examples where it falls. That is the case in both the exogenous and endogenous settings.

To bring some order to this diversity, we divide examples (Σ, R, C) into classes. To do so, we consider the effect of a drop in t on the example's best effort \hat{x} and on the example's endogenous-case best share r^* . A higher share stimulates the Agent to work harder, but the strength of the stimulus depends on t . Consider any pair of shares (r_L, r_H) , where $0 < r_L < r_H < 1$, and suppose that t drops. In some examples the effort increase $\hat{x}(r_H, t) - \hat{x}(r_L, t)$ rises and in other examples it falls. In some examples, moreover, the Principal's best share r^* rises when t drops, and in other examples it falls. So we have four classes of examples.

For each class we examine the *effort gap* — the amount by which decentralized effort falls short of the “first-best” effort. The effort gap is $\hat{x}(1, t) - \hat{x}(r, t)$ when r is exogenous and it is $\hat{x}(1, t) - \hat{x}(r^*(t), t)$ in the endogenous case. The effect of a drop in t on the effort gap is again tricky — just as it was for the Decentralization Penalty (the surplus gap). Under broad conditions, both terms of the gap rise when t drops, but the gap itself may rise or fall.

The effect of a drop in t on the effort gap is interesting in itself. There are classes of examples, moreover, in which the effort gap *tracks* the surplus gap (the Decentralization Penalty): when t drops, the two gaps move in the same direction. Imagine that first-best effort $\hat{x}(1, t)$ has been studied for many triples (R, C, t) . Then for an impending new technology t , first-best effort is already known but the welfare effects of decentralizing remain to be discovered. If we indeed have tracking, then it suffices to observe the Agent's work to see whether, with the new technology, her effort has moved closer to first-best effort or further away from it. In the former case we know — if we indeed have tracking — that the new technology has shrunk the Decentralization Penalty, so it has made perfect monitoring/policing less attractive. In the latter case it has increased the Penalty.

We obtain results about tracking. We do so first for the exogenous case² under the assumption that R and C are thrice differentiable. We consider the effectiveness of a share increase in stimulating higher effort and we classify examples according to the change in effectiveness when t drops, i.e., the sign of the cross partial $\hat{x}_{rt}(r, t)$. We find that we indeed have tracking, provided

²In Theorem 4

that the following monotonicity condition holds: either $\hat{x}_{rt}(r, t) > 0$ at all (r, t) or $\hat{x}_{rt}(r, t) < 0$ at all (r, t) . That gives us conditions³ on the signs of C'' , R'' , C''' , R''' under which the Decentralization Penalty rises when technology improves (t drops) and conditions under which the Penalty falls.

As one would expect, the tracking question is more difficult in the endogenous case. We again classify examples with regard to the effect of technical improvement on effectiveness (the sign of $\hat{x}_{rt}(r, t)$) but we also classify them with regard to the effect of technical improvement on the Principal's endogenous-case "generosity", i.e., the sign of the derivative $r^*(t)$. We find⁴ that we have tracking if either of the following conditions hold: (i) for every possible (r, t) , $\hat{x}_{rt}(r, t) < 0$ and $r^*(t) \geq 0$; (ii) for every possible (r, t) , $\hat{x}_{rt}(r, t) > 0$ and $r^*(t) < 0$. In contrast to the exogenous case, these results do not give us conditions on the the signs of the second and third derivatives of R and C under which the Penalty falls (rises) when technology improves.

3. Related literature

A great many Principal/Agent papers, starting with the earliest ones, use a framework that allows the Agent's effort to have a cost. The Agent has a utility function on her actions and rewards. In many papers Agent utility for the action a and the reward y takes the form $V(y) - g(a)$. Among the early papers where this occurs are Holmstrom (1979), (1982) and Grossman and Hart (1983). The action a might be effort and $g(a)$ could be its cost. Welfare loss also appears very early in the literature. Ross (1973), for example, finds conditions under which the solution to the Principal's problem maximizes welfare (as measured by the sum of Agent's utility and Principal's utility) and notes that these conditions are very strong. But Principal/Agent papers whose main concern is the relation between effort cost and welfare loss are scarce.

The Principal/Agent papers closest to ours are Balmaceda, Balseiro, Correa, and Stier-Moses (2016) and Nasri, Bastin, and Marcotte (2015). They study an Agent who has m possible efforts. Each effort has a cost, which the Agent pays. There are n possible revenues. For a given effort, the probability of each of the possible revenues is common knowledge, but the revenue actually realized only becomes known after the Agent's effort choice has been made. The Principal announces a vector of n nonnegative wages. For each of the n possible revenues, the vector specifies a wage received by the Agent when that revenue is realized. Both Principal and Agent are risk-neutral. Surplus for a given effort equals expected revenue minus the effort's cost. The socially preferred effort maximizes surplus. In the decentralized (Principal/Agent) mode, on the other hand, surplus is not maximal. Instead it is the surplus when the effort is the one the Principal chooses to induce. The papers study a fraction. Its numerator is maximal surplus and its denominator is "worst-case" Principal/Agent surplus. (When the Principal is indifferent between several efforts, the denominator of the fraction selects the one that is socially worst). The fraction is a measure of the welfare loss due to decentralizing. It is shown, under standard assumptions on the probabilities and on the possible (revenue, effort-cost) pairs, that the ratio cannot exceed m , the number of efforts. That upper bound does not depend on the effort costs, so the papers are silent on the effect of a drop in those costs on welfare loss.

³Stated in a Corollary to Theorem 4

⁴In Theorem 5

Note that welfare loss is also defined as a fraction in a larger literature, initially developed by computer scientists. Typically the object of study is a game. The fraction studied is often called “the price of anarchy”. Its numerator is the payoff sum in the “socially worst” equilibrium of the game. Its denominator — attainable when the players cooperate — is the largest possible payoff sum.⁵ In our setting it is natural to use the surplus gap rather than a ratio in defining the Penalty (welfare loss) due to decentralizing. Perfect monitoring (centralization) would eliminate the gap but it would be expensive. If its cost exceeds the gap then decentralization is the preferred mode.

If we allow more than one Agent, then parts of the large literature on the design of organizations become relevant. The designer has a goal, say surplus (profit) maximization, and can choose between a structure where a single member commands the choices made by all the others, and a structure where everyone is autonomous. The latter structure might be modeled as a game. A rather small piece of the design literature studies the communication and computation costs of each structure and the trade-off between those costs and some measure of gross performance (e.g., gross expected surplus, before the costs are subtracted). The problem is far more complex than the one we consider here and the results remain scarce and specialized.⁶

4. Plan of the remainder of the paper.

In Section 5 we examine five examples where the set of possible efforts and the set of possible values of t are not finite. The examples provide a preview of our general results. In Section 6 we develop basic results, presented in two theorems. They do not require differentiability, so finite examples are covered. The results concern the exogenous case in the first theorem and the endogenous case in the second. Section 7 presents two exogenous-case theorems which require differentiability. Section 8 presents two endogenous-case theorems which again require differentiability. Section 9 considers the shape of the function which relates r to the Principal’s gain. A concave shape implies a proposition about the negotiation set when the two parties bargain about the size of r . Section 10 explores an alternative model, often studied in the Principal/Agent literature. Revenue for a given effort is now uncertain, though the probabilities are common knowledge. the Agent’s reward is no longer a fixed share of revenue. Rather, the Principal chooses a vector of wages, one for each of the possible realized revenues. Both parties are risk-neutral. We ask whether we again have one of our basic results: the Agent never works less when technology improves, We then consider the effect of improved technology on the Decentralization Penalty. Section 11 provides concluding remarks about extensions and variations of the model.

5. Some examples

⁵A variety of social situations are studied from this point of view. One of them concerns optimal versus “selfish” routing in transportation (Roughgarden (2005)). Others are found in Nissan, Roughgarden, Tardos, and Vazirani (eds.) (2007). Many of these studies develop bounds on the price of anarchy. Several of them (e.g., Babaioff, Feldman, and Nissan (2009)) consider a Principal/Agent setting.

⁶Surveys of the design literature with communication and computation costs are found in Garicano and Prat (2013) and Marschak (2006). A model in which revenue is shared by a group of game-playing Agents is studied in Courtney and Marschak (2009). Each player chooses effort and bears its cost. Equilibria of the game are compared with the welfare-maximizing efforts. The paper finds conditions under which the welfare loss drops (rises) when effort costs shift down.

Our model, simple as it is, turns out to have quite diverse results. To illustrate the diversity, we now consider five examples. In all five of them the effort set and the set of possible values of the technology parameter t are continua and calculus methods are used to study them. In the simplest finite example, on the other hand, there would be just two values of t and two values of effort. We can construct simple finite examples yielding a variety of answers to the questions we just listed, but they are not presented here. In each of our five non-finite examples we provide some statements that we shall subsequently generalize.

5.1 A “Classic monopoly” example where marginal revenue falls and marginal cost is flat.

For convenient reference we shall call this our *Classic monopoly* example — or, for brevity, our *Classic* example. It is suggested by the introductory monopoly diagram in the typical text, where marginal revenue drops and marginal cost is flat or rising. We may think of the Principal as a monopolist who delegates the choice of product quantity to the Agent. Quantity will be our “effort”. At effort x , price is $A - Bx$, where $A > 0, B > 0$. Cost is $t \cdot C(x) = tx$ and revenue is $R(x) = Ax - Bx^2$. Marginal revenue becomes negative at $x = \frac{A}{2B}$. To keep price and marginal revenue positive, our set of possible efforts will be

$$\Sigma = \left(0, \frac{A}{2B}\right).$$

We consider a set Γ of pairs (r, t) , where (i) every $r \in (0, 1)$ belongs to one and only one pair, and (ii) for every pair $(r, t) \in \Gamma$ there is a positive effort $\hat{x}(r, t)$ which belongs to Σ and is the unique maximizer of $r \cdot R(x) - t \cdot C(x)$. That is the case for

$$\Gamma \equiv \{(r, t) : 0 < r < 1; 0 < t < Ar\}$$

and

$$\hat{x}(r, t) = \frac{A}{2B} - \frac{t}{2Br}.$$

For a given t , the Agent’s best response to the share r — if $(r, t) \in \Gamma$ — is the effort $\hat{x}(r, t)$, which belongs to Σ .

In every example that we study we will specify a similar set Γ , having the properties (i) and (ii). Note that in the Classic example, Γ is the interior of a triangle. In a diagram with r on the horizontal axis and t on the vertical axis the triangle has vertices at $(0, 0)$, $(1, 0)$ and $(1, A)$. In other examples Γ might be a rectangle, as in the example which follows (in 5.2). In still other examples one of the boundaries of Γ might have curvature. We shall let $\tilde{\Gamma}$ denote the set of values of t which we consider. Thus

$$\tilde{\Gamma} \equiv \{t : (r, t) \in \Gamma \text{ for some } r \in (0, 1)\}.$$

In our Classic example, $\tilde{\Gamma} = (0, A)$.

Now for every $(r, t) \in \Gamma$ consider the derivative of \hat{x} with respect to r , the derivative with respect to t , and the cross derivative. They will be denoted by \hat{x}_r, \hat{x}_t and \hat{x}_{rt} . We have the following results. Some of them will be generalized to wider classes of examples.

- $\hat{x}_r(r, t) = \frac{t}{2Br^2}$, which is positive. For a given t , increasing the share evokes more effort. We shall show⁷ that in any example, finite or nonfinite, increasing the share never evokes less effort.

- $\hat{x}_t(r, t) = -\frac{1}{2Br}$, which is negative. When r is fixed and technology improves (when t drops), the Agent works harder. We shall show⁸ that in any example, finite or nonfinite, the Agent never works less when t drops.

- We have $\hat{x}_{rt}(r, t) = \frac{-1}{2B} \cdot \frac{-1}{r^2} > 0$. So technology improvement (a drop in t) diminishes effectiveness (the effort increase evoked by a small rise in r).

- When the Agent uses the best effort $\hat{x}(r, t)$, he receives $r \cdot R(\hat{x}(r, t)) - t \cdot \hat{x}(r, t)$. In our Classic example, the derivative of that expression with respect to t turns out to be negative.⁹ So in the exogenous case, technology improvement is good news for the Agent. We shall provide¹⁰ a simple proof that this statement holds in any example, finite or nonfinite.

- We find that

$$\text{surplus} = W(r, t) = R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t)) = \frac{1}{4B^2r^2} \cdot [(Ar - t) \cdot (BAr + Bt - 2Brt)].$$

The derivative with respect to t of the expression in square brackets is

$$-2BAr^2 - 2Bt + 4Brt.$$

Our requirement that $t < Ar$ implies that this is negative.¹¹ Thus, for a fixed $r < 1$, decentralized exogenous-case surplus rises when technology improves (t drops). We shall see¹² that this always holds, for both finite and infinite message sets, as long as R and C are strictly increasing.

- For all $t \in \tilde{\Gamma}$, we have¹³ $W_t(1, t) < 0$. Maximal surplus¹⁴ rises when technology improves (t drops). As we shall see¹⁵, a trivial argument shows that this always holds in both finite and non-finite examples.

- We have $W_{rt}(r, t) = \frac{(1-r) \cdot t}{Br^3} > 0$.

So $W_{rt}(r, t)$ and $\hat{x}_{rt}(r, t)$ have the same sign. That implies, as we shall see, that the exogenous Decentralization Penalty (surplus gap) $W(1, t) - W(r, t)$ and the exogenous effort gap $\hat{x}(1, t) -$

⁷In Part (a) of Theorem 1.

⁸In Part (b) of Theorem 1.

⁹The derivative is $\hat{x}_t(r, t) \cdot [rR'(\hat{x}(r, t)) - t \cdot C'(\hat{x}(r, t))] - C(\hat{x}(r, t))$. That is negative, since $0 < r < 1$ and $\hat{x}(r, t)$ satisfies the first-order condition $0 = rR' - tC'$.

¹⁰In Part (f) of Theorem 1.

¹¹The derivative is negative if $Ar^2 > t \cdot (2r - 1)$. That is the case at $r = 0$ and at $r = 1$ (since $t < A$). At all $r \in (0, 1)$ our requirement $t < Ar$ implies that $2Ar$, the derivative of the left side of the inequality with respect to r exceeds $2t$, the derivative of the right side. So at all $(r, t) \in \Gamma$ the inequality holds.

¹²In Part (g) of Theorem 1.

¹³For derivatives, we shall use the symbols W_r, W_t, W_{rt} , which are analogous to our symbols $\hat{x}_r, \hat{x}_t, \hat{x}_{rt}$.

¹⁴In the monopoly setting our general term “surplus” should not be confused with consumers’ surplus. Our “Surplus” is the same as the monopolists’ profit.

¹⁵In Part (d) of Theorem 1

$\hat{x}(r, t)$ move in the same direction when technology improves, i.e., the exogenous surplus gap **tracks** the exogenous effort gap. There are finite examples where that is not the case. But we shall show¹⁶ that if R and C are thrice differentiable then it *must* be the case, because, as we shall prove, $W_{rt} \cdot \hat{x}_{rt} \geq 0$.

We now turn to the endogenous case. The Principal chooses a best share r , but excludes $r = 1$, which would give the Agent all of the revenue. To study the consequence of choosing $r = 0$, we would have to specify how the Agent responds to $r = 0$. The natural answer is zero effort, but there would then be some technical difficulties in our analysis of certain examples. So, as already noted, we confine attention to the case where the Principal chooses a share in the open interval $(0, 1)$ and the Agent's response is a positive effort. We will show¹⁷ that under simple conditions (which are satisfied in the Classic example), the Principal's gain $(1 - r) \cdot (R(\hat{x}(r, t)))$ is positive for all $r \in (0, 1)$ and is concave on $(0, 1)$. That implies that there is a share in $(0, 1)$ which solves the first-order condition

$$0 = \frac{d}{dr} [(1 - r) \cdot R(\hat{x}(r, t))] = -R(\hat{x}(r, t)) + (1 - r) \cdot R'(\hat{x}(r, t)) \cdot \hat{x}_r(r, t)$$

and maximizes the Principal's gain on the set $(0, 1)$. That share is our $r^*(t) \in (0, 1)$. In our Classic example the Principal's first-order condition turns out to be the cubic equation

$$0 = A^2 r^3 + r t^2 - 2 t^2.$$

When we graph the implicit function $r^*(t)$, we obtain the following figure for the case $A = 2, B = 3$:

FIGURE 1 HERE

The graph reveals that r^* is increasing in t . *The share-choosing Principal becomes less generous when technology improves.* But we can establish this fact — for a very wide class of examples that includes the Classic example — *without* any graphing. We shall show¹⁸ that it must hold whenever $R'' < 0$ and $\hat{x}_{rt} < 0$ (as in the Classic example). Thus *a drop in t makes the Principal less generous if it makes effectiveness drop and in addition the Agent's increased effort (in response to the lower t) makes marginal revenue drop.*

Next consider the endogenous effort $\hat{x}(r^*(t), t)$. Figure 2 shows that in the Classic example the endogenous effort rises when technology improves (t drops). We shall show¹⁹ that *this must happen, for the endogenous case, in every example, finite or nonfinite.*

FIGURE 2 HERE

The endogenous decentralization Penalty (surplus gap) is $W(1, t) - W(r^*(t), t)$. That can be graphed as a function of t , even without an explicit expression for $r^*(t)$. We do so in Figure 3

¹⁶In Theorem 4.

¹⁷In Theorem 7.

¹⁸In Theorem 6.

¹⁹In Part (b) of Theorem 2.

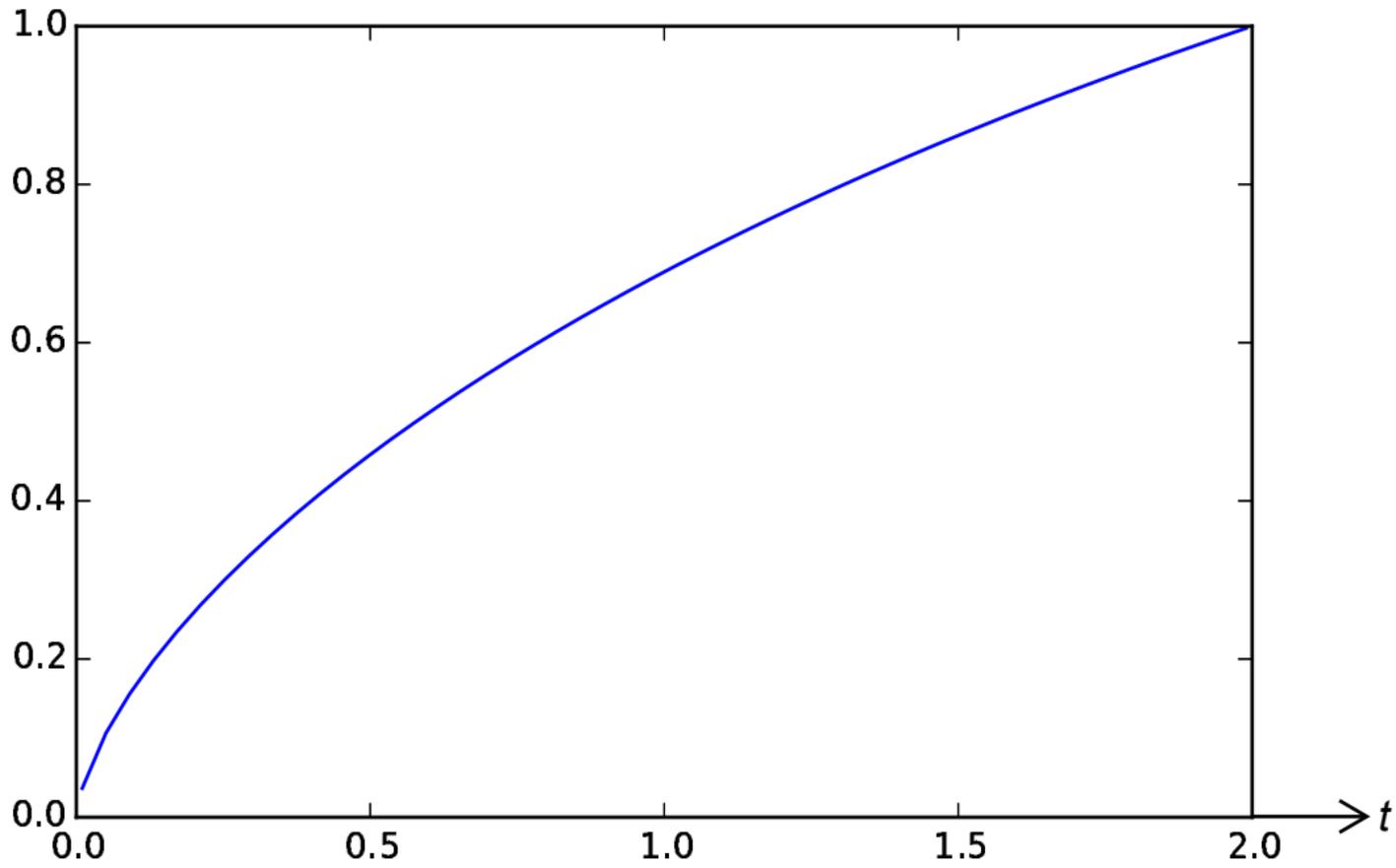


Fig. 1 graph of $r^*(t)$ for the Classic example with $A = 2, B = 3$

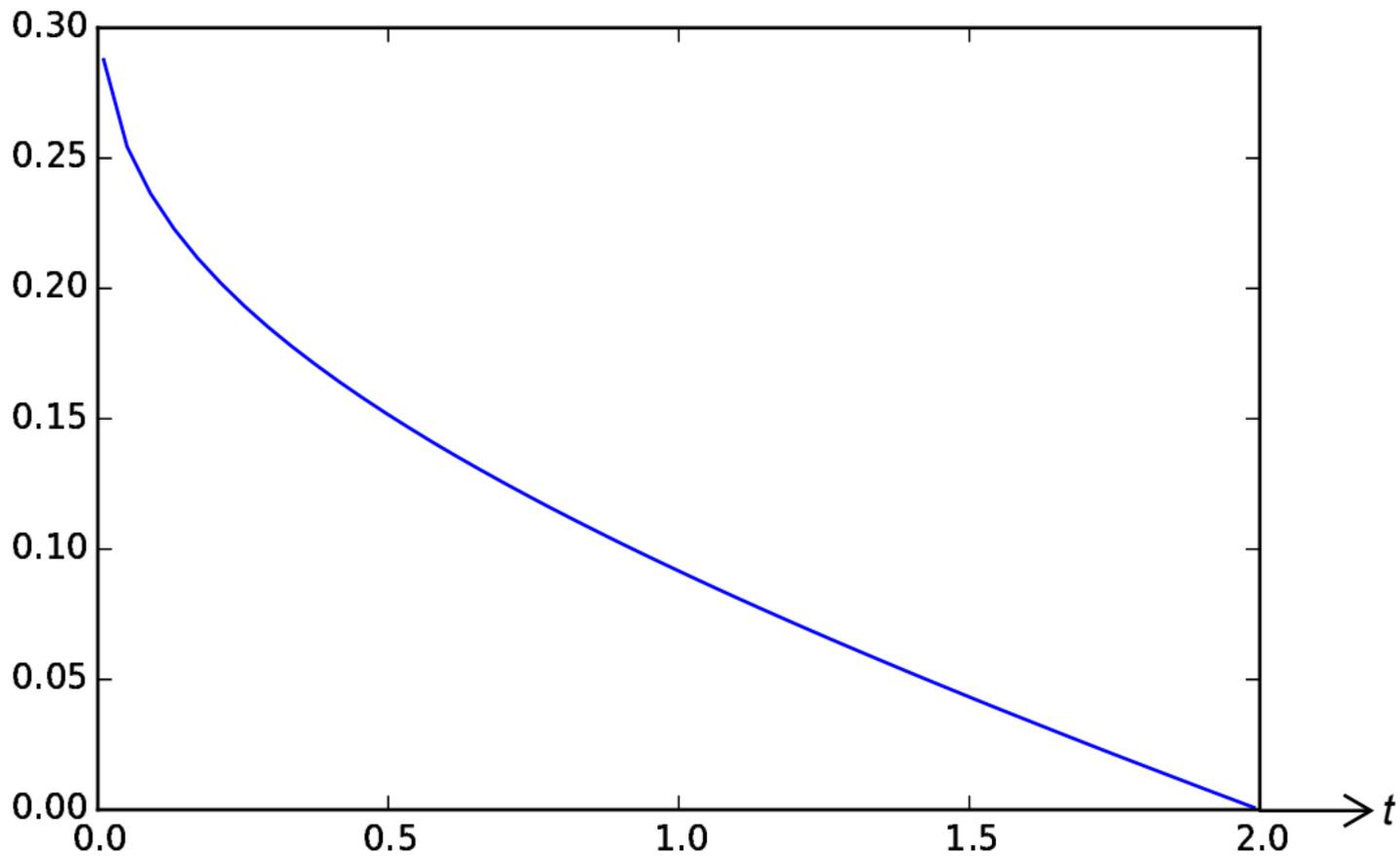


Fig. 2 graph of $\hat{x}(r^*(t), t)$ for the Classic example with $A = 2, B = 3$

for the Classic example. We see that for large t further technical improvement (a further drop in t) *raises* the Penalty, but for small t further improvement lowers it.

FIGURE 3 HERE

We now turn to the tracking question. For the Classic example, Figure 4 shows both the Penalty (surplus gap) and the effort gap $\hat{x}(1, t) - \hat{x}(r^*(t), t)$. When t increases each gap first rises and then falls and for each t the gaps move in the same direction, so we indeed have tracking.

FIGURE 4 HERE

5.2 A “Cubic-revenue” example, where, just as in the Classic example, technology improvement diminishes effectiveness and generosity, but now we do *not* have tracking.

In this example:

$R(x) = x^3 - x^2$, $C(x) = x$, and the set of possible (r, t) pairs is $\Gamma = \{(r, t) : r \in (0, 1); t \leq r\}$,

so the set of possible values of t is $\tilde{\Gamma} = (0, 1)$. We find — just as in the Classic example — that for all (r, t) in Γ , effectiveness diminishes when t drops, i.e., $\hat{x}_{rt}(r, t) > 0$.²⁰ Now consider $r^*(t)$, the implicit function which solves the Principal’s first-order equation for $t \in (0, 1)$. That function turns out to satisfy a cumbersome polynomial equation.²¹ Figure 5 graphs the implicit function r^* .

²⁰We have

$$\hat{x}(r, t) = \frac{1}{\sqrt{3}} \cdot \left(1 - \frac{t}{r}\right)^{1/2}.$$

Next we obtain

$$\hat{x}_r(r, t) = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} \cdot \frac{d}{dr} \left[\left(1 - \frac{t}{r}\right)^{1/2} \right] = \frac{1}{4\sqrt{3}} \cdot \left(1 - \frac{t}{r}\right)^{-1/2} \cdot \frac{t}{r^2}.$$

We then have:

$$\begin{aligned} 4\sqrt{3} \cdot \hat{x}_{rt}(r, t) &= \frac{d}{dt} \left[\left(1 - \frac{t}{r}\right)^{-1/2} \cdot \frac{t}{r^2} \right] \\ &= \left(1 - \frac{t}{r}\right)^{1/2} \cdot \left(\frac{1}{r^2}\right) + \frac{t}{r^2} \cdot \left(\frac{-1}{2} \cdot \frac{d}{dt} \left[\left(1 - \frac{t}{r}\right)^{-1/2} \right]\right). \end{aligned}$$

But

$$\frac{d}{dt} \left[\left(1 - \frac{t}{r}\right)^{-1/2} \right] = \frac{-1}{2} \cdot \left(1 - \frac{t}{r}\right)^{-3/2} \cdot \frac{t}{r^2} < 0.$$

So we indeed have $\hat{x}_{rt}(r, t) > 0$.

²¹The equation is

$$0 = 16[r^*(t)]^6 - 12t^2 - 27t^4 \cdot [r^*(t)]^4 - 4t^3 + 108t^4 \cdot [r^*(t)]^3 - 162t^4 \cdot [r^*(t)]^2 + 108t^4 \cdot r^*(t) - 27t^4.$$

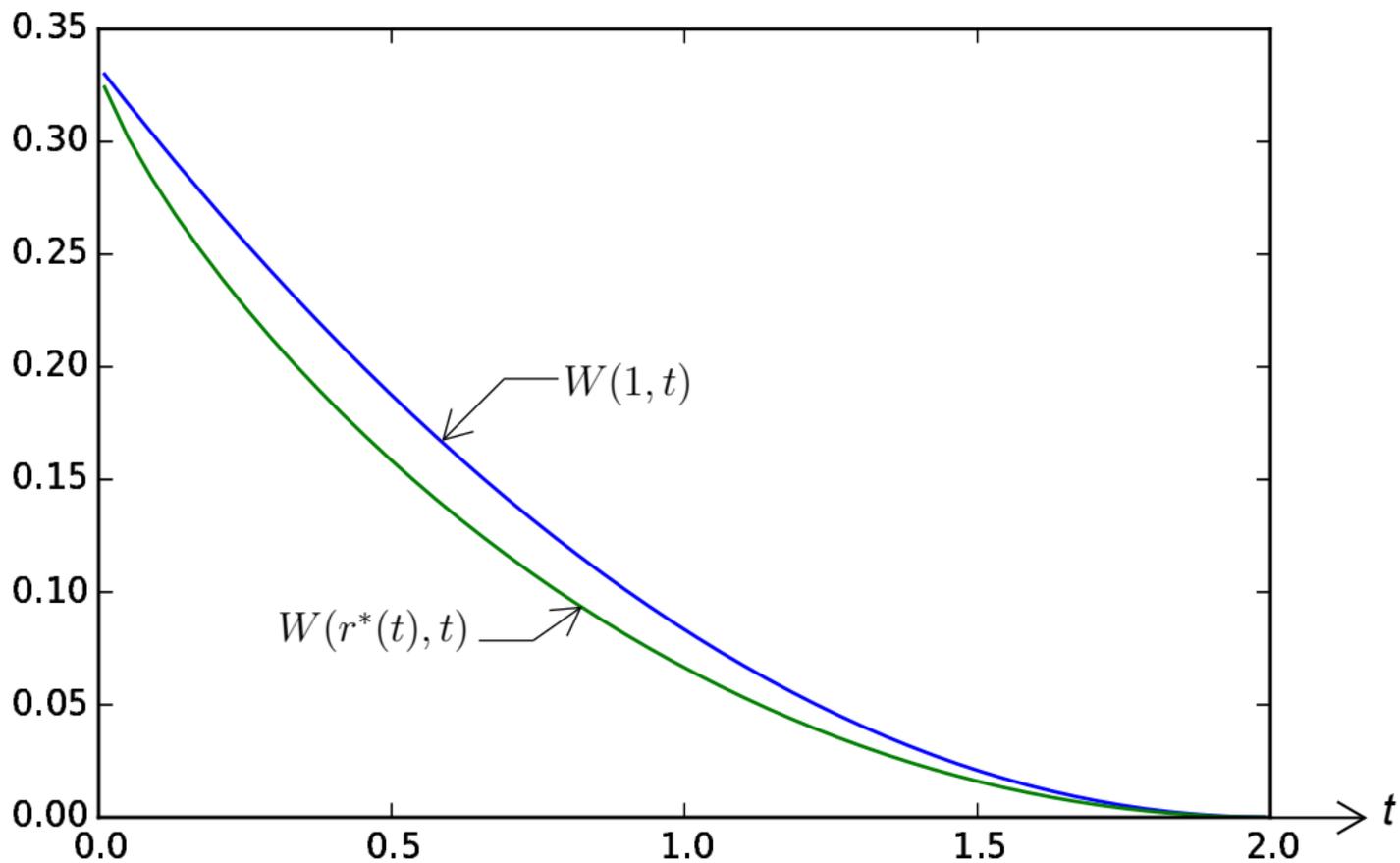


Figure 3: $W(1, t)$ and $W(r^*(t), t)$ for the Classic case, with $A = 2, B = 3$

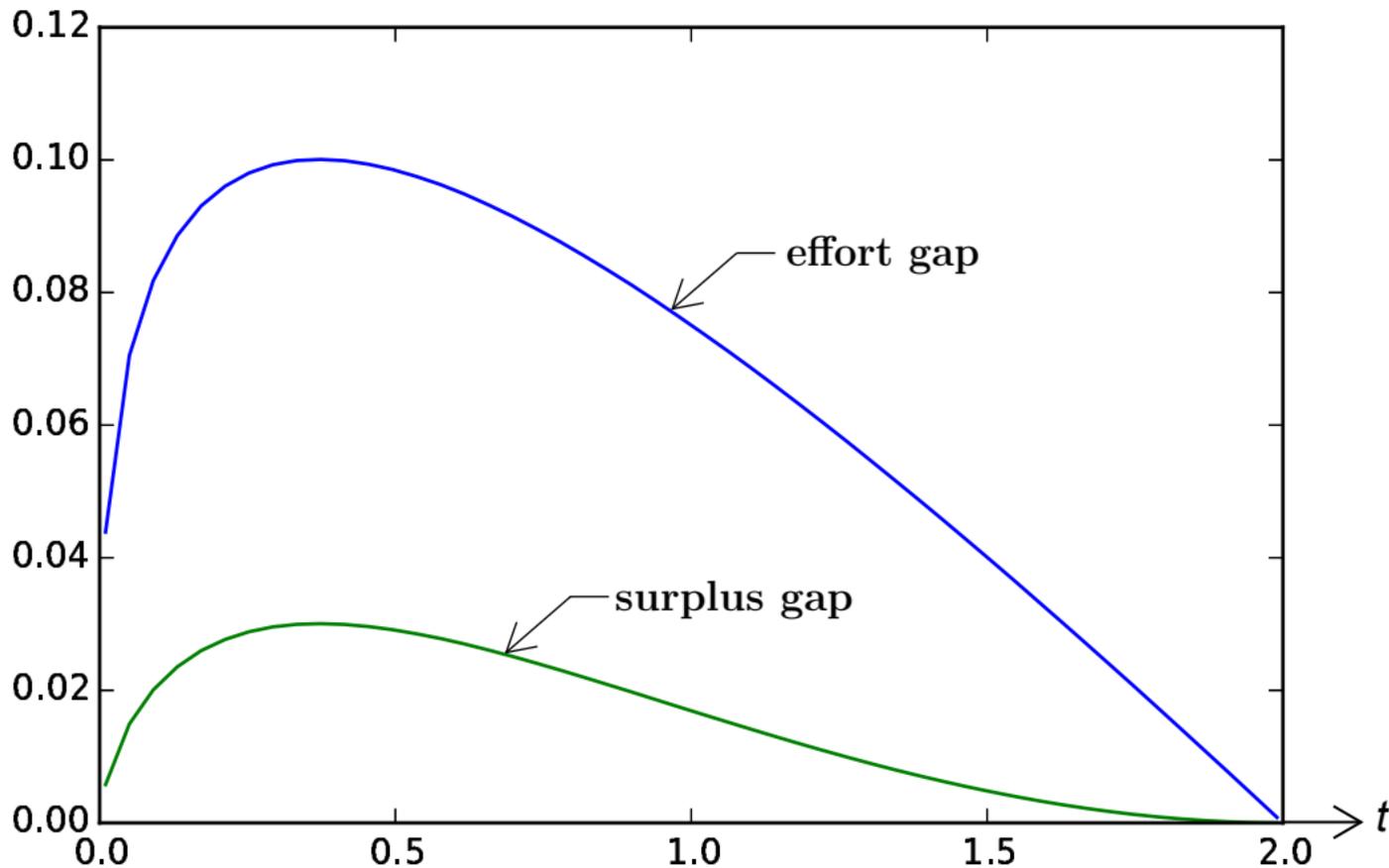


Figure 4: The two gaps: surplus gap ($W(1, t) - W(r^*(t), t)$) and effort gap ($\hat{x}(1, t) - \hat{x}(r^*(t), t)$) for the Classic case, with $A = 2, B = 3$

FIGURE 5 HERE

Just as in the Classic example, r^* rises at every possible value of t (all $t \in (0, 1)$). Finally, we plot the surplus gap and the effort gap.

FIGURE 6 HERE

We find that for t in the interval (.48, .63), the effort gap rises but the surplus gap falls. Unlike the Classic example, we do *not* have tracking.

5.3 A “Price-taker” example where marginal cost rises and marginal revenue is flat

In this example we may think of the Principal as a price-taker (with price equal to one). He delegates quantity choice to the Agent and the Agent’s cost function is quadratic. “Price-taker” is a convenient label for this example. The example is defined as follows:

- The set of possible efforts is $\Sigma = \mathbb{R}^+$.
- $R(x) = x$.
- $C(x) = \frac{1}{2}(x - 1)^2$.
- The set of possible pairs (r, t) is the rectangle $\Gamma = \{(r, t) : 0 < r < 1; 0 < t < 1\}$.

We find the following:

- Given r , the Agent chooses $\hat{x}(r, t) = \frac{r}{t} + 1$. We then have $\hat{x}_{rt}(r, t) = \frac{-1}{t^2} < 0$. *Effectiveness rises when technology improves (when t drops).*
- When the Agent uses the best effort $\hat{x}(r, t)$, he receives $r + \frac{r^2}{2t}$. The derivative with respect to t is $-\frac{r^2}{2t^2} < 0$. So, just as in the Classic example *technology improvement is good news for the Agent in the exogenous case*. As we already noted, we will provide a simple (calculus-free) proof that this “good news” statement holds, for the exogenous case, in any example, finite or nonfinite.
- Exogenous surplus is $R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t)) = \frac{r}{t} + 1 - \frac{r^2}{2t}$.
- Surplus-maximizing effort is $\frac{1}{t} + 1$ and maximal surplus is $1 + \frac{1}{2t}$.
- The exogenous surplus gap (the Penalty) is $\frac{1}{2t} - \frac{r}{t} + \frac{r^2}{2t}$. Its derivative with respect to t is negative. The exogenous effort gap is $\frac{1-r}{t}$, which also has a negative derivative. So *the exogenous surplus gap tracks the exogenous effort gap*, just as in the Classic example. As already noted, this will be proved to hold, in the exogenous case, for any differentiable example.

We now turn to the endogenous case. We find that:

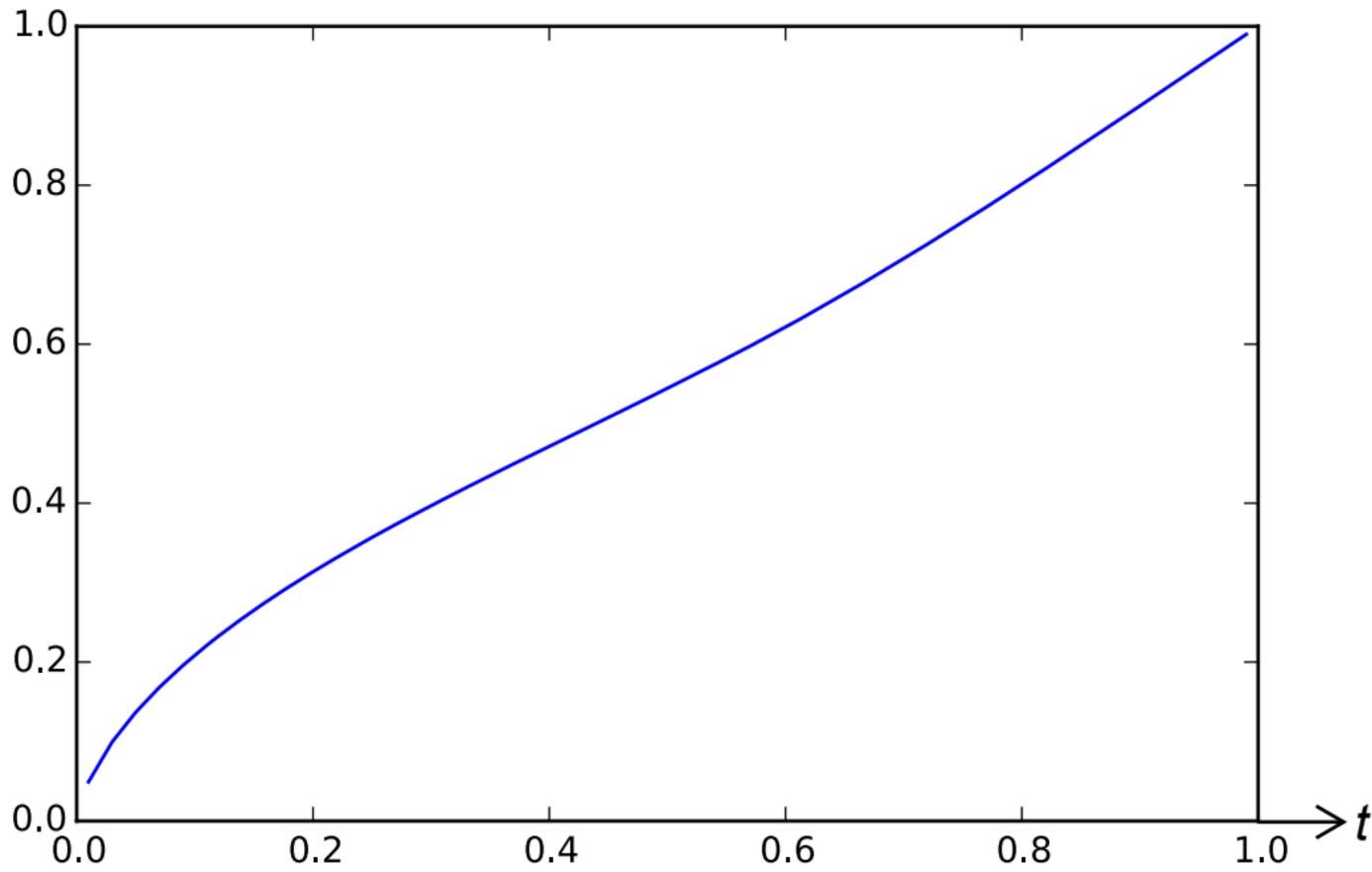


Fig. 5. Graph of $r^*(t)$ in the Cubic-revenue example.

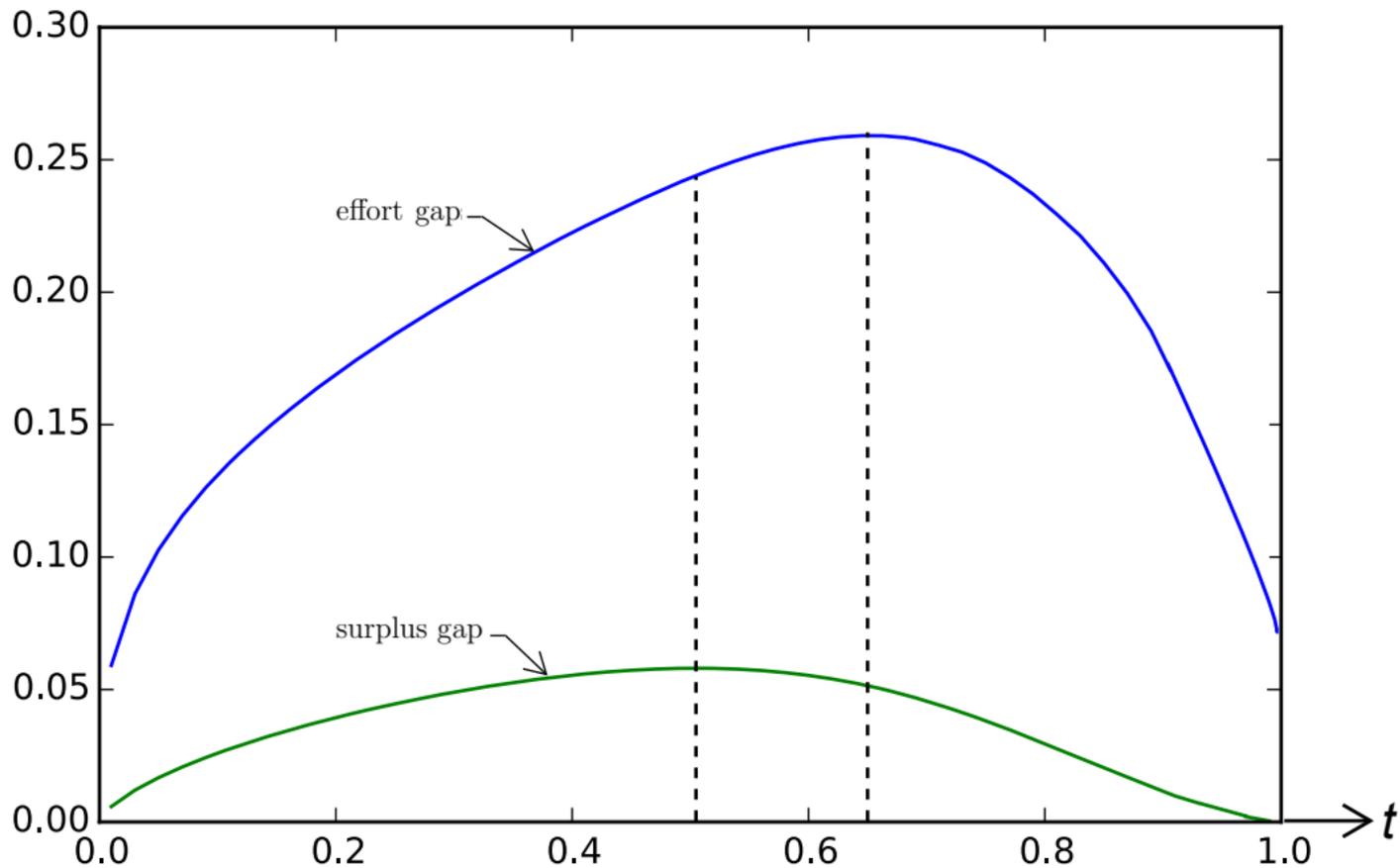


Fig. 6 Surplus and effort gaps in the Cubic-revenue example. For $t \in (.48, .63)$, effort gap rises but surplus gap falls.

- The solution to the Principal’s first order condition $0 = \frac{d}{dr}[R(\hat{x}(r, t)) - t \cdot C(\hat{x}(r, t))]$ is $r^*(t) = \frac{1-t}{2}$. So we have $r^{*'}(t) < 0$ at every possible t . (Recall that $t < 1$). So, in sharp contrast to the Classic monopoly example, *the Principal becomes more generous when technology improves*.
- We find that — just as in the exogenous case — *a drop in t is good news from the welfare point of view*. This must be the case²² whenever — as in the Price-taker example and the Rising Marginals example which we consider next — the Principal becomes more generous (or stays just as generous) when t drops.²³
- The endogenous Penalty (the endogenous surplus gap) is $\frac{1}{4} + \frac{1}{8t} + \frac{t}{8}$. Its derivative with respect to t is $\frac{1}{8t^2} \cdot (t^2 - 1)$, which is negative, since $t < 1$. *The Penalty rises when technology improves*. Again, note the contrast with the Classic example, where the Penalty *drops* when technology improves, once t has dropped below a critical value.
- The endogenous effort gap is $\frac{1+t}{t} - \frac{r^*(t)+t}{t} = \frac{1}{2t} + \frac{1}{2}$. That also has a negative derivative.
- So *the endogenous effort gap tracks the exogenous surplus gap*. But that is NOT implied, as we shall see, by the fact that $\hat{x}_{rt} < 0$ and $r^{*'}(t) < 0$.

5.4 A “Cubic-cost” example, where, just as in the Price-taker example, technology improvement increases both effectiveness and generosity, but now we have “opposite directions” rather than tracking.

In this example

$$R(x) = \frac{1}{2}x^2$$

and

$$C(x) = \frac{1}{3}x^3 + \frac{a}{2}x^2 - \epsilon x,$$

where $\epsilon > 0$ and $a > 0$. The numbers a, ϵ and the set Σ of possible efforts will be chosen as we proceed. The triple (a, ϵ, Σ) will have the property that $C(x) > 0$ for all $x \in \Sigma$.

The Agent’s first-order condition for given r, t is

$$rx = t \cdot (x^2 + ax - \epsilon).$$

²²Shown in Part (d) of Theorem 2.

²³The argument is as follows: We have

$$\frac{d}{dt} [R(\hat{x}(r^*(t), t)) - t \cdot C(\hat{x}(r^*(t), t))] = [\hat{x}_r \cdot r^{*'} + \hat{x}_t] \cdot (R' - tC') - C.$$

We have $R' - tC' > 0$ (because of the first-order condition $rR' - tC' = 0$, where $0 < r < 1$). Since $\hat{x}_t < 0$ and $r^{*'} \leq 0$, we conclude that the derivative is negative, so we indeed have “good news” from the welfare point of view.

This is solved by

$$\hat{x}(r, t) = \frac{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon} - (a - \frac{r}{t})}{2} > 0.$$

Note that

$$\begin{aligned} \hat{x}_r &= \frac{1}{2} \cdot \left(\frac{1}{t} - \frac{1}{2} \left[\left(a - \frac{r}{t} \right)^2 + 4\epsilon \right]^{-1/2} \cdot \frac{2}{t} \cdot \left(a - \frac{r}{t} \right) \right) \\ &= \frac{1}{2t} \cdot \left(1 - \frac{a - \frac{r}{t}}{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon}} \right) \\ &= \frac{1}{2t} \cdot \frac{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon} - (a - \frac{r}{t})}{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon}}. \end{aligned}$$

Our set Γ of possible (r, t) pairs will be

$$\Gamma = \left\{ (r, t) ; t \in \left(\frac{1}{a}, \frac{2}{\sqrt{a^2 + 4\epsilon}} \right) ; 0 < r < 1 \right\}.$$

Now assume that

- $t \geq \frac{1}{a}$
- $\epsilon < \frac{3}{4}a^2$.

Then $\frac{1}{a} < \frac{2}{\sqrt{a^2 + 4\epsilon}}$, so Γ is not empty. Moreover $a - r/t \geq 0$ for all $r \in (0, 1)$. Under these assumptions we have $\hat{x}_{rt}(r, t) < 0$. Effectiveness increases when technology improves. ²⁴

²⁴Consider the fraction

$$\hat{x}_r = \frac{\sqrt{(a - \frac{r}{t})^2 + 4\epsilon} - (a - \frac{r}{t})}{2t \cdot \sqrt{(a - \frac{r}{t})^2 + 4\epsilon}}.$$

Multiply numerator and denominator by $\sqrt{(a - \frac{r}{t})^2 + 4\epsilon} + (a - \frac{r}{t})$. The new fraction simplifies to

$$\frac{4\epsilon}{2t \cdot \left[(a - \frac{r}{t})^2 + \sqrt{(a - \frac{r}{t})^2 + 4\epsilon} \right]}.$$

Since $a > \frac{r}{t}$, the denominator is strictly increasing in t . Hence the whole fraction is strictly decreasing in t . So, as claimed, we have $\hat{x}_{rt}(r, t) < 0$.

In the endogenous case the first-order condition satisfied by the Principal's chosen share r^* satisfies the first-order condition

$$\begin{aligned}
r &= 1 - \frac{R}{R'\hat{x}_r} \\
&= 1 - \frac{\frac{1}{2}\hat{x}^2}{\hat{x}\hat{x}_r} \\
&= 1 - \frac{\hat{x}}{2\hat{x}_r} \\
&= 1 - \frac{\frac{\sqrt{(a-\frac{r}{t})^2+4\epsilon}-(a-\frac{r}{t})}{2}}{\frac{1}{t}\frac{\sqrt{(a-\frac{r}{t})^2+4\epsilon}-(a-\frac{r}{t})}{((a-\frac{r}{t})^2+4\epsilon)^{1/2}}}.
\end{aligned}$$

So, if $r^*(t)$ is a maximizer of $(1-r) \cdot R(\hat{x}(r,t))$, it satisfies

$$(+)\quad r^*(t) = 1 - \frac{t\sqrt{\left(a - \frac{r^*(t)}{t}\right)^2 + 4\epsilon}}{2}.$$

We can show the following for every t in our set $\tilde{\Gamma} = \left(\frac{1}{a}, \frac{2}{\sqrt{a^2+4\epsilon}}\right)$ of possible values of t :

- (1) There is a unique value of r , denoted $r^*(t)$, such that for every $t \in \tilde{\Gamma}$, $r^*(t)$ satisfies the first-order condition and hence $r^*(t)$ is the unique maximizer of $(1-r) \cdot R(\hat{x}(r,t))$ on the interval $(0,1)$.
- (2) $r^{*\prime}(t) < 0$ (the Principal becomes more generous when technology improves).²⁵

²⁵We first prove (2). To do so, we use a very general result, which holds even without differentiability. It will be proved below, in Part (a) of Theorem 2. It says that if $t_H > t_L$, then $\frac{r^*(t_H)}{t_H} \leq \frac{r^*(t_L)}{t_L}$, where $r^*(t)$ is any maximizer of $(1-r) \cdot R(\hat{x}(r,t))$ on $(0,1)$. So when t rises, the right side of (+) falls, i.e., $r^{*\prime}(t) < 0$.

Next rewrite the first-order condition satisfied by the Principal's chosen share as

$$0 = f(r,t) \equiv 1 - r - \frac{\sqrt{(at-r)^2 + 4\epsilon t^2}}{2}.$$

Then $f(0,t) < 0$, i.e., $2 > t\sqrt{a+4\epsilon}$ for all of the possible values of t . That is the case since

$$2 = \frac{2}{\sqrt{a^2+4\epsilon}} \cdot \sqrt{a^2+4\epsilon}$$

and $t < \frac{2}{\sqrt{a^2+4\epsilon}}$. We also have $f(1,t) < 0$. Hence, by the Intermediate Value Theorem, for any $t \in \left(0, \frac{2}{\sqrt{a^2+4\epsilon}}\right)$, we have $f(r^*(t),t) = 0$ for some $r^*(t) \in (0,1)$. Moreover, since $r^{*\prime}(t) < 0$, that $r^*(t)$ is the only solution to $f(r,t) = 0$ in $(0,1)$. That establishes (1).

Now consider the case where $a = 1$ and $\epsilon = 0.6$. That meets our requirement $\epsilon < \frac{3}{4}a^2$. Define our set of possible efforts to be

$$\Sigma = (1, \infty].$$

Then $\tilde{\Gamma} = (1, 1.084)$ and $C(x) > 0$ for every $x \in \Sigma$. Figure 7 graphs the surplus gap, which falls when technology improves (t drops). Figure 8 graphs the effort gap, which rises when technology improves. We have “opposite directions” rather than tracking.

FIGURES 7 AND 8 HERE

5.5 An example where marginal revenue rises but marginal cost rises faster.

It will be convenient to call this the “Rising Marginals” example. We have:

- The set of possible efforts is $\Sigma = \mathbb{R}^+$.
- $R(x) = x^a, C(x) = x^b, 0 < a < b$.
- The set of possible pairs (r, t) is $\Gamma = \{(r, t) : 0 < r < 1; t > 0\}$.

We obtain the following:

- $\hat{x}(r, t) = \left(\frac{tb}{ra}\right)^{\frac{1}{a-b}}$.
- $\hat{x}_{rt}(r, t) = -\frac{1}{(a-b)^2} \cdot t^{1/(a-b)-1} \cdot \left(\frac{b}{a}\right)^{1/(a-b)} \cdot r^{1/(b-a)-1}$. That is negative. So *when technology improves, effectiveness increases*.
- In the endogenous case the Principal chooses the share $r^*(t) = \frac{a}{b}$. *The Principal’s generosity remains unchanged when technology changes.*²⁶

²⁶The first-order condition satisfied by r^* can be written

$$r = 1 - \frac{R(\hat{x}(r, t))}{R'(\hat{x}(r, t)) \cdot \hat{x}_r(r, t)}.$$

In the example we obtain:

$$\frac{R(x)}{R'(x)} = \frac{x}{a},$$

$$\hat{x}_r = \left(\frac{tb}{a}\right)^{1/(a-b)} \cdot \frac{1}{b-a} \cdot r^{1-1/(b-a)},$$

and

$$\frac{\hat{x}}{\hat{x}_r} = r \cdot (b-a).$$

So the first-order condition is $r = 1 - \frac{r \cdot (b-a)r}{a}$. That is solved by $r^* = \frac{a}{b}$.

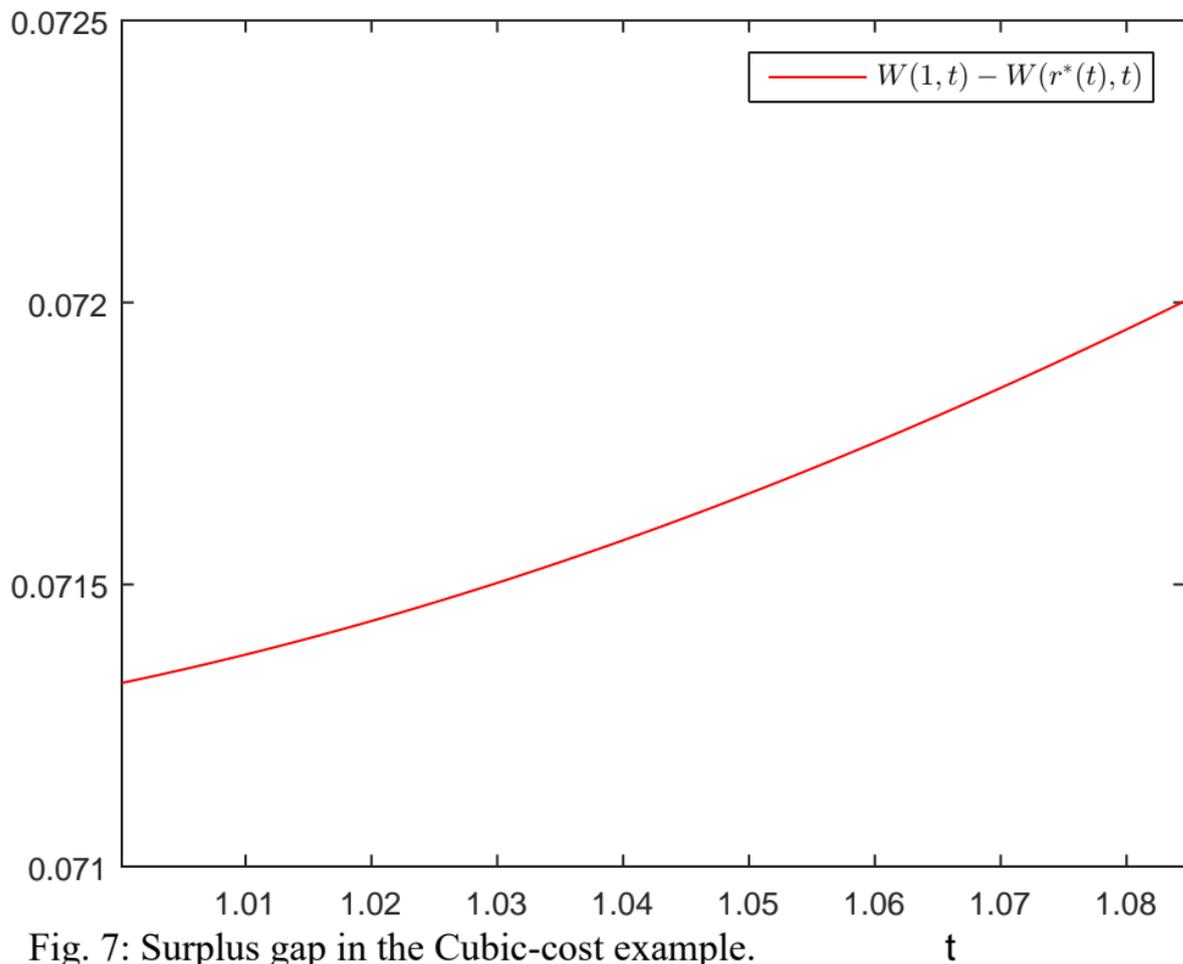


Fig. 7: Surplus gap in the Cubic-cost example.

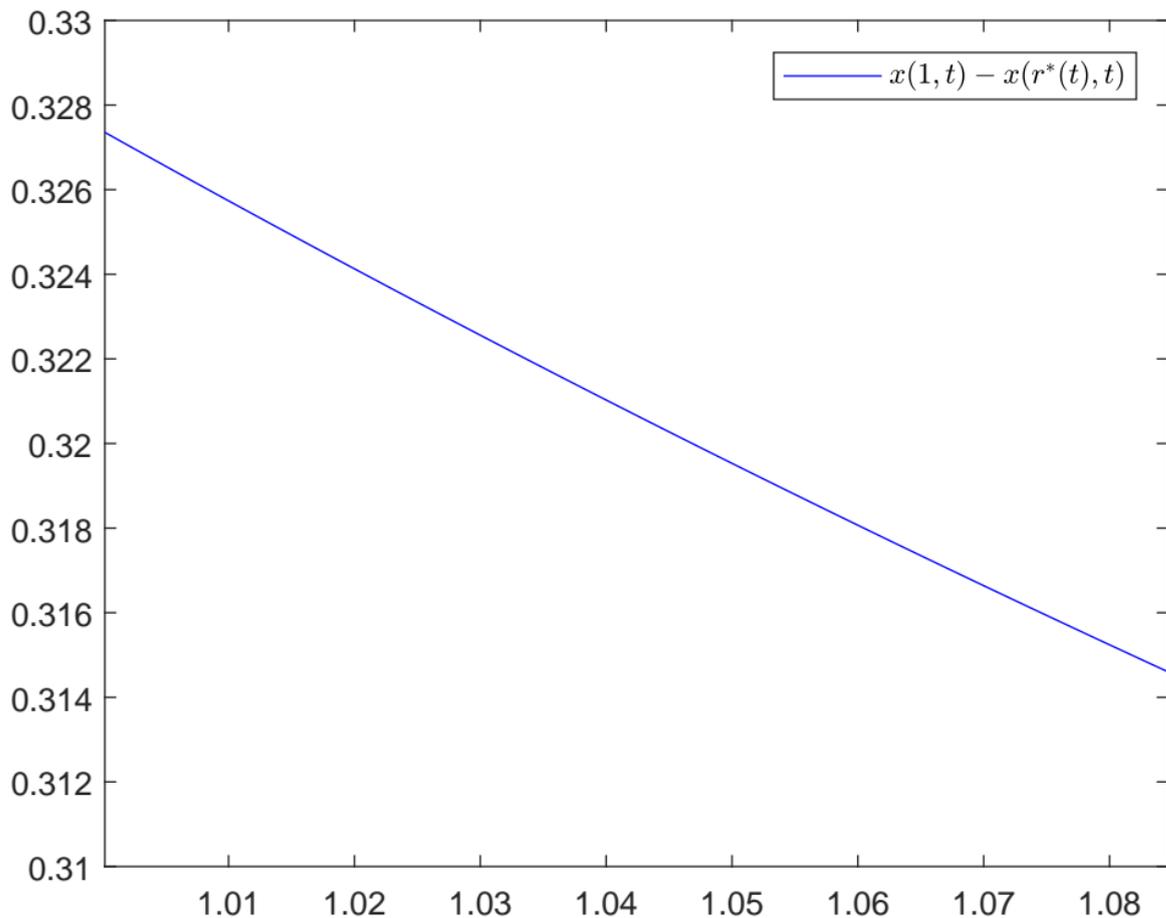


Fig. 8: Effort gap in the Cubic-cost example. t

- Just as in the Price-taker example, *Improvement in technology is good news for the share-choosing Principal*. That is the case because $r^{*'} = 0$.
- Even though we have an explicit expression for r^* , computing the derivative of endogenous effort gap with respect to t and the derivative of endogenous surplus gap (Penalty) with respect to t is cumbersome. It turns out that both are negative. So *the endogenous surplus gap tracks the endogenous effort gap*. This is true as well in the Classic and Price-taker examples (but not in the Cubic-revenue example). The fact that it is true in the Classic and Price-taker examples does *not* follow from the signs of \hat{x}_{rt} and $r^{*'}$ in those examples. In contrast, we shall show²⁷ that, in the endogenous case, *the surplus gap tracks the effort gap whenever (as in the Rising Marginals example) $\hat{x}_{rt} < 0$ and $r^{*' } \geq 0$* .

6. Basic results that do not require differentiability.

We now state two theorems which are proved without requiring differentiability of R or C . Theorem 1 concerns the exogenous case and Theorem 2 concerns the endogenous case. Both theorems hold for all examples in which R and C are strictly increasing. An example is defined by a set Σ of possible positive efforts, the functions R and C , and a set Γ of possible pairs (r, t) . Recall that for every $r \in (0, 1)$, Γ contains some pair (r, t) . The set of values of t such that $(r, t) \in \Gamma$ for some r is again denoted $\tilde{\Gamma}$. Recall that the function \hat{x} has the property that for all $(r, t) \in \Gamma$, we have $\hat{x}(r, t) \in \Sigma$ and $r \cdot R(\hat{x}(r, t)) - tC(\hat{x}(r, t)) \geq r \cdot R(x) - tC(x)$ for all $x \in \Sigma$.

The exogenous-case Theorem 1 has eight parts. Part **(a)** says that the Agent never works less when the share r rises (while remaining less than one), and strictly prefers the higher share. Part **(b)** says that the Agent never works less hard when t drops (technology improves) Part **(c)** says that the surplus-maximizing effort cannot fall when t drops. Part **(d)** says that maximal surplus must rise when t drops. Part **(e)** says that a drop in t is never bad news for the Principal and Part **(f)** says that it must be good news for the Agent. Part **(g)** says that a drop in t is never bad news from the welfare point of view. Part **(h)** says that a rises in the share r is never bad news from the welfare point of view and is good news if and only if the Agent's effort changes after the drop. Thus, in the exogenous case, *it is in the "social" interest for the Principal to be more generous*.

Theorem 1

Let R and C be strictly increasing on Σ . Then:

- (a)** $\hat{x}(r_H, t) \geq \hat{x}(r_L, t)$ and $R(\hat{x}(r_H, t)) - tC(\hat{x}(r_H, t)) > R(\hat{x}(r_L, t)) - tC(\hat{x}(r_L, t))$ whenever $(r_L, t) \in \Gamma, (r_H, t) \in \Gamma$, and $0 < r_L < r_H < 1$.
- (b)** $\hat{x}(r, t_L) \geq \hat{x}(r, t_H)$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma$, and $0 < t_L < t_H$.
- (c)** $\hat{x}(1, t_L) \geq \hat{x}(1, t_H)$ whenever $t_L, t_H \in \tilde{\Gamma}$, and $0 < t_L < t_H$.
- (d)** $W(1, t_L) > W(1, t_H)$ whenever $t_L, t_H \in \tilde{\Gamma}$ and $0 < t_L < t_H$.

²⁷In Part **(b)** of Theorem 5.

- (e) $(1 - r) \cdot R(\hat{x}(r, t_L)) \geq (1 - r) \cdot R(\hat{x}(r, t_H))$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma$, and $0 < t_L < t_H$.
- (f) $rR(\hat{x}(r, t_L)) - t_L C(\hat{x}(r, t_L)) > rR(\hat{x}(r, t_H)) - t_H C(\hat{x}(r, t_H))$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma$, and $0 < t_L < t_H$.
- (g) $W(r, t_L) > W(r, t_H)$ whenever $(r, t_L) \in \Gamma, (r, t_H) \in \Gamma$, and $0 < t_L < t_H$.
- (h) $W(r_H, t) \geq W(r_L, t)$ whenever $(r_H, t) \in \Gamma, (r_L, t) \in \Gamma$, and $0 < r_L < r_H < 1$. The inequality is strict if and only if $\hat{x}(r_H, t) \neq \hat{x}(r_L, t)$.

The proof of Theorem 1, like all the subsequent proofs, is found in the Appendix. In proving Parts (e),(f),(g), (h) we use the simple observation that when t drops or r rises, the Agent could continue to use the same effort as before the change. In proving Parts (a),(b),(c),(d) we use a basic proposition from monotone comparative statics.²⁸

Theorem 2 concerns the endogenous case. Part (a) says that the ratio of the Principal's chosen share to the technology parameter t cannot fall when t drops. But, as we have already seen in the examples, *the chosen share itself may rise or fall or stay the same*. Nevertheless Part (b) says that in the endogenous case the Agent never works less hard when t drops. Part (c) says that in the endogenous case a drop in t is never bad news for the Principal. (That is the endogenous counterpart of Part (h) of theorem 1). Part (d) says that in the endogenous case a drop in t must be good news from the welfare point of view.

Theorem 2

Let R and C be strictly increasing on Σ . Let $r^*(t)$ denote a maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$ on the interval $(0, 1)$. Then

- (a) $\frac{r^*(t_L)}{t_L} \geq \frac{r^*(t_H)}{t_H}$ whenever $t_L, t_H \in \tilde{\Gamma}$ and $0 < t_L < t_H$.
- (b) $\hat{x}(r^*(t_L), t_L) \geq \hat{x}(r^*(t_H), t_H)$ whenever $t_L, t_H \in \tilde{\Gamma}$ and $0 < t_L < t_H$.
- (c) $(1 - r^*(t_L)) \cdot R(\hat{x}(r^*(t_L), t_L)) \geq (1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_H))$ whenever $t_L, t_H \in \tilde{\Gamma}$ and $0 < t_L < t_H$.
- (d) $W(r^*(t_L), t_L) > W(r^*(t_H), t_H)$ whenever $t_L \in \tilde{\Gamma}, t_H \in \tilde{\Gamma}$, and $0 < t_L < t_H$.

While Part (c) of Theorem 2 tells us that in the endogenous case technical improvement can never be bad news for the Principal, the situation is different for the Agent. Figure 9 is a graph

²⁸See, for example, Sundaram (1996). The proposition concerns a function $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ which displays strictly increasing differences. [For such a function we have $h(u_H, v_H) - h(u_L, v_H) > h(u_H, v_L) - h(u_L, v_L)$ whenever $u_H > u_L, v_H > v_L$]. The proposition is as follows:

If a function $h(u, v)$ displays strictly increasing differences, and if u_H maximizes $h(u, v_H)$ while u_L maximizes $h(u, v_L)$, then $u_H \geq u_L$ if $v_H > v_L$.

of the Agent's endogenous-case net earnings $r^*(t) \cdot R(\hat{x}(r^*(t)), t) - t \cdot C(\hat{x}(r^*(t)), t)$ in the Classic example. Once t has dropped to a critical value that is close to 0.5, a further drop is *Bad* news for the Agent. Informally: *in the endogenous case, the Principal is never the enemy of technical progress but the Agent might be.*

FIGURE 9 HERE

7. Two exogenous-case theorems which require differentiability.

7.1 Effectiveness and the effort gap move in the same direction when t changes.

Theorem 3

Let Γ be an open set in \mathbb{R}^{2+} . Suppose that the functions R and C are thrice differentiable. Suppose that the following *monotonicity* condition is met:

we either have

$$\hat{x}_{rt} > 0 \text{ for all } (r, t) \in \Gamma$$

or

$$\hat{x}_{rt} < 0 \text{ for all } (r, t) \in \Gamma.$$

Suppose, in addition, \hat{x}_t is continuous with respect to r at all points in $(0, 1]$.

Then $\hat{x}_{rt}(r, t) > 0$ (< 0) at every $(r, t) \in \Gamma$ if and only if

$$\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] > (< 0) \text{ at every } (r, t) \in \Gamma.$$

The assumptions of the theorem are met in all the examples we have presented. Note that the pair $(r^*(t), t)$ belongs to Γ , so the theorem applies, in particular, to $\hat{x}_{rt}(r^*(t), t)$ and the endogenous effort gap $\hat{x}(1, t) - \hat{x}(r^*(t), t)$. The proof (in the Appendix) is very simple.

7.2. A theorem about the effort gap and the surplus gap.

We first formally define *exogenous tracking* in examples where R and C are twice differentiable on the effort set Σ and, for fixed $r \in (0, 1]$, the Agent's effort choice $\hat{x}(r, t) \in \Sigma$ solves the the first-order condition $0 = rR'(x) - tC'(x)$.

Definition 1²⁹

²⁹Note that we could state a more general definition, not requiring differentiability. There we would say that the example has the exogenous tracking property if we have

$$(+) \quad \begin{cases} \hat{x}(1, t_L) - \hat{x}(r, t_L) > \hat{x}(1, t_H) - \hat{x}(r, t_H) & (\hat{x}(1, t_L) - \hat{x}(r, t_L) < \hat{x}(1, t_H) - \hat{x}(r, t_H)) \\ \text{whenever } (r, t_L), (r, t_H) \in \Gamma \text{ and } 0 < t_L < t_H \end{cases}$$

if and only if we also have

$$(++) \quad \begin{cases} W(1, t_L) - W(r, t_L) > W(1, t_H) - W(r, t_H) & (W(1, t_L) - W(r, t_L) < W(1, t_H) - \\ W(r, t_H)) & \text{whenever } (r, t_L), (r, t_H) \in \Gamma \text{ and } 0 < t_L < t_H. \end{cases}$$

For "opposite directions" the appropriate inequalities are reversed. Using this definition, one could explore the tracking question for finite examples.

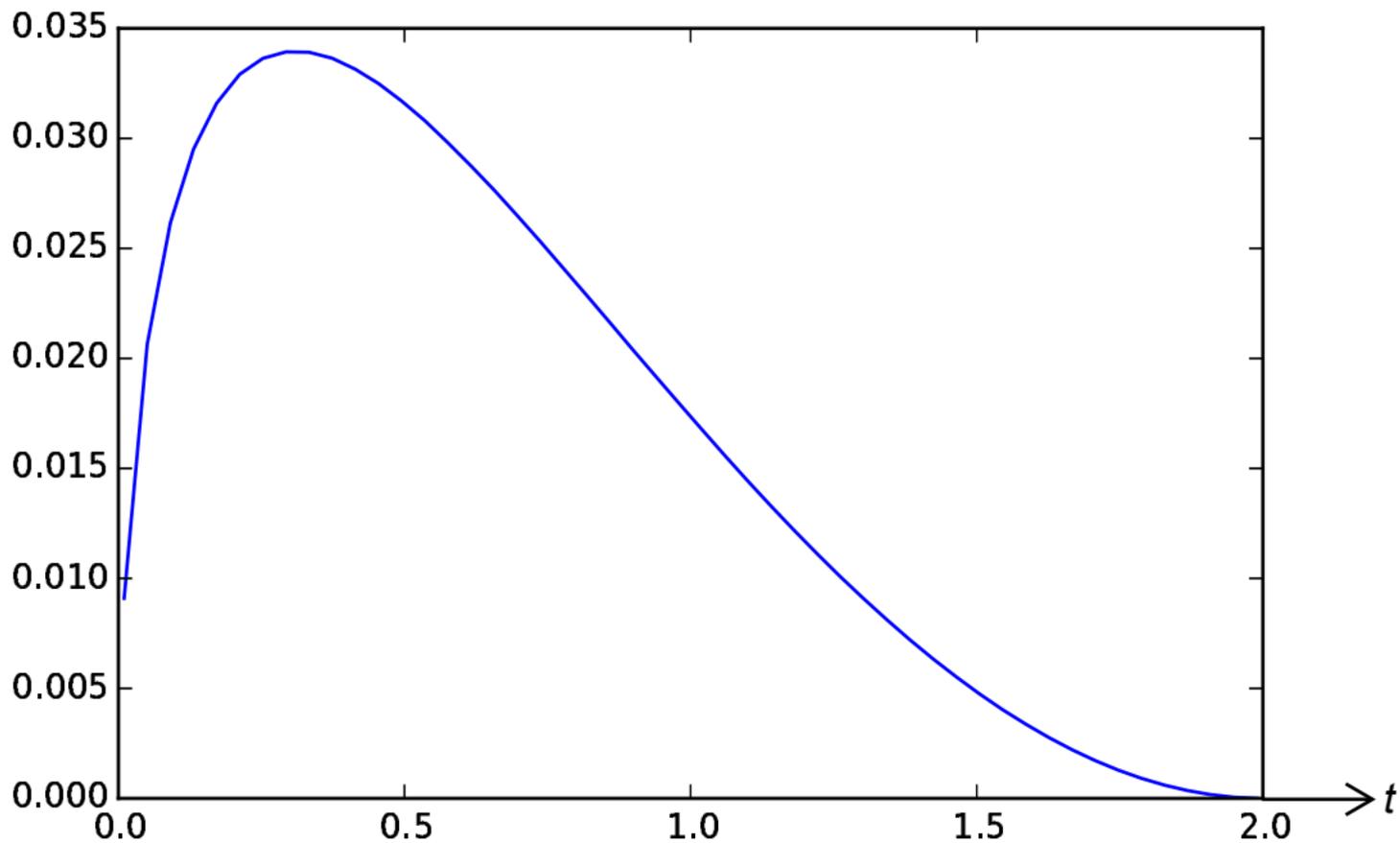


Fig. 9 The Agent's net earnings for the Classic example with $A = 2, B = 3$

An example (R, C, Γ, Σ) , with R and C thrice differentiable, has the *exogenous tracking (opposite directions)* property if

$$\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] \cdot \frac{d}{dt} [W(1, t) - W(r, t)] > 0 (< 0) \text{ at all } (r, t) \in \Gamma.$$

The next theorem concerns exogenous tracking in *Interior* examples. In an Interior example the Agent's best effort is the unique solution to a first-order equation and the same is true for the Principal's chosen share. Before providing the definition, we recall that for every $(r, t) \in \Gamma$ we have $0 < r < 1$. Recall also that $\tilde{\Gamma}$ denotes the set of possible values of t . ($\tilde{\Gamma} = \{t : (r, t) \in \Gamma \text{ for some } r\}$).

Definition 2

An example (Σ, R, C, Γ) is Interior if

- $\Sigma \subseteq \mathbb{R}^+$, and $\Gamma \subseteq \mathbb{R}^{2+}$, are open sets.
- R, C are thrice differentiable on Σ and $R' > 0, C' > 0$.
- There exists a twice differentiable function $\hat{x} : (0, 1] \times \tilde{\Gamma} \rightarrow \Sigma$ such that for $r \in (0, 1]$, $\hat{x}(r, t)$ satisfies the first-order condition $0 = rR'(x) - tC'(x)$ and is the unique maximizer of $rR(x) - tC(x)$ on Σ .
- For every $t \in \tilde{\Gamma}$, there exists a share $r^*(t) \in (0, 1)$ which satisfies the first-order condition $0 = \frac{d}{dr} [(1 - r) \cdot R(\hat{x}(r, t))]$ and is the unique maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$ on $(0, 1)$.

All the examples we have discussed satisfy these conditions.³⁰

Theorem 4

An interior example has the exogenous tracking property if the effort set is $\Sigma = (0, J)$, where $J > 0$, and the monotonicity condition of Theorem 2 holds (we either have $\hat{x}_{rt} > 0$ for all $(r, t) \in \Gamma$ or $\hat{x}_{rt} < 0$ for all $(r, t) \in \Gamma$).

Straightforward calculation yields the following Corollary, proved (together with theorem 4) in the Appendix.

Corollary

The following hold for an interior example in which the monotonicity condition of Theorem 2 is satisfied, the effort set is $\Sigma = (0, J)$ (where $J > 0$), and $\hat{x}_r(r, t) > 0, \hat{x}_t(r, t) < 0$ for all $(r, t) \in \Gamma$:

- (i) the Decentralization Penalty (surplus gap) is decreasing in t (so the Penalty grows when technology improves) if at every effort $x \in (0, J)$ we have $R''(x) \geq 0, R'''(x) = C'''(x) = 0$.

³⁰Consider the condition we imposed in Theorem 3. We require that the function \hat{x}_t is continuous with respect to r at every $r \in (0, 1]$. The third item in our Interior Example definition insures that this is indeed the case.

- (ii) the Decentralization Penalty (surplus gap) is increasing in t (so the Penalty shrinks when technology improves) if at every effort $x \in (0, J)$ we have $R''(x) < 0, C'''(x) = 0, R'''(x) \leq 0$.

Even though we are in the relatively straightforward exogenous case, the Corollary's sufficient conditions for the Penalty to grow (shrink) when technology improves are restrictive but simple. When we turn to the endogenous case, we find *no* similarly simple conditions on R and C which tell us, all by themselves, the direction in which the Penalty moves when technology improves.

8. Endogenous-case results which require differentiability.

Classifying the endogenous-case results.

We have seen that in the endogenous case there are examples where the Decentralization Penalty (surplus gap) rises when technology improves and there are examples where it falls. There are examples where we have “endogenous tracking” (surplus and effort gaps move in the same direction when t changes), but there other examples where that is not true. There is no endogenous analog of Theorem 4 in which we again have tracking under very general assumptions. Here is our tracking definition for the endogenous case.

Definition 3

An interior example (R, C, Γ, Σ) has the *endogenous tracking (opposite directions)* property if

$$\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r^*(t), t)] \cdot \frac{d}{dt} [W(1, t) - W(r^*(t), t)] > 0 (< 0) \text{ at all } t \in \tilde{\Gamma} = \{t : (r, t) \in \Gamma \text{ for some } r\}.$$

An instructive way to bring order to the rich variety of endogenous results is to characterize the way that (a) effectiveness $\hat{x}_{rt}(r, t)$ and (b) the Principal's chosen share r^* move when t changes. For a drop in t , we consider four combinations: (a) and (b) both rise; they both fall; (a) rises and (b) falls; (a) falls and (b) rises. In particular we find — in Theorem 5 — that when one rises and the other falls, then we indeed have endogenous tracking. In Theorem 6 we find that if $R'' < 0$ then a drop in t cannot lead to more effectiveness and greater Principal's generosity. Before proceeding to these theorems we present a four-box endogenous-case table which serves as a guide to those theorems and their relation to the examples we have considered.

	<p>WHEN t DROPS, EFFECTIVENESS OF A SHARE INCREASE <u>FALLS</u>. HENCE SO DOES THE EXOGENOUS EFFORT GAP (SEE THEOREM 3).</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $\hat{x}_{rt} > 0 \text{ and hence } \frac{d}{dt}[\hat{x}(1, t) - \hat{x}(r, t)] > 0$ </div>	<p>WHEN t DROPS, EFFECTIVENESS OF A SHARE INCREASE <u>RISES</u>. HENCE SO DOES THE EXOGENOUS EFFORT GAP (SEE THEOREM 3).</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $\hat{x}_{rt} < 0 \text{ and hence } \frac{d}{dt}[\hat{x}(1, t) - \hat{x}(r, t)] < 0$ </div>
<p>WHEN t DROPS, PRINCIPAL BECOMES <u>LESS</u> GENEROUS OR GENEROSITY STAYS THE SAME.</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $r^* \geq 0$ </div>	<p>1 SEE "CLASSIC" AND "CUBIC-REVENUE" EXAMPLES. WE HAVE TRACKING IN THE CLASSIC EXAMPLE BUT IN THE CUBIC-REVENUE EXAMPLE WE HAVE "OPPOSITE DIRECTIONS" (IF THE SET OF POSSIBLE VALUES OF t IS PROPERLY CHOSEN).</p>	<p>2 SEE "RISING MARGINALS" EXAMPLE. EVERY EXAMPLE THAT LIES IN THIS BOX HAS THE TRACKING PROPERTY. (See Theorem 5, Part (a)).</p>
<p>WHEN t DROPS, PRINCIPAL BECOMES <u>MORE</u> GENEROUS.</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $r^* < 0$ </div>	<p>3 SEE "EXPLODING MARGINALS" EXAMPLE. EVERY EXAMPLE THAT LIES IN THIS BOX HAS THE TRACKING PROPERTY. (See Theorem 5, Part (b)). AN EXAMPLE WITH $R'' < 0$ CANNOT BE IN THIS BOX. (See Theorem 6).</p>	<p>4 SEE THE "PRICE-TAKER" EXAMPLE, WHERE WE HAVE TRACKING AND THE "CUBIC-COST" EXAMPLE, WHERE WE HAVE "OPPOSITE DIRECTIONS".</p>

The example in Box **3**, which we call the "Exploding Marginals" example, has functions R and C such that $C'' > R''$ and both C'' and R'' are very large. It appears difficult to construct Box-**3** examples where that is not the case.³¹ The Exploding Marginals example is as follows:

- $\Sigma = (0, 1)$.
- $\Gamma = \{(r, t) : 0 < r < 1; \frac{r}{t} \in (e, e^e)\}$ (e is the base of the natural logarithms).
- $R(x) = e^{x^2}$.
- $C(x) = \int_0^x [2e^{e^p} \cdot e^p \cdot p] dp$.

The proof that the Exploding Marginals example indeed lies in Box **3** is provided in the Appendix.

We now have Theorem 5, a two-part theorem, which concerns Box **2** and Box **3**.

³¹One can prove, for example, that if $R = \frac{1}{2}x^2$, so that $R'' = 1$, then we cannot be in Box **3**.

Theorem 5

Consider an interior example (Σ, Γ, R, C) .

(a). Suppose the following holds:

for every $t \in \tilde{\Gamma}$ we have $r^(t) \geq 0$ and for every $(r, t) \in \Gamma$ we have $\hat{x}_{rt}(r, t) < 0$.*

Then we have endogenous tracking.

(b). Suppose the following holds:

for every $t \in \tilde{\Gamma}$ we have $r^(t) < 0$ and for every $(r, t) \in \Gamma$ we have $\hat{x}_{rt}(r, t) > 0$.*

Then we have endogenous tracking.

The next theorem does not directly concern the two gaps. But it implies that if marginal revenue is decreasing or constant ($R'' \leq 0$) in an interior example and the Principal has a unique best share, then the example cannot be in Box **3**.

Theorem 6

Suppose that in the interior example (Σ, Γ, R, C) we have:

- $R''(x) \leq 0$ at every $x \in \Sigma$.
- $\hat{x}_{rt}(r, t) \geq 0$, $\hat{x}_t(r, t) < 0$ and $\hat{x}_r(r, t) > 0$ at every $(r, t) \in \Gamma$.
- $r^*(t)$ is the unique maximizer of $(1 - r) \cdot R(\hat{x}(r, t))$ on $(0, 1)$,

Then $r^*(t) \geq 0$ for all $t \in \Gamma$.

It is difficult to give a clear intuition for Theorems 4 and 5. That is a little easier for Theorem 6, which says that if marginal revenue is decreasing, and effectiveness drops when technology improves, then when technology improves, the Principal does not become more generous ($r^*(t) \geq 0$), i.e., we cannot be in Box **3**. Intuitively one might say: *when t drops, increasing the share above its previous level would damage the Principal, because the extra revenue due to extra effort has dropped (marginal revenue has declined) and at the same time the extra effort evoked by a share increase has dropped as well.*

8.2 A summary.

Return to our original puzzle: when does technical improvement lower the Penalty and when does it raise the Penalty? Here is a summary of what Theorems 5 and 6 have told us about the puzzle in the endogenous case.

Consider any interior example. If, in that example, technical improvement makes the Principal less generous or keeps his generosity unchanged, while at the same time it raises the effectiveness of a share increase ($r^*(t) \geq 0$ and $\hat{x}_{rt}(r, t) < 0$), then the improvement raises the Penalty or keeps it unchanged. If technical improvement makes the Principal more generous, while at the same time it decreases the effectiveness of a share increase or leaves it unchanged ($r^*(t) < 0$ and $\hat{x}_{rt}(r, t) \geq 0$) — which cannot happen if marginal revenue is nonincreasing — then the improvement lowers the Penalty or keeps it unchanged.

Unfortunately there are no simple conditions on R and C , similar to those in the Corollary to the exogenous-case Theorem 4, which imply, all by themselves, that the Penalty rises (falls) when technology improves.

9. Finding the Principal's best share for a given t .

The function $r^*(t)$ may be increasing on the set $\tilde{\Gamma}$ of possible values of t . It may also be decreasing or constant. We have discussed the implications of each case. But we have not yet studied, in a general way, the shape of the Principal's gain *as a function of $r \in (0, 1)$ when t is fixed*. The gain for fixed t is $(1 - r) \cdot R(\hat{x}(r, t))$. The graph of the non-negative values of $(1 - r) \cdot R(\hat{x}(r, t))$, with r on the horizontal axis, starts at zero and ends at zero. The graph coincides with the Principal's gain curve except at $r = 0$ and $r = 1$, since the Principal confines attention to the open interval $(0, 1)$. It would be particularly helpful if the gain curve rises and then falls, achieving its maximum height at $r^*(t)$. More generally, the curve could rise until $r = r^*(t)$ and could then be flat for an interval before descending. Let us call such a gain curve *single-peaked*. As long as the gain is positive at some $r \in (0, 1)$ the curve is single-peaked if it is concave on $(0, 1)$. The following theorem provides conditions under which the gain is indeed concave. The theorem has two parts. The first part does not require differentiability with respect to r , but the second part does. Informally, the second part says that we have concavity if marginal revenue drops ($R'' < 0$) and in addition the effectiveness of a share increase drops when the share increases ($\hat{x}_{rr} < 0$).

Theorem 7

(a) If, for a fixed t , $R(\hat{x}(r, t))$ is concave on $(0, 1)$, then the Principal's gain $(1 - r) \cdot R(\hat{x}(r, t))$ is also concave on $(0, 1)$.

(b) Consider an interior example (Σ, Γ, R, C) where $\Sigma = (0, J)$, with $J > 0$. Then R is concave on $(0, J)$ if for all $x \in (0, J)$ we have $R''(x) < 0$, and for all $(r, t) \in \Gamma$ we have $\hat{x}_{rr}(r, t) < 0$. If $R''(x) < 0$, then a sufficient condition for $\hat{x}_{rr} < 0$ is

$$r \cdot R'''(x) - t \cdot C'''(x) \leq 0.$$

Suppose that the Principal's gain curve is indeed single-peaked, and suppose that the share the Principal uses is determined by bargaining between the Principal and the Agent. In the

interval between zero and the peak (the interval $(0, r^*(t))$), the Agent strictly benefits from a rise in r , as we established in Part **(a)** of Theorem 1. The Principal prefers a higher r as well (since the gain curve is rising in the interval). But in the interval between the peak and zero (the interval $(r^*(t), 1)$), where the gain curve is falling, the Agent prefers a higher r and the Principal prefers a lower r . (If $r^*(t)$ is followed by a flat interval, then the Principal is indifferent between shares in the flat interval, but is damaged by share increases beyond the flat interval). So the negotiation set, where bargaining occurs, is the interval $(r^*(t), 1)$. Now consider the outcome of the bargaining from the welfare point of view. Assume that R and C are differentiable and that $\hat{x}(r, t)$ is the solution to the first-order equation $0 = r \cdot R(x) - t \cdot C(x)$. Then we know, from Part **(h)** of Theorem 1, that for a fixed t , welfare increases when the exogenous share increases. So, informally speaking, *increasing the Agent's bargaining strength increases welfare*. A formal model of the bargaining process is needed to make “bargaining strength” precise. Such a model might also reveal the welfare implications of the fact that if r^* is increasing in t , then the negotiation interval $(0, r^*(t))$ shrinks when t drops.

10. Examining our main questions in a different model

In this model revenue is uncertain, there is no informational asymmetry, and the Agent's reward for a given revenue is a wage which need not be a fixed share of revenue.

Consider the simplest case. There are just two possible efforts: x_L and x_H . (As usual, the subscripts L and H mean “low” and “high”). The low effort costs $tC_L > 0$ and the high effort costs $tC_H > tC_L$, where $t > 0$ is our technology parameter. For each effort, there are two possible revenues, R_L and R_H . For the effort x_L , the probability of R_H is p and the probability of R_L is $1 - p$. For the effort x_H the revenue probabilities are $q, 1 - q$. We assume that $q > p > 0$. Before the Agent chooses effort, the Principal announces a wage pair, say (w_H, w_L) . The Agent receives w_H if revenue turns out to be R_H and w_L if revenue turns out to be R_L . If the Agent chooses x_H , then the Agent's average gain is $qw_H + (1 - q) \cdot w_L - tC_H$ and the Principal's average gain is $q \cdot (R_H - w_H) + (1 - q) \cdot (R_L - w_L)$. If the Agent chooses x_L , then the Agent's average gain is $pw_H + (1 - p) \cdot w_L - tC_L$ and the Principal's average gain is $p \cdot (R_H - w_H) + (1 - p) \cdot (R_L - w_L)$. Both parties are risk neutral. Each wants average gain to be maximal.

We make the customary limited-liability assumption. The Agent cannot assume debt, so the wage paid to the Agent is nonnegative. We shall say that the Principal's nonnegative wage pair (w_H, w_L) *induces* the effort x if the Agent finds that (1) average wage is not less than the cost $tC(x)$, and (2) her net gain (average wage minus $tC(x)$) is not less for the effort x than for the alternative effort. The first of these constraints is the Individual Rationality (IR) constraint and the second is the Incentive Compatibility (IC) constraint. Note that the IC constraint is a weak inequality. If the Agent's net gain is the same for both efforts, then we have equality and the two efforts are tied. Pursuing a standard approach, we shall say that the Agent always breaks a tie between two efforts by choosing the effort that the Principal seeks to induce. Thus the Agent knows *both* the Principal's chosen wage pair and the effort the Principal seeks to induce.

If a wage pair induces the effort x and does so with an average wage that is not larger than the average for any other wage pair which induces x , then we shall say that the pair *optimally*

induces x . Given a wage pair which optimally induces x , and another wage pair which optimally induces an effort x^* , we shall say that the Principal *weakly prefers the first wage pair* — or, equivalently, *weakly prefers x* — if average revenue minus average wage is not lower at the first pair.

We shall verify that a key result in the fixed-share model holds again: it can never happen that the Agent works less when technology improves. We conclude by examining the Decentralization Penalty.

We first consider the optimal inducement of x_L and then the optimal inducement of x_H .

10.1 Optimally inducing x_L .

We let IR-L and IC-L denote the IR and IC constraints which must be satisfied if the Agent chooses x_L . The set of nonnegative wage pairs satisfying IR-L is

$$A_L = \{(w_H, w_L) : w_H \geq 0, w_L \geq 0; pw_H + (1 - p) \cdot w_L \geq tC_L\}.$$

The IC-L constraint is

$$p \cdot w_H + (1 - p) \cdot w_L - tC_L \geq q \cdot w_H + (1 - q) \cdot w_L - tC_H,$$

which simplifies to

$$(q - p) \cdot (w_H - w_L) \leq t \cdot (C_H - C_L).$$

So the set of nonnegative wage pairs satisfying IC-L is

$$B_L = \{(w_H, w_L) : w_H \geq 0, w_L \geq 0; (q - p) \cdot (w_H - w_L) \leq t \cdot (C_H - C_L)\}.$$

Any wage pair which lies in both A_L and B_L induces the Agent to choose x_L . If the inequality in B_L becomes an equality for a given pair, then the Agent breaks the tie between the two efforts by choosing x_L , which the Principal seeks to induce.

Consider the wage pair $(w_H, w_L) = \left(0, \frac{tC_L}{1 - p}\right)$, which lies in A_L . For this pair the last inequality in B_L becomes

$$(q - p) \cdot \frac{-tC_L}{1 - p} \leq t \cdot (C_H - C_L),$$

which holds. So this wage pair also lies in B_L and hence it induces the Agent to choose x_L . At this pair, moreover, the last inequality in A_L becomes an equality, i.e., IR-L is binding. The average wage at the pair is

$$p \cdot w_H + (1 - p)w_L = tC_L.$$

No wage pair in A_L has a lower average than tC_L . Thus our pair $(w_H, w_L) = \left(0, \frac{tC_L}{1 - p}\right)$ optimally induces x_L and makes IR-L binding.

Since effort determines average revenue, it follows that:

(§) All wage pairs which optimally induce a given effort have the same average wage.

So if we want to know the average wage required to induce a given effort, it suffices to exhibit just one of the wage pairs that optimally induce it. In particular, consider any pair which — like the pair $(w_H, w_L) = \left(0, \frac{tC_L}{1-p}\right)$ — optimally induces x_L for t . In view of (§), its average wage must again equal tC_L and hence it again makes IR-L binding.

To summarize, we have shown:

For all t , IR-L is binding at any wage pair which optimally induces x_L .

10.2 Optimally inducing x_H .

We let IR-H and IC-H denote the IR and IC constraints which must be satisfied when the Agent chooses x_H . The set of nonnegative wage pairs satisfying IR-H is

$$A_H = \{(w_H, w_L) : w_H \geq 0, w_L \geq 0; qw_H + (1-q) \cdot w_L \geq tC_H\}.$$

The set of nonnegative wage pairs satisfying IC-H is

$$B_H = \{(w_H, w_L) : w_H \geq 0, w_L \geq 0; (q-p) \cdot (w_H - w_L) \geq t \cdot (C_H - C_L)\}.$$

Any wage pair which lies in both A_H and B_H induces the Agent to choose x_H . If the inequality in B_H becomes an equality for a given pair, then the Agent breaks the tie between the two efforts by choosing x_H , which the Principal seeks to induce.

Now consider a specific t , a wage pair $(\tilde{w}_H, \tilde{w}_L)$ which optimally induces x_L , and a specific wage pair that optimally induces x_H . Suppose the Principal weakly prefers inducing x_H by using the latter pair to inducing x_L by using $(\tilde{w}_H, \tilde{w}_L)$. Then we can conclude — using the statement (§) in the preceding section — that for our specific t , the Principal weakly prefers *any* pair which optimally induces x_H to *any* pair which optimally induces x_L . We shall follow those steps in our analysis. We divide the possible quadruples (p, q, C_H, C_L) into two cases. In each case we exhibit a specific pair that optimally induces x_H . In each case we then show that if, at a specified t , the Principal weakly prefers optimally inducing x_H to optimally inducing x_L , then he continues to do so if we replace t by $t^* < t$.

Case (i): $pC_H \leq qC_L$

For this case, our x_H -inducing wage pair will be

$$(w_H, w_L) = \left(\frac{tC_H}{q}, 0\right).$$

That pair lies in A_H . Moreover, for that pair the last inequality in A_H becomes an equality, i.e., IR-H is binding. Our pair also lies in B_H since the last inequality in B_H — namely $(q-p) \cdot (w_H - w_L) \geq t \cdot (C_H - C_L)$ — can be written $pC_H \leq qC_L$ (our Case-(i) inequality). Thus the pair induces x_H . But the average wage paid by the Principal when he uses the pair is

$qw_H + (1 - q) \cdot w_L = tC_H$. For our specific t , that is the smallest average wage among all the pairs in A_H and hence it is indeed true that $\left(\frac{tC_H}{q}, 0\right)$ *optimally* induces x_H . Now consider $(\tilde{w}_H, \tilde{w}_L)$, which optimally induces x_L . The Principal weakly prefers optimally inducing x_H to optimally inducing x_L if and only if

$$(+) \quad [qR_H + (1 - q) \cdot R_L] - [pR_H + (1 - p) \cdot R_L] \geq [qw_H + (1 - q) \cdot w_L] - [p\tilde{w}_H + (1 - p) \cdot \tilde{w}_L].$$

The expression on the left equals $(q - p) \cdot (C_H - C_L)$. Since $[qw_H + (1 - q) \cdot w_L] = tC_H$ and $[p\tilde{w}_H + (1 - p) \cdot \tilde{w}_L] = tC_L$, we have

$$(++) \quad t \leq \frac{(q - p) \cdot (R_H - R_L)}{C_H - C_L}.$$

That holds *a fortiori* if t is replaced by $t^* < t$. The Principal continues to weakly prefer x_H when t drops.

Case (ii): $pC_H > qC_L$

For this case, our x_H -inducing pair will be $(w_H, w_L) = \left(\frac{t \cdot (C_H - C_L)}{q - p}, 0\right)$. For this pair the final inequality in A_H — namely $qw_H + (1 - q) \cdot w_L \geq tC_H$ — becomes

$$\frac{C_H - C_L}{q - p} \geq \frac{C_H}{q},$$

which is equivalent to $pC_H > qC_L$ (our Case-(ii) inequality). The final inequality in B_H is also satisfied. In fact it becomes an equality. The average wage at our pair is

$$q \cdot \frac{t \cdot (C_H - C_L)}{q - p}.$$

But the final inequality in B_H can be rewritten

$$w_H \geq \frac{t \cdot (C_H - C_L)}{q - p}.$$

Since that becomes an equality for our pair $\left(\frac{t \cdot (C_H - C_L)}{q - p}, 0\right)$, it follows that any other wage pair that induces x_H has a higher average wage than our pair. Hence it is indeed true that $\left(\frac{t \cdot (C_H - C_L)}{q - p}, 0\right)$ *optimally* induces x_H .

In contrast to Case (i), IR-H is not binding. We have

$$[qw_H + (1 - q) \cdot w_L] = t \cdot \frac{q \cdot (C_H - C_L)}{q - p}.$$

The condition (+) now becomes

$$(++++) \quad (q-p) \cdot (R_H - R_L) \geq t \left[\frac{q \cdot (C_H - C_L)}{q-p} - C_L \right].$$

But the Case-(ii) inequality $pC_H > qC_L$ implies³²

$$q \cdot (C_H - C_L) > C_L(q-p).$$

So (+++) is equivalent to

$$(+++++) \quad t \leq \frac{(q-p) \cdot (R_H - R_L)}{[q \cdot (C_H - C_L)] / (q-p) - C_L}.$$

This again holds *a fortiori* for $t^* < t$.

To summarize, we have established the following proposition.

If the Principal weakly prefers optimally inducing x_H to optimally inducing x_L at t , then he cannot strongly prefer x_L to x_H at $t^* < t$. (The Agent never works less when technology improves).

10.3 The Decentralization Penalty.

We shall say that we have a positive Decentralization Penalty at a given t if and only if highest attainable (social) surplus exceeds the surplus at an effort which the Principal strictly prefers to (optimally) induce. We take surplus at a given effort to be the expected revenue which the effort yields minus the effort's cost. At the effort x_H , surplus is

$$qR_H + (1-q)R_L - tC_H.$$

At the effort x_L , surplus is

$$pR_H + (1-p)R_L - tC_L.$$

We first show that in Case (i), the Penalty is zero at every t . Recall that when the Principal optimally induces x_L then IR-L must be binding and the average wage is tC_L . For Case (i) recall also that when the Principal optimally induces x_H , IR-H is binding and the average wage is tC_H . Thus the Principal's net gain at the effort x_H is $qR_H + (1-q)R_L - tC_H$. At the effort x_L the net gain is $pR_H + (1-p)R_L - tC_L$. So the Principal strictly prefers to optimally induce x_H if and only if surplus is higher at x_H . The Principal strictly prefers to optimally induce x_L if and only if surplus is higher at x_L . Thus we have indeed shown:

³²That is the case since

$$qC_H - qC_L > pC_H - pC_L,$$

and hence (using the Case-(ii) inequality)

$$qC_H - qC_L > qC_L - pC_L.$$

In Case (i) the Decentralization Penalty is zero at every t .

Now what can be said about the Penalty if we are in case (ii)? We saw, in our Case-(ii) analysis in 10.2, that the pair $(w_H, w_L) = \left(\frac{t(C_H - C_L)}{q - p}, 0 \right)$ optimally induces x_H . The Principal strictly prefers optimally inducing x_H if and only if (+) is a strict inequality, which means that (++++) is also a strict inequality. Thus the Principal strictly prefers optimally inducing x_H if and only if

$$t < J, \text{ where } J \equiv \frac{(q - p)^2 \cdot (R_H - R_L)}{q \cdot (C_H - C_L) - C_L \cdot (q - p)}.$$

The Principal strictly prefers optimally inducing x_L if and only if $t > J$.

On the other hand, surplus is strictly higher at x_H if and only if

$$t < K \text{ where } K \equiv \frac{(q - p) \cdot (R_H - R_L)}{C_H - C_L}.$$

It is easily verified that $J < K$ whenever $pC_H > qC_L$ (i.e. whenever the Case-(ii) condition holds). So if we are in Case (ii) then we have a positive Penalty for all t in the open interval (J, K) . For t in that interval the Principal strictly prefers to optimally induce x_L , but ‘‘Society’’ strictly prefers x_H . Thus the Decentralization Penalty for $t \in (J, K)$ is

$$[qR_H + (1 - q) \cdot R_L - tC_H] - [pR_H + (1 - p) \cdot R_L - tC_L] = (q - p) \cdot (R_H - R_L) + t \cdot (C_L - C_H).$$

The Penalty is strictly decreasing in t . It rises (linearly) when technology improves.

To summarize, we have shown the following:

- In Case (i) the Decentralization Penalty is zero at every t .
- In Case (ii) there exists an open interval (J, K) with $0 < J < K$ such that for all t in the interval the Principal strictly prefers x_L to x_H while ‘‘Society’’ strictly prefers x_H , so the Penalty is positive. When t drops within the interval (technology improves), the Penalty rises. At t outside the interval the Penalty is zero.³³

11. Concluding remarks.

Recall our central question: does technical improvement strengthen the case for full Agent autonomy or does it weaken it so much that perfect monitoring and policing has now become attractive? One might have reasonably hoped for a straightforward answer since our revenue-sharing Principal/Agent model is so simple. Specifically one might have hoped that a natural condition like rising marginal cost and falling marginal revenue unambiguously implies that the Decentralization Penalty rises (or falls) when technology improves. Instead we have found that

³³At $t < J$, both Society and the Principal prefer x_H and at $t > K$ both prefer x_L . At $t = J$, Society prefers x_H and (+++++) becomes an equality, so the Principal is indifferent between the two efforts. At $t = K$, the Principal prefers x_L and Society is indifferent between the two efforts.

there is no easy answer to our central question. On the other hand, we have found a rich array of other results. One of them is that in both the exogenous case and the endogenous case, an advance in technology increases welfare. Another is that an advance in technology causes the Agent to work harder. That is obvious in the exogenous case, since the Agent benefits from the advance even if he continues to use his previous effort. It is not obvious in the endogenous case.

Other interesting results for the challenging endogenous case concern the tracking question. If the effort gap always moves in the same direction as the surplus gap (the Penalty), then to see whether a technical advance has strengthened or weakened the case for autonomy, it suffices to observe (but not police) the Agent's effort before and after the advance and to compare it with first-best effort. We saw that two key properties of an example are the sign of r^* and the sign of \hat{x}_{rt} . We *must* have tracking if $r^* \geq 0, \hat{x}_{rt} > 0$ or $r^* < 0, \hat{x}_{rt} > 0$ — if a drop in t decreases generosity (or leaves it unchanged) and increases the effectiveness of a share increase in eliciting higher effort, or the drop increases generosity and decreases effectiveness. For the other combinations of the two signs, we may have tracking but we may also have “opposite directions”.

Can we obtain an easier answer to our central question if we vary or complicate the model? There are many ways to do so. Here are a few of them.

- **Change the definition of “Decentralization Penalty”.** As we have already noted, one could let the Penalty be the *ratio* $\frac{W(\hat{x}(r,t))}{W(\hat{x}(1,t))}$ (or, in the endogenous case, $\frac{W(\hat{x}(r^*(t),t))}{W(\hat{x}(1,t))}$), rather than the *difference*, which we have been considering. Our central question becomes technically harder and preliminary exercises suggest that it again has no simple answer. It appears, again, that there are no simple conditions on R and C implying that the redefined Penalty rises or falls when t drops. In the Classic example, for instance, we again find (in the endogenous case) that when t rises, the redefined Penalty first rises and then falls. Moreover, we can easily find propositions which are reversed when we move from the difference definition of Penalty to the ratio definition.³⁴

³⁴In the exogenous case, with r fixed, consider the derivative of Penalty with respect to t . For the difference definition we have

$$\frac{d}{dt} [W(1, t) - W(r, t)] = W_t(1, t) - W_t(r, t),$$

which is *negative* if

$$(+) \quad W_t(1, t) < W_t(r, t).$$

For the ratio definition we have

$$\frac{d}{dt} \left[\frac{W(r, t)}{W(1, t)} \right] = \left[\frac{1}{W(1, t)} \right]^2 \cdot [W(1, t) \cdot W_t(r, t) - W(r, t) \cdot W_t(1, t)].$$

But that is *positive* if (+) holds, since we know that $W(1, t) \geq W(r, t)$. If (+) fails to hold, then whether the two derivatives have opposite signs remains open. We have to look at the functions R and C .

Now consider the endogenous case. We have

$$\frac{d}{dt} [W(1, t) - W(r^*(t), t)] = W_t(1, t) - [W_r \cdot r^{*'} + W_t].$$

- **Generalize the results of Section 10.** There are now m possible efforts and n possible revenues, where m, n are unrestricted. In a further generalization, the Agent would be risk-averse rather than risk-neutral. Does that change the effect of improved technology on the Decentralization Penalty?
- **Introduce uncertainty about the technology parameter t .** This variation is trivial if Principal and Agent are risk-neutral and if W becomes an expected value. Simply replace t , by its expected value, say \bar{t} . If we abandon risk neutrality, then it is conceivable that there are specific probability distributions of t , and specific utility functions for the Principal and the Agent under which we get a simple answer to our central question.
- **Introduce informational asymmetry.** Let t be a random variable observed by only one of the two parties. The other knows the probability distribution of t . If it is the Agent who observes t , then his best effort $\hat{x}(r, t)$ is a random variable for the Principal and so is the revenue $R(\hat{x}(r, t))$. Even if the Principal is risk-neutral and we retain our linear sharing scheme, it appears difficult to find simple conditions on R, C which imply that the Penalty rises (or falls) when technology improves. That is especially true for the endogenous case.
- **Many agents.** In the easiest case there are two Agents, the parameter t is known to all three parties, both Agents have the same function C , and we retain linear sharing. The realized revenue R is a function of t and the Agents' efforts. The Principal chooses two shares whose sum must lie between zero and one. Then for every given t we have a three-player game. Each Agent chooses an effort x_i^t and the Principal chooses the two shares, r_1^t, r_2^t . Agent i 's payoff is $r_i^t \cdot R(x_1^t, x_2^t) - t \cdot C(x_i^t)$ and the Principal's payoff is $(1 - r_1^t - r_2^t) \cdot R(x_1^t, x_2^t)$. Suppose that for every t the game has a pure-strategy equilibrium where the Principal chooses $(\tilde{r}_1^t, \tilde{r}_2^t)$ and Agent i chooses the effort \tilde{x}_i^t , and suppose that for a given t , surplus is maximized by the efforts $(\bar{x}_1^t, \bar{x}_2^t)$. The Decentralization Penalty at the equilibrium is

$$\left[R(\bar{x}_1^t, \bar{x}_2^t) - t \cdot C(\bar{x}_1^t) - t \cdot C(\bar{x}_2^t) \right] - \left[R(\tilde{x}_1^t, \tilde{x}_2^t) - t \cdot C(\tilde{x}_1^t) - t \cdot C(\tilde{x}_2^t) \right].$$

When t drops, does the equilibrium Penalty rise or fall?

It was natural to start with our stripped-down model, where we already saw the unexpected challenges posed by our central question. The question of the effect of improved technology on the merits of alternative modes of organizing is well motivated but has seldom been the focus

(Here W_r, W_t are abbreviations for $W_r(r^*(t), t)$ and $W_t(r^*(t), t)$). We know from Part **(d)** of Theorem 1 that $W_t \leq 0$. Hence the derivative for the difference definition is *negative* if

$$W_r \cdot r^{*'} + W_t > 0.$$

On the other hand, for the ratio definition we have

$$\frac{d}{dt} \left[\frac{W(r^*(t), t)}{W(1, t)} \right] = \left[\frac{1}{W(1, t)} \right]^2 \cdot \left(\left[W_r \cdot r^{*'} + W_t \right] \cdot W(1, t) - W(r^*(t), t) \cdot W_t(1, t) \right).$$

Since $W_t \leq 0$, the whole expression is *positive* if $(++)$ holds.

of previous research. The variations and extensions that we have noted, and numerous others, merit further attention.

APPENDIX

Proof of Theorem 1

In proving Parts **(a)**, **(b)**, **(c)**, **(d)**, we shall use the standard proposition from monotone comparative statics which we summarized in the text and state more completely here.

Consider sets $U \in \mathbb{R}, V \in \mathbb{R}$ and a function $h : U \times V \rightarrow \mathbb{R}$. The two arguments of h are denoted u, v . The function h displays *strictly increasing differences in the variables u, v* if

$$h(u_H, v_H) - h(u_L, v_H) > h(u_H, v_L) - h(u_L, v_L)$$

whenever $u_H, u_L \in U, v_H, v_L \in V, u_H > u_L$, and $v_H > v_L$.

(*) $\left\{ \begin{array}{l} \text{Suppose that for every } v \in V, \text{ the problem} \\ \qquad \qquad \qquad \text{maximize } h(u, v) \text{ subject to } u \in U \\ \text{has at least one solution. Suppose also that } h \text{ satisfies strictly increasing differences in } u, v. \\ \text{Consider } v_H, v_L \in V \text{ with } v_H > v_L. \text{ Let } u_H \text{ be a maximizer of } h(u, v_H) \text{ on } U \text{ and let } u_L \text{ be} \\ \text{a maximizer of } h(u, v_L) \text{ on } U. \text{ Then } u_H \geq u_L. \end{array} \right.$

Note the following:

- (α) If h takes the form $h(u, v) = f(u, v) + g(u)$, then h displays strictly increasing differences in u, v if and only if f displays strictly increasing differences in u, v .
- (β) If h takes the form $h(u, v) = f(u) \cdot g(v)$ and f is strictly increasing while g is nondecreasing, then h displays strictly increasing differences in u, v .
- (γ) If h takes the form $h(u, v) = u \cdot g(v)$ and g is nondecreasing, then h displays strictly increasing differences in u, v .

Proof of Part **(a)**

By (α), the function $r \cdot R(x) - tC(x)$, where t is fixed, displays strictly increasing differences in r, x if $r \cdot R(x)$ displays strictly increasing differences in r, x . But, in view of (γ), that is the case, since R is nondecreasing. Since, for fixed t , the effort $\hat{x}(r, t)$ maximizes $r \cdot R(x) - tC(x)$ on the effort set Σ , Proposition (*) implies $\hat{x}(r_H, t) \geq \hat{x}(r_L, t)$, as **(a)** asserts. Part **(a)** also asserts that the Agent strictly prefers the higher share. That is the case since $\hat{x}(r_H, t)$ is a maximizer of $r_H \cdot R(x) - t \cdot C(x)$, so we have

$$r_H \cdot R(\hat{x}(r_H, t)) - t \cdot C(\hat{x}(r_H, t)) \geq r_H \cdot R(\hat{x}(r_L, t)) - t \cdot C(\hat{x}(r_L, t)) > r_H \cdot R(\hat{x}(r_L, t)) - t \cdot C(\hat{x}(r_L, t)).$$

Proof of Part (b)

By (α) , the function $r \cdot R(x) - tC(x)$, where $r \in (0, 1)$ is fixed, displays strictly increasing differences in $-t, x$ if $-t \cdot C(x)$ displays strictly increasing differences in $-t, x$. By (γ) , that is the case, since C is nondecreasing. Since, for fixed r , the effort $\hat{x}(r, t)$ maximizes $r \cdot R(x) - tC(x)$ on Σ , Proposition $(*)$ implies $\hat{x}(r, t_L) \geq \hat{x}(r, t_H)$, as **(b)** asserts.

Proof of Part (c)

By (α) and (γ) , $R(x) - t \cdot C(x)$ displays strictly in $-t, x$. The effort $\hat{x}(1, t)$ is a maximizer of $R(x) - t \cdot x$. Hence, by Proposition $(*)$, $\hat{x}(1, t_L) \geq \hat{x}(1, t_H)$, as Part **(c)** asserts.

Proof of Part (d)

Recall that $W(1, t)$ is the maximal surplus for a given t . Using **(c)**, and the fact that R and C are nondecreasing, we have

$$\begin{aligned} W(1, t_L) = R(\hat{x}(1, t_L)) - t_L \cdot C(\hat{x}(1, t_L)) &\geq R(\hat{x}(1, t_H)) - t_L \cdot C(\hat{x}(1, t_H)) \\ &\geq R(\hat{x}(1, t_H)) - t_H \cdot C(\hat{x}(1, t_H)) = W(1, t_H), \end{aligned}$$

as **(d)** asserts.

Proof of Part (e)

This follows immediately from **(b)** and the fact that R is nondecreasing.

Proof of Part (f)

Since $\hat{x}(r, t)$ is a maximizer of $R(x) - t \cdot C(x)$, we have

$$rR(\hat{x}(r, t_L)) - t_L \cdot C(\hat{x}(r, t_L)) \geq rR(\hat{x}(r, t_H)) - t_L \cdot C(\hat{x}(r, t_H)) > rR(\hat{x}(r, t_H)) - t_H \cdot C(\hat{x}(r, t_H)).$$

That implies **(f)**.

Proof of Part (g)

Part **(g)** says:

$$W(r, t_L) > W(r, t_H) \text{ whenever } t_L, t_H \in \tilde{\Gamma} \text{ and } 0 < t_L < t_H.$$

The effort $\hat{x}(r, t_L)$ is a maximizer of $rR(x) - t_H \cdot C(x)$. Hence

$$r \cdot R(\hat{x}(r, t_L)) - t_L \cdot C(\hat{x}(r, t_L)) \geq r \cdot R(\hat{x}(r, t_H)) - t_L \cdot C(\hat{x}(r, t_H))$$

or

$$r \cdot [R(\hat{x}(r, t_L)) - R(\hat{x}(r, t_H))] \geq t_L \cdot [C(\hat{x}(r, t_L)) - C(\hat{x}(r, t_H))].$$

That implies — since $0 < r < 1$ — that

$$R(\hat{x}(r, t_L)) - R(\hat{x}(r, t_H)) > t_L \cdot [C(\hat{x}(r, t_L)) - C(\hat{x}(r, t_H))]$$

or

$$R(\hat{x}(r, t_L)) - t_L \cdot C(\hat{x}(r, t_L)) > R(\hat{x}(r, t_H)) - t_L \cdot C(\hat{x}(r, t_H))$$

and hence (since $t_H > t_L$)

$$R(\hat{x}(r, t_L)) - t_L \cdot C(\hat{x}(r, t_L)) > R(\hat{x}(r(t_H), t_H)) - t_H \cdot C(\hat{x}(r, t_H))$$

The term on the left of the inequality is $W(r, t_L)$ and the term on the right is $W(r, t_H)$. That completes the proof of Part **(g)**.

Proof of Part **(h)**:

When the Agent's share is r_H , he chooses an effort $\hat{x}(r_H, t)$ which satisfies

$$r_H R(\hat{x}(r_H, t)) - t C(\hat{x}(r_H, t)) \geq r_H R(\hat{x}(r_L, t)) - t C(\hat{x}(r_L, t)),$$

or equivalently

$$(1) \quad r_H \cdot [R(\hat{x}(r_H, t)) - R(\hat{x}(r_L, t))] \geq t \cdot [C(\hat{x}(r_H, t)) - C(\hat{x}(r_L, t))].$$

Part **(a)** of Theorem 1 tells us that $\hat{x}(r_H, t) \geq \hat{x}(r_L, t)$. Since R is strictly increasing, that means that the left side of (1) is either positive or zero. First suppose that it is positive. Then, since $r_H < 1$, (1) implies that

$$(2) \quad R(\hat{x}(r_H, t)) - R(\hat{x}(r_L, t)) > t \cdot [C(\hat{x}(r_H, t)) - C(\hat{x}(r_L, t))],$$

or equivalently

$$(3) \quad R(\hat{x}(r_H, t)) - t \cdot C(\hat{x}(r_H, t)) > R(\hat{x}(r_L, t)) - t \cdot C(\hat{x}(r_L, t)),$$

i.e.,

$$(4) \quad W(r_H, t) > W(r_L, t).$$

If $\hat{x}(r_H, t) \neq \hat{x}(r_L, t)$, then, since R is strictly increasing, the left side of (1) is indeed positive, so (4) holds. If, on the other hand, $\hat{x}(r_H, t) = \hat{x}(r_L, t)$, then both sides of (1) equal zero and (2),(3),(4) become equalities. So, as claimed, $W(r_H, t) \geq W(r_L, t)$ and the inequality is strict if and only if $\hat{x}(r_H, t) \neq \hat{x}(r_L, t)$.

That concludes the proof of Theorem 1. □

Proof of Theorem 2

Proof of Part **(a)**

We note first that the Agent's chosen effort $\hat{x}(r, t)$ depends only on the ratio $\frac{r}{t}$, which we shall call ρ . The set of possible values of ρ is $(0, \frac{1}{t}]$. The Agent's effort is a value of x which maximizes $t \cdot (\rho R(x) - C(x))$ on the effort set Σ , and is therefore a maximizer of $\rho R(x) - C(x)$. We shall use a new symbol, namely $\phi(\rho)$ to denote the Agent's chosen effort when the ratio is ρ .

So $\phi(\rho) = \hat{x}(r, t)$. The function $\rho R(x) - C(x)$ displays strictly increasing differences with respect to ρ, x . Hence the maximizer $\phi(\rho)$ is nondecreasing in ρ , so we have

$$(+) \quad \phi(\rho_H) \geq \phi(\rho_L) \text{ whenever } 0 < \rho_L < \rho_H.$$

We can now reinterpret the Principal as the chooser of a ratio. For a given t , he chooses the ratio $\rho^*(t) = \frac{r^*(t)}{t}$, where

$$\rho^*(t) = \min\{\operatorname{argmax}_{\rho \in (0, 1/t)} M(\rho, -t)\},$$

and

$$M(\rho, -t) = (1 - t\rho) \cdot R(\phi(\rho)) = R(\phi(\rho)) - t \cdot \rho \cdot R(\phi(\rho)).$$

The function M has strictly increasing differences in $\rho, -t$ if the function $-t \cdot \rho \cdot R(\phi(\rho))$ has strictly increasing differences in $\rho, -t$. But that is the case, since R is nondecreasing, which implies (using (+)) that $R(\phi(\cdot))$ is also nondecreasing. Since $\rho^*(t)$ is a maximizer of $M(\rho, -t)$, we conclude that

$$\frac{r^*(t_L)}{t_L} = \rho^*(t_L) \geq \rho^*(t_H) = \frac{r^*(t_H)}{t_H} \text{ whenever } 0 < t_L < t_H,$$

as Part (a) asserts.

Proof of Part (b)

We use the terminology just used in the proof of Part (a). Since $\phi\left(\frac{r^*(t)}{t}\right) = \hat{x}(r^*(t), t)$, we have, using (+), $\hat{x}(r^*(t_L), t_L) \geq \hat{x}(r^*(t_H), t_H)$, as (b) asserts.

Proof of Part (c)

Part (c) says:

$$(1 - r^*(t_L)) \cdot R(\hat{x}(r^*(t_L), t_L)) > (1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_H)) \text{ whenever } t_L, t_H \in \tilde{\Gamma} \text{ and } 0 < t_L < t_H.$$

When t drops from t_H to t_L , the Principal could continue to use the share $r^*(t_H)$. It suffices to show that if he does so, the Principal's gain cannot be less than it was at $t = t_H$. Then, *a fortiori*, it cannot be less when he uses $r^*(t_L)$, which is his best share when $t = t_L$. Part ((b)) tells us that $\hat{x}(r^*(t_H), t_L) \geq \hat{x}(r^*(t_H), t_H)$. Since R is nondecreasing, we have

$$(1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_L)) \geq (1 - r^*(t_H)) \cdot R(\hat{x}(r^*(t_H), t_H)),$$

as (c) asserts.

Proof of Part (d)

Part (d) says:

$$W(r^*(t_L), t_L) > W(r^*(t_H), t_H) \text{ whenever } t_L, t_H \in \tilde{\Gamma} \text{ and } 0 < t_L < t_H.$$

The effort $\hat{x}(r^*(t_L), t_L)$ is a maximizer of $r^*(t_L) \cdot R(x) - t_L \cdot C(x)$. Hence

$$r^*(t_L) \cdot R(\hat{x}(r^*(t_L), t_L)) - t_L \cdot C(\hat{x}(r^*(t_L), t_L)) \geq r^*(t_L) \cdot R(\hat{x}(r^*(t_H), t_H)) - t_L \cdot C(\hat{x}(r^*(t_H), t_H))$$

or

$$r^*(t_L) \cdot [R(\hat{x}(r^*(t_L), t_L)) - R(\hat{x}(r^*(t_H), t_H))] \geq t_L \cdot [C(\hat{x}(r^*(t_L), t_L)) - C(\hat{x}(r^*(t_H), t_H))].$$

That implies — since $0 < r^*(t_L) < 1$ — that

$$R(\hat{x}(r^*(t_L), t_L)) - R(\hat{x}(r^*(t_H), t_H)) > t_L \cdot [C(\hat{x}(r^*(t_L), t_L)) - C(\hat{x}(r^*(t_H), t_H))]$$

or

$$R(\hat{x}(r^*(t_L), t_L)) - t_L \cdot C(\hat{x}(r^*(t_L), t_L)) > R(\hat{x}(r^*(t_H), t_H)) - t_L \cdot C(\hat{x}(r^*(t_H), t_H))$$

and hence (since $t_H > t_L$)

$$R(\hat{x}(r^*(t_L), t_L)) - t_L \cdot C(\hat{x}(r^*(t_L), t_L)) > R(\hat{x}(r^*(t_H), t_H)) - t_H \cdot C(\hat{x}(r^*(t_H), t_H))$$

The term on the left of the inequality is $W(r^*(t_L), t_L)$ and the term on the right is $W(r^*(t_H), t_H)$. That establishes **(d)**.

That concludes the proof of Theorem 2. \square

Proof of Theorem 3

Recall that if $(r, t) \in \Gamma$, then $0 < r < 1$. For $\epsilon > 0$, $1 - \epsilon > r$, and $0 < r < 1$, we have

$$(+) \quad \frac{d}{dt} [\hat{x}(1 - \epsilon, t) - \hat{x}(r, t)] = \hat{x}_t(1 - \epsilon, t) - \hat{x}_t(r, t).$$

Now suppose that $\hat{x}_{rt} > 0$ at all $(r, t) \in \Gamma$. Then the difference on the right of the equality in (+) is positive. But the same is true if $\epsilon = 0$, since, by assumption, the function \hat{x}_t is continuous with respect to t at all $r \in (0, 1]$. Hence $\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] > 0$ at all $(r, t) \in \Gamma$.

An analogous argument shows that if $\hat{x}_{rt} < 0$ at all $(r, t) \in \Gamma$, then $\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] < 0$ at all $(r, t) \in \Gamma$.

\square

Proof of Theorem 4

An argument analogous to the proof of Theorem 3 tells us that

$$(1) \quad \begin{cases} \text{if } W_{rt}(r, t) > 0 \text{ at all } (r, t) \in \Gamma, \text{ then } \frac{d}{dt} [W(1, t) - W(r, t)] > 0 \text{ at all } (r, t) \in \Gamma; \\ \text{if } W_{rt}(r, t) < 0 \text{ at all } (r, t) \in \Gamma, \text{ then } \frac{d}{dt} [W(1, t) - W(r, t)] < 0 \text{ at all } (r, t) \in \Gamma. \end{cases}$$

We will now show that

$$(2) \quad \hat{x}_{rt}(r, t) \cdot W_{rt}(r, t) > 0 \text{ at all } (r, t) \in \Gamma.$$

Recall our assumption that we either have $\hat{x}_{rt} > 0$ at all $(r, t) \in \Gamma$ or $\hat{x}_{rt} < 0$ at all $(r, t) \in \Gamma$. Using that assumption as well as Theorem 3 and (1) and (2), we see that:

- If $\hat{x}_{rt}(r, t) > 0$ at all $(r, t) \in \Gamma$, then at all $(r, t) \in \Gamma$ we have $\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] > 0$ and $\frac{d}{dt} [W(1, t) - W(r, t)] > 0$.
- If $\hat{x}_{rt}(r, t) < 0$ at all $(r, t) \in \Gamma$, then at all $(r, t) \in \Gamma$ we have $\frac{d}{dt} [\hat{x}(1, t) - \hat{x}(r, t)] < 0$ and $\frac{d}{dt} [W(1, t) - W(r, t)] < 0$.

So we indeed have exogenous tracking.

To establish (2), we shall use the fact that the cross partials W_{rt}, W_{tr} are equal. Part (a) of Theorem 1 tells us that $\hat{x}_r(r, t) \geq 0$. Since $rR'(\hat{x}(r, t)) - tC'(\hat{x}(r, t)) = 0$ and $0 < r < 1$, we have

$$(3) \quad W_r(r, t) = \hat{x}_r(r, t) \cdot [R'(\hat{x}(r, t)) - tC'(\hat{x}(r, t))] \geq 0.$$

Since, by assumption, $tC'(\hat{x}(r, t)) = rR'(\hat{x}(r, t))$, the equality in (3) can be rewritten

$$(4) \quad W_r(r, t) = (1 - r) \cdot R'(\hat{x}(r, t)) \cdot \hat{x}_r(r, t).$$

Now differentiate both sides of (3) with respect to t . We have:

$$(5) \quad W_{rt}(r, t) = (1 - r) \cdot [R'(\hat{x}(r, t)) \cdot \hat{x}_{rt}(r, t) + \hat{x}_r(r, t) \cdot R''(\hat{x}(r, t)) \cdot \hat{x}_t(r, t)].$$

Next we differentiate W in the reverse order: first with respect to t and then with respect to r . We obtain the following, using condensed notation when convenient.³⁵

$$(6) \quad W_t(r, t) = R' \cdot \hat{x}_t - [t \cdot C' \cdot \hat{x}_t + C(\hat{x}(r, t))] = \hat{x}_t \cdot [R' - tC'] - C(\hat{x}(r, t)).$$

Differentiating the final expression with respect to r , we obtain:

$$(7) \quad W_{tr}(r, t) = \hat{x}_t \cdot [R'' \cdot \hat{x}_r - t \cdot C'' \cdot \hat{x}_r] + [R' - t \cdot C'] \cdot \hat{x}_{tr} - C' \cdot \hat{x}_r.$$

We now rewrite (5) and (7). We identify separate terms so that cancellations in the equality $W_{rt} = W_{tr}$ can be easily detected. For (5) we obtain

$$W_{rt} = \underbrace{R' \cdot \hat{x}_{rt}}_{\boxed{1}} + \underbrace{\hat{x}_r \cdot R'' \cdot \hat{x}_t}_{\boxed{2}} - r \cdot R' \cdot \hat{x}_{rt} - r \cdot \hat{x}_r \cdot R'' \cdot \hat{x}_t.$$

For (7) we obtain

$$W_{tr} = \underbrace{\hat{x}_t \cdot R'' \cdot \hat{x}_r}_{\boxed{2}} - t \cdot C'' \cdot \hat{x}_r \cdot \hat{x}_t + \underbrace{R' \cdot \hat{x}_{tr}}_{\boxed{1}} - t \cdot C' \cdot \hat{x}_{tr} - C' \cdot \hat{x}_r.$$

³⁵Thus $R, R', R'', C, C', C'', \hat{x}_t, \hat{x}_r, \hat{x}_{rt}, W_r, W_t, W_{rt}$ denote, respectively, $R(\hat{x}(r, t)), R'(\hat{x}(r, t)), \dots, C(\hat{x}(r, t)), \dots, \hat{x}_t(r, t), \hat{x}_r(r, t), \hat{x}_{rt}(r, t), W_r(r, t), W_t(r, t), W_{rt}(r, t)$.

Deleting the terms $\boxed{1}$ and $\boxed{2}$ and multiplying both sides by -1 , we can now write the equality $W_{rt} = W_{tr}$ as

$$r \cdot R'' \cdot \hat{x}_{rt} + r \cdot \hat{x}_r \cdot R'' \cdot \hat{x}_t = t \cdot C''' \cdot \hat{x}_r \cdot \hat{x}_t + t \cdot C' \cdot \hat{x}_{tr} + C' \cdot \hat{x}_r$$

or

$$\hat{x}_{rt} \cdot \underbrace{[rR' - tC']}_{=0} + \hat{x}_r \hat{x}_t r \cdot [R'' - t \cdot C''] = C' \cdot \hat{x}_r.$$

So

$$(8) \quad C' \cdot \hat{x}_r = \hat{x}_t \cdot \hat{x}_r \cdot (R'' - tC'').$$

Now, using (8), we can rewrite (7) as

$$W_{tr} = \hat{x}_t \cdot \hat{x}_r \cdot [R'' - tC''] + [R' - t \cdot C'] \cdot \hat{x}_{tr} - \hat{x}_t \cdot \hat{x}_r \cdot [R'' - tC''] = [R' - t \cdot C'] \cdot \hat{x}_{tr}.$$

But $R' - t \cdot C' > 0$. So W_{rt} has the same sign as \hat{x}_{rt} . That establishes (2) and completes the proof. \square

Proof of Corollary to Theorem 4

Since we have exogenous tracking, the Decentralization Penalty (surplus gap) is decreasing in t if $\hat{x}_{rt}(r, t) < 0$ at all $(r, t) \in \Gamma$, and is increasing in t if $\hat{x}_{rt}(r, t) > 0$ at all $(r, t) \in \Gamma$. So Part (i) of the Corollary is proved if we establish that $\hat{x}_{rt} < 0$, and Part (ii) is proved if we establish that $\hat{x}_{rt} > 0$.

Proof of Part (i)

The first-order condition satisfied by $\hat{x}(r, t)$ is, in abbreviated form (with arguments deleted):

$$rR' - tC' = 0.$$

Differentiating with respect to r on both sides:

$$(+) \quad R' + rR'' \cdot \hat{x}_r - tC'' \cdot \hat{x}_r = 0.$$

Differentiating with respect to t on both sides of (+):

$$(++) \quad R'' \cdot \hat{x}_t + [rR'' - tC''] \cdot \hat{x}_{rt} + [rR''' \hat{x}_t - tC''' \hat{x}_t - C'''] \cdot \hat{x}_r = 0.$$

Since, by assumption, $R''' = C''' = 0$, we obtain:

$$(+++) \quad \frac{R'(\hat{x})}{\hat{x}_r} \hat{x}_{rt} = R'' \hat{x}_t - C'' \hat{x}_r.$$

By assumption, $\hat{x}_r > 0$ and $\hat{x}_t < 0$. By assumption, $R' > 0, C'' > 0$ and $R'' \geq 0$. So the right side of (+++) is negative. That implies $x_{rt} < 0$, which establishes part (i).

Proof of Part (ii)

In this part we assume $R' > 0, R'' < 0, 0 = C''' = C'''' , R''' \leq 0$. We first establish that under these assumptions:

$$(*) \quad \frac{R'(\hat{x})}{\hat{x}_r} \hat{x}_{rt} = R'' \hat{x}_t + r R''' \hat{x}_r \hat{x}_t.$$

Statement (+) in the Part-(i) proof can now be written

$$(**) \quad \hat{x}_r = \frac{-R'}{r R''}.$$

So the left side of (*) can be written

$$R' \cdot \frac{-r R''}{R'} \cdot \hat{x}_{rt} = -r R'' \hat{x}_{rt}.$$

Statement (++) in the Part-(i) proof can now be written

$$-r R'' \hat{x}_{rt} = R'' \hat{x}_t + r R''' \hat{x}_r \hat{x}_t.$$

So the left side of (*) indeed equals the right side. Under our Part-(ii) assumptions, we have (since $\hat{x}_r > 0, \hat{x}_t < 0$)

$$R'' \hat{x}_t > 0, R''' \hat{x}_r \hat{x}_t \geq 0.$$

So the right side of (*) is positive. Since $R' > 0, \hat{x}_r > 0$, the equality (*) implies that $\hat{x}_{rt} > 0$, as required.

That concludes the proof of the Corollary. □

Proof of Theorem 5

Part (a)

We have

$$(1) \quad \frac{\partial}{\partial t} \left[W(1, t) - W(r^*(t), t) \right] = W_t(1, t) - W_r(r^*(t), t) \cdot r^{*'}(t) - W_t(r^*(t), t).$$

Our exogenous Theorem 4 tells us that for all $(r, t) \in \Gamma$ — including the pair $(r^*(t), t)$ — we have $W_{rt}(r, t) \cdot \hat{x}_{rt}(r, t) > 0$. That implies (since we assume that $\hat{x}_{rt}(r, t) < 0$ at all $(r, t) \in \Gamma$) that $W_{rt}(r^*(t), t) < 0$. Now consider (1) the interval $[r^*(t), 1]$, (2) the function W_t on the possible shares, and (3) the derivative of that function, namely W_{rt} . In an interior example the function \hat{x}_t is continuous with respect to t at all $r \in [0, 1]$ and hence the same is true for the function $W_t = \frac{d}{dt} [R(\hat{x}(r, t) - tC(\hat{x}(r, t))]$. By the Mean Value Theorem, there exists $r_0 \in (r^*(t), 1)$ such that

$$W_t(1, t) - W_t(r^*(t), t) = W_{rt}(r_0, t) \cdot (1 - r^*(t)) < 0$$

and hence

$$(2) \quad W_t(1, t) - W_t(r^*(t), t) < 0.$$

Since we know, from Part (h) of Theorem 1, that $W_r(r, t) \geq 0$ at all (r, t) , including $(r^*(t), t)$, and since we assume $r^{*\prime}(t) \geq 0$, we conclude, using (1) and (2), that

$$(3) \quad \frac{\partial}{\partial t} \left[W(1, t) - W(r^*(t), t) \right] < 0.$$

We now turn to $\frac{\partial}{\partial t} \left[\hat{x}(1, t) - \hat{x}(r^*(t), t) \right]$. We have

$$(4) \quad \frac{\partial}{\partial t} \left[\hat{x}(1, t) - \hat{x}(r^*(t), t) \right] = \hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) - \hat{x}_r(r^*(t), t) \cdot r^{*\prime}(t).$$

Our assumption that $\hat{x}_{rt}(r^*(t), t) < 0$ implies, using Theorem 3, that $\hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) < 0$. Since we assume that $r^{*\prime}(t) > 0$, and since we know from Part (a) of Theorem 1, that $\hat{x}_r(r, t) \geq 0$ at all (r, t) , we conclude that $\frac{\partial}{\partial t} \left[\hat{x}(1, t) - \hat{x}(r^*(t), t) \right] < 0$. So we indeed have

$$\frac{\partial}{\partial t} \left[\hat{x}(1, t) - \hat{x}(r^*(t), t) \right] \cdot \frac{\partial}{\partial t} \left[W(1, t) - W(r^*(t), t) \right] > 0.$$

Part (b)

First consider again the three terms at the right of the equality (1). Since we now assume that $\hat{x}_{rt}(r, t) > 0$ at all $(r, t) \in \Gamma$, Theorem 4 tells us that we now have $W_{rt}(r, t) > 0$. Hence — using the Mean-Value-Theorem argument again — we now have

$$W_t(1, t) - W_t(r^*(t), t) < 0.$$

Since we now assume $r^{*\prime}(t) < 0$, we conclude (using (1)) that

$$\frac{\partial}{\partial t} \left[W(1, t) - W(r^*(t), t) \right] > 0.$$

Now consider the three terms on the right of the equality in (4). Our assumption that $\hat{x}_{rt}(r^*(t), t) > 0$ now implies, using Theorem 3, that $\hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) > 0$. Since we now assume that $r^{*\prime}(t) < 0$, we now conclude, using (4), that $\hat{x}_r(r, t) \geq 0$ and hence $\hat{x}_t(1, t) - \hat{x}_t(r^*(t), t) < 0$. So we again have

$$\frac{\partial}{\partial t} \left[\hat{x}(1, t) - \hat{x}(r^*(t), t) \right] \cdot \frac{\partial}{\partial t} \left[W(1, t) - W(r^*(t), t) \right] > 0.$$

□

Proof of Theorem 6

The theorem concerns the Principal's gain (residual revenue), which we now denote $H(r, t)$. Thus

$$H(r, t) \equiv (1 - r) \cdot R(\hat{x}(r, t)).$$

We use the following fact about implicit functions: If we have $f(\bar{u}, \bar{v}) = 0$, where f is twice differentiable, then there exists a neighborhood of (\bar{u}, \bar{v}) and a twice differentiable function g such that for all (u, v) in the neighborhood we have $f(g(u), v) = 0$ and, if $f_v(g(\bar{u}), \bar{v}) \neq 0$, we have

$$g'(v) = \frac{-f_u(g(v), v)}{f_v(g(v), v)}.$$

To apply this, let r play the role of u , let t play the role of v , let H_r play the role of f and consider the first-order equation satisfied by $r^*(t)$:

$$H_r(r, t) = 0.$$

Under our assumptions, H_r is differentiable with respect to r and t at every $(r, t) \in \Gamma$ and $r^*(t)$ belongs to the open interval $(0, 1)$ and is the unique solution to $H_r(r, t) = 0$. The role of the twice differentiable function g is now played by r^* .

Since $r^*(t)$ is the unique interior maximizer of $H(r, t)$ it satisfies the second-order condition

$$(+) \quad H_{rr}(r, t) \leq 0.$$

For every $t \in \tilde{\Gamma}$, we have

$$r^{*'}(t) = \frac{-H_{rr}(r^*(t), t)}{H_{rt}(r^*(t), t)} \text{ if } H_{rt}(r^*(t), t) \neq 0.$$

In view of the second-order condition (+), the numerator in this fraction is nonnegative. Hence, if $H_{rt}(r^*(t), t) \geq 0$, then

$$(++) \quad r^{*'}(t) > 0 \text{ if } H_{rr}(r, t) < 0; \quad r^{*'}(t) = 0 \text{ if } H_{rr}(r, t) = 0.$$

We now claim that $H_{rt}(r^*(t), t) > 0$. To see this, note that

$$H_{rt} = (1 - r) \cdot [R' \cdot \hat{x}_{rt} + \hat{x}_r \cdot R'' \cdot \hat{x}_t] - R' \cdot \hat{x}_t.$$

By assumption we have $\hat{x}_t < 0$, $\hat{x}_r > 0$, $R' > 0$, $R'' < 0$, and $\hat{x}_{rt} \geq 0$. Hence $-R' \cdot \hat{x}_t > 0$ and

$$(1 - r) \cdot [R' \cdot \hat{x}_{rt} + \hat{x}_r \cdot R'' \cdot \hat{x}_t] \geq 0.$$

We conclude that $H_{rt} > 0$ at every $(r, t) \in \Gamma$. That implies, in view of (++), that $r^{*'}(t) \geq 0$ at every $t \in \tilde{\Gamma}$, as the Theorem asserts. \square

Proof of Theorem 7.

Part (a)

We first establish the following general proposition:

Let $h(x) = (1 - x) \cdot g(x)$. If g is increasing and concave on $(0, 1)$, then h is concave on $(0, 1)$

Here is the proof:

Consider $x_1 > 0$ and $x_2 < 1$. Consider $\lambda \in (0, 1)$ and let x_0 denote $\lambda x_1 + (1 - \lambda) \cdot x_2$. We shall prove that

$$h(x_0) = (1 - x_0) \cdot g(x_0) \geq \lambda \cdot (1 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (1 - x_2) \cdot g(x_2).$$

We have

$$\begin{aligned} 0 &= \lambda \cdot [(x_0 - x_1) + (1 - \lambda) \cdot (x_0 - x_2)] \cdot g(x_2) \\ &\geq \lambda \cdot (x_0 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (x_0 - x_2) \cdot g(x_2) \\ &= \lambda \cdot (x_0 - 1 + 1 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (x_0 - 1 + 1 - x_2) \cdot g(x_2). \end{aligned}$$

Moving parts of the last expression to the left of the inequality we obtain:

$$\begin{aligned} \lambda \cdot (1 - x_1) \cdot g(x_1) + (1 - \lambda) \cdot (1 - x_2) \cdot g(x_2) &\leq \lambda \cdot (1 - x_0) \cdot g(x_1) + (1 - \lambda) \cdot (1 - x_0) \cdot g(x_2) \\ &= (1 - x_0) \cdot [\lambda \cdot g(x_1) + (1 - \lambda) \cdot g(x_2)] \\ &\leq (1 - x_0) \cdot g(x_0) \\ &= h(x_0). \end{aligned}$$

The final inequality holds because g is concave.

Now apply this general proposition to our case, where r plays the role of “ x ” and $R(\hat{x}(r, t))$ plays the role of “ $g(x)$ ”. That establishes Part **(a)**.

Part (b)

$R(\hat{x}(r, t))$ is concave in r if its second derivative is negative, i.e.,

$$R' \cdot \hat{x}_{rr} + \hat{x}_r \cdot R'' < 0.$$

That is the case, as claimed, if $R'' < 0$ and $\hat{x}_{rr} < 0$. To check the claimed sufficient condition for $\hat{x}_{rr} < 0$, start by writing the first-order condition satisfied by $\hat{x}(r, t)$:

$$r \cdot R'(\hat{x}(r, t)) - t \cdot C'(\hat{x}(r, t)) = 0.$$

Differentiating with respect to r we obtain

$$\hat{x}_r = \frac{R'}{tC'' - rR''}.$$

Since $\hat{x}(r, t)$ is an interior maximizer, the denominator is negative. Differentiating both sides with respect to r we obtain:

$$\hat{x}_{rr} = \left(\frac{1}{tC'' - rR''} \right)^2 \cdot [R'' \cdot (tC''' - rR''') + R' \cdot (rR'''' - tC'''')] + \frac{R' \cdot R''}{(tC'' - rR'')^2}.$$

Since $tC'' - rR'' < 0$ and, by assumption, $R'' < 0$, we see, as claimed, that $\hat{x}_{rr} < 0$ if $rR''' - tC''' \leq 0$. \square

Proof that the “exploding marginals” example lies in Box 3 of the Interior Examples table.

We shall show this by examining the *construction* of the example. Recall that Box 3 requires that for every $(r, t) \in \Gamma$ we have $r^*(t) < 0$ and $\hat{x}_{rt}(r, t) \geq 0$.

As long as $R' \neq 0$, the first-order condition $r \cdot R'(\hat{x}(r, t)) = t \cdot C(\hat{x}(r, t))$ can be written

$$\frac{C'(\hat{x}(r, t))}{R'(\hat{x}(r, t))} = \frac{r}{t}.$$

That defines an implicit function g which satisfies

$$\hat{x}(r, t) = g\left(\frac{r}{t}\right).$$

It will be convenient to let S denote the ratio $\frac{r}{t}$. So

$$\hat{x}(r, t) = g(S) \text{ and } \frac{C'(g(S))}{R'(g(S))} = S.$$

We have:

$$\hat{x}_r = g' \cdot \frac{1}{t}; \quad \hat{x}_t = g' \cdot \frac{-r}{t^2}.$$

$$\hat{x}_{rt} = \hat{x}_{tr} = g'' \cdot \frac{-r}{t^3} - \frac{1}{t^2} \cdot g'.$$

$$(a) \quad \hat{x}_{rt} > 0 \iff g'' \cdot \frac{r}{t} + g' < 0.$$

Since r^* satisfies the first-order condition $0 = \frac{d}{dr} [(1-r) \cdot R(\hat{x}(r, t))]$, we have

$$(b) \quad 1 - r^* = \frac{R}{R' \cdot \hat{x}_r} = \frac{R}{R'} \cdot \frac{t}{g'}.$$

Using (b), we obtain:

$$r^{*'}(t) = -\Delta,$$

where

$$\begin{aligned}
\Delta &= \frac{d}{dt} \left[\frac{R}{R'} \cdot \frac{t}{g'} \right] \\
&= \left[\frac{d}{dt} \left(\frac{R}{R'} \right) \right] \cdot \frac{t}{g'} + \left[\frac{d}{dt} \left(\frac{t}{g'} \right) \right] \cdot \frac{R}{R'} \\
&= \frac{1}{(R')^2} \cdot \frac{t}{g'} \cdot ((R')^2 - R'' \cdot R) \cdot \hat{x}_t + \frac{R}{R'} \cdot \frac{g' - g'' \cdot t \cdot \frac{-r}{t^2}}{(g')^2}
\end{aligned}$$

Since $\hat{x}_t \cdot \frac{t}{g'} = -S$ and $\frac{r}{t} = S$, we have:

$$(c) \quad \Delta = -S \cdot \frac{(R')^2 - R'' \cdot R}{(R')^2} + \frac{R}{R'} + \frac{g' - g'' \cdot (-S)}{(g')^2}.$$

To construct our example, we now let $g(S) = \ln \ln S$. Then

$$g' = \frac{1}{S \cdot \ln S}; \quad g'' = -(\ln S + 1) \cdot \frac{1}{S^2 \cdot (\ln S)^2}.$$

[We can then verify that the condition in (a) is satisfied, and hence $\hat{x}_{rt} > 0$, as Box **3** requires]. Hence

$$(d) \quad g' + g'' \cdot S = \frac{1}{S \cdot \ln S} + S \cdot \left[\frac{-(\ln S + 1)}{S^2 \cdot (\ln S)^2} \right] = \frac{\ln S - (\ln S + 1)}{S \cdot (\ln S)^2} = \frac{-1}{S \cdot (\ln S)^2}.$$

Using (c),(d), and the fact that $(g')^2 = \frac{1}{S^2 \cdot (\ln S)^2}$, we obtain

$$\Delta = -S \cdot \frac{(R')^2 - R'' \cdot R}{(R')^2} - S \cdot \frac{R}{R'} = \frac{-S}{(R')^2} \cdot [(R')^2 - R'' \cdot R + RR'].$$

Thus (recalling that $r^{*'} = -\Delta$) we have

$$(e) \quad r^{*'} < 0 \Leftrightarrow (R')^2 - R'' \cdot R' + R \cdot R' < 0.$$

To continue our construction, we now suppose that

$$R = e^{kx^2}, \text{ where } k > 0.$$

We now claim that

$$(f) \quad (R')^2 - R'' \cdot R' + R \cdot R' = e^{2kx^2} \cdot (2kx - 2k).$$

To show this we first note that

$$R' = e^{kx^2} \cdot 2kx$$

and

$$R'' = e^{kx^2} \cdot 2k + 2kx \cdot e^{kx^2} \cdot 2kx.$$

We then factor out the term e^{2kx^2} in writing the following expressions.

$$R \cdot R' = e^{kx^2} \cdot e^{kx^2} \cdot 2kx = e^{2kx^2} \cdot 2kx.$$

$$(R')^2 = e^{2kx^2} \cdot 4k^2x^2.$$

$$R'' \cdot R = 2k \cdot e^{kx^2} \cdot [1 + 2kx^2] \cdot e^{kx^2} = e^{2kx^2} \cdot [2k + 4k^2x^2].$$

So

$$(R')^2 - R'' \cdot R + R \cdot R' = e^{2kx^2} \cdot [4k^2x^2 - 2k - 4k^2x^2 + 2kx] = e^{2kx^2} \cdot (2kx - 2k)$$

and (f) is verified.

So, in view of (e), we have

$$r^{*'} < 0 \text{ as Box } \mathbf{3} \text{ requires, if } e^{2kx^2} \cdot (2kx - 2k) < 0.$$

But if $e^{2kx^2} \cdot (2kx - 2k) < 0$, then $x < 1$. So our set of available efforts will be $\Sigma = [0, 1)$.

Summarizing, we have

- $R = e^{kx^2}$.
- $\hat{x}(r, t) = g(s) = \ln \ln s < 1$ and hence $S < e^e$.

We need to specify our set Γ of possible pairs (r, t) . It will be the set

$$\{(r, t) : 0 < r \leq 1; \frac{r}{t} \in (e, e^e)\}.$$

It remains to specify the function C . We seek a function C with the following property:

$$\text{for every } S \text{ we have } C'(g(S)) = S \cdot R'(g(S)).$$

That can be rewritten as:

$$\text{for every } S \text{ we have } C'(g(S)) = g(g^{-1}(S)) \cdot R'(g(S)).$$

Now let M denote $g(s)$. Since $C(M) = \int C'(M) dM$, we have:

$$(g) \quad C(M) = \int \left[g(g^{-1}(M)) \cdot R'(M) \right] dM.$$

In our example

- $g(S) = \ln \ln S$.
- Hence, for any M , we have $g^{-1}(M) = e^{e^M}$.

- $R(x) = e^{kx^2}$ and $R'(x) = e^{kx^2} \cdot 2kx$.

Thus, in our example, the equality (g) becomes:

$$C(M) = \int \left[e^{e^M} \cdot e^{kM} \cdot 2kM \right] dM.$$

So for every x in our effort set $\sigma = (0, 1]$ we have

$$C(x) = \int_0^x \left[e^{e^p} \cdot e^{kp} \cdot 2kp \right] dp.$$

To summarize, our Box **3** example is as follows.

- $\Sigma = (0,]$.
- $\Gamma = \{(r, t) : 0 < r \leq 1; \frac{r}{t} \in (e, e^e)\}$.
- $R(x) = e^{kx^2}$, where $k > 0$.
- $C(x) = \int_0^x \left[e^{e^p} \cdot e^{kp} \cdot 2kp \right] dp$.

In the example provided in the text we have $k = 1$.

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ACKNOWLEDGMENT:

We are grateful to Junjie Zhou for sketching the monotone-comparative-statics proofs of parts **(a)** and **(b)** in Theorem 2.