Bidding with Securities: Auctions and Security Design†

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ABSTRACT. We study *security-bid auctions* in which bidders compete for an asset by bidding with securities. That is, they offer payments that are contingent on the realized value of the asset being sold. Standard auction mechanisms (such as first price and second price auctions) are not well-defined unless the set of securities is restricted to an ordered set. For example, the seller may require bids to be exclusively an equity share, or exclusively a debt payment. Given such a restriction, we first ask whether revenue equivalence holds; i.e., whether expected revenues depend upon the auction format. We show that this principle holds if the set of permissible securities is convex. Otherwise, this need not be true. For example, when bidders offer standard debt securities, a second price auction is superior. On the other hand, if bidders compete on the conversion ratio of convertible debt, a first price auction yields higher revenues. We then consider joint problem of choosing both the security and auction design. We show that the optimal mechanism yielding the highest possible expected revenues for the seller is a first price auction with levered equity. On the other hand, a first price auction with debt contracts is the worst possible mechanism for the seller. Finally, we examine the case in which the seller cannot commit to a formal mechanism. Instead, he chooses the ex-post most attractive bid based on his beliefs. We show that this procedure yields the lowest possible revenues across all mechanisms.

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1. Introduction

Auction theory and its application is an area of research of growing importance over the last twenty years. As a result, we have now a better understanding of how the structure of an auction affects its outcome. Almost all the existing literature studies the case when bidders use cash payments, so that the value of a bid is not contingent on future events. Similarly, an important topic in financial economics is that of security design. A great deal of progress has been made in answering questions like: (i) why do parties choose to divide future cash flows (and hence risks) in standard forms, such as debt and equity; and (ii) does this choice matter? Here, the existing finance literature primarily considers the case of a single issuer selling a security to competing investors.

There are many important applications that combine auctions with security design: there are multiple competing bidders, who bid using securities rather than cash. More generally, in many auctions bidders bid for an asset by offering payments that are contingent on the realized cash flow. For example, bidders may offer a fixed future payment (a debt obligation), combined with a royalty agreement (an equity share). Such auctions are commonly used to sell oil leases, the rights to publish books, the rights to broadcast the Olympic games, FCC auctions, and in mergers and acquisitions.

In this paper we consider the joint design of securities and auctions in this setting. We study a model in which several agents compete for the right to undertake a project. Bidders are endowed with private signals regarding the value they can expect from the project. The structure is similar to an independent private values model, so that different bidders expect different payoffs upon winning. The model differs from standard auction models in that bids are securities. Bidders offer derivatives in which the underlying value is the future payoff of the project.

One may conjecture that since we can compute the value of each security, this has no real effect. That is, perhaps the results from standard auction theory carry over, replacing each security by its cash value. However, unlike cash bids, the value of a security-bid depends upon the bidder's private information. This difference can have important consequences as the following simple example demonstrates:[1]

> Consider an auction in which two bidders Alice and Bob compete for a project. The project requires an initial fixed and non-random investment that is equivalent to \$1M. Alice expects that if she undertakes the project then on average it would yield revenues of \$3M; Bob expects that future revenues will equal only \$2M. Hence, Alice sees a profit of \$2M while Bob sees a profit of \$1M. Assuming these estimates are private values then in a standard second price auction, it is a dominant strategy for bidders to bid their reservation values. As a result, Alice would win the auction and pay Bob's bid, \$1M.
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> Now suppose that rather than bidding with cash, the bidders compete by offering a fraction of the future revenues. As we later discuss, it is again a dominant strategy for bidders to bid their reservation values. Alice offers 2/3 of future revenues while Bob offers 1/2. As a result Alice wins the auction and pays

[1] This example is based on Hansen (1985).

according to Bob's, half of future revenues. This yields a higher payoff for the auctioneer; (1/2)×\$3M = \$1.5M > \$1M.

Note that in general, security-based auctions cannot be immediately approached as a standard auction. Auctions are generally defined by specifying in advance a decision mechanism that determines the allocation and payments based on the bids received. Standard mechanisms include first price auctions, in which the highest bidder wins and pays his bid, and second price auctions, in which the highest bidder wins and pays the second highest bid. When bids are securities, however, the notion of a "highest" bid is in general not well-defined, and so these standard mechanisms cannot be used.

A common solution to this problem is for the seller to accept bids only from a pre-specified, well-ordered set of securities. In the above example, the seller restricted bidders to bid by specifying the equity share of the asset that the seller will retain. By restricting bids to an ordered set, the determination of the highest bid is clear, and both first and second price auctions are meaningful mechanisms.

Of course, to use such a mechanism the seller must commit not to consider bids outside the restricted set, even if they are more attractive than the other bids received. Without such a commitment, the seller will choose the bid that maximizes his ex-post payoff. However, this choice is complicated because the value of each security depends crucially on the seller's beliefs regarding the types of bidders that may use a particular security. In this case, the auction contains the elements of a standard signaling game.

In this paper, we extend both security design and auction theory to this setting, and consider both the case in which the seller can and cannot commit to a decision mechanism. First we analyze the case when the seller can commit to restrict bids to an ordered set of securities. We begin by characterizing equilibria in first and second price auctions. We then ask whether the Revenue Equivalence principle holds. This principle is a classic result of auction theory.[2] It states that the choice of auction mechanism (e.g., first price or second price) has no effect on expected revenues. When bidders use securities, this principle need not be true; it depends on the properties of the set of securities that are permitted. Finally, we find the optimal and worst mechanisms. We establish the following results:

- We characterize super modularity conditions under which monotone equilibrium is the unique outcome for the first and second price auctions.
- We show that if the set of securities is convex (that is, if convex combinations of securities in the set are also in the set), then the Revenue Equivalence principle holds. This is true for important classes of securities, such as simple equity. However, many other standard sets of securities do not satisfy convexity.
- We define two other important categories of sets: flat sets and steep sets. For flat sets – which include, for example, the set of standard debt securities – we show that a second price auction yields higher expected revenues than a first price

[2] See for example Myerson (1981), Samuelson and Riley (1981)

auction. Alternatively, if the set is steep, the reverse conclusion holds and first price auctions are superior.[3]

- Having ranked auction mechanisms fixing the security design, we then fix the mechanism and compare security designs. We first focus on the second price auction format. There we show that the seller's expected revenues are positively related to the "steepness" of the securities. As a result, debt contracts minimize the seller's expected payoffs while levered equity maximizes it.
- We then show that the first price auction with levered equity maximizes the seller revenues while the first price format with debt contract minimizes it, *over all ordered sets of securities and general auction mechanisms.*[4]

In the second part of the paper, we consider the case in which the seller is unable to commit ex-ante to an auction mechanism. Instead he accepts all bids and chooses the security that is optimal ex-post. As mentioned above, in this case the task of selecting the winning bid is not trivial; it involves a signaling game in which the seller uses his beliefs to rank the different securities and choose the most attractive one. Our main result in this section is as follows:

- In the unique equilibrium satisfying standard refinements of off-equilibrium beliefs, bidders use only debt securities. Moreover, the outcome is equivalent to a first price auction. As a result we conclude that this ex-post maximization yields the worst possible outcome for the seller!

The intuition is that debt provides the cheapest way for a high type to signal his quality. Thus, bidders find it optimal to compete using debt.

In the final section of the paper we generalize the model further and discuss bidding with combinations of cash and securities, as well as the introduction of moral hazard. We demonstrate that the main insights of our analysis carry over to these settings. In particular, we show that

- If bidders can combine cash payments with their bids, this effectively "flattens" their bids and reduces the expected revenues of the seller.
- Moral hazard on the part of the bidder will in general decrease the steepness and flatness of the set of feasible securities, reducing the range of payoffs for the seller. As a result, moral hazard will reduce the expected revenues for a seller who can commit to an optimal auction, but can enhance the expected revenues of a seller in an unrestricted auction.

Related Literature

Hansen (1985) was the first to examine the use of securities in an auction setting. He shows that a second price mechanism that is based on equity payments yields higher

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[3] Examples of steep sets include: (i) when bidders compete based on the amount of debt they will *retain* (i.e., the seller receives junior equity), and (ii) when bidders compete on the conversion ratio of convertible debt.

[4] This fits our previous results; since the set of debt contracts is flat and the set of levered equity is steep, from our earlier result we know the second price format can neither maximize nor minimize payoffs.

expected revenues than a cash-based auction. Cremer (1987) shows how a seller may be able to extract the whole surplus if he can "buy" the winning bidder. Samuelson (1985) points to some problems in the implementation of such a mechanism.

In a more recent paper, Rhodes-Kropf and Viswanathan (2000) focus on the first price auction format in a setup that is similar to the model we study in the first part of the paper. They generalize Hansen (1985) by showing that securities yield higher revenues than a cash-based auction. However, this is conditional on the existence of a separating equilibrium in which a higher type bids a higher security. In their model there always exists a pooling equilibrium and in some cases it is the unique outcome. This is because they assume that the project does not require any costly inputs – thus the lowest type can offer 100% of the proceeds to the seller and break-even. Thus, a low type is always willing to imitate the bid of a high-type. We use a framework that is closer to Hansen (1985), in which the project requires costly inputs. In this case, we show that under certain conditions the first price auction has a unique separating equilibrium.

Other related literature includes McAfee and McMillan (1987), who solve for the optimal mechanism in a model with a moral hazard problem. The optimal mechanism is a combination of debt and equity, with the mixture depending on the distribution of types. Laffont and Tirole (1987) examine a similar model.

Some of our results are also related to the security design literature. DeMarzo and Duffie (1999) consider the ex-ante security design problem faced by an issuer who will face a future liquidity need. They show that debt securities are optimal because they have the greatest liquidity. DeMarzo (2002) extends this result to the case in which the issuer learns his private information prior to the design of the security, as is the case here. The security design results of this paper are also related to the results of Nachman and Noe (1994). They consider a situation in which the seller is obligated to raise a fixed amount of capital, which leads to a pooling equilibrium using debt securities. None of these models consider security design in a competitive setting like the auction environment considered here.

2. The Model

Signals and Values

There are n risk neutral bidders who compete for the rights to a project. The project requires an up-front investment of $X > 0$. For tractability, we assume that this up-front cost is non-random and equal across bidders. We assume, for now, that no agents have access to cash now; they can only transfer property rights regarding future payoffs. Hence one should interpret X as the opportunity cost of assets employed in the project. We will generalize the setting to include cash payments in Section 5.1.

Conditional on being undertaken by bidder i, the project yields a stochastic future payoff Z_i. Bidders have private signals regarding Z_i, which we denote by V_i. The seller is also risk neutral, and cannot undertake the project independently. The interest rate is normalized to zero.

We make the following standard economic assumptions on the signals and payoffs:

ASSUMPTION A. The private signals $V = (V_1, \ldots, V_n)$ and payoffs $Z = (Z_1, \ldots, Z_n)$ satisfy the following properties:

1. The private signals V_i are i.i.d. with density $f(v)$ with support $[v_L, v_H]$,

2. Conditional on $V = v$, the payoff Z_i has density $h(z|v_i)$ with full support $[0, \infty)$,

3. (Z_i, V_i) satisfy the strict Monotone Likelihood Ratio Property (SMLRP); that is, the likelihood ratio

$$L(z,v,v') = \frac{h(z \mid v)}{h(z \mid v')}$$

is strictly increasing in z if $v > v'$.[5]

The important economic assumptions contained above are, first, that the private signals of other bidders are not informative regarding the signal or payoff of bidder i. Second, because Z_i is not bounded away from zero, the project payoff cannot be used to provide a completely riskless payment to the seller. Finally, the private signal V_i is "good news" about the project payoff Z_i. This is the standard strict version of the affiliation assumption (see Milgrom and Weber (1982)).

Given the above assumptions, we normalize (without loss of generality) the private signals so that

$$E[\, Z_i \mid V_i \,] - X = V_i \,.$$

Thus, we can interpret the signal as the NPV of the project.

To simplify our analysis, we make several additional technical assumptions regarding differentiability and integrability:

ASSUMPTION B. The conditional density function $h(z|v)$ is twice differentiable in z and v. In addition, the functions $z\, h(z|v)$, $|\, z\, h_v(z|v)\, |$ and $|\, z\, h_{vv}(z|v)\, |$ are integrable on $z \in (0, \infty)$.

These assumptions are weak, and allow us to take derivatives "through" expectation operators. As a concrete example, we can consider the following payoff structure:

$$Z_i = \theta\,(X + V_i), \tag{1}$$

where θ is independent of V and log-normal with a mean of 1.[6] Here we can interpret θ as the "market risk" associated with the project.

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[5] This is equivalent to the log-supermodularity of h, which can be written as $\frac{\partial^2}{\partial z \partial v} \log h(z \mid v) > 0$ assuming differentiability.

[6] More generally, what is required for the SMLRP is that $\log(\theta)$ have a log-concave density function.

The Auction Environment

The focus of this paper is on the case in which bids are securities. In particular, we assume that the bidders do not have cash with which to bid for the project. Rather, they compete for the project by offering the seller a share of the final payoff. That is, the bids are in terms of derivative securities, in which the underlying asset is the future payoff of the project Z_i. Bids can be described as function $S(z)$, indicating the payment to the seller when the project has final payoff z.

We make the following assumptions regarding the set of feasible bids:

> **DEFINITION.** A *feasible* security bid is described by a function $S(z)$, such that S is non-decreasing, $z - S(z)$ is non-decreasing, and $0 \le S(z) \le z$.

The assumption that $S(z) \le z$ implies limited liability for the bidder; only the underlying asset can be used to pay the seller. On the other hand, requiring $S(z) \ge 0$ implies limited liability for the seller; the seller cannot commit to pay the bidder except through a share of the project payoff.[7] Finally, we require both the seller's and the bidder's payment to be non-decreasing in the payoff of the project. Absent this, parties will have an incentive to "sabotage" the project and destroy output, or alternatively artificially inflate the output.[8] Together, these requirements are equivalent to $S(0) = 0$, S is continuous, and $S'(z) \in [0, 1]$ almost everywhere. Thus, we admit standard sets of securities such as equity and debt contracts.[9] For example, some standard contracts that we consider in our analysis include:

1. Equity: The seller receives some fraction a $\in$ [0,1] of the payoff: $S(z)$ = a z.
2. Debt: The seller is promised a face value $d \ge 0$, secured by the project: $S(z) = \min(z, d)$.
3. Convertible Debt: The seller is promised a face value $d \ge 0$, secured by the project, or a fraction a $\in$ [0,1] of the payoff: $S(z)$ = max (a z, $\min(z, d)$). (This is equivalent to a debt plus royalty rate contract.)
4. Levered Equity: The seller receives a fraction a $\in$ [0,1] of the payoff, after debt with face value $d \ge 0$ is paid: $S(z) = \alpha \max(z - d, 0)$. (This is equivalent to a royalty agreement in which the bidder recoups some costs upfront.)

Given any security S, we define $ES(v) \equiv E[S(Z_i) \mid V_i = v]$ to denote the excepted payoff of security S conditional on the bidder having value $V_i = v$. Thus, the expected payoff to seller if the bid S is accepted from bidder i is $ES(V_i)$. On the other hand, the bidder's expected payoff is given by $V_i - ES(V_i)$. Thus, we can interpret V_i as the independent,

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[7] In many instances, the seller may not have the resources to do so, which may in fact be the motivation for selling off the project.

[8] For example, if $S(z)$ is somewhere decreasing so that $S(z_0) > S(z_1)$ for $z_0 < z_1$, the bidder can artificially inflate the cash flows of the project from z_0 to z_1 via a short-term loan from a third party and get the payoff $z_0 - S(z_1)$.

[9] Aside from these motivations, the assumptions above on the set of bids are typical of those made in the security design literature (see, e.g., DeMarzo and Duffie (1999), Hart and Moore (1995), and Nachman and Noe (1994)). These assumptions therefore make it easier to compare our results to the prior literature.

private value for bidder i, and $ES(V_i)$ as the payment offered. The key difference from a standard auction, of course, is that the seller does not know the value of the bids, but only the security bid, S. The seller must infer the value of this security. Since the security S is monotone, the value of the security is increasing with the signal V_i of the bidder, as we show below:

> **LEMMA 1.** The value of the security $ES(v)$ is twice differentiable. For $S \neq 0$, $ES'(v) > 0$, and for $S \neq Z$, $ES'(v) < 1$.

Mergers and Acquisitions

Thus far we have interpreted the setting as one in which bidders compete for the right to undertake a project. We remark, however, that the model can also be applied to mergers and acquisitions. In this case, the bidders are rival firms, each competing to takeover the target company (the seller). We interpret X as the stand-alone value of the acquiring firm plus any acquisition related costs, and V_i as the bidder's estimate of the synergy value of the acquisition (i.e., the value of the target once acquired). The bids in this case represent the securities offered to the target shareholders.

3. Auctions with Ordered Sets of Securities

In many auctions, bidders compete by offering "more" of a certain security. For example, they compete by offering more debt or more equity. We begin our analysis by examining a case in which the seller restricts the bids to elements of a well-ordered set of securities. Bidders compete by offering a higher security.

There are two main reasons why sellers restrict the set of securities that are admissible as bids in the auction. First, it allows them to use standard mechanisms – such as first or second price auction formats – to allocate the object and to determine the payments. Without an imposed structure, ranking different securities is very difficult and depends on the beliefs of the seller. There is therefore no objective notion of the "highest" bid.

Second main reason a seller may want to restrict the set of securities is that it can be revenue enhancing for the seller. We will demonstrate this result by first, in this section, studying the revenues from auctions with ordered sets of securities and then, in the next section, comparing this to the revenues from auctions in which the seller cannot commit to a restricted set and bidders can bid using any feasible security.

Formally, a collection of securities can be defined by a function $S(s,z)$, where $s \in [s_0, s_1]$ is the "index" of the security, and $S(s,\cdot)$ is a feasible security. That is, $S(s,z)$ is the payment of security s when the output of the project has value z.

For the collection of securities to be ordered, we require that $S(s,z)$ is increasing in its first argument. Then, a bid of s dominates a bid of s' if $s > s'$. We would also like S to increase "nicely" and non-trivially as s increases. Finally, we would like to allow for a sufficient range of bids so that for the lowest bid, every bidder will earn a non-negative profit, while for the highest bid, no bidder will earn a positive profit. This leads to the following formal requirements for an ordered set of securities:

DEFINITION. The function $S(s,z)$ for $s \in [s_0,s_1]$ defines an *ordered set of securities* if

1. $S(s,\cdot)$ is a feasible security,
2. S is continuous, piecewise differentiable, and non-decreasing in s, and for any $s > s'$, the set $\{z : S(s,z) > S(s',z)\}$ has positive measure,
3. For all v, $ES(s_0,v) \le v$ and $ES(s_1,v) \ge v$.

Natural examples of ordered sets include the sets of (levered) equity and (convertible) debt described earlier, indexed by the parameters α or d.

3.1. Auctions and Mechanisms

Given an ordered set of securities, it is straightforward to generalize the standard definitions of a first and second price auction to our setting:

FIRST PRICE AUCTION: Each agent submits a security. The bidder who submitted the highest security (highest s) wins and pays according to his security.

SECOND PRICE AUCTION: Each agent submits a security. The bidder who submitted the highest security (highest s) wins and pays according to the second highest security (second highest s).

Note that in our private value setup, the second price auction is equivalent to an English auction.

In this section, we characterize the equilibria for both types of auction formats. We are interested in the case for which these equilibria are efficient; that is, the case for which the highest value bidder wins the auction. For second price auctions this is straightforward, as the standard characterization of the second price auction with private values generalizes to:

LEMMA 2. The unique equilibrium in weakly undominated strategies in the second price auction is for a bidder i who has value $V_i = v$ to submit security $s(v)$ such that $ES(s(v),v) = v$. The equilibrium strategy $s(v)$ is strictly increasing.

The above lemma implies that similar to a standard second price auction, each bidder submits bids according to his true value. We now turn our attention to the first price auction. Here, additional assumptions are required to guarantee the existence of a well-behaved equilibrium. In this case, we introduce the following sufficient condition:

ASSUMPTION C. For all (s, v) such that bidder earns a positive expected profit, i.e. $v - ES(s,v) > 0$, the profit function is log-supermodular:

$$\tfrac{\partial^2}{\partial v \partial s} \log\left[v - ES(s,v)\right] > 0 .$$

This assumption allows for the following generalization of the standard characterization of the first price auction to our setting:

> **LEMMA 3.** There exists a unique symmetric equilibrium for the first price auction. It is strictly monotone, differentiable, and it is the unique solution to the following differential equation:
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> $$s'(v) = \frac{(n-1)\,f(v)}{F(v)} \times \frac{\left[v - ES(s(v),v)\right]}{ES_1(s(v),v)}$$
>
> together with the boundary condition $ES(s(v_L), v_L) = v_L$.

Thus, given Assumption C, Lemma 3 characterizes the first price auction and shows that it is efficient. Of course, the question remains regarding how restrictive is Assumption C.[10] It is a joint restriction on the set of securities and the conditional distribution of Z. Below we show that it holds generally for several important cases; it can be established numerically for other types of securities under suitable parameter restrictions.

> **LEMMA 4.** Suppose Z is given by (1), so that $Z_i = \theta\,(X+V_i)$, where θ is lognormal. Then Assumption C holds and the first price auction is efficient if
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> 1. Bidders compete using standard debt, or standard equity,
> 2. Bidders compete using the share α of levered equity, and $d < X$.

The first and second price auctions are two standard auction mechanisms. They share the features that the highest bid wins, and only the winner pays. The first property is necessary for efficiency, and the second is natural in our setting, since only the winner can use the assets of the project to collateralize the payment.

One can construct many other auction mechanisms, however, that share these properties. For example, one can consider third price auctions, or auctions where the winner pays an average of the bids, etc. Below we define a broad class of mechanisms that will encompass these examples:

> **DEFINITION.** A *General Symmetric Mechanism* (GSM) is a symmetric incentive compatible mechanism in which the highest type wins, and pays a security chosen at random from a given set ***S***. The randomization can depend on the realization of types, but not on the identity of the bidders (so as to be symmetric).

The first price auction fits this description, with no randomization (the security is a function of your type). In the second price auction, the security you pay depends upon

[10] Assumption C is the same as a standard condition on the utility function used in other papers on auctions: for example Maskin and Riley (1984) use it to show existence and uniqueness of equilibria with risk averse bidders.

the realization of the second highest type. GSM's also allow for more complicated payment schemes that depend on all of the bids.

It will be useful in what follows to derive a basic characterization of the incentive compatibility condition for a GSM. We show that any GSM can be converted into an equivalent mechanism in which the winner pays a security that depends only on his reported type without further randomization. This observation will be crucial in comparing revenues across mechanisms.

LEMMA 5. Incentive compatibility in a GSM implies the existence of securities $\hat{S}_v$ in the convex hull of ***S*** such that

$$v \in \arg\max_{v'} F^{n-1}(v')\left(v - E\hat{S}_{v'}(v)\right).$$

Thus, it is equivalent to a GSM in which the winner pays the *non-random* security $\hat{S}_v$.

This result will allow us to compare revenues across different mechanisms by studying the relationship between the set of securities ***S*** and its convex hull.

3.2. Revenue Equivalence for Convex Sets of Securities

In our setting of independent private values and risk neutrality, a well-known and important result for cash auctions is the Revenue Equivalence Principle. It implies that the choice of the auction format is irrelevant when the ultimate allocation is efficient.[11] In this section, we look at whether this result generalizes to security-bid auctions.

One feature of cash auctions is that the space of bids is convex; that is, any convex combination of bids is also a feasible bid. This is not necessarily the case for an arbitrary ordered set of securities. For example, a convex combination of standard debt contracts with different face values of debt is not a standard debt contract. On the other hand, the set of standard equity contracts is convex. Formally, we call an ordered set convex if it satisfies the following:

DEFINITION. An ordered set of securities S is *convex* if for any two securities s, s' and $\lambda \in [0, 1]$ there exists a security s'' such that for all z,[12]

$$S(s'', z) = \lambda S(s, z) + (1-\lambda) S(s', z).$$

In fact, convex sets of securities have a simple characterization – each security is a convex combination of the lowest security s_0 and the highest security s_1.[13] Thus, each

[11] Vickery (1961), Myerson (1981), Riley and Samuelson (1981).

[12] In fact, given our assumption that the securities are well-ordered, we could without loss of generality index the securities so that $s'' = \lambda s + (1-\lambda) s'$.

[13] To see why, note that since the set is convex, for each λ there exists a mapping $s:[0, 1] \to [s_0, s_1]$ such that $S(s(\lambda), z) = (1-\lambda) S(s_0, z) + \lambda S(s_1, z)$. Then $s(0) = s_0$, $s(1) = s_1$ and since the set is ordered and $s_0 \neq s_1$, $s(\lambda)$ is strictly increasing. Thus, the result follows if $s(\lambda)$ is continuous. But since $ES(s(\lambda),v) = (1-\lambda) ES(s_0, v) + \lambda ES(s_1, v)$ is continuous, so is $s(\lambda)$ since $ES(s,v)$ is strictly increasing in s.

security can be thought of as s_0 plus some "equity shares" of the security ($s_1 - s_0$), and so it can be thought of as a generalization of a standard equity auction. Our main result in this section is that under convexity, the Revenue Equivalence Principle for auctions continues to hold.

> **PROPOSITION I.** Every efficient equilibrium of a general symmetric mechanism (GSM) with securities from an ordered convex set yields the same expected revenues.

Before proving our main result we note that the standard envelope argument behind Revenue Equivalence does not extend easily to security auctions. This is because the expected payment will depend on the true type V_i (through Z_i), even if a false announcement is made. Such dependence on the true type is the reason why, for example, the Revenue Equivalence Principle does not extend to correlated types in a simple cash auction. Hence, to prove our result we instead use a "replication" argument.

PROOF: In a GSM, the winner pays according to a random security. From Lemma 5, the expected payment by type v reporting v' can be written as $E\hat{S}_{v'}(v)$, where $\hat{S}_{v'}$ is in the convex hull of the ordered set of securities S. Since S is convex, we can define $s^*(v')$ such that

$$S(s^*(v'),\cdot) = \hat{S}_{v'}(\cdot)\,.$$

Because S is ordered, incentive compatibility implies $s^*(v)$ must be strictly increasing; otherwise a bidder could raise the probability of winning without increasing the expected payment. Thus, $s^*(v)$ defines an efficient equilibrium for the first price auction. The result then follows from the uniqueness of equilibrium in the first price auction. ◆

This result yields the following immediate corollary establishing revenue equivalence for first and second price auctions:

> **COROLLARY.** Given a convex ordered set of securities, the first and second price auctions yield the same expected revenues.

The proof of the result also allows us to weaken our condition for the existence of an efficient equilibrium in the first price auction:

> **COROLLARY.** Even absent Assumption C, given a convex ordered set of securities, there exists an efficient symmetric equilibrium in a first price auction with the same expected revenues as in a second price auction.

PROOF: Assumption C is not required for existence of the second price auction. Then, we can use the construction in the proof to generate an equivalent equilibrium for the first price auction. ◆

3.3. Non-Convex Sets of Securities

Our proof for revenue equivalence in the previous section critically depends on the set securities being convex. However, many important classes of securities, such as debt, are not convex. In this section we explore conditions that allow us to compare auction mechanisms for non-convex sets of securities.

When the set of securities is not convex, lotteries over the set of securities are not equivalent to any fixed security from the set. As we will show, the revenues of different mechanisms will depend upon the "steepness" of the equilibrium payment schedules. In order to compare the steepness of securities, we introduce the following definitions:

> **DEFINITION.** Security S_1 strictly crosses security S_2 from below if $ES_1(v^*) = ES_2(v^*)$ implies $ES_1'(v^*) > ES_2'(v^*)$. S_1 strictly crosses S_2 from above if S_2 strictly crosses S_1 from below.

The following technical lemma is useful in identifying a strict crossing by relating it to the shape of the underlying securities.

> **LEMMA 6.** (Single Crossing) A sufficient condition for S_1 to strictly cross S_2 from below is that $S_1 \neq S_2$, and there exists z^* such that $S_1(z) \leq S_2(z)$ for $z < z^*$ and $S_1(z) \geq S_2(z)$ for $z > z^*$.

Using the idea of strict crossing, we can identify two important types of sets of securities, steep and flat, for which we can rank the revenues from first price and second price auctions:

> **DEFINITION.** An ordered set of securities $\boldsymbol{S} = \{S(s,\cdot) : s \in [s_0, s_1]\}$ is *steep* if for all $S_1 \in \boldsymbol{S}$ and S_2 in the convex hull of $\boldsymbol{S}$ with $S_1 \neq S_2$, S_1 strictly crosses S_2 from below. The set S is *flat* if under the same conditions, S_1 strictly crosses S_2 from above.

Not every non-convex set falls into one of the above categories. A set may be steep in some securities and flat in others. Still, there are some important examples of flat and steep sets. We discuss some examples below.

<u>**EXAMPLE** 1**:** Debt securities (a flat set)</u>

> Consider a 50-50 combination of two debt securities, on with face value 50, and one with face value 100. In the range $z \in (50, 100)$, this security will have slope between 0 and 1, and so is not a debt security. However, a debt security with face value between 50 and 75 will cross this security from above. As we will show, the set of debt securities is flat.

<u>**EXAMPLE** 2**:** Levered Equity (a steep set)</u>

> Consider levered equity securities when bidders compete over the amount of retained debt; that is, $S(d, z) = \alpha \max(z - d, 0)$. Consider a 50-50 combination of these securities, one with debt of 50 and one with debt of 100. In the range (50, 100), this security has slope between 0 and α. A levered equity security with face value between 50 and 75 will cross this security from below. As we will show, the set of levered equity, when indexed by debt, is steep.

For the general securities we defined earlier, we have the following characterizations:

LEMMA 7. The set of standard debt contracts is flat. The set of convertible debt contracts indexed by the equity share α, and the set of levered equity contracts indexed by the retained debt d are steep sets.

3.4. Ranking Auctions for Flat and Steep Sets

Based on the above characterization, an argument based on the "Linkage Principle"[14] shows the expected revenues of the first and second price auctions can be ranked. The typical use of the linkage principle is in settings for which bidders' signals are affiliated. Hence it is interesting that the same argument can be applied to rank security auctions when types are independent. The reason is that the expected payment of the winner depends on his true type, not just his bid.

PROPOSITION II. If the ordered set of securities is flat, then the first price auction yields strictly lower expected revenues than the second price auction. If the ordered set of securities is steep, the first price auction yields strictly higher expected revenues than the second price auction.

PROOF: Consider the direct revelation game corresponding to the two auctions. Let S_v^1 be the security bid in the first price auction, and let S_v^2 be the expected security payment in the second price auction for a winner with type v, defined according to the construction (8) in the proof of Lemma 5. Adopting the notation $S^j(v \mid v') = S_{v'}^j(v)$, the first order condition for both mechanisms implies that

$$\tfrac{\partial}{\partial v'} P(v')(v - ES^j(v \mid v'))\Big|_{v'=v} = 0$$

for $j = 1, 2$ and where $P(v) = F(v)^{n-1}$ is the probability of type v winning. This implies

$$P'(v)\big(v - ES^j(v \mid v)\big) = P(v)\big(v - ES_2^j(v \mid v)\big)\,. \tag{2}$$

Note first that for both mechanisms, the lowest type bids his value: $ES^j(v_L \mid v_L) = v_L$ (see Lemma 2 and Lemma 3). Next note that if $ES^1(v \mid v) = ES^2(v \mid v)$, then from (2)

$$ES_2^1(v \mid v) = ES_2^2(v \mid v)\,.$$

Also, from the definition of flat and steep sets,

$$ES_1^1(v \mid v) < ES_1^2(v \mid v) \qquad \text{if the set is flat, and}$$

$$ES_1^1(v \mid v) > ES_1^2(v \mid v) \qquad \text{if the set is steep.}$$

Combining these results, we have that if $ES^1(v \mid v) = ES^2(v \mid v)$, if the set is flat then

$$\tfrac{d}{dv} ES^1(v \mid v) = ES_1^1(v \mid v) + ES_2^1(v \mid v) < ES_1^2(v \mid v) + ES_2^2(v \mid v) = \tfrac{d}{dv} ES^2(v \mid v)\,.$$

Thus, starting from the same expected payment for type v_L, the expected equlibrium payment for any type in the first price auction is always below that in the second price

[14] See Milgrom and Weber (1982), and Krishna (2002) for nice summary and discussion.

auction. For a steep set the above inequality is reversed, and the expected payment for each type is higher in the first price than in the second price auction. ◆

3.5. Security Design

Having examined the impact of the mechanism, we turn our attention to the issue of security design. That is, we compare the expected revenues from auctions using different ordered sets of securities. This will have implications for the seller's choice of security design.

We begin with the following result, which provides a useful characterization of the revenues from a second price auction:

> **LEMMA 8.** Let V^1 and V^2 denote the highest and second highest bidder values. Given an ordered set of securities S, expected revenues in a second price auction can be ranked according to
>
> $$\Delta_S \equiv E\left[ES(s(V^2),V^1) - ES(s(V^2),V^2)\right].$$

PROOF: From Lemma 2, the winner is the highest type and he pays the second highest security. Therefore, the winner pays $ES(s(V^2), V^1)$. Since $ES(s(v), v) = v$ in a second price auction,

$$E\left[ES(s(V^2),V^1)\right] = E\left[V^2\right] + E\left[ES(s(V^2),V^1) - ES(s(V^2),V^2)\right],$$

and only the second term varies with the security set. ◆

We can interpret Δ_S as a measure of the expected "steepness" of the security chosen by the second highest type. The lemma shows that the seller's revenues are positively related to this steepness. As a result, we might expect that flat securities, like debt, would lead to low expected revenues, and steep securities, like levered equity, would lead to high expected revenues. In fact, the steepest possible levered equity security has equity share $\alpha = 1$, and can be described as

> <u>Retained Debt:</u> The seller receives the residual payoff, after debt with face value $d \geq 0$ is paid: $S(z) = \max(z - d, 0)$. That is, the bidder retains a debt claim with face value d.

We verify this intuition with the following result for second price auctions:

> **PROPOSITION III.** For the second price auction, standard debt yields the lowest possible expected revenues. Retained debt yields the highest possible expected revenues.

PROOF: Let S^d represent the set of standard debt securities, and S some other ordered set, and s^d and s the corresponding equilibrium strategies for the second price auction. Since $ES(s(v), v) = v$ in a second price auction,

$$\Delta_S - \Delta_{S^d} = E\left[ES(s(V^2),V^1) - ES^d(s^d(V^2),V^1)\right]$$

But for any realization of V^1, V^2, from Lemma 6 the debt security strictly crosses any other security from above. Since $ES(s(V^2), V^2) = ES^d(s^d(V^2), V^2) = V^2$ and $V^1 > V^2$, this implies that

$$ES(s(V^2), V^1) - ES^d(s^d(V^2), V^1) > 0,$$

which proves the result for debt. The result for levered equity follows similarly, noting that levered equity with $\alpha = 1$ crosses any other security from below. ◆

Note that, since debt is a flat set, from Proposition II the first price auction is even worse for the seller. Also, since levered equity is a steep set, the first price auction is even better for the seller. The following proposition establishes that a first price auction with debt and with levered equity bound the range of outcomes for the seller for a broad class of mechanisms.

> **PROPOSITION IV.** A first price auction with retained debt yields the highest expected revenues amongst all general symmetric mechanisms. A first price auction with standard debt yields the lowest expected revenues amongst all general symmetric mechanisms.

PROOF: The proof is identical to that of Proposition II, except that now for the second price auction we consider a general symmetric mechanism over some feasible set of securities. The result follows from the fact that the feasible set is convex, and that a retained debt contract is steeper than any feasible security and a standard debt contract is flatter than any feasible security. ◆

The preceding result is related to a result of Cremer (1987). He shows that if the seller can pay the winner, he can choose a set of securities with slope arbitrarily close to $Z_i - X$ and extract the entire surplus. This negative payment is even steeper than retained debt, in which the bidder holds default risk.

At the other extreme, if bidders can pay cash, the same methodology establishes that a cash auction is the worst possible auction for the seller. This is because cash, which is insensitive to type, is even flatter than standard debt securities (see Section 5.1).

We conclude this section with a numerical example:

EXAMPLE 3: Comparison of Revenues Across Mechanisms and Securities

> Suppose $Z_i = \theta\,(X + V_i)$, where θ is lognormal with mean 1 and volatility 50%. Let $X = 100$, and V_i be uniform with $v_L = 20$ and $v_H = 110$.[15] Suppose there are $N = 2$ bidders. Then the expected value, $E[\max(V_1, V_2)] = 80$. This is the maximum expected revenue achievable by any auction. The expected revenue from a standard cash auction is given by $E[\min(V_1, V_2)] = 50$ (which is the same for first and second price auctions by revenue equivalence). The revenues for different security auction types are calculated numerically and given in Figure 1 below. Note the failure of revenue equivalence for standard debt and retained debt, while it holds for equity, in accord with Proposition I and Proposition II. Note also the

[15] For these parameter values, Assumption C also holds for retained debt.

strong dependence of the revenues on the security type, in accord with Proposition III and Proposition IV.

	Expected Seller Revenues	
Security Type	First Price Auction	Second Price Auction
Standard Debt	50.05	50.14
Equity	58.65	58.65
Retained Debt (Levered Equity)	74.53	74.49

Figure 1. Revenue Comparisons for Example 3.

4. Unrestricted Bidding: The Signaling Game

In the previous section we have restricted bidders to choose securities from a specific well-ordered set. In reality, there is often no such a restriction. That is, the seller is unable to commit ex-ante not to consider all bids. In this case, the "security design" is in the hands of the bidders, who can choose to submit a bid using any feasible security.

Without the structure of a well-ordered set, once the bids are submitted there is no obvious notion of a "highest" bid. In this case, the seller faces the task of choosing one of the submitted bids. Since there is no ex-ante commitment by the seller to a decision rule, the seller will choose the winning bid that offers the highest expected payoff. Since the payoff of the security depends on the bidder's type, the seller's choice will depend upon his beliefs regarding the bid each type submits in equilibrium. Thus, this setting has the features of a classic signaling game.

Formally, in this section we examine an auction setting in which:

1. Bidders submit simultaneous bids that are feasible securities,
2. The seller chooses a winning bid from the set submitted,
3. The winner pays his bid and runs the project.

In a sequential equilibrium of this game, the seller will choose the bid that he believes has the highest expected payoff. Thus, this is a generalization of the standard first price auction, where the ranking now depends upon the seller's beliefs.

Refining Beliefs – The D1 Criterion

As with general signaling games, there are many equilibria of this game if we do not impose any restrictions on the beliefs of the seller when an "unexpected" bid is observed. The standard refinement of beliefs in the signaling literature is the notion of strategic stability, introduced by Kohlberg and Mertens (1986). For our purposes, a weaker refinement, known as D1, is sufficient to identify a unique equilibrium. The D1 refinement, introduced by Cho and Sobel (1990), is a refinement commonly used in the security design literature.[16] Intuitively, the D1 refinement criterion requires that if the seller observes an out-of-equilibrium bid, the seller should believe the bid came from the type "most eager" to make the deviation.

[16] See, e.g., Nachman and Noe (1994) and DeMarzo and Duffie (1999).

In order to define the D1 criterion in our context, we introduce the following notation. First, let S^i be the random variable representing the security bid by bidder i, which will depend on V_i. For any feasible security S, let $R^i(S)$ be the *scoring rule* assigned by the seller, representing the expected revenues the seller anticipates from that security given his beliefs. Along the equilibrium path, the seller's beliefs are correct, so that the scoring rule satisfies

$$R^i(S) = E\left[ES(V_i) \,\middle|\, S^i = S \right]. \tag{3}$$

Given the seller's scoring rule, R^i, it must also be the case in equilibrium that bidders are bidding optimally. That is, conditional on $V_i = v$, S^i solves

$$U^i(v) = \max_S P^i(R^i(S))(v - ES(v)), \tag{4}$$

where $P^i(r)$ is the probability that r is the highest score.[17] Thus, $U^i(v)$ is the equilibrium expected payoff for bidder i with type v.

Suppose the seller observes an out-of-equilibrium bid, so that the score is not determined by (3). Which types would be most likely to gain from such a bid? For each type v, we can determine the minimum probability of acceptance, $B^i(S,v)$, that would make bidding S attractive:

$$B^i(S,v) = \min\left\{ p : p\,(v - ES(v)) \geq U^i(v) \right\}.$$

Then the D1 criterion requires that the seller believe that a deviation to security S came from the types which would find S attractive for the lowest probability:[18]

$$R^i(S) \in ES(\arg\min_v B^i(S,v)). \tag{5}$$

Thus, a sequential equilibrium satisfying the D1 criterion for the auction game can be described by scoring rules R^i and bidding strategy S^i for all i satisfying (3)-(5).

Equilibrium Characterization

We now turn to characterizing the equilibrium of the auction game. As is standard in the auction setting, we will focus on symmetric equilibria.[19] Our main result is that if the bidders are unrestricted, the resulting symmetric equilibrium is equivalent to a first price auction with standard debt securities. Note that, from Proposition IV, this implies that the seller's expected revenues are the lowest possible from any general auction mechanism.

The intuition for standard debt is as follows. Consider any equilibrium security S_v bid by type v. If type v were instead to offer a standard debt contract with the same expected payoff, how would the seller respond? Because the debt contract is flatter than any non-debt contract, types $v' > v$ would find the debt contract cheaper, whereas types $v' < v$

[17] If there are ties, we require that P^i be consistent with *some* tie-breaking rule.
[18] We have economized on notation here. If the set of minimizers is not unique, the score is in the convex hull of $ES(v)$ for v in the set of minimizers.
[19] That is, we restrict attention to equilibria in which the bidders use symmetric strategies and the seller uses the same scoring rule for all players.

would find the debt contract more expensive than S_v. That is, only those types $v' > v$ would benefit from such a deviation. By D1, this implies that the score can only increase if the bidder switches to debt, so that debt is an optimal bid.

We now proceed with a formal statement of our results. We continue to maintain **ASSUMPTION C** for standard debt, so that existence of an efficient equilibrium of the first price auction is assured. Then we have

> **PROPOSITION V.** Given symmetric strategies, there is a unique equilibrium of the unrestricted auction satisfying D1. This equilibrium is equivalent, in both payoffs and strategies, to the equilibrium of a first price auction in standard debt contracts.

Again, we can now combine this result with the result of the previous section to formalize the value of the seller's ability to commit to a restricted set of securities.

> **COROLLARY.** If the seller can commit to an ordered set of securities other than debt contracts, then expected revenues are strictly greater than without such commitment.

PROOF: Follows immediately from Proposition IV and Proposition V. ◆

5. Extensions

We finish this section with a discussion of several extensions of the basic model. First, we consider the implications of both the bidders and the seller having access to cash. Second we discuss the implications of moral hazard on the part of the bidders.

5.1. Cash Payments

Thus far, we have assumed the all transfers between bidders and sellers are restricted to claims over the future cash flows of the project. Suppose in addition that the bidders and seller have cash available that can also be used. Specifically, suppose bidders have cash C_b, and the seller has cash C_s.

In terms of the assumptions of our model, the effect of adding cash is to relax the limited liability restrictions on the set of feasible securities. Rather than requiring $0 \le S(z) \le z$, we require

$$-C_s \le S(z) \le z + C_b.$$

This weaker constraint reflects the possibility that the seller can pay the bidder (up to C_s) and that the bidder can pay cash (up to C_b) in addition to a security claim on the project.

How does this relaxation of the constraint affect our results? Essentially, cash allows us to make the flattest securities even flatter (by adding a risk-free claim) and the steepest securities even steeper (by increasing leverage). Specifically, the flattest securities are now debt claims on the total assets of the bidder (cash + project), defined by

$$S^D(d, z) = \min(d, C_b + z) = \min(d, C_b) + \min((d - C_b)^+, z).$$

As the decomposition above reveals, we can think of this security as an immediate cash payment plus a standard debt claim on the project.

Similarly, the steepest securities involve a cash payment from the seller. Retained debt generalizes to

$$S^{RD}(d, z) = -C_s + \max(z - d, 0).$$

That is, the bidder *receives* a cash payment, and gives up an equity claim on the project.

Since these new securities are respectively flatter and steeper than in the absence of cash, the results in the paper imply that the expected revenues will be, respectively, lower and higher than before.

If C_b is large enough, a pure cash auction is possible. This has the lowest possible expected revenues for the seller. For this to be the case, it is necessary that

$$C_b \geq E\left[V_{n-1}^*\right] \qquad \text{for a first price auction,}$$

and

$$C_b \geq v_H > E\left[V_{n-1}^*\right] \qquad \text{for a second price auction,}$$

where V_{n-1}^* is the maximum type for n–1 bidders.

Note that this requirement is stricter for a second price auction, so that a first price auction yields strictly lower revenues for the seller as long as $C_b < v_H$.

On the other hand, if $C_s \geq X$, it is possible for the seller to "buy" the participation of the bidders. As Cremer (1987) shows, in this case it is possible for the seller to extract the full surplus. For example, the seller can pay the bidders $X - \varepsilon$ and have them compete using equity (or retained debt). As $\varepsilon \to 0$, the bids will become arbitrarily close to giving 100% of the project cash flows to the seller. As a practical matter, however, this extreme case quite strongly depends on the absence of moral hazard on the part of the bidder, which we consider next.

5.2. Moral Hazard

Thus far, we have included minimal moral hazard restrictions in the model by restricting the payoffs to the bidder and the seller to be weakly increasing. Absent this, either party would have an incentive to sabotage or artificially inflate the project cashflows.

Depending on the environment, more stringent moral hazard constraints might be appropriate. For example, the bidder might have the opportunity to divert cash flows from the project. Suppose that each dollar that the bidder diverts leads to a private payoff for the bidder of $\delta < 1$. This is a standard agency setting in which the bidder can siphon off resources inefficiently, but enjoy a private gain.[20]

By the usual revelation principle type of argument, we can restrict attention to securities that do not induce the bidder to divert cash flows. This is equivalent to adding the additional constraint,

$$S'(z) \leq 1 - \delta,$$

[20] This is a special case of Lacker and Weinberg's (1989) model of "costly state falsification."

to the requirements for a feasible security.[21] Note that this implies that the payoff of the winning bidder is bounded below by

$$E[\, Z_i - (1-\delta)\, Z_i \,] - X = \delta\,(X + V_i) - X.$$

To be sure that this constraint does not bind and the set of securities is sufficiently rich, we require that

$$\delta < \frac{X}{X + v_H}\,.$$

What is the effect of this additional moral hazard consideration? First, it reduces the maximal steepness of a feasible security. In this case the security type which maximizes the seller's expected revenue is levered equity, with an equity share $\alpha = 1 - \delta$. Because this is less steep than retained debt, competition between bidders is reduced and the maximum possible revenues the seller can achieve using the optimal mechanism will be strictly lower as a result of this moral hazard.

On the other hand, if the seller cannot commit to a restricted set of securities, this constraint prevents bidders from using standard debt. The flattest securities are now securities of the form

$$S(d, z) = \min(d, (1-\delta)\, z).$$

Since these securities are not as flat as standard debt, the revenues of the seller are in fact enhanced by this restriction. In other words, moral hazard benefits the seller by allowing him to commit not to accept standard debt contracts.

Similar arguments can be used to analyze the impact of other forms of moral hazard. In general, moral hazard concerns will limit the steepness of the securities that can be used by bidders, reducing the revenues of an optimal auction. For example, in the costly state verification framework of Townsend (1979) or the costly effort model of Innes (1990), standard debt is an optimal security design in mitigating the moral hazard problem.[22] In an auction framework, our results show that there is a tradeoff in terms of auction revenues.

On the other hand, some forms of moral hazard can be beneficial to a seller who cannot commit to a restricted set of securities. The framework above is one example. For another, suppose the bidder can increase the risk of the project (asset substitution). This problem may prevent the use of standard debt by bidders; for example, if non-positive NPV projects with arbitrary distributions are available to the bidder, then we can restrict the set of feasible securities to be convex (so that the bidder's payoff is concave).[23] In

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[21] The security S need not be everywhere differentiable – in this case we mean that $(1-\delta)$ is a supergradient of S.

[22] McAfee and McMillan (1987) also consider an optimal mechanism with full commitment from the seller; without the constraint $0 < S(z) < z$ and a costly effort model of moral hazard. The optimal mechanism is under some conditions a linear function that consists of a fixed fee and an equity share. However, the menu of securities they propose depends critically on the underlying distribution of types and the effort cost function (in particular, it is not detail-free), and hence is difficult to implement.

[23] See Ravid and Spiegel (1997) for a model of this sort.

this case, in the unrestricted auction the bidder will use equity (the flattest convex security), enhancing the seller's expected revenues.

6. Conclusion

We have examined an aspect of bidding relatively ignored in auction theory – the fact that bidders' payments often depend on the realization of future cash flows. This embeds a security design problem within the auction setting. We analyzed two environments. In the first, the seller chooses the security design and restricts bidders to bid using securities in an ordered set. This enables a simple ranking of the securities and the use of standard mechanisms. We show conditions for which 'Revenue Equivalence' holds, and what are the optimal and worst mechanisms and security designs.

In the second environment, the seller does not restrict the set of securities or the mechanism ex ante, but chooses the most attractive bid ex post. In this case, security design is in the hands of the bidders. We show that this yields the lowest possible expected revenues for the seller. Thus it gives incentives for the seller to be actively involved in the auction design and select the securities that can be used.

There are several important potential extensions of our results that are likely to be important in applications. First, it would be useful to allow for asymmetries, both in bidders' valuations and costs. While we believe that our result that debt is the outcome for the unrestricted auction generalizes to this setting, for restricted mechanisms we are confronted with the relative lack of theoretical results in the presence of asymmetries even in the case of cash auctions.

Also, for some applications such as mergers and acquisitions, other considerations such as taxes and accounting treatment are likely to be important considerations. For example, the deferral of taxes possible with an equity-based transaction may give rise to the use of equity bids even in an unrestricted setting. Our results are useful, in that they imply that this tax preference can also lead to enhanced revenues for the seller.

7. Appendix

PROOF OF LEMMA 1: Differentiability follows from dominated convergence under our assumptions. Then, for any z^*,

$$ES'(v) = \int S(z)\, h_v(z\,|\,v)\, dv = \int \left[S(z) - S(z^*) \right] \left[\frac{h_v(z\,|\,v)}{h(z\,|\,v)} \right] h(z\,|\,v)\, dv\,,$$

since

$$\int S(z^*)\, h_v(z\,|\,v)\, dv = S(z^*) \tfrac{\partial}{\partial v} \int h(z\,|\,v)\, dv = 0\,.$$

From SMLRP, [h_v / h] is strictly increasing in z. Therefore, we can choose z^* so that

$$\left[\frac{h_v(z\,|\,v)}{h(z\,|\,v)} \right] \begin{matrix} > \\ < \end{matrix} 0 \quad \text{if and only if} \quad z \begin{matrix} > \\ < \end{matrix} z^*.$$

Then, since S is weakly increasing,

$$\left[S(z)-S(z^*)\right]\left[\frac{h_v(z\,|\,v)}{h(z\,|\,v)}\right]\geq 0,$$

and the inequality is strict for z such that $S(z) \neq S(z^*)$. This set has positive measure since $S \neq 0$ and Z has full support conditional on v. Hence, $ES'(v) > 0$. The proof of $ES'(v) < 1$ is identical, substituting $Z - S(Z)$ for S. ◆

PROOF OF LEMMA 2: The proof that $s(v)$, which solves $ES(s,v) = v$, is the unique weakly undominated strategy is standard. Differentiating $ES(s(v),v) = v$ yields,

$$s'(v)=\frac{\frac{\partial}{\partial v}\left[v-ES(s,v)\right]}{\frac{\partial}{\partial s}ES(s,v)}.$$

Thus s is strictly increasing in v follows since S is increasing in s, and from Lemma 1, $\frac{\partial}{\partial v}\left[v-ES(s,v)\right]>0$ as long as $S(s,Z) \neq Z$ (which is not possible in equilibrium since $X > 0$). ◆

PROOF OF LEMMA 3: Let $P(s)$ be the probability of winning with a bid of s, and $\pi(s, v) = \log(P(s)) + \log(v - ES(s, v))$. Then

$$s(v)\in \arg\max_s P(s)(v-ES(s,v)) = \arg\max_s \pi(s,v)$$

By Assumption C the objective in the second expression is strictly supermodular, and so by Topkis (1978), any selection $s(v)$ is increasing in v. If $s(v)$ were constant on an interval, then the highest type in that interval can increase his bid marginally and increase his probability of winning, and thus his payoff, by a discrete amount. Thus, $s(v)$ is strictly increasing. This implies $P(s(v)) = F(v)^{n-1}$. Assuming differentiability (proved by a standard argument TBA) the first order condition for the above maximization leads to the differential equation for s. For the boundary condition, note that $P(s(v_L)) = 0$, and since all types earn non-negative profits $ES(s(v_L), v_L) \leq v_L$. But if the inequality were strict, the lowest type could raise his bid and earn positive profits with positive probability.

Having established uniqueness, it remains to verify existence by establishing the sufficiency of the bidder's first order condition. Consider any s' such that $s(v_L) < s' < s(v)$. There exists $v_L < v' < v$ such that $s(v') = s'$. Thus, by Assumption C,

$$\pi_s(s', v) > \pi_s(s', v') = 0.$$

A similar argument shows that for $s(v) < s' < s(v_H)$, $\pi_s(s', v) < 0$. Hence, π is quasiconcave in s and the first order condition is sufficient. ◆

PROOF OF LEMMA 4: First consider levered equity. In this case, the securities are indexed by the equity share α. Thus,

$$v - ES(\alpha, v) = v - \alpha\, C(X+v, d),$$

where C is the value of equity as a call option given the asset has present value $X + v$ and the face value of the debt is d. Therefore,

$$\tfrac{\partial}{\partial\alpha}\log\left[v - ES(\alpha, v)\right] = \frac{-C(X+v,d)}{v-\alpha C(X+v,d)}.$$

Differentiating with respect to v, we find that Assumption C holds if and only if

$$C(X+v, d) > v\,\Delta(X+v, d), \tag{6}$$

where $\Delta = C_1$, the derivative of the call value with respect to the underlying asset value. Using Black-Scholes,

$$C(X+v,d) = (X+v)\Delta - d\ \Pr(Z > d \mid v)$$

$$\Delta(X+v,d) = \Pr(Z > e^{-\sigma^2} d \mid v)$$

where σ^2 is the variance of $\log(\theta)$. Thus, (6) can be rewritten as

$$X > d\ \frac{\Pr(Z > d \mid v)}{\Pr(Z > e^{-\sigma^2} d \mid v)}, \tag{7}$$

which is satisfied if $X > d$. Note that this also implies the result for standard equity, since then $d = 0$. (As an aside, given MLRP, condition (7) is most restrictive for $v = v_H$.)

In the case of debt, the securities are indexed by the face value d. Since the bidder retains the equity, in this case

$$v - ES(d, v) = C(X+v, d) - X.$$

Hence,

$$\tfrac{\partial}{\partial v}\log\left[v - ES(d,v)\right] = \frac{\Delta(X+v,d)}{C(X+v,d)-X} = \frac{\Delta / C}{1 - X/C}.$$

Since C is decreasing in d, so is the denominator. Thus, we only need to show that Δ/C is increasing in d. This follows from the well-known result that the effective leverage of a call option is increasing with the strike price k (i.e., the leverage is greatest for the deepest out-of-the-money options). ◆

PROOF OF LEMMA 5: Using the revelation principle, note that if type v reports v' he will win with probability $F^{n-1}(v')$. His expected payoff conditional on winning is equal to $(v - T(v, v'))$, where $T(v, v')$ is the expected payment by type v when he reports v'. Thus, type v will choose v' to maximize $F^{n-1}(v')\left(v - T(v,v')\right)$. Thus, we need to establish the correct form for T.

Letting V_{-i}^* be the highest type excluding i, bidder i wins with report v' if $V_{-i}^* < v'$. Let $\tilde{S}_{v'} \in \boldsymbol{S}$ be the random security that he will pay if he wins. Then define

$$\hat{S}_{v'}(z) = E\left[\tilde{S}_{v'}(z) \,\middle|\, V_{-i}^* \le v'\right], \tag{8}$$

a security in the convex hull of $\boldsymbol{S}$ (which does not depend on i by symmetry). This is the "expected security" paid with a report of v'. Then we have, using the fact that types are independent and that Z_i and V_{-i} are independent given V_i (private values):

$$\begin{aligned}T(v,v') &= E\left[\tilde{S}_{v'}(Z_i)\,\middle|\, V_i = v, V_{-i}^* \le v'\right] \\ &= E\left[E\left[\tilde{S}_{v'}(Z_i)\,\middle|\, Z_i, V_i = v, V_{-i}^* \le v'\right]\middle|\, V_i = v, V_{-i}^* \le v'\right] \\ &= E\left[E\left[\tilde{S}_{v'}(Z_i)\,\middle|\, Z_i, V_{-i}^* \le v'\right]\middle|\, V_i = v, V_{-i}^* \le v'\right] \\ &= E\left[\hat{S}_{v'}(Z_i)\,\middle|\, V_i = v, V_{-i}^* \le v'\right] \\ &= E\hat{S}_{v'}(v)\end{aligned}$$

This completes the proof. ◆

PROOF OF LEMMA 6: Let $H(z) = S_1(z) - S_2(z)$. Then if $EH(v^*) = 0$,

$$\begin{aligned}EH'(v^*) &= \int H(z)\, h_v(z\,|\,v^*)\, dv = \int H(z)\left[\frac{h_v(z\,|\,v^*)}{h(z\,|\,v^*)}\right] h(z\,|\,v^*)\, dv \\ &= \int H(z)\left[\frac{h_v(z\,|\,v^*)}{h(z\,|\,v^*)} - \frac{h_v(z^*\,|\,v^*)}{h(z^*\,|\,v^*)}\right] h(z\,|\,v^*)\, dv\end{aligned}$$

From SMLRP, [h_v / h] is strictly increasing in z. Therefore,

$$H(z)\left[\frac{h_v(z\,|\,v^*)}{h(z\,|\,v^*)} - \frac{h_v(z^*\,|\,v^*)}{h(z^*\,|\,v^*)}\right] \ge 0\,,$$

and the inequality is strict on the set $\{z : S_1(z) \neq S_2(z)\}$. Thus, $EH'(v^*) > 0$. ◆

PROOF OF LEMMA 7: For standard debt securities, consider any feasible security S_2. If $S_2(z) > \min(d, z)$, then $z > d$ and so $S_2(z') > \min(d, z')$ for all $z' > z$. Hence $\min(d, z)$ crosses S_2 from above.

For levered equity, note that a convex combination of these securities for different levels of retained debt is a security $S_2(z)$ that is convex in z with maximum slope α. Thus, any levered equity security crosses S_2 from below. A similar argument applies to convertible debt when indexed by the equity share α. ◆

PROOF OF PROPOSITION V: We prove our result in several steps.

STEP 1: The equilibrium from a first price debt auction is a D1 equilibrium in the unrestricted auction.

To show this, we need to show that given the strategies from the debt auction, there is a set of beliefs satisfying D1 that support this equilibrium in the unrestricted auction. We construct the beliefs R using (3) and (5). If (5) does not produce a unique score, we can choose the lowest one. We now show that this supports the equilibrium.

STEP 1A: For debt contracts, the score is increasing in the face value of the debt. That is, $R(S^d)$ is strictly increasing in d, where $S^d(z) = \min(d,z)$.

Note that

$$\operatorname{argmin}_v B(S^d, v) = \arg\max_v \frac{v - ES^d(v)}{U(v)}.$$

Because the objective function is strictly log-supermodular, from Topkis (1978) we know that v is weakly increasing with d. Thus, (5) implies $R(S^d)$ is strictly increasing in d.

STEP 1B: R supports an equilibrium in the unrestricted auction.

Consider any deviation to a debt contract. From Step 1a, the probability of winning the auction is the same as in a first price auction. Since we have a first-price equilibrium, there is no gain to the deviation.

Consider a deviation to a non-debt contract S. Let v be the highest type in the set that minimizes $B(S,v)$. Then $ES(v) \geq R(S)$. It is sufficient to show that type v does not find the deviation to S profitable; i.e., to show that $P(R(S)) \leq B(S,v)$.

Find d such that $ES^d(v) = ES(v)$. Then $B(S^d,v) = B(S,v)$. From Lemma 6, types $v' < v$ find S^d more expensive than S, so that $B(S^d,v') > B(S,v') \geq B(S,v)$. Therefore $\operatorname{argmin}_{v'} B(S^d, v') \geq v$, and so from (5),

$$R(S^d) \geq ES^d(v) \geq R(S).$$

Thus, if a deviation to S is profitable, so is a deviation to S^d. But this contradicts the fact that no deviation to a debt contract is profitable.

STEP 2: A symmetric D1 equilibrium in the unrestricted auction has the same payoffs as the equilibrium of the first price debt auction.

Our method of proof is to show that any non-debt bids can be replaced with an equivalent debt bid without changing the equilibrium.

STEP 2A: If S is not a debt contract, then at most one type uses this security.

Suppose not, so that $v_1 < v_2$ are the lowest and highest types that use S. Then, by (3), $R(S) = ES(v^*)$ for some $v_1 < v^* < v_2$. Consider the debt contract S^d with the same cost for type v^*; i.e., such that $ES^d(v^*) = ES(v^*)$. From Lemma 6, types $v < v^*$ find the S^d more expensive than S, so that $B(S^d, v) > P(R(S))$. Therefore, $\arg\min_v B(S^d, v) \geq v^*$. Thus, from (5), $R(S^d) \geq ES^d(v^*) = R(S)$. This implies that v_2 bidding S contradicts (4), since S^d is strictly cheaper and yields a weakly higher score.

STEP 2B: If type v uses contract S, and if $ES^d(v) = ES(v)$, then $P(R(S^d)) = P(ES^d(v)) = P(R(S))$ and S^d is also optimal for type v.

Note first that $P(R(S^d)) \leq P(R(S))$ by (4). However, by the same argument used in the previous step, $B(S^d, v') > P(R(S))$ for $v' < v$, and hence $R(S^d) \geq ES^d(v) = R(S)$. Thus, $P(R(S^d)) = P(R(S))$. Since the cost and probability of acceptance are the same, bid S^d is also optimal for type v.

STEP 2C: $R(S^d)$ is strictly increasing in d. (Same proof as Step 1a.)

STEP 2D: Let $d(v)$ be an optimal face value of debt for each type v in Step 2b. Then $d(v)$ is strictly increasing in v.

From Step 2b, there exists $d(v)$ that solves

$$U(v) = P(R(S^{d(v)}))(v - ES^{d(v)}(v)) = \max_d P(R(S^d))(v - ES^d(v))$$

From Assumption C, the objective function is strictly log-supermodular, so that $d(v)$ is weakly increasing in v.

Suppose $d(v)$ were constant on an interval. Then, from step 2b, all types on that interval have the same probability of winning. Since strategies are symmetric, this implies they must tie. But then the highest type can increase his debt marginally, increase his score (by Step 2c), and raise his probability of winning by a discrete amount. This contradicts the optimality of $d(v)$.

STEP 2E: If type v uses contract S, then $R(S^{d(v)}) = R(S)$.

From Step 2b, $R(S^d) \geq R(S)$. But if $R(S^d) > R(S)$, this implies d is optimal for some type $v' > v$, contradicting Step 2d.

STEP 2F: Bidding $d(v)$ is an equilibrium in the unrestricted auction, and in a first price debt auction.

From Step 2d and 2e, payoffs and scores are unchanged. Thus, it is an equilibrium in the unrestricted auction.

Since $R(S^d)$ is strictly increasing, $P(R(S^d))$ = Prob(d is highest debt bid). Thus, this is also an equilibrium of a first price auction.

STEP 3: In a symmetric D1 equilibrium in the unrestricted auction, almost every bid is a debt contract.

From Step 2 and Lemma 3, the equilibrium payoff from type v is given by

$$U(v) = F^{n-1}(v)\,(v - ES^{d(v)}(v)),$$

and so U is differentiable. Suppose type $v \in (v_L, v_H)$ bids S in equilibrium. Then by a standard envelope argument,

$$U'(v) = F^{n-1}(v)(1 - ES'(v)).$$

Thus, $ES(v) = ES^{d(v)}(v)$ and $ES'(v) = ES^{d(v)'}(v)$. Thus, from Lemma 6, $S = S^{d(v)}$. ◆

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