

Optimal asset management contracts with hidden savings

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Abstract

We characterize optimal contracts for delegated asset management with hidden savings. Incentive provision requires exposing the agent to risk, which induces a precautionary motive for saving that tightens the incentive constraints. The optimal contract manipulates the agent's precautionary motive by promising lower risk in the future and after underperformance. CRRA utility makes the problem easy to characterize with a single state variable that captures the principal's commitment to reduce the agent's precautionary motive. We provide a sufficient analytic condition for the validity of the first-order approach: if the agent's precautionary motive falls after underperformance, the contract is globally incentive compatible. This condition holds in the optimal contract because it is less costly to insure the agent, and reduce his precautionary motive, after underperformance.

1 Introduction

Delegated asset management plays an important role in modern economies, from financial intermediaries such as fund managers, to CEOs or entrepreneurs who manage real capital assets. However, financial frictions can limit the efficient allocation of capital to its most productive users. In particular, in the presence of moral hazard it will be necessary to expose the asset manager to risk in order to provide incentives, and this will make it costly to delegate capital. Providing incentives is particularly difficult when the agent has access to hidden savings, which he can use to undo the incentive scheme. This paper studies

the impact of hidden savings on the optimal dynamic contract for an agent who manages capital.

We consider a classical investment setting. An agent with CRRA preferences over consumption can continuously invest in risky capital. He would like to raise funds and share risk with a complete financial market, but he faces a moral hazard problem: he can secretly divert funds and has access to hidden savings. The full commitment contract specifies compensation and capital under management contingent on the agent's returns.

The optimal contract can be characterized with a single state variable that captures the agent's precautionary motive. Delegating capital to the agent is attractive because the agent can get an excess return, but it requires exposing him to risk to provide incentives. With hidden savings, exposing the agent to risk induces a precautionary motive for saving and postponing consumption. This is costly because it both distorts the intertemporal consumption profile and worsens the risk sharing problem. Indeed, if the agent postpones consumption the marginal utility of diverting funds and consuming them is high, and therefore requires a high exposure to risk to deter fund diversion. The principal must therefore take into account the agent's precautionary motive, and manipulate it to his advantage.

First, the principal can relax the agent's precautionary motive by promising him a less risky continuation contract in the future. This reduces the cost of giving capital to the agent today in exchange for an inefficiently low amount of capital in the future. As a result, the optimal contract gives less capital to the agent over time, and therefore becomes less risky and less efficient. Second, the principal can further relax the precautionary motive by promising the agent a less risky contract after bad outcomes. Intuitively, the agent is most concerned about the risk he will face when his consumption is low and his marginal utility high. By promising him a less risky continuation contract after bad outcomes, the principal insures the agent's consumption, and instead punishes him with lower consumption growth. In exchange he can give the agent a more risky contract after good outcomes. Since more risky continuation contracts are more efficient, the principal finds it attractive to use them when he must deliver more utility to the agent - after good outcomes.

One of the main methodological contributions of this paper is to provide an analytical verification of the validity of the first order approach. Contractual environments where the agent has access to hidden savings are often difficult to analyze because we need to ensure incentive compatibility with respect to double deviations.¹ Dealing with single deviations

¹See for example Kocherlakota (2004).

is relatively straightforward. By giving the agent some skin in the game we can deter him from stealing and immediately consuming the proceeds. Likewise, incorporating the agent's Euler equation as a constraint in the contract design problem ensures that he won't secretly save his recommended consumption for later.² But what if the agent both steals and saves the proceeds for later? Since stealing makes bad outcomes more likely, the agent expects to be punished with lower consumption in the future. In other words, he expects a high marginal utility in the future, so stealing and secretly saving the proceeds for later could potentially be an attractive double deviation.

We prove the validity of the first order approach analytically by establishing an upper bound on the agent's continuation utility from any valid deviation, after any history. The crucial ingredient is that after bad outcomes the agent's precautionary motive becomes weaker. To understand why, note that if after some history the agent expects his legitimate consumption (which is his income in his hidden savings problem) to be very risky looking forward, he will have a large precautionary motive for saving and will place a large marginal value on hidden savings that can help him self insure. As a result, if after bad outcomes the contract became more risky, and the agent's precautionary motive larger, stealing and saving the proceeds for later would be an attractive double deviation, since the agent would expect to have hidden savings precisely when they are most valuable to him. It is here that the contract's dynamic behavior helps ensure incentive compatibility. After bad outcomes, the contract becomes less risky and the precautionary motive weaker, so hidden savings become less valuable to the agent. It's important to note that this verification argument applies beyond the optimal contract, to any contract under which the precautionary motive becomes weakly smaller after bad outcomes.

The optimal contract can be implemented as a consumption-portfolio problem where the agent consumes and invests in risky capital, and must keep a fraction of the risk on his balance sheet - i.e. he must keep some "skin in the game". In order to relax the agent's precautionary motive, the optimal contract reduces the agent's portfolio weight on capital over time, and after bad outcomes. Indeed, the agent is punished not only with less net worth, but also with a less efficient continuation portfolio. There is no endogenous termination. Because capital can be continuously adjusted, after a very good history the agent will be managing a large amount of capital, but he will not be retired nor outgrow the moral hazard problem as in Sannikov (2008) or Hopenhayn and Clementi (2006) respectively. Since

²See Werning (2001).

there is no outside option and the project can be scaled down, neither will he retire after sufficiently bad outcomes as in DeMarzo and Sannikov (2006): the contract just gives him a very small amount of capital and consumption, but there's always the chance that he will recover.

In fact, the long-run behavior of the contract features a non-degenerate stationary distribution. In the absence of shocks, contract dynamics would lead to a "steady state" with constant growth rate and volatility. However, the "steady state" can be a very misleading guide to the contract's long run behavior. The contract spends most of the time in the low risk region. The reason for this is that with low risk the only way for the contract to get out of this region is to slowly move towards the steady state. On the other hand, when the contract finally gets to a high risk region, even small shocks will quickly move the contract away and it will quickly fall back into the low risk region.

Literature Review. This paper fits within the literature on dynamic agency problems, such as DeMarzo and Fishman (2007), Sannikov (2008), He (2011), Biais et al. (2007), and Hopenhayn and Clementi (2006). The agency problem is one of cash flow diversion, as in DeMarzo and Sannikov (2006) and DeMarzo and Sannikov (2006), but unlike their models we have CRRA (as opposed to risk neutral) preferences and no investment frictions, i.e. there are no costs to adjusting the size of the project. CRRA utility allows us to study the effects of risk aversion and elasticity of intertemporal substitution. With risk-neutral preferences, the optimal contracts with and without hidden savings are the same. Once concave preferences are introduced, the principal has incentives to front load consumption in order to reduce the private benefit of cash diversion and relax the risk sharing problem.

The combination of CRRA preferences and frictionless investment technology affords a scale invariance properties that make the optimal contract particularly tractable. In fact, in the version of our model without hidden savings, the optimal contract can be characterized in closed form. For more details, see Di Tella (2014) which adopts a version of our model without hidden savings and develops it to allow for aggregate shocks and Epstein-Zin preferences. The focus of Di Tella (2014) is on optimal financial regulation in a general equilibrium setting. Without hidden savings and in a stationary environment such as the one considered here, the optimal contract would be characterized by a constant growth rate and volatility, and front loaded consumption.³

³When the agent does not have access to hidden savings, the principal can relax the agent's incentive constraint significantly by front-loading the agent's consumption, thus reducing the benefits the agent

Several papers deal with hidden savings. Werning (2001) introduces the first order approach in the context of unemployment insurance. He (2011) characterizes the optimal contract when the agent must put in unobservable effort and has access to hidden savings. He obtains incentive compatibility in the context of a fixed project size and CARA preferences. Williams (2013) studies a similar environment with fixed project size, CARA preferences, hidden effort and hidden savings. CARA preferences play important roles in ensuring incentive compatibility in both papers because they eliminate the wealth effect. Edmans et al. (2011) obtains a tractable incentive scheme in a setting where the agent can take a hidden action and has access to hidden savings. They allow for CRRA preferences, but assume the hidden action enters the agent's utility function in a multiplicative way with consumption. In addition, the size of the project is also fixed. In contrast to these papers, here we allow CRRA preferences, continuous trading of capital, and the hidden action consists of diverting funds.

TO BE COMPLETED

2 The model

Let (Ω, P, \mathcal{F}) be a complete probability space equipped with filtration \mathbb{F} generated by a brownian motion Z , with the usual conditions. Throughout, all stochastic processes are adapted to \mathbb{F} . There is a complete financial market with equivalent martingale measure Q . The risk-free interest rate is $r > 0$ and Z is idiosyncratic risk and therefore not priced by the market. In this setting there is no aggregate risk, so $Q = P$, but in general P and Q could differ.

The agent can manage capital to obtain a risky return that exceeds the required return of r , but he may also get a private monetary benefit by diverting returns. If the diversion rate is a_t , the observed return is

$$dR_t = (r + \alpha - a_t) dt + \beta dZ_t$$

where $\alpha > 0$ is the maximal excess return and $\beta > 0$ is the volatility. Notice Z is agent-specific idiosyncratic risk. If we think of the agent as a fund manager, it represents the

derives from diverting cash to consume. In fact, if the agent's elasticity of intertemporal substitution is sufficiently low, the principal is able to design a contract that attains infinite profits, thereby creating arbitrage. Introducing hidden savings constraints the principal's power and eliminates this possibility.

outcome of his particular trading activity.⁴ If we take the agent to be an entrepreneur it represents the outcome of his particular project.

If assets under management are k_t , then diversion of a_t (per dollar invested) gives the agent $\phi a_t k_t$. For each stolen dollar, the agent keeps only fraction $\phi \in (0, 1)$. If the agent also receives payments c_t from the principal and consumes \tilde{c}_t , then his hidden savings evolve according to⁵

$$dh_t = (rh_t + c_t - \tilde{c}_t + \phi k_t a_t) dt.$$

The agent wants to raise funds and share risk with the market, which we refer to as the principal. The principal observes realized returns dR_t but not the agent's diversion action a_t , consumption \tilde{c}_t or savings. The principal can commit to a fully history-dependent contract $\mathcal{C} = (c, k)$ that specifies payments to the agent $c_t > 0$ as well as assets under management $k_t \geq 0$ as a function of the history of realized returns up to time t . After signing the contract \mathcal{C} the agent can choose the strategy (\tilde{c}, a) that specifies \tilde{c}_t and a_t for each time t .

The agent has CRRA preferences. Given contract \mathcal{C} , under strategy (\tilde{c}, a) the agent gets utility

$$U_0^{\tilde{c}, a} = \mathbb{E}^a \left[\int_0^\infty e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \quad (1)$$

where superscript a indicates that the expectation is computed under the distribution over returns that is induced by the stealing strategy a . We assume that $\rho > r(1 - \gamma)$ to make the agent's hidden saving problem well defined. Given contract \mathcal{C} , we say a strategy (\tilde{c}, a) is *feasible* if 1) utility $U_0^{\tilde{c}, a}$ is finite, and 2) $h_t \geq 0$ always.⁶ Let $\mathbb{S}(\mathcal{C})$ be the set of feasible strategies (\tilde{c}, a) given contract \mathcal{C} .

The principal pays for the agent's consumption, but keeps the excess return α on the capital that the agent manages. He tries to minimize the cost of delivering utility u_0 to the agent

$$J_0 = \mathbb{E}^Q \left[\int_0^\infty e^{-rt} (c_t - k_t \alpha) dt \right] \quad (2)$$

⁴If we give \$1 to invest to two fund managers, they will obtain different returns depending on exactly which assets they buy or sell, and the exact timing and price of their trades.

⁵The agent gets the risk free interest rate for his hidden savings, but he cannot invest them in capital which is contractible. If the agent could enter into secret contracts whose payoffs are contingent on his return, he would be able to undo any incentives.

⁶The agent can have hidden savings but not hidden debt. This is without loss of generality if the contract can exhaust the agent's credit capacity.

A standard argument in this setting implies that the optimal contract must implement no stealing, i.e. $a = 0$. In addition, without loss of generality and for analytic convenience, we can restrict attention to contracts in which $h_0 = 0$ and $\tilde{c} = c$, i.e. the principal saves for the agent.⁷ Of course, the optimal contract has many equivalent and more natural forms, in which the agent maintains savings, but all these forms can be deduced easily from the optimal contract with $\tilde{c} = c$.

We say a contract $\mathcal{C} = (c, k)$ is *admissible* if 1) utility $U_0^{c,0}$ is finite, and 2)

$$\mathbb{E}^Q \left[\int_0^\infty e^{-rt} |c_t + k_t \alpha| dt \right] < \infty \quad (3)$$

We say an admissible contract \mathcal{C} is *incentive compatible* if⁸

$$(c, 0) \in \arg \max_{(\tilde{c}, a) \in \mathbb{S}(\mathcal{C})} U_0^{\tilde{c}, a}$$

Let \mathbb{IC} be the set of incentive compatible contracts. For an initial utility u_0 for the agent, an incentive compatible contract is *optimal* if it minimizes the cost of delivering initial utility u_0 to the agent, that is

$$\begin{aligned} v_0 &= \min_{(c, k)} J_0 \\ \text{st : } & U_0^{c,0} \geq u_0 \\ & (c, k) \in \mathbb{IC} \end{aligned}$$

By changing u_0 we can trace the Pareto frontier for this problem.

Remark. About admissibility condition (3): this assumption plays the role of a no-Ponzi scheme condition, making sure the principal's objective function is well defined. It rules out exploding strategies where the present value of consumption and capital is infinity.

⁷Given any contract that implements $(\tilde{c}, s) \neq (c, 0)$, the principal can do weakly better by giving the agent $c = \tilde{c} + \phi a k$ to the agent legitimately, saving for him, and asking him to not steal instead. See DeMarzo and Sannikov (2006).

⁸Notice $\mathbb{S}(\mathcal{C}) \neq \emptyset$ because $(c, 0) \in \mathbb{S}(\mathcal{C})$ for an admissible contract.

Incentive compatibility

As usual, we use the continuation utility of the agent as a state variable for the contract

$$U_t^{c,0} = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right]$$

However, we also need the marginal utility of the agent as a state variable in order to keep track of the agent's incentives to save. The marginal utility of consumption also affects the incentives to steal.

First we obtain the law of motion for the agent's continuation utility.

Lemma 1. *For any admissible contract $\mathcal{C} = (c, k)$, the agent's continuation utility $U^{c,0}$ satisfies*

$$dU_t^{c,0} = \left(\rho U_t^{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma} \right) dt + \Delta_t \underbrace{(dR_t - (\alpha + r) dt)}_{\beta dZ_t \text{ if } a_t=0} \quad (4)$$

for some stochastic process Δ .

Faced with this contract, the agent might consider stealing and immediately consuming the proceeds, i.e. following a strategy $(c + \phi k a, a)$ for some a , which results in savings $h_t = 0$. The agent gets to add $\phi k_t a_t$ to his consumption, but reduces the realized returns. For stealing to be unattractive we need to make sure that the agent's continuation utility goes down sufficiently after bad outcomes. Incentive compatibility therefore requires

$$0 \in \arg \max_{a \geq 0} \frac{(c_t + \phi k_t a)^{1-\gamma}}{1-\gamma} - \Delta \frac{a}{\beta} \quad (5)$$

Taking FOC yields

$$\Delta \geq c_t^{-\gamma} \phi k_t \quad (6)$$

which is positive. In other words, we need to give the agent some "skin in game". This is costly because the principal is risk-neutral with respect to Z so he would like to provide full insurance to the agent, but cannot because of the moral hazard problem.

Notice how the private benefit of the hidden action depends on the marginal utility of consumption, so the principal would like to front load his consumption. If the agent could not save, the principal would distort the intertemporal consumption profile to reduce the private benefit of the hidden action and therefore relax the risk-sharing constraint.

However, with hidden savings this is not possible. If the principal tries to front load the agent's consumption, the agent will secretly save and consume when his marginal utility is higher, and this will in addition affect his incentives to steal.

The optimal contract must therefore respect the agent's Euler equation: the discounted marginal utility $e^{-(\rho-r)t}c_t^{-\gamma}$ must be a supermartingale. The economic intuition is that if the expected discounted marginal utility in the future is higher than the marginal utility today, the agent would save today in order to consume later. If instead the expected discounted marginal utility is lower than the marginal utility today, he would like to borrow to consume more today, but he cannot. The following lemma summarizes all the necessary conditions for incentive compatibility and their implication on the path of the agent's consumption. We present sufficient conditions in the next subsection.

Lemma 2. *If $\mathcal{C} = (c, k)$ is an incentive compatible contract, then (6) must hold and the agent's consumption must satisfy*

$$\frac{dc_t}{c_t} = \left(\frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma_t^c)^2 \right) dt + \sigma_t^c dZ_t + dL_t \quad (7)$$

for some σ^c and a weakly increasing process L .

Notice that, by Ito's lemma, $e^{-(\rho-r)t}c_t^{-\gamma}$ would have drift 0 if $dL = 0$ and negative drift if $dL > 0$.⁹ In fact, it turns out that in the optimal contract, $e^{-(\rho-r)t}c_t^{-\gamma}$ is a proper martingale, so $dL = 0$ and the agent does not want to borrow.

Hidden savings restrict the agent's incentive compatible consumption paths. It is easy to make the agent consume with a high expected growth rate or even have his consumption "jump up" (because the agent cannot borrow against future consumption). However, (7) imposes a lower bound on the agent's consumption growth, beyond which the agent would secretly save to achieve a better intertemporal smoothing.

State space

In the case without hidden savings, we would only need to work with the agent's continuation utility $U^{c,0}$. However, in light of (7), we also need to keep c as a state variable, and we need to identify the domain of $(U^{c,0}, c)$ for which incentive compatible contracts exist. Because the drift in the agent's consumption has a lower bound, it may very well be impossible to give an agent a very high c combined with a very low $U^{c,0}$. For easy comparison of

⁹If the agent could have hidden debt then dL_t must be 0.

utility and consumption, it is convenient to work with the following transformation of the state variables

$$x_t = \left((1 - \gamma) U_t^{c,0} \right)^{\frac{1}{1-\gamma}} > 0$$

$$\hat{c}_t = \frac{c_t}{x_t} \geq 0$$

Variable x is just a monotone transformation of continuation utility, but it is measured in consumption units (up to a constant). As a result, \hat{c} measures how front loaded the agent's consumption is. While x_t can take any positive value, \hat{c}_t has an upper bound.

Lemma 3. *For any incentive compatible contract \mathcal{C} , at all times t , $\hat{c}_t \in (0, \hat{c}_h]$, where*

$$\hat{c}_h \equiv \left(\frac{\rho - r(1 - \gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} > 0$$

If ever $\hat{c}_t = \hat{c}_h$, then the continuation contract satisfies $k_{t+s} = 0$ and $\hat{c}_{t+s} = \hat{c}_h$ at all future times $t + s$ and gives the agent a unique deterministic consumption path with growth $(r - \rho)/\gamma$. The contract has cost $\hat{v}_h x_t$ to the principal, where $\hat{v}_h \equiv \hat{c}_h^\gamma$.

This upper bound has a simple interpretation. If the agent manages capital he must be exposed to risk, which induces a precautionary motive for savings. The agent will therefore back load consumption. The most a contract can front load the agent's consumption is if it gives the agent no capital to manage, and thus does not expose the agent to any risk. This is precisely the continuation contract after hitting \hat{c}_h . The deterministic growth rate of $(r - \rho)/\gamma$ results in the best way to deliver utility to the agent without any capital. As we will see, this is an incentive compatible contract, but not optimal because while it minimizes the cost of the agent's consumption, it misses out on the excess return of capital $\alpha > 0$.

Using Ito's lemma we can obtain laws of motion for x_t and \hat{c}_t from (4) and (7) Using the normalization $\Delta_t \beta / U_t = (1 - \gamma) \sigma_t^x$, we obtain

$$\frac{dx_t}{x_t} = \underbrace{\left(\frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma_t^x)^2 \right)}_{\mu_t^x} dt + \sigma_t^x dZ_t \quad (8)$$

and

$$\frac{d\hat{c}_t}{\hat{c}_t} = \underbrace{\left(\frac{r - \rho}{\gamma} + \frac{(\sigma_t^x)^2}{2} + \gamma \sigma_t^x \sigma_t^{\hat{c}} + \frac{1 + \gamma}{2} (\sigma_t^{\hat{c}})^2 - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right)}_{\mu_t^{\hat{c}}} dt + \sigma_t^{\hat{c}} dZ_t + dL_t \quad (9)$$

for some $\sigma_t^{\hat{c}} = \sigma_t^c - \sigma_t^x$. The constraint (6) can be rewritten as

$$\sigma_t^x \geq \hat{c}_t^{-\gamma} \hat{k}_t \phi \beta, \quad (10)$$

where $\hat{k}_t = \frac{k_t}{x_t}$. It will always be binding because conditional on σ_t^x it is always better to give the agent more capital to manage. The minimal cost flow that the principal can attain is

$$c_t - k_t \alpha = x_t \left(\hat{c}_t - \hat{c}_t^\gamma \frac{\alpha}{\phi \beta} \sigma_t^x \right). \quad (11)$$

The principal benefits by exposing the agent to greater risk - choosing higher σ_t^x - but he has to pay the cost of compensating the agent as σ_t^x raises the required drift of the transformed continuation value x . Risk exposure also raises the agent's precautionary motive. This lowers the current consumption ratio \hat{c} , which both distorts the intertemporal consumption profile and tightens the incentive constraint (10), reducing the benefits of risk exposure in (11).

To understand the trade-offs precisely, it is useful to interpret the variable \hat{c}_t as promised safety. By Lemma 3, the upper bound \hat{c}_h corresponds to maximal safety and a continuation contract that may not expose the agent to any risk. Otherwise, the recursive equation (9) provides the promise-keeping condition for \hat{c}_t . The principal wants to promise a degree of safety early on in order to reduce the agent's precautionary motive and relax the incentive constraint, but these promises create distortions later on as they limit the agent's risk exposure - and promises must be kept.

Specifically, notice that with $\sigma_t^{\hat{c}} = 0$, the drift of \hat{c}_t always increases in σ_t^x . The principal has to respect promised safety - and if he chooses to expose the agent to more risk now, he has to give the agent even safer continuation contract in the future. This effect can be offset somewhat by also choosing $\sigma_t^{\hat{c}} < 0$. Intuitively, the agent is concerned the most about risk exposure in bad states. Thus, the principal can reduce the agent's precautionary motive by giving him greater safety - raising \hat{c}_t - in the event that x goes down. This happens if $\sigma_t^{\hat{c}}$ has the opposite sign from σ_t^x . In fact, we will see that $\sigma_t^{\hat{c}} \leq 0$ in the optimal contract also for

another reason - as (11) indicates, the benefits of exposing the agent to risk through σ_t^x rise when x goes up, so the principal wants to expose the agent who performs well to greater risk to raise managed assets, even though this raises the agent's precautionary motive.

It turns out that the property $\sigma_t^{\hat{c}} \leq 0$, that the principal reduces the agent's precautionary motive after poor performance, is also a *sufficient* condition for the contract to be *globally* incentive compatible if the necessary conditions of Lemma 2 hold. While our paper is the first to our knowledge to prove this form of a result analytically, the intuition is simple. Notice that the marginal value of hidden savings rises in the agent's precautionary motive, i.e. savings become more valuable when the agent is exposed to risk going forward and therefore \hat{c} is lower. Therefore, whenever a contract reduces the precautionary motive after bad outcomes - these outcomes become more likely when the agent steals - stealing and saving is an unattractive deviation. The agent expects to have hidden savings when they are less valuable to him. This intuition is formalized below in Lemma 5.

The HJB equation

Because preferences are homothetic and the principal's objective is linear, we know the principal's cost function takes the form $v(x, \hat{c}) = \hat{v}(\hat{c}_t)x_t$ with $\hat{v}(\hat{c}_h) = \hat{v}_h > 0$ because of Lemma 3. Notice that since we could always raise \hat{c}_t using dL_t , we know that $\hat{v}(\hat{c})$ must be weakly increasing. In fact, we show below that $\hat{v}(\hat{c})$ increases strictly over the interval in which \hat{c}_t stays over the course of the optimal contract, so we can drop the term dL_t from what follows. We will also sometimes write $\hat{v}_t = \hat{v}(\hat{c}_t)$, and \hat{v} instead of $\hat{v}(\hat{c})$.

The HJB equation associated with this problem is

$$r\hat{v}x = \min_{\sigma^x, \sigma^{\hat{c}}} (\hat{c} - \hat{k}\alpha)x + \mathbb{E}_t^Q [d(\hat{v}_t x_t)]$$

subject to (8), (9), and (10), and $\sigma_t^x \geq 0$.

Using Ito's lemma and canceling the x on both sides, we get

$$\begin{aligned} r\hat{v} = \min_{\sigma^x, \sigma^{\hat{c}}} & \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\beta} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) \\ & + \hat{v}' \hat{c} \left(\frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma)\sigma^x \sigma^{\hat{c}} + \frac{1 + \gamma}{2} (\sigma^{\hat{c}})^2 - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} \right) + \frac{\hat{v}''}{2} \hat{c}^2 (\sigma^{\hat{c}})^2 \end{aligned} \quad (12)$$

Notice how even though we have two state variables, \hat{c} and x , the HJB equation boils

down to a second order ODE in \hat{c} . This is a feature of homothetic preferences and linear technology which makes the problem more tractable.

Shape of the cost function. The following lemma characterizes the shape of the principal's cost function and the range of \hat{c}_t under the optimal contract.

Theorem 1. *The principal's cost function $\hat{v}(\hat{c})$ has a flat portion on $[0, \hat{c}_l]$ and a strictly increasing portion on $[\hat{c}_l, \hat{c}_h]$, for some $\hat{c}_l \in (0, \hat{c}_h)$. The HJB equation (12) holds with equality above \hat{c}_l and with inequality below \hat{c}_l , i.e.*

$$r\hat{v} < \min_{\sigma^x} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\beta} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) \quad \forall \hat{c} < \hat{c}_l. \quad (13)$$

Below \hat{c}_l , $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l)$, and at \hat{c}_l the cost function $\hat{v}(\hat{c})$ satisfies the smooth-pasting condition $\hat{v}'(\hat{c}_l) = 0$, and $\hat{v}''(\hat{c}_l) > 0$.

The optimal contract starts at $\hat{c}_0 = \hat{c}_l$, where σ_0^x is chosen without taking into account its effect on the agent's precautionary motive, to maximize

$$\sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\beta} - \hat{v}(\hat{c}) \frac{\gamma}{2} (\sigma^x)^2 \quad (14)$$

At \hat{c}_l we have $\mu^{\hat{c}}(\hat{c}_l) > 0$ and $\sigma^{\hat{c}}(\hat{c}_l) = 0$. For all $t > 0$, $\hat{c}_t > \hat{c}_l$, $\sigma_t^{\hat{c}} \leq 0$ and $\sigma_t^x \geq 0$, but at a level lower than that which maximizes (14).

Figures 1 and 2 show the cost function and the drift and volatility of state variables x and \hat{c} for a numerical solution. To understand why the cost function has a flat portion and an increasing portion, consider the problem of the optimal choice of \hat{c}_0 . Choosing $\hat{c}_0 = \hat{c}_h$ is suboptimal, because then the principal cannot give the agent any capital. It is beneficial to give the agent capital and expose him to risk because capital generates an excess return of α . However, risk exposure is costly because the agent is risk averse. In addition, with hidden savings risk exposure also generates a precautionary motive which lowers \hat{c} , distorting the intertemporal consumption profile and further tightening the incentive constraint. Eventually, the costs of risk exposure outweigh the benefits. Under these trade-offs, we denote by \hat{c}_l the value that minimizes the cost of delivering utility to the agent, indicated with a blue dot in Figure 1. For $\hat{c}_0 < \hat{c}_l$, the principal has the option to raise \hat{c}_0 to \hat{c}_l immediately using the process dL_t . Hence, the cost function is flat over $[0, \hat{c}_l]$. For $\hat{c}_0 > \hat{c}_l$, he would be forced to give the agent inefficiently too little risk exposure to reduce the agent's

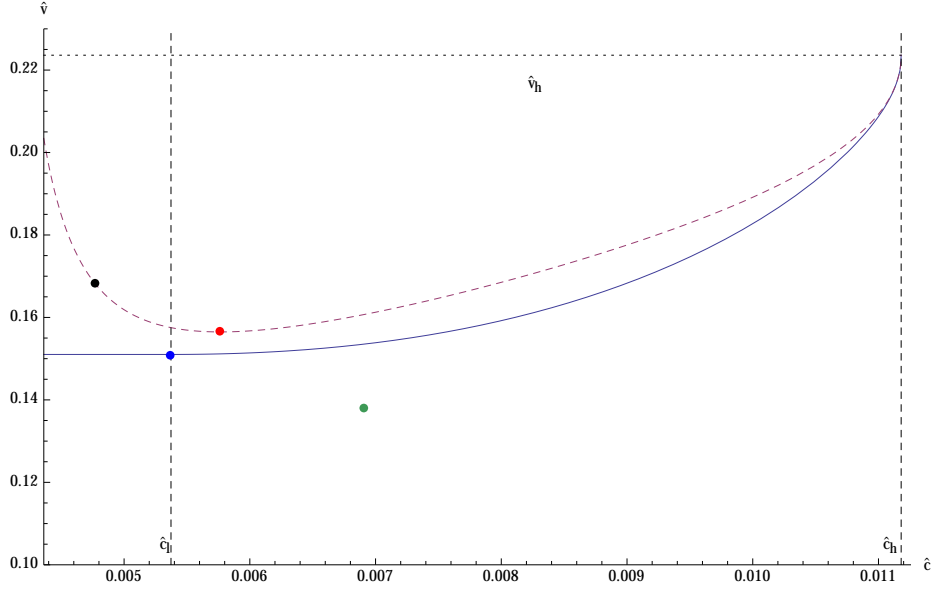


Figure 1: The cost function $\hat{v}(\hat{c})$ solid in blue for the optimal contract, and dashed in red for stationary contracts. The the starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. *Parameters:* $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi\beta = 0.2$.

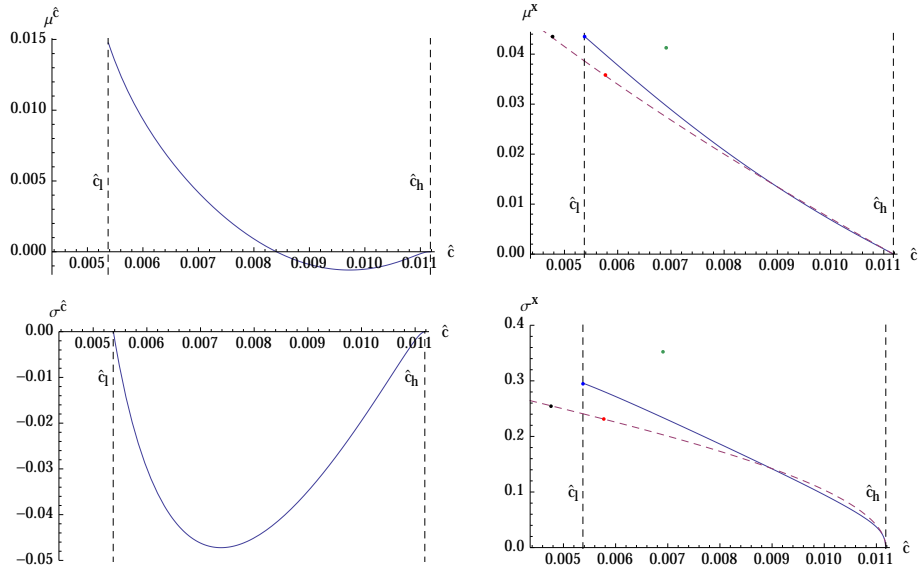


Figure 2: The drift, $\mu^{\hat{c}}$ and μ^x , and volatility, $\sigma^{\hat{c}}$ and σ^x , of the state variables \hat{c} and x . The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. *Parameters:* $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi\beta = 0.2$.

precautionary motive. As a result the cost function $\hat{v}(\hat{c})$ is increasing over $[\hat{c}_l, \hat{c}_h]$, and the optimal contract starts at $\hat{c}_0 = \hat{c}_l$. Notice also that it is conceivable theoretically for the function $\hat{v}(\hat{c})$ to have flat portions also above the cost-minimizing point \hat{c}_l . Theorem 1 rules out this possibility analytically.

The optimal contract starts at the optimal \hat{c}_l but becomes less risky over time - \hat{c} drifts up up to a steady state, as show in Figure 2. Why does \hat{c}_t not stay at the cost-minimizing level of \hat{c}_l forever? Because the principal can give more capital to the agent without increasing his precautionary motive if he promises a less risky contract in the future (a higher \hat{c} in the future). To see this, consider the first-order condition¹⁰ for σ^x

$$\alpha = \underbrace{\gamma (\hat{v}\hat{c}^{-\gamma} \phi \beta \sigma^x)}_{\text{risk sharing}} + \underbrace{\phi \beta \hat{v}' \hat{c}^{1-\gamma} \left((1 + \gamma) \sigma^{\hat{c}} + \sigma^x \right)}_{\text{intertemporal tradeoff}} \quad (15)$$

The left hand side and the first term on the right capture the tradeoff between excess return and risk sharing. The second term captures the intertemporal tradeoff. The principal can expose the agent to more risk today without a larger precautionary motive if he promises a less risky contract in the future, so that the drift of \hat{c} is increasing in σ^x . Since less risky contracts require an inefficiently low level of capital - $\hat{v}(\hat{c})$ is strictly increasing in \hat{c} as described above - this is a potentially costly tradeoff for the principal. At the optimal point \hat{c}_l , however, the principal is indifferent about small changes in \hat{c} because $\hat{v}'(\hat{c}_l) = 0$, so the intertemporal trade-off is absent. The principal gives the agent a high level of risk exposure σ^x to maximize (14), and doesn't care that he is picking a positive drift of \hat{c} . As a result, the drift of \hat{c} is initially positive. As \hat{c}_t moves away from \hat{c}_l , the cost of the contract goes up. The principal would benefit from reducing \hat{c}_t - giving the agent a more risky contract - but he cannot do so because he must keep his promise to the agent. The principal then chooses σ^x taking into account this intertemporal tradeoff.

Notice that above \hat{c}_l , when $\sigma^x > 0$ and $\sigma^{\hat{c}} = 0$, the derivative of the right-hand side of the HJB equation with respect to $\sigma^{\hat{c}}$ is positive. Hence, the principal can reduce the costs of distortions by setting $\sigma^{\hat{c}} < 0$, as shown in Figure 2. After bad outcomes the contract

¹⁰For this first-order condition characterizes a minimum (rather than a maximum) because $\hat{v}\gamma + \hat{v}'\hat{c} > 0$, since $\hat{v} > 0$ and $\hat{v}' \geq 0$.

becomes less risky. This has two benefits, as shown by the FOC

$$\underbrace{\hat{v}' \left(\gamma \sigma^x + (1 + \gamma) \sigma^{\hat{c}} \right)}_{\text{relax precautionary motive}} + \underbrace{\left(\hat{v}' \sigma^x + \hat{v}'' \hat{c} \sigma^{\hat{c}} \right)}_{\text{hedging}} = 0 \quad (16)$$

The first term captures that by setting $\sigma^{\hat{c}} < 0$ the principal can reduce the agent's precautionary motive. The principal promises the agent less risk after bad outcomes, reflected in higher \hat{c} after bad outcomes, and as a result the agent worries less about risk exposure in the worst states. What is going on is that the agent's consumption is somewhat insured, and instead he is punished with a reduction in the growth rate of his consumption. As a result, the agent's precautionary motive is weaker and the drift of \hat{c} lower. The second term in the FOC captures a hedging motive for the principal: he also prefers to use the relatively costly contracts with high \hat{c} when he must deliver less utility x , because then he can use the relatively less costly contracts with low \hat{c} when he must deliver more utility x .

Candidate optimal contract and verification theorem. We know that the principal's cost function satisfies the HJB equation, but how do we know that the equation has no other solutions? The following theorem shows that if an appropriate solution has been found, e.g. numerically, then it must be the true cost function.

Theorem 2 (Verification Theorem). *Let $\hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \rightarrow [\hat{v}_l, \hat{v}_h]$ be a strictly increasing C^2 solution to the HJB equation (12) for some $\hat{c}_l \in (0, \hat{c}_h)$, such that $\hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h]$, $\hat{v}'(\hat{c}_l) = 0$, $\hat{v}''(\hat{c}_l) > 0$ and $\hat{v}(\hat{c}_h) = \hat{v}_h$. If $\gamma < \frac{1}{2}$, we also need to check that*

$$1 - \hat{v}_l (\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \alpha^2 (\phi\beta)^{-2} \hat{v}_l^{-2}) \leq 0 \quad (17)$$

Then,

1) For any incentive compatible contract $\mathcal{C} = (c, k)$ that delivers at least utility u_0 to the agent, we have $\hat{v}(\hat{c}_l) ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \leq J_0(\mathcal{C})$.

2) Let \mathcal{C}^* be a contract generated by the policy functions of the HJB. Specifically, the state variables x^* and \hat{c}^* are solutions to (8) and (9) (with potential absorption at \hat{c}_h), with initial values $x_0^* = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$ and $\hat{c}_0^* = \hat{c}_l$. If \mathcal{C}^* is admissible, and $\sigma^{\hat{c}^*}$ is bounded, then \mathcal{C}^* is an optimal contract, with cost $J_0(\mathcal{C}^*) = \hat{v}(\hat{c}_l) ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$.

The HJB equation can be solved as an ODE by plugging in the FOCs. We only need to verify condition (17) in case $\gamma < \frac{1}{2}$, and that the contract generated by the HJB \mathcal{C}^* is admissible. The following result is useful here.

Lemma 4. *If the candidate contract \mathcal{C}^* constructed in Theorem 2 has $\mu^{x^*} < r$, then \mathcal{C}^* is admissible and delivers utility u_0 to the agent.*

Verifying incentive compatibility. An important step in the proof of verification Theorem 2 is making sure that the candidate optimal contract \mathcal{C}^* is incentive compatible. While (6) and (7) will ensure that neither stealing and immediately consuming, nor secretly saving without stealing are attractive on their own, they leave open the possibility that a double deviation (stealing and saving the proceeds for later) could be attractive to the agent. To see how this can happen, notice that while stealing reduces the continuation utility of the agent under good behavior, it might increase the expected marginal utility of consumption $\mathbb{E}_t^a \left[e^{(r-\rho)u} c_{t+u}^{-\gamma} \right]$ to such an extent that saving the stolen funds for consumption later could be very attractive.¹¹

We show that the candidate optimal contract is indeed globally incentive compatible by deriving an upper bound for the agent's continuation utility after any deviation.

Lemma 5. *Let $\mathcal{C} = (c, k)$ be an admissible contract with associated processes x and \hat{c} satisfying (8) and (9) and (10), with bounded μ^x , $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$, and with \hat{c} uniformly bounded away from zero and bounded above by \hat{c}_h . Suppose that the contract satisfies the following property*

$$\sigma_t^{\hat{c}} \leq 0 \tag{18}$$

Then for any feasible strategy (\tilde{c}, a) , with associated hidden savings h , we have the following upper bound on the utility

$$U_t^{\tilde{c}, a} \leq \left(1 + \frac{h_t}{x_t} \hat{c}_t^{-\gamma} \right)^{1-\gamma} U_t^{c, 0} \tag{19}$$

In particular, since $h_0 = 0$, for any feasible strategy $U_0^{\tilde{c}, a} \leq U_0^{c, 0}$, and the contract \mathcal{C} is therefore incentive compatible.

Notice how hidden savings can allow the agent to achieve more utility than he should have had he always behaved (however remember this is just an upper bound on achievable utility). In particular, if the agent has somehow accumulated hidden savings in the past,

¹¹In other words, even if $e^{(r-\rho)t} c_t^{-\gamma}$ is a martingale under P , it might be a submartingale under P^a .

he will want to deviate from good behavior $(c, 0)$ in the future, at the very least to increase his consumption. However, if he does not have any hidden savings $h_t = 0$ (regardless of whether he stole in the past), the most utility he could get is $U_t^{c,0}$.

The dynamic behavior described above plays an important role in ensuring incentive compatibility of the optimal contract. Recall that after bad outcomes the agent receives a less risky continuation contract. To see why this is important, note that from the point of view of the agent's hidden savings problem, the legitimate consumption he receives is his income, and since he cannot insure against this risk, the agent faces an incomplete markets problem. If after some history his future consumption is very volatile, the agent will place a very large marginal value on hidden savings that can help him self-insure. This will be captured by a low \hat{c}_t : to the extent that the agent faces risk, he will have a precautionary motive to postpone consumption which will result in a back loaded consumption path. As a result of this, if after bad outcomes the contract becomes more risky for the agent, then stealing could be attractive since the agent would expect to have hidden dollars precisely when they are most valuable. An important condition to prevent this from happening is therefore that after bad outcomes the contract exposes the agent to less volatility and he is consequently willing to consume more up front, i.e. $\sigma^{\hat{c}} \leq 0$. As a result, if the agent steals he expects to have hidden dollars when they are less valuable to him.

As shown above, the optimal contract has the property that $\sigma^{\hat{c}} \leq 0$ because the principal wants to reduce the agent's precautionary motive and the most efficient way to do this is by reducing the agent's risk exposure after bad outcomes. As it happens it is the same property that is sufficient for global incentive compatibility.

We would like to point out that Lemma 5 identifies $\sigma^{\hat{c}} \leq 0$ as a general sufficient condition for incentive compatibility, without even assuming that the contract is recursive in variables x and \hat{c} . These variables are well-defined for an arbitrary contract, and their laws of motion do not need to be Markov for Lemma 5 to apply. Condition (18) can be used to verify full incentive compatibility of other contracts, for example, it implies that *stationary* contracts that we discuss below are also all globally incentive compatible.

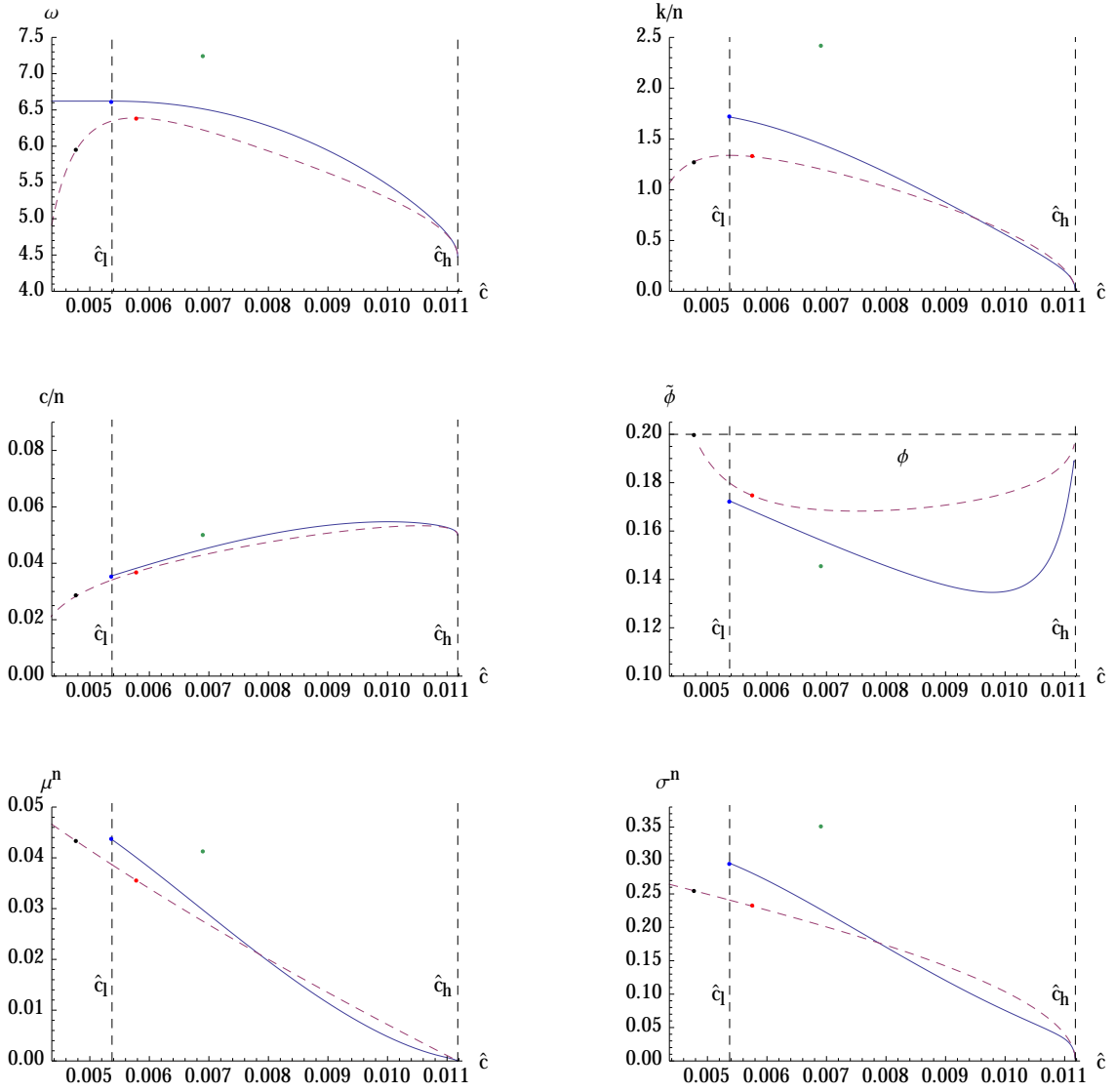


Figure 3: The implementation of the optimal contract. The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. *Parameters:* $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi\beta = 0.2$.

Implementation

The optimal contract can be implemented as a constrained portfolio problem. The agent starts with legitimate net worth n_0 , which then follows the dynamic budget constraint

$$\frac{dn_t}{n_t} = \underbrace{\left(r + \frac{k_t}{n_t}\alpha - \frac{c_t}{n_t}\right)}_{\mu_t^n} dt + \underbrace{\tilde{\phi}_t \frac{k_t}{n_t} \beta}_{\sigma_t^n} dZ_t \quad (20)$$

where

$$\tilde{\phi}_t \equiv (\hat{v}_t \hat{c}_t^{-\gamma} \phi) + \hat{v}'_t \hat{c}_t \hat{k}_t^{-1} \beta^{-1} \sigma_t^{\hat{c}} \quad (21)$$

The contract controls the portfolio weight on capital k_t/n_t , the agent's consumption c_t/n_t , and the fraction of risk the agent must keep on his balance sheet $\tilde{\phi}_t$ as functions of the history of returns, summarized by the state variable \hat{c}_t . The value function of the agent under the prescribed portfolio plan can be written

$$U_t^0 = \frac{(\omega_t n_t)^{1-\gamma}}{1-\gamma}$$

The process $\omega_t = \hat{v}^{-1}(\hat{c}_t)$ also depends on the history of returns and indicates how efficiently the agent can use his net worth to get utility under the continuation portfolio plan prescribed by the contract. It is highest for \hat{c}_l , when $\omega(\hat{c}_l) = \hat{v}_l^{-1}$, and decreases with \hat{c} . Indeed, with high \hat{c} the principal must distort the optimal portfolio plan in order to deliver a higher degree of safety to the agent. In the extreme case, with \hat{c}_h the contract must put portfolio weight on risky capital $k/n = 0$, and the agent can only use his net worth to consume. As a result, the utility he can get out of his net worth $\omega = \omega_h = \hat{v}_h^{-1}$ is lowest at this point.

Figure 3 shows the implementation of the optimal contract. It starts at $\hat{c}_0 = \hat{c}_l$, with a large portfolio weight on capital k/n and low consumption c/n , and a high growth rate μ^n and volatility σ^n of net worth. This is the best continuation portfolio plan, with high ω . Because $\mu^{\hat{c}}(\hat{c}_l) > 0$, the portfolio plan quickly becomes suboptimal, and ω falls. By promising more safety in the future - higher \hat{c} - the principal can give the agent a higher initial portfolio weight on capital without increasing his precautionary motive. As a result, the portfolio plan becomes less risky over time, with lower weight on capital k/n , and lower growth μ^n and volatility σ^n of net worth.

In addition, because $\sigma^{\hat{c}} < 0$, after bad returns the agent is punished not only with less net worth according to (20), but also with a worse continuation portfolio reflected in a

lower ω . With higher promised safety \hat{c} the continuation portfolio plan has a lower weight on capital k/n , and lower growth μ^n and volatility σ^n of net worth. As explained above, this dynamic behavior reduces the agent's precautionary motive, and allows the principal to hedge and use the better portfolio plans after good outcomes. Notice how this reduces the fraction of risk $\tilde{\phi} \leq \hat{v}_t \hat{c}_t^{-\gamma} \phi \beta$ that the agent must keep on his balance sheet.

Three benchmarks

In order to understand the behavior of the optimal contract, it is useful to compare it to other suboptimal alternatives.

Stationary contracts. Stationary contracts set $\mu^{\hat{c}} = \sigma^{\hat{c}} = 0$, so that \hat{c} , \hat{k} , σ^x and μ^x are all constant as well. In order to get $\mu^{\hat{c}} = 0$ we must set σ^x so that

$$\frac{1}{2}(\sigma^x)^2 = \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \quad (22)$$

where, using $\hat{c}_h^{1-\gamma} = \frac{\rho - r(1-\gamma)}{\gamma} > 0$, we see the right hand side is non-negative for $\hat{c} \leq \hat{c}_h$. We can then build a stationary contract for a given \hat{c} , using (8) with $x_0 = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$ for any initial utility level u_0 for the agent. Lemma 5 is general enough to ensure global incentive compatibility. The HJB equation (12) yields the cost of the stationary contract.

Lemma 6. *Assume*

$$\alpha < \bar{\alpha} \equiv \frac{\phi \beta \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}} \quad (23)$$

Take any $\hat{c} \in (\hat{c}_*, \hat{c}_h]$, where

$$\hat{c}_* = \left(\frac{2\gamma}{1 + \gamma} \frac{\rho - r(1 - \gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} \in (0, \hat{c}_h) \quad (24)$$

There is an incentive compatible stationary contract for this \hat{c} , where $\sigma_r^x(\hat{c})$ is given by (22) and the cost $\hat{v}_r(\hat{c})x_0$ is given by

$$\hat{v}_r(\hat{c}) = \frac{\hat{c} - \frac{\alpha}{\phi \beta} \hat{c}^\gamma \sqrt{2} \sqrt{\frac{\rho - r(1 - \gamma)}{(1 - \gamma)\gamma} \left(1 - \left(\frac{\hat{c}}{\hat{c}_h} \right)^{1 - \gamma} \right)}}{2r - \rho - \frac{1 + \gamma}{1 - \gamma} \rho + \frac{\hat{c}^{1 - \gamma}}{1 - \gamma} (1 + \gamma)} \quad (25)$$

For $\hat{c} \leq \hat{c}_*$ the growth rate $\mu_r^x(\hat{c}) > r$ and the corresponding stationary contract violates the No-Ponzi condition (3) and is therefore not admissible. Since stationary contracts are not necessarily optimal, we have $\hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c})$.

Remark. Notice that $\hat{v}_r(\hat{c}_h) = \hat{v}_h = \hat{c}_h^\gamma$, which makes sense because \hat{v}_h corresponds to a stationary contract with $\sigma^x = 0$. If $\alpha \geq \bar{\alpha}$ the principal can deliver utility to the agent and attains unboundedly large profits using stationary contracts, so the optimal contract doesn't exist.

Since stationary contracts give the agent a constant exposure to risk σ^x we can see the tradeoff between risk exposure and the precautionary motive clearly. Equation (22) says that the only way to give the agent a permanently high risk exposure is to accept that he will postpone his consumption for precautionary motives - i.e. low \hat{c} . Figures 1 and 2 show the cost $\hat{v}_r(\hat{c})$ and the law of motion of x for stationary contracts. Giving capital to the agent is attractive at first because capital pays excess return α , so $\hat{v}_r(\hat{c})$ falls and σ_r^x goes up as we move back from \hat{c}_h . However, at some point the distortions in risk sharing and intertemporal consumption become too large so the cost of stationary contracts goes up below \hat{c}_r^{min} . The stationary contract with minimum cost is indicated by a red point.

What the principal can do to further reduce the cost is to move away from stationary contracts. By promising the agent less risky continuation contracts in the future and after bad returns - i.e. $\mu(\hat{c}_l) > 0$ and $\sigma_t < 0$ - the principal can relax the agent's precautionary motive and hedge the cost of delivering utility to the agent. Figure 3 shows the optimal contract initially gives the agent more capital k/n , and higher growth μ^n and volatility σ^n of net worth.

Optimal portfolio with $\tilde{\phi}_t = \phi$. A stationary contract of particular interest arises if we set $\tilde{\phi}_t = \phi$, and we solve the optimal portfolio problem. This is a well known portfolio problem, with consumption and capital portfolio weight given by

$$\left(\frac{c}{n}\right)^{op} = \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{\gamma} \left(\frac{1}{2\gamma} \frac{\alpha^2}{(\phi\beta)^2} \right) \quad (26)$$

$$\left(\frac{k}{n}\right)^{op} = \frac{1}{\gamma} \frac{1}{\phi\beta} \underbrace{\frac{\alpha}{\phi\beta}}_{\text{Sharpe ratio}} \quad (27)$$

We can build a contract using (20) with (26), (27), and $\tilde{\phi} = \phi$, with initial n_0 taken as given. The agent then gets utility

$$u_0^{op} = \frac{(\omega^{op} n_0)^{1-\gamma}}{1-\gamma}$$

where $\omega^{op} \equiv \left(\frac{\rho-r(1-\gamma)}{\gamma} - \frac{1-\gamma}{\gamma} \left(\frac{1}{2\gamma} \frac{\alpha^2}{(\phi\beta)^2} \right) \right)^{\frac{\gamma}{\gamma-1}}$.

Is this contract incentive compatible? First, consumption under the optimal portfolio problem will naturally satisfy the agent's Euler equation, so the agent won't find it optimal to postpone consumption. The "skin in the game" constraint ensures that every dollar that the agent steals, he loses the same amount in legitimate net worth. If the agent immediately consumes the stolen dollar, he gets the marginal utility of consumption, $c_t^{-\gamma}$. On the other hand, a dollar in legitimate net worth has marginal value $(\omega^{op})^{1-\gamma} n_t^{-\gamma}$. In the optimal portfolio problem the marginal value of net worth must equal the marginal utility of consumption, because the agent could always choose to adjust his consumption, i.e.

$$c_t^{-\gamma} = \left(\left(\frac{c}{n} \right)^{op} \right)^{-\gamma} n^{-\gamma} = (\omega^{op})^{1-\gamma} n^{-\gamma}$$

As a result, exposing the agent to exactly a fraction ϕ of his return ensures that he doesn't find stealing and immediately consuming attractive. In other words, the resulting contract satisfies the necessary conditions (8), (9), and (10). Lemma 5 then ensures it is globally incentive compatible.

Lemma 7. *Assume $\alpha < \bar{\alpha}$ defined by (23). The contract generated by the optimal portfolio plan with $\tilde{\phi} = \phi$ is incentive compatible and delivers utility $u_0^{op} = \frac{(\omega^{op} n_0)^{1-\gamma}}{1-\gamma}$ to the agent.*

The solution is indicated with a black dot in Figures 1, 2, and 3. The best stationary contract improves upon this portfolio plan by front loading the agent's consumption in order to relax the risk sharing problem. Indeed, the "skin in the game" constraint under the best stationary contract is

$$\tilde{\phi}_r \equiv \hat{v}_r (\hat{c}^{min})^{-\gamma} \phi = \omega_r^{\gamma-1} \left(\left(\frac{c}{n} \right)_r \right)^{-\gamma} \phi < \phi$$

Instead of consuming up to the point where the marginal utility of consumption equals the marginal value of net worth, the agent receives higher consumption, which reduces $\tilde{\phi}_r < \phi$. The agent is happy to increase his consumption up front because he faces less risk, so his

precautionary motive is weaker. By internalizing the effect of consumption on his ability to offload risk onto the market, the best stationary contract improves on the optimal portfolio problem. Figure 3 shows the best stationary contract can give the agent more capital k/n and at the same time a smaller volatility of net worth σ^n . If the principal could force the agent to consume even more, he could do even better. This is the case when the agent doesn't have access to hidden savings, to which we now turn.

No hidden savings. To understand the role of hidden savings, we can take the optimal contract without hidden savings as a benchmark.¹² The contract takes a simple stationary form, with a constant growth rate μ^x , volatility σ^x , and intertemporal consumption profile \hat{c} . The solution is indicated with a green dot in Figures 1, 2, and 3. Without hidden savings, the principal can force the agent to consume in a front loaded way. This reduces his marginal utility from consumption, and therefore makes stealing and immediately consuming less attractive. Consequently, by front loading the agent's consumption the principal can relax the risk sharing problem. As a result, the principal can deliver utility at a lower cost compared to the hidden savings case. And because it is not necessary to take into account the agent's precautionary motive, there is no need for the dynamic behavior. The principal always uses the best contract.

We can characterize the optimal contract without hidden savings using similar tools, ignoring the agent's Euler equation (7). Because of homothetic preferences, the cost function takes the form $v_n(x) = \hat{v}_n x$, and satisfies an HJB equation

$$r\hat{v}_n = \min_{\hat{c}, \sigma^x} \hat{c} - \frac{\alpha \hat{c}^\gamma}{\phi \beta} \sigma^x + \hat{v}_n \left(\frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right)$$

where notice that \hat{c} is now a choice variable, instead of a state. The FOC for σ^x is

$$\alpha = \hat{v}_n \gamma \hat{c}^{-\gamma} \phi \beta \sigma^x$$

This is similar to equation (15), except that it is missing the last term that comes from the presence of hidden savings. The FOC for \hat{c} is

$$1 = \hat{v}_n \hat{c}^{-\gamma} + \hat{v}_n \gamma^2 (\sigma^x)^2 \hat{c}^{-1}$$

¹²See Di Tella (2014) for details and extensions to the cases of aggregate shocks to investment opportunities and Epstein-Zin preferences.

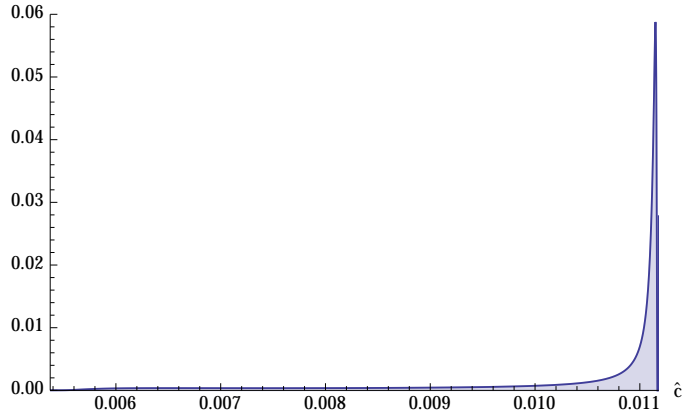


Figure 4: The stationary distribution of \hat{c} . *Parameters:* $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi\beta = 0.2$.

On the right hand side we have the cost of giving consumption to the agent. On the left hand side we have the benefit. First, if the agent receives more consumption today, the principal must deliver less utility in the future, which reduces the cost. This is the standard tradeoff we would expect to see. But in addition, the principal knows that by front loading the agent he can relax the risk sharing problem. The second term captures this benefit. As a result, the principal has an extra incentive to front load the agent’s consumption. The optimal contract without hidden savings is able to give the agent more capital and better risk sharing, and therefore deliver utility at a lower cost.

The difference between the optimal contract with and without hidden savings can be very large. Without hidden savings if the elasticity of intertemporal substitution of the agent is low, the principal can achieve an arbitrage (the \hat{k} is “infinity” and there is no optimal contract). With EIS less than 2, by front loading the agent’s consumption, the principal can relax the risk sharing problem to the point that in the limit he can get the excess return of capital without exposing the agent to any risk. What is going on is that because the agent has a low EIS, the principal has a lot of power over him and can eliminate the moral hazard problem in the limit. The power of the principal is limited if the elasticity of intertemporal substitution is higher (here we considered EIS = 3 to ensure the optimal contract exists both with and without hidden savings), or if the agent has access to hidden savings, which limit the principal’s ability to manipulate the agent’s consumption.

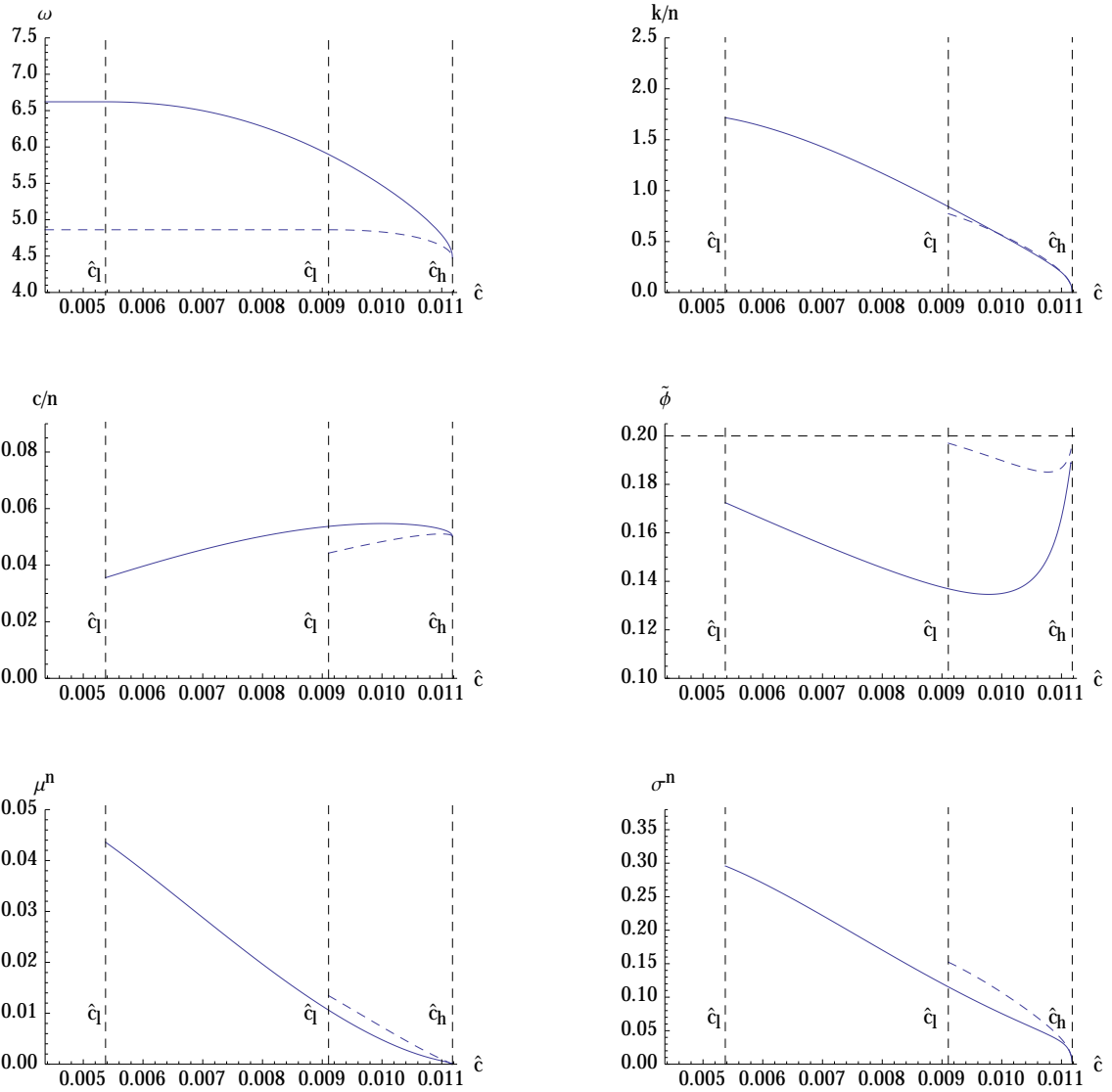


Figure 5: The portfolio implementation for $\alpha = 1.7\%$ (solid) and $\alpha = 1\%$ (dashed). *Other parameters:* $\rho = r = 5\%$, $\gamma = 1/3$, $\phi\beta = 0.2$.

Long-run behavior

The contract has a scale invariance property for x . The drift of \hat{c} , on the other hand, takes the contract to a “steady state” $\hat{c}_{ss} \in (\hat{c}_l, \hat{c}_h)$. However, the volatility $\sigma^{\hat{c}}$ is relatively high in that area, so the contract actually spends very little time near the “steady state”. Figure 4 shows the stationary distribution of \hat{c} . The contract spends most close to \hat{c}_h , with consumption that is close to intertemporally optimal but very little capital. While the drift $\mu^{\hat{c}} < 0$ pushes \hat{c} down towards the “steady state”, it is very weak and so is the volatility $\sigma^{\hat{c}}$. As a result, the contract can become “trapped” in this high cost region for very long times. When it finally approaches the “steady state”, the volatility is higher, and the contract quickly moves away and falls back into the high cost region near \hat{c}_h . At the other extreme, when \hat{c} is close to the lower bound \hat{c}_l , the drift $\mu^{\hat{c}}$ is large and pushes the system back towards the steady state. As a result, while the contract starts at \hat{c}_l in the “high growth/high volatility” region, it quickly moves away and spends most of the time in the “low growth/low volatility” region.

Comparative Statics

Figure 5 shows the portfolio implementation of the optimal contract for $\alpha = 1.7\%$ and $\alpha = 1\%$. When the excess return of capital is lower, the agent can get less utility for his net worth. Since capital is less attractive the initial k/n is lower, and the optimal contract is less willing to expose the agent to risk to give him capital, so \hat{c}_l is higher - the contract is safer. This is reflected in a lower growth rate μ^n and volatility σ^n of net worth. The difference is significant: with $\alpha = 1.7\%$ the portfolio weight on capital k/n is around 1.7 so the agent is leveraged, while with $\alpha = 1\%$ it's less than 1. Perhaps surprisingly, in the case with lower α the agent must keep a larger fraction of his return, $\tilde{\phi}$. The optimal contract is less willing to distort the intertemporal consumption decision in order to relax the risk sharing problem. In the limit, if $\alpha = 0$, the optimal contract would coincide with the deterministic consumption path without any capital corresponding to \hat{c}_h .

Aggregate Risk

We can incorporate aggregate risk into the environment and extend all our results in a natural way with appropriate modifications. Appendix B has the formal results. Here we'll go over the main economic implications. Let \tilde{Z} be an independent Brownian motion that

represents aggregate risk, with price π in the market. Let the return on capital be

$$dR_t = (r + \pi\tilde{\beta} + \alpha - a_t) dt + \beta dZ_t + \tilde{\beta} d\tilde{Z}_t$$

The agent can now invest his hidden savings in the market, so they follow¹³

$$dh_t = (rh_t + \pi\tilde{\sigma}_t^h h_t + c_t - \tilde{c}_t + \phi k_t a_t) dt + \tilde{\sigma}_t^h h_t d\tilde{Z}_t$$

where $\tilde{\sigma}_t^h$ is the fraction of his hidden savings invested in the risky market, for which he receives a premium π . Since aggregate shocks are observable, the contract (c, k) specifies consumption and capital as functions of the history of not only realized returns R , but also aggregate shocks \tilde{Z} . After signing the contract the agent can choose a strategy $(\tilde{c}, a, \tilde{\sigma}_t^h)$ to maximize his utility, which can also depend on the history of both R and \tilde{Z} . The agent's utility and the principal's objective function are still given by (1) and (2).¹⁴ Since there is now aggregate risk that pays a premium, we need to slightly modify the parameter restrictions

$$\hat{c}_h \equiv \left(\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2}(1 - \gamma) \left(\frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} > 0 \quad (28)$$

and

$$\alpha < \bar{\alpha} \equiv \frac{\phi\beta\gamma\sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2}(1 - \gamma) \left(\frac{\pi}{\gamma} \right)^2} \quad (29)$$

We can check that with $\pi = 0$ we recover the formulas without aggregate risk.

Since the contract can depend on the history of aggregate shocks \tilde{Z} , so can his continuation utility $U^{c,0}$ and his consumption c . However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings in aggregate risk, his Euler equation needs to be modified appropriately. His discounted marginal utility $\exp\left(\int_0^t r - \rho + \pi_s \tilde{\sigma}_s^h - \frac{1}{2}(\tilde{\sigma}_s^h)^2 ds + \int_0^t \tilde{\sigma}_s^h d\tilde{Z}_s\right) c_t^{-\gamma}$ must be a supermartingale *for any* $\tilde{\sigma}^h$, since otherwise he could save a dollar, investing it in aggregate risk, and consume it later when the marginal utility is expected to be larger. Because aggregate risk pays a premium π , the agent would like to invest in it and expose his consumption to aggregate risk with loading π/γ . An incentive compatible contract must allow this.

¹³As before, the agent cannot secretly invest in risky capital.

¹⁴The definitions of feasible strategy, and admissible, incentive compatible, and optimal contracts are unchanged.

Lemma 8. *If $C = (c, k)$ is an incentive compatible contract, the agent's continuation utility $U^{c,0}$ and consumption c satisfy the laws of motion*

$$dU_t^{c,0} = \left(\rho U_t^{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma} \right) dt + \Delta_t \beta dZ_t + \tilde{\sigma}_t^u d\tilde{Z}_t \quad (30)$$

$$\frac{dc_t}{c_t} = \left(\frac{r-\rho}{\gamma} + \frac{1+\gamma}{2} (\sigma_t^c)^2 + \frac{1+\gamma}{2} (\tilde{\sigma}_t^c)^2 \right) dt + \sigma_t^c dZ_t + \tilde{\sigma}_t^c d\tilde{Z}_t + dL_t \quad (31)$$

for some Δ , $\tilde{\sigma}^u$, σ^c , $\tilde{\sigma}^c$, and a weakly increasing processes L , such that

$$\Delta_t \geq c_t^{-\gamma} \phi k_t \quad (32)$$

$$\tilde{\sigma}_t^c = \frac{\pi_t}{\gamma} \quad (33)$$

In light of this, we must modify the laws of motion of the state variables x and \hat{c} . Using Ito's lemma:

$$\frac{dx_t}{x_t} = \left(\frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma_t^x)^2 + \frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 \right) dt + \sigma_t^x dZ_t + \tilde{\sigma}_t^x d\tilde{Z}_t \quad (34)$$

$$\begin{aligned} \frac{d\hat{c}_t}{\hat{c}_t} = & \left(\frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{(\sigma_t^x)^2}{2} + \gamma \sigma_t^x \sigma_t^{\hat{c}} + \frac{1+\gamma}{2} (\sigma_t^{\hat{c}})^2 \right. \\ & \left. + \frac{(\tilde{\sigma}_t^x)^2}{2} + \gamma \tilde{\sigma}_t^x \tilde{\sigma}_t^{\hat{c}} + \frac{1+\gamma}{2} (\tilde{\sigma}_t^{\hat{c}})^2 \right) dt + \sigma_t^{\hat{c}} dZ_t + \tilde{\sigma}_t^{\hat{c}} d\tilde{Z}_t + dL_t \end{aligned} \quad (35)$$

and the incentive compatibility constraints can be written $\sigma_t^x = \hat{c}_t^{-\gamma} \phi \hat{k}_t \beta$ and $\tilde{\sigma}_t^{\hat{c}} = \frac{\pi}{\gamma} - \tilde{\sigma}_t^x$. As before, \hat{c} has an upper bound \hat{c}_h , given by (28). Notice that there is no conflict between the principal and the agent regarding aggregate risk sharing. In the first-best without moral hazard¹⁵ the agent would get an exposure to aggregate risk $\tilde{\sigma}_{fb}^x = \pi/\gamma$, which together with (33) implies $\tilde{\sigma}^{\hat{c}} = 0$. Indeed, this is also the case in the optimal contract.

Lemma 9. *The optimal contract has first-best aggregate risk sharing $\tilde{\sigma}^x = \pi/\gamma$ and $\tilde{\sigma}^{\hat{c}} = 0$.*

The agent's ability to invest his hidden savings in aggregate risk makes the verification of global incentive compatibility potentially more difficult. The agent could find it attractive to steal and save the proceeds for later, while investing them in aggregate risk. However,

¹⁵Without moral hazard, $\phi = 0$, $\bar{\alpha} = 0$, since otherwise there is an arbitrage.

using the result of Lemma 9, we can extend the results of verification Lemma 5 to deal with aggregate risk.

Lemma 10. *Let $\mathcal{C} = (c, k)$ be an admissible contract with associated processes x and \hat{c} satisfying (34) and (35), and (32) and (33), with bounded μ^x , $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$, and with \hat{c} uniformly bounded away from zero and bounded above by \hat{c}_h . Suppose that the contract satisfies the following properties*

$$\begin{aligned}\sigma_t^{\hat{c}} &\leq 0 \\ \tilde{\sigma}_t^x &= \frac{\pi}{\gamma}\end{aligned}$$

Then for any feasible strategy $(\tilde{c}, a, \tilde{\sigma}^h)$, with associated hidden savings h , we have the following upper bound on the utility

$$U_t^{\tilde{c}, a} \leq \left(1 + \frac{h_t}{x_t} \hat{c}_t^{-\gamma}\right)^{1-\gamma} U_t^{c, 0}$$

In particular, since $h_0 = 0$, for any feasible strategy $U_0^{\tilde{c}, a} \leq U_0^{c, 0}$, and the contract \mathcal{C} is therefore incentive compatible.

The optimal contract can then be characterized with an HJB equation as in the case without aggregate risk, with small modifications to account for the risk premium π . Appendix B has all the formal results and proofs.

3 Conclusions

This paper develops a tractable contractual framework for delegated asset management that is well suited to macro and asset pricing applications. In particular, hidden savings are an important realistic feature, but can be technically challenging to deal with. We show that the introduction of hidden savings creates rich dynamics in an otherwise stationary environment, and that these in turn ensure incentive compatibility of the optimal contract. The optimal contract has a scale invariance property and can be characterized by an ODE, which makes it easy to embed in general equilibrium models.

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Appendix A - Omitted Proofs

Lemma 1

Consider

$$Y_t = \mathbb{E}_t \left[\int_0^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] = \int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} U_t^{c,0}$$

Since Y is an \mathbb{F} -adapted P -martingale, and \mathbb{F} is generated by Z , we can apply a martingale representation theorem to obtain

$$dY_t = e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho t} dU_t^{c,0} - \rho e^{-\rho t} U_t^{c,0} dt = e^{-\rho t} \Delta_t dZ_t$$

for some stochastic process Δ also adapted to \mathbb{F} . Dividing by $e^{-\rho t}$ and rearranging we get (4).

Lemma 2

First, take the strategy $(c + \phi ka, a)$. Here a is zero where (5) is satisfied and $a = a_t^* \wedge \bar{a}$ where a_t^* achieves the maximum in (5) and $\bar{a} > 0$ is an arbitrary bound. Because the objective in (5) is concave and it's $\frac{c_t^{1-\gamma}}{1-\gamma}$ for $a = 0$, then for $a_t^* \wedge \bar{a}$ it is strictly greater when (5) fails. Notice $h_t = 0$ by construction, and we can follow strategy (\bar{c}^n, a^n) until some stopping time τ^n and then revert to good behavior, where the stopping time $\tau^n \rightarrow \infty$ a.s., ensures it's feasible and reduces the stochastic integral. We can compare the utility from this strategy $U^{\bar{c}^n, a^n}$ with the utility from good behavior $U^{c,0}$:

$$U^{\bar{c}^n, a^n} - U_0^{c,0} = \mathbb{E}_0^a \left[\int_0^{\tau^n} e^{-\rho t} \left(\frac{(c_t + \phi k_t s_t)^{1-\gamma}}{1-\gamma} - \frac{c_t^{1-\gamma}}{1-\gamma} - \Delta_t \frac{a_t}{\beta} \right) dt \right] \quad (36)$$

where we have used $Z_t^a = Z_t + \int_0^t \frac{a_s}{\beta} ds$ to express $U_0^{c,0}$ as an expected integral under P^a , and also the fact that $U_{\tau^n}^{\bar{c}^n, a^n} = U_{\tau^n}^{c,0}$. Taking $n \rightarrow \infty$ and using the monotone convergence theorem (the integrand is always positive), if (5) fails we get $U_0^{\bar{c}^n, a^n} > U_0^{c,0}$ for some large n , and \mathcal{C} is not incentive compatible. (6) is simply the FOC necessary condition for (5) (and sufficient because of concavity).

For (7) this is the result of $e^{-(\rho-r)t} c_t^{-\gamma}$ being a supermartingale, which is a standard necessary condition in a savings/consumption problem. Note this doesn't involve stealing a , just hidden consumption c . We can then write it $e^{-(\rho-r)t} c_t^{-\gamma} = M_t - A_t$, where $M_t =$

$\int_0^t \sigma_t^M dZ_t$ is a local martingale, and A a weakly increasing process. Using Ito's lemma, we get the desired expression.

Lemma 3

For the bound, since both $c_t \geq 0$ and $x_t \geq 0$, we only need to show that $\hat{c}_t \leq \hat{c}_h$. Marginal utility of consumption is $m_t = c_t^{-\gamma}$ and the utility flow $\frac{c_t^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} m_t^{\frac{\gamma-1}{\gamma}}$. This is a convex and decreasing function of m_t . From Lemma 2, we have by (7) that $\mathbb{E}_t[m_{t+u}] \leq e^{(\rho-r)u} m_t$. Given any m_0 we have by Jensen's inequality

$$\mathbb{E} \left[\frac{c_t^{1-\gamma}}{1-\gamma} \right] \geq \frac{1}{1-\gamma} \mathbb{E} [m_t]^{\frac{\gamma-1}{\gamma}} \geq \frac{1}{1-\gamma} m_0^{\frac{\gamma-1}{\gamma}} e^{(\rho-r)\frac{\gamma-1}{\gamma}t} = \frac{c_0^{1-\gamma}}{1-\gamma} e^{(\rho-r)\frac{\gamma-1}{\gamma}t}$$

with equality only if c is deterministic. So

$$\begin{aligned} U_t^{c,0} &= \mathbb{E}_t \left[\int_t^\infty e^{-\rho(u-t)} \frac{c_u^{1-\gamma}}{1-\gamma} du \right] \\ &\geq \int_t^\infty e^{-\rho(u-t)} \frac{c_t^{1-\gamma}}{1-\gamma} e^{(\rho-r)\frac{\gamma-1}{\gamma}(u-t)} ds = \frac{c_t^{1-\gamma}}{1-\gamma} \frac{\gamma}{\rho-r(1-\gamma)} \end{aligned}$$

where the second equality uses the fact that the termination contract must deliver \bar{U} to the agent. This bound implies

$$x_t = \left((1-\gamma)U_t^{c,0} \right)^{\frac{1}{1-\gamma}} \geq c_t \left(\frac{\gamma}{\rho-r(1-\gamma)} \right)^{\frac{1}{1-\gamma}}$$

So $\hat{c}_t = c_t/x_t$ has an upper bound

$$\hat{c}_t \leq \left(\frac{\rho-r(1-\gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} = \hat{c}_h$$

In addition, since the upper bound can only be achieved with a deterministic consumption, $U^{c,0}$ is deterministic too. This implies that both c_t and x_t grow at rate $\frac{r-\rho}{\gamma}$, so \hat{c}_h is an absorbing state. In light of (6), we must have $k_{t+u} = 0$ in the continuation contract, so we have the “retirement” contract with cost $\hat{v}_h x_t$. This completes the proof.

Theorem 1

The proof is split into parts.

1) The cost function must be bounded above by \hat{v}_h since we can always just give consumption to the agent without any capital, and obtain cost \hat{v}_h . It must be strictly positive because if $\hat{v}(\hat{c}) = 0$ for any $\hat{c} \in [0, \hat{c}_h]$, then we can scale up the contract and give infinite utility to the agent at zero cost, or else achieve infinite profits.

2) Because we can always move \hat{c} up using dL_t , we know that \hat{v} must be weakly increasing. It is useful to define

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \frac{\left(\frac{\alpha\hat{c}^\gamma}{\phi\beta}\right)}{\hat{v}\gamma} + \hat{v} \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} \quad (37)$$

which is the HJB equation when $\hat{v}(\hat{c})$ is flat. The region where the HJB equation holds cannot have any flat parts, because this would mean that $A(\hat{c}, \hat{v}(\hat{c})) = 0$. We know however from Lemma 11 (with $\pi = 0$) that this function can have at most two roots, so \hat{v} must be strictly increasing in the region where the HJB holds.

The only way the HJB could not hold is if the optimal contract never spends any time there, i.e. if we ever found ourselves there, it would be optimal to immediately jump out using dL_t . The value in that region then must be constant and equal to the value at the destination point (the upper end of the flat region). The HJB should hold as an inequality with $A(\hat{c}; \hat{v}(\hat{c})) \geq 0$, since otherwise we could improve by lingering in the flat region for a while before jumping. Since $\hat{v}(\hat{c})$ is strictly increasing when the HJB equation holds, the contract will start with \hat{c}_0 at the upper end of a flat region. It cannot be that $\hat{c}_0 = 0$ because of Inada conditions, so we must have at least one flat interval $[0, \hat{c}_l]$.

3) We would like to show that this is the only flat interval, and the HJB equation holds as an equality in the strictly increasing region $[\hat{c}_l, \hat{c}_h]$, and both regions are connected with smooth pasting, i.e. $\hat{v}'(\hat{c}_l) = 0$. Suppose then that there is a region $[\hat{c}_1, \hat{c}_2] \subset (0, \hat{c}_h)$ where the cost function is flat, and it's strictly increasing immediately above it (possibly, $\hat{c}_1 = 0$). Let's show that as $\hat{c} \searrow \hat{c}_2$, $A(\hat{c}, \hat{v}(\hat{c})) \rightarrow 0$ and $\hat{v}'(\hat{c}) \rightarrow 0$ (i.e. we have smooth pasting). Towards contradiction, imagine there is a kink at \hat{c}_2 , i.e. the right-derivative of $\hat{v}(\hat{c})$ is strictly positive. This can only happen if $\sigma^{\hat{c}}(\hat{c} + \epsilon) \rightarrow 0$ and $\mu^{\hat{c}}(\hat{c}_2 + \epsilon) \geq 0$ as $\epsilon \rightarrow 0$, since otherwise we would cross into the flat region where the HJB doesn't hold (with $\hat{v}'(\hat{c}) > 0$ we must have $dL = 0$). First consider the case with $\liminf \mu^{\hat{c}}(\hat{c}_2 + \epsilon) > 0$. We

can contemplate the following deviation: start at $\hat{c}_2 - \delta$, with the same σ^x and $\sigma^{\hat{c}} = 0$: by continuity $\mu_t^{\hat{c}} > 0$ along this plan, so after some time we will end up at \hat{c}_2 and we can go back to the optimal contract and obtain continuation cost $\hat{v}(\hat{c}_2)$. The value of this strategy for $\hat{c} < \hat{c}_2$ extends the cost function $\hat{v}(\hat{c})$ below \hat{c}_2 and satisfies the HJB equation (with $\sigma^{\hat{c}} = 0$, so it's a first order ODE with boundary condition given by $\hat{v}(\hat{c}_2)$ at \hat{c}_2). However, because $\hat{v}'(\hat{c}_2) > 0$, we obtain a lower cost at $\hat{c}_2 - \delta$. This is therefore an attractive deviation, which is a contradiction, so $\mu^{\hat{c}}(\hat{c}_2 + \epsilon) \rightarrow 0$ is the only option left. In this case, since we also have $\sigma^{\hat{c}}(\hat{c} + \epsilon) \rightarrow 0$, it means we have a stationary contract, and because we are at a local minimum of the cost function, we must also be at a local minimum of the cost of stationary contracts $\hat{v}_r(\hat{c})$ given by (25), implying $\hat{v}'_r(\hat{c}_2) = 0$. However, because $\hat{v}'_+(\hat{c}_2) > 0$, we get $\hat{v}(\hat{c} + \epsilon) > \hat{v}_r(\hat{c} + \epsilon)$, which cannot be. We conclude that $\hat{v}'(\hat{c}_2) = 0$.

We still need to show that $A(\hat{c}, \hat{v}(\hat{c})) \rightarrow 0$ as $\hat{c} \searrow \hat{c}_2$. To see this it's useful to use the FOC for σ^x conditional on $\sigma^{\hat{c}}$ to obtain

$$\sigma^x = \frac{\frac{\alpha \hat{c}^\gamma}{\phi \beta} - \hat{v}' \hat{c} (1 + \gamma) \sigma^{\hat{c}}}{\hat{v} \gamma + \hat{v}' \hat{c}} \quad (38)$$

and plug it into the HJB equation. We can then re-write the HJB

$$0 = \min_{\sigma^{\hat{c}}} \bar{A} + \bar{B} \sigma^{\hat{c}} + \frac{1}{2} \bar{C} (\sigma^{\hat{c}})^2$$

with

$$\begin{aligned} \bar{A} &= \hat{c} - r \hat{v} - \frac{1}{2} \frac{\left(\frac{\alpha \hat{c}^\gamma}{\phi \beta}\right)^2}{\hat{v} \gamma + \hat{v}' \hat{c}} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \right) - \hat{v}' \hat{c} \left(\frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \right) \\ \bar{B} &= \hat{v}' \hat{c} (1 + \gamma) \frac{\frac{\alpha \hat{c}^\gamma}{\phi \beta}}{\hat{v} \gamma + \hat{v}' \hat{c}} \geq 0 \\ \bar{C} &= \gamma \hat{v}' \hat{c} (1 + \gamma) \frac{\hat{v} - \hat{v}' \hat{c}}{\hat{v} \gamma + \hat{v}' \hat{c}} + \hat{v}'' \hat{c}^2 \end{aligned}$$

For the HJB to have a minimum, it must be that $\bar{C} \geq 0$. If $\bar{C} = 0$ we can only have a minimum if $\bar{B} = 0$ as well, in which case $\bar{A} = 0$, and with $\hat{v}'(\hat{c}_2) = 0$ we get $A(\hat{c}, \hat{v}(\hat{c})) \rightarrow 0$ as $\hat{c} \searrow \hat{c}_2$ as desired. If instead $\bar{C} > 0$, then

$$\sigma^{\hat{c}} = -\frac{\bar{B}}{\bar{C}} \leq 0 \quad (39)$$

With $\hat{v}'(\hat{c}_2) = 0$ we must have $\hat{v}''(\hat{c} + \epsilon) \geq 0$ for small ϵ (or else $\hat{v}'(\hat{c} + \epsilon) < 0$). This means $\frac{\bar{B}}{C} \leq \frac{\alpha \hat{c}^\gamma}{\phi \beta} \frac{1}{\hat{v} - \hat{v}'\hat{c}}$ at that point, which implies $\frac{\bar{B}^2}{C} \rightarrow 0$ and therefore $\bar{A} \rightarrow 0$, which with $\hat{v}'(\hat{c}_2) = 0$ implies in turn $A(\hat{c}, \hat{v}(\hat{c})) \rightarrow 0$ as $\hat{c} \searrow \hat{c}_2$, as desired. Since this is true in particular when $\hat{c}_1 = 0$ and $\hat{c}_2 = \hat{c}_l$, we have proven the smooth pasting condition.

Now since at \hat{c}_2 we have $A(\hat{c}, \hat{v}(\hat{c})) = 0$, this is a root of A . Below that we have $A(\hat{c}, \hat{v}(\hat{c})) \geq 0$. From Lemma 11 we know that this can only be the case if \hat{c}_2 is the first root of $A(\hat{c}, \hat{v}(\hat{c}_2))$. We would like to show that it cannot be the case that for $\hat{c} < \hat{c}_1 < \hat{c}_2$ the cost function is lower, i.e. $\hat{v}(\hat{c}_1 - \delta) < \hat{v}(\hat{c}_1) = \hat{v}(\hat{c}_2)$. To see this, imagine the same problem with a smaller $\alpha' < \alpha$, and cost function $\hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c})$. We can pick α' small enough that $\hat{v}_{\alpha'}(\hat{c}_l^{\alpha'}) = \hat{v}(\hat{c}_2)$, where $\hat{c}_l^{\alpha'}$ is the upper end of the first flat region for the contract with α' (and it minimizes $\hat{v}_{\alpha'}(\hat{c})$). We can do this because $\hat{v}_{\alpha'}(\hat{c})$ is continuously increasing in α' for any \hat{c} , and has $\hat{v}_{\alpha'}(\hat{c}) = \hat{v}_h$ for all $\hat{c} \leq \hat{c}_h$ if $\alpha' = 0$. It must be that $\hat{c}_l^{\alpha'} \leq \hat{c}_2$, because to the right of \hat{c}_2 $\hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}) > \hat{v}(\hat{c}_2)$ for all $\hat{c} > \hat{c}_2$. However, looking at $A(\hat{c}, \hat{v}(\hat{c}_2))$ we notice it is decreasing in α , so $A_{\alpha'}(\hat{c}_l^{\alpha'}, \hat{v}(\hat{c}_2)) > A(\hat{c}_l^{\alpha'}, \hat{v}(\hat{c}_2)) \geq 0$. But the previous argument shows that $A_{\alpha'}(\hat{c}_l^{\alpha'}, \hat{v}(\hat{c}_2)) = A_{\alpha'}(\hat{c}_l^{\alpha'}, \hat{v}_{\alpha'}(\hat{c}_l^{\alpha'})) = 0$ because $\hat{c}_l^{\alpha'}$ is the upper end of the first flat region of the optimal contract for α' . This is a contradiction, so we cannot have $\hat{v}(\hat{c}_1 - \delta) < \hat{v}(\hat{c}_1) = \hat{v}(\hat{c}_2)$.

Putting all of this together, we only have one flat region, $[0, \hat{c}_l]$, where the HJB equation holds as an inequality $A(\hat{c}, \hat{v}(\hat{c})) > 0$ (the inequality is strict because of Lemma 11 and the fact that \hat{c}_l must be the first root), and a strictly increasing region $[\hat{c}_l, \hat{c}_h]$ where the HJB equation holds with equality, and $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l)$ for all $\hat{c} \leq \hat{c}_l$ and $\hat{v}'(\hat{c}_l) = 0$.

4) Now we want to show that $\hat{v}''(\hat{c}_l) > 0$. First suppose the $A(\hat{c}, \hat{v}(\hat{c}_l))$ is strictly positive below \hat{c}_l and strictly negative above it, so that $A_{\hat{c}}(\hat{c}_l, \hat{v}_l) < 0$ (we will show that this must indeed be the case below). Consider the first order ODE that results from setting $\sigma^{\hat{c}} = 0$ in the HJB equation:

$$A(\hat{c}, \hat{v}) + \hat{v}'\hat{c} \left(\frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \right) = 0$$

and let $\hat{v}_{fo}(\hat{c})$ be the solution to this ODE with boundary condition $\hat{v}_{fo}(\hat{c}_l) = \hat{v}(\hat{c}_l)$. Since we already know that $A(\hat{c}_l, \hat{v}(\hat{c}_l)) = 0$, it must have $\hat{v}'_{fo}(\hat{c}_l) = 0$ because the term in parenthesis is $\mu^{\hat{c}}$ if $\sigma^{\hat{c}} = 0$, and Lemma 13 shows this is strictly positive if $A(\hat{c}, \hat{v}(\hat{c})) = 0$ and $\sigma^{\hat{c}} = 0$. Furthermore, we must have $\hat{v}''_{fo}(\hat{c}_l) \leq \hat{v}''(\hat{c}_l)$. To see this, if $\hat{v}''(\hat{c}_l) > \hat{v}''(\hat{c}_l) \geq 0$, then $\hat{v}'_{fo}(\hat{c}_l + \epsilon) > \hat{v}'(\hat{c}_l + \epsilon)$, while $\hat{v}_{fo}(\hat{c}_l + \epsilon) = \hat{v}(\hat{c}_l + \epsilon) + o(\epsilon)$ (because both have first

derivatives equal to zero) for some small ϵ . By continuity, it will still be the case that $\mu^{\hat{c}} > 0$ for $\hat{c}_l + \epsilon$ if we set $\sigma^{\hat{c}} = 0$, so starting at $\hat{c}_l + \epsilon - \delta$ we will eventually get to $\hat{c}_l + \epsilon$. We can then solve the first order ODE backwards with boundary condition $\hat{v}_{f_o}(\hat{c}_l + \epsilon) = \hat{v}(\hat{c}_l + \epsilon)$ and we will obtain a lower cost for $\hat{c}_l + \epsilon - \delta$, i.e. $\hat{v}_{f_o}(\hat{c}_l + \epsilon - \delta) < \hat{v}(\hat{c}_l + \epsilon - \delta)$, because by continuity $\hat{v}'_{f_o}(\hat{c}_l + \epsilon) > \hat{v}'(\hat{c}_l + \epsilon)$ for this solution as well. This cannot be, so we must have $\hat{v}''_{f_o}(\hat{c}_l) \leq \hat{v}''(\hat{c}_l)$.

Differentiating the first order ODE with respect to \hat{c} we obtain

$$0 = \partial_{\hat{c}} A(\hat{c}, \hat{v}_l)|_{\hat{c}=\hat{c}_l} + \hat{v}''(\hat{c}_l) \hat{c}_l \left(\frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho - \hat{c}_l^{1-\gamma}}{1-\gamma} \right)$$

where we have used $\hat{v}'(\hat{c}_l) = 0$. It follows that since $A_{\hat{c}}(\hat{c}_l, \hat{v}_l) < 0$, then either $\hat{v}''(\hat{c}_l) > 0$ and $\frac{r-\rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho-\hat{c}_l^{1-\gamma}}{1-\gamma} > 0$, or both are strictly negative. But we already know that $\hat{v}''(\hat{c}_l) \geq 0$, so this leaves only $\hat{v}''(\hat{c}_l) > 0$. Notice that (39) then implies $\sigma^{\hat{c}}(\hat{c}_l) = 0$, and therefore $\mu^{\hat{c}}(\hat{c}_l) = \frac{r-\rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho-\hat{c}_l^{1-\gamma}}{1-\gamma} > 0$ in the optimal contract. It only remains to rule out $A_{\hat{c}}(\hat{c}_l, \hat{v}(\hat{c}_l)) = 0$ which means $A(\hat{c}, \hat{v}(\hat{c}_l)) > 0$ for all $\hat{c} \neq \hat{c}_l$ (from Lemma 11). Since $A(\hat{c}, \hat{v}) = 0$ is the HJB equation if \hat{v} is flat, by a martingale verification argument as in Theorem 2 we obtain that the cost of any incentive compatible contract \mathcal{C} is strictly larger than $\hat{v}(\hat{c}_l)$, unless it has $\hat{c}_t = \hat{c}_l$ always so that $A(\hat{c}_t, \hat{v}_l) = 0$ always. This requires $\mu^{\hat{c}} = \sigma^{\hat{c}} = 0$. But Lemma 13 shows that for $\sigma^{\hat{c}} = 0$ and $A(\hat{c}_l, \hat{v}_l) = 0$ we get $\mu^{\hat{c}}(\hat{c}_l) > 0$. Since the optimal contract must be incentive compatible and achieve cost $\hat{v}(\hat{c}_l)$, $A_{\hat{c}}(\hat{c}_l, \hat{v}(\hat{c}_l)) = 0$ cannot be. So we have $\hat{v}''(\hat{c}_l) > 0$.

5) $\hat{v}''(\hat{c}_l) > 0$ implies $\sigma^{\hat{c}}(\hat{c}_l) = 0$ and $\mu^{\hat{c}}(\hat{c}_l) > 0$. This in turn proves that $\hat{c}_t > \hat{c}_0 = \hat{c}_l$. From (39) we get $\sigma^{\hat{c}}(\hat{c}) \leq 0$, and (38) implies $\sigma^x(\hat{c}) \geq 0$. It also implies that at \hat{c}_l , since $\hat{v}'(\hat{c}_l) = 0$, the choice of σ^x maximizes (14).

6) Finally, it remains to show that $\sigma^x(\hat{c})$ is less than that value that maximizes (14) for $\hat{c} > \hat{c}_l$.

Theorem 2

First let's extend the function $\hat{v}(\hat{c})$ as described above, with $\hat{v}(\hat{c}) = \hat{v}_l \equiv \hat{v}(\hat{c}_l)$ for all $\hat{c} < \hat{c}_l$ (we always have $\hat{c} \in [0, \hat{c}_h]$). The HJB holds as an equality for $\hat{c} \geq \hat{c}_l$, but we need to check that it holds as an inequality for $\hat{c} < \hat{c}_l$. Using the definition of $A(\hat{c}, \hat{v})$ in (37), notice $A(\hat{c}_l, \hat{v}_l) = 0$, so \hat{c}_l is a root of $A(\hat{c}; \hat{v}_l)$. If $\gamma \geq \frac{1}{2}$, Lemma 11 (with $\pi = 0$) says that it can

have at most one root, and it's positive for small \hat{c} , so $A(\hat{c}; \hat{v}_l) \geq 0$ for all $\hat{c} < \hat{c}_l$. For $\gamma < \frac{1}{2}$, it's convex and can have at most two roots. Condition (17) guarantees that the derivative is negative, so \hat{c}_l is the smaller root, and we also have $A(\hat{c}; \hat{v}_l) \geq 0$ for all $\hat{c} < \hat{c}_l$. Notice that we only need to check (17) if $\gamma < \frac{1}{2}$.

Consider any incentive compatible contract $\mathcal{C} = (c, k)$ that delivers utility u_0 to the agent, with associated state variables x and \hat{c} . Because $\hat{v}'(\hat{c}_l) = 0$ we can use Ito's lemma¹⁶ and the HJB equation to obtain

$$\begin{aligned} e^{-r\tau^n} \hat{v}(\hat{c}_{\tau^n}) x_{\tau^n} &\geq \hat{v}(\hat{c}_0) x_0 - \int_0^{\tau^n} e^{-rt} (\hat{c}_t - \hat{k}_t \alpha) x_t dt \\ &+ \int_0^{\tau^n} e^{-rt} \hat{v}(\hat{c}_t) x_t \left(\frac{\hat{v}'(\hat{c}_t)}{\hat{v}(\hat{c}_t)} \hat{c}_t \sigma_t^{\hat{c}} + \sigma_t^x \right) dZ_t \end{aligned}$$

for the localizing sequence of stopping times $\{\tau^n\}_{n \in \mathbb{N}}$:

$$\tau^n = \inf \left\{ T \geq 0 : \int_0^T \left| e^{-rt} \hat{v}(\hat{c}_t) x_t \left(\frac{\hat{v}'(\hat{c}_t)}{\hat{v}(\hat{c}_t)} \hat{c}_t \sigma_t^{\hat{c}} + \sigma_t^x \right) \right|^2 dt \geq n \right\}$$

The stopped stochastic integrals are therefore martingales, so take expectations under Q to obtain

$$\mathbb{E}_0^Q [e^{-r\tau^n} \hat{v}(\hat{c}_{\tau^n}) x_{\tau^n}] \geq \hat{v}(\hat{c}_0) x_0 - \mathbb{E}_0^Q \left[\int_0^{\tau^n} e^{-rt} (c_t - k_t \alpha) dt \right] \quad (40)$$

Now we would like to take the limit $n \rightarrow \infty$, but we need to use the dominated convergence theorem. First,

$$\left| \int_0^{\tau^n} e^{-rt} (c_t - k_t \alpha) dt \right| \leq \int_0^{\infty} e^{-rt} |c_t - k_t \alpha| dt \leq \int_0^{\infty} e^{-rt} (|c_t| + |k_t \alpha|) dt$$

which is integrable because the contract is admissible. Second, for an admissible contract

$$0 \leq \lim_{n \rightarrow \infty} \mathbb{E}_0^Q [e^{-r\tau^n} \hat{v}(\hat{c}_{\tau^n}) x_{\tau^n}] \leq \lim_{n \rightarrow \infty} \mathbb{E}_0^Q [e^{-r\tau^n} \hat{v}_h x_{\tau^n}] = 0$$

To see why the last equality holds, notice that since $\hat{v}_h x$ is the cheapest way of delivering

¹⁶Notice \hat{v}'' is discontinuous at \hat{c}_l , but this doesn't change Ito's formula.

utility to the agent without capital, the cost of consumption on the contract is

$$\infty > \mathbb{E}^Q \left[\int_0^\infty e^{-rt} c_t dt \right] \geq \mathbb{E}^Q \left[\int_0^{\tau^n} e^{-rt} c_t dt + e^{-r\tau^n} \hat{v}_h x_{\tau^n} \right]$$

Taking the limit $n \rightarrow \infty$ and using the monotone convergence theorem, we obtain $0 \leq \lim_{n \rightarrow \infty} \mathbb{E}^Q [e^{-r\tau^n} \hat{v}_h x_{\tau^n}] \leq 0$.

Upon taking the limit $n \rightarrow \infty$ in (40), we obtain $\tau^n \rightarrow \infty$ a.s. and therefore

$$\mathbb{E}_0^Q \left[\int_0^\infty e^{-rt} (c_t - k_t \alpha) dt \right] \geq \hat{v}(\hat{c}_0) x_0$$

Using $\hat{v}(\hat{c}_l) \leq \hat{v}(\hat{c}_0)$ we obtain the first result.

For the second part, first let's show that \mathcal{C}^* is incentive compatible. We already know that for the HJB to have a solution with well defined policy functions it must be the case that $\bar{B} \geq 0$ and $\bar{C} > 0$, and therefore $\sigma^{\hat{c}^*} \leq 0$ and $\sigma^{x^*} \geq 0$. If $\hat{c}_t^* \in [\hat{c}_l, \hat{c}_h]$, because $\sigma^{\hat{c}^*}$ is bounded, so is σ^{x^*} and therefore so is μ^{x^*} and $\mu^{\hat{c}^*}$. We can then use Lemma 5 and the fact that $h_0 = 0$ to show that

$$U_0^{\tilde{c}, a} \leq U_0^{c^*, 0} = u_0$$

for any feasible strategy (\tilde{c}, a) , and therefore \mathcal{C}^* is indeed incentive compatible. To show that the cost of the contract is $\hat{v}_l x_0^*$, we can use the HJB. If $\hat{c}_t^* \in [\hat{c}_l, \hat{c}_h]$ always, where the HJB holds, then the same argument as in the first part shows the desired result. Notice that with $\hat{v}'(\hat{c}_l) = 0$ and $\hat{v}''(\hat{c}_l) > 0$ we get that as $\hat{c} \searrow \hat{c}_l$, $\bar{B} \rightarrow 0$ and $\bar{C} \rightarrow \bar{C}_l > 0$, so $\sigma^{\hat{c}^*} \rightarrow 0$ and $A(\hat{c}_l; \hat{v}_l) = 0$. From the first part we know $\hat{v}(\hat{c})$ is weakly below the true cost function, which is also bounded above by \hat{v}_h , so there is a finite cost function and \hat{v}_l is weakly below it. Lemma 13 shows that under these conditions $\mu^{\hat{c}}(\hat{c}_l) > 0$ and therefore $\hat{c}_t^* \in [\hat{c}_l, \hat{c}_h]$. Notice that the candidate contract does indeed deliver utility u_0 to the agent. To see this let $U^* = \frac{(x^*)^{1-\gamma}}{1-\gamma}$, so using the law of motion of x^* , (8), we get

$$U_0^* = \mathbb{E} \left[\int_0^{\tau^n} e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau^n} U_{\tau^n}^* \right]$$

with some sequence $\tau^n \rightarrow \infty$ a.s. Use the monotone convergence theorem and notice that

$$\lim_{n \rightarrow \infty} \mathbb{E} [e^{-\rho \tau^n} U_{\tau^n}^*] = 0$$

because $\rho - (1 - \gamma)(\mu^{x^*} - \frac{\gamma}{2}(\sigma^{x^*})^2) = \hat{c}^{1-\gamma} \geq \min\{\hat{c}_l^{1-\gamma}, \hat{c}_h^{1-\gamma}\} > 0$. We then get that $U_0^{c^*,0} = U_0^* = u_0$. This completes the proof.

Lemma 4

We know from Lemma 13 that $\hat{c}_t^* \in [\hat{c}_l, \hat{c}_h]$ and recall that $\hat{c}_l > 0$. Then an upper bounded $\mu^{x^*} < r$ implies a bounded $0 \leq \sigma^{x^*} \leq \bar{\sigma}_X$. Then

$$\mathbb{E}^Q \left[\int_0^\infty e^{-rt} (|c_t^*| + |k_t^* \alpha|) dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\bar{\sigma}_X \hat{c}_h^\gamma}{\phi \beta} \alpha \right\} \mathbb{E}^Q \left[\int_0^\infty e^{-rt} x_t^* dt \right] < \infty$$

where the last inequality follows from $\mu^{x^*} < r$. Let $U^* = \frac{(x^*)^{1-\gamma}}{1-\gamma}$, so using the law of motion of x^* , (8), we get

$$U_0^* = \mathbb{E} \left[\int_0^{\tau^n} e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau^n} U_{\tau^n}^* \right]$$

with $\tau^n \rightarrow \infty$ a.s. Use the monotone convergence theorem and notice that

$$\lim_{n \rightarrow \infty} \mathbb{E} [e^{-\rho \tau^n} U_{\tau^n}^*] = 0$$

because $\rho - (1 - \gamma)(\mu^{x^*} - \frac{\gamma}{2}(\sigma^{x^*})^2) = \hat{c}^{1-\gamma} \geq \min\{\hat{c}_l^{1-\gamma}, \hat{c}_h^{1-\gamma}\} > 0$. We then get that $U_0^{c^*,0} = U_0^* = u_0$. We conclude that the contract is indeed admissible.

Lemma 11. *Define the function*

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \frac{\left(\frac{\hat{c}^\gamma \alpha}{\phi \beta}\right)^2}{\hat{v} \gamma} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

For any $\hat{v} \in (0, \hat{v}_h)$, we have $A(\hat{c}; \hat{v}) > 0$ for \hat{c} near 0. In addition, if $\gamma \geq \frac{1}{2}$ then $A(\hat{c}; \hat{v})$ has at most one root in $[0, \hat{c}_h]$, where $\hat{c}_h = \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma) \left(\frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}}$. If instead $\gamma < \frac{1}{2}$, $A(\hat{c}; \hat{v})$ is convex and has at most two roots.

Proof. First, for $\gamma < 1$ $\lim_{\hat{c} \rightarrow 0} A(\hat{c}; \hat{v}) = \hat{v} \frac{\rho - r(1-\gamma)}{1-\gamma} > 0$. For $\gamma > 1$, $\lim_{\hat{c} \rightarrow 0} A(\hat{c}; \hat{v}) = \infty$.

For $\gamma \geq 1/2$, to show that $A(\hat{c}; \hat{v})$ has at most one root in $[0, \hat{c}_h]$ for any $\hat{v} \in (0, \hat{v}_h)$, we will show that $A'_\hat{c}(\hat{c}; \hat{v}) = 0 \implies A(\hat{c}; \hat{v}) > 0$ for all $\hat{c} < \hat{c}_h$. Compute the derivative

(dropping the arguments to avoid clutter)

$$A'_c = 1 - \hat{v}\hat{c}^{-\gamma} - \hat{c}^{2\gamma-1} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}}$$

So

$$A'_c = 0 \implies \hat{c} - \hat{v}\hat{c}^{1-\gamma} = \hat{c}^{2\gamma} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}}$$

Plug this into the formula for A to get

$$\begin{aligned} A &= \hat{c} - r\hat{v} + \hat{v} \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\hat{c}^{2\gamma}}{2\hat{v}\gamma} \left(\frac{\alpha}{\phi\beta} \right)^2 - \frac{\hat{v}}{2} \frac{\pi^2}{\gamma} \\ A &= \hat{c} - r\hat{v} + \hat{v} \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2\gamma} (\hat{c} - \hat{v}\hat{c}^{1-\gamma}) - \frac{\hat{v}}{2} \frac{\pi^2}{\gamma} \\ &= \frac{2\gamma-1}{2\gamma} \hat{c} + \frac{1-3\gamma}{2\gamma} \hat{v} \frac{\hat{c}^{1-\gamma}}{1-\gamma} + \hat{v} \frac{\rho - r(1-\gamma)}{1-\gamma} - \frac{\hat{v}}{2} \frac{\pi^2}{\gamma} \equiv B(\hat{c}, \hat{v}) \end{aligned}$$

$B(\hat{c}, \hat{v})$ is convex in \hat{c} because $1 - 3\gamma < 0$ for $\gamma \geq \frac{1}{2}$, so it's minimized in \hat{c} when $B'_c = 0$:

$$\frac{2\gamma-1}{3\gamma-1} = \hat{v}\hat{c}^{-\gamma} \tag{41}$$

and it is strictly decreasing before this point. Now we have two possible cases:

CASE 1: The minimum of B is achieved for $\hat{c} \geq \hat{c}_h$, so in the relevant range, it is minimized at \hat{c}_h . So let's plug in \hat{c}_h into $B(\hat{c}, \hat{v})$:

$$\begin{aligned} 2\gamma B(\hat{c}_h, \hat{v}) &= (2\gamma-1)\hat{c}_h + \frac{\hat{v}}{1-\gamma} \left((\rho - r(1-\gamma))2\gamma + (1-3\gamma)\hat{c}_h^{1-\gamma} \right) - \hat{v}\pi^2 \\ &= (2\gamma-1)\hat{c}_h + \frac{\hat{v}}{1-\gamma} \frac{(\rho - r(1-\gamma))}{\gamma} (2\gamma^2 + (1-3\gamma)) - \frac{\hat{v}}{1-\gamma} (1-3\gamma) \frac{1}{2} (1-\gamma) \left(\frac{\pi}{\gamma} \right)^2 - \hat{v}\pi^2 \\ &= (2\gamma-1)\hat{c}_h + \hat{v} \left(\frac{(\rho - r(1-\gamma))}{\gamma} \right) (1-2\gamma) - \frac{1}{2} (1-\gamma) \left(\frac{\pi}{\gamma} \right)^2 \hat{v} \left(\frac{1-3\gamma}{1-\gamma} + \frac{2\gamma^2}{1-\gamma} \right) \\ &= (2\gamma-1)\hat{c}_h + \hat{v} \left(\frac{(\rho - r(1-\gamma))}{\gamma} \right) (1-2\gamma) - \frac{1}{2} (1-\gamma) \left(\frac{\pi}{\gamma} \right)^2 \hat{v} (1-2\gamma) \end{aligned}$$

$$(2\gamma - 1) \left(\hat{c}_h - \hat{v} \left(\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2}(1 - \gamma) \left(\frac{\pi}{\gamma} \right)^2 \right) \right) \geq 0$$

and the inequality is strict if $\hat{v} < \hat{v}_h$. So $A(\hat{c}, \hat{v}) = B(\hat{c}, \hat{v}) > B(\hat{c}_h, \hat{v}) \geq 0$ for any $\hat{c} < \hat{c}_h$.

CASE 2: If the minimum is achieved for $\hat{c}_m \in [0, \hat{c}_h)$ it must be that $\gamma > 1/2$. Then plugging in (41) into B :

$$\begin{aligned} B(\hat{c}, \hat{v}) &\geq \frac{2\gamma - 1}{2\gamma} \hat{c}_m - \frac{2\gamma - 1}{2\gamma} \frac{\hat{c}_m}{1 - \gamma} + \hat{v} \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2 \gamma} \\ &= \frac{1 - 2\gamma}{2} \frac{\hat{c}_m}{1 - \gamma} + \hat{v} \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2 \gamma} \\ &= \frac{1 - 2\gamma}{2} \frac{\hat{c}_m}{1 - \gamma} + \frac{2\gamma - 1}{3\gamma - 1} \hat{c}_m^\gamma \left(\frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) \end{aligned}$$

and dividing throughout by $2\gamma - 1 > 0$

$$= -\frac{1}{2} \frac{\hat{c}_m}{1 - \gamma} + \frac{\hat{c}_m^\gamma}{3\gamma - 1} \left(\frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

and multiplying by $\hat{c}_m^{-\gamma} > 0$ and using $\frac{\hat{c}_m^{1-\gamma}}{1-\gamma} < \frac{\hat{c}_h^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned} &> -\frac{1}{2} \frac{\hat{c}_h^{1-\gamma}}{1 - \gamma} + \frac{1}{3\gamma - 1} \left(\frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) \\ &= \left(\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2}(1 - \gamma) \left(\frac{\pi}{\gamma} \right)^2 \right) \left(-\frac{1}{2} \frac{1}{1 - \gamma} + \frac{\gamma}{(3\gamma - 1)(1 - \gamma)} \right) \\ &\left(\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2}(1 - \gamma) \left(\frac{\pi}{\gamma} \right)^2 \right) \frac{1 - 3\gamma + 2\gamma}{(1 - \gamma)(3\gamma - 1)2} = \left(\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2}(1 - \gamma) \left(\frac{\pi}{\gamma} \right)^2 \right) \frac{1}{(3\gamma - 1)2} > 0 \end{aligned}$$

So $A(\hat{c}; \hat{v}) \geq B(\hat{c}, \hat{v}) > 0$ for all $\hat{c} \in [0, \hat{c}_h]$.

For the case with $\gamma < \frac{1}{2}$, the second derivative of A is

$$A''_{\hat{c}} = \gamma \hat{v} \hat{c}^{-\gamma-1} - (2\gamma - 1) \hat{c}^{2\gamma-2} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}} > 0$$

So $A(\hat{c}; \hat{v})$ is strictly convex and so can have at most two roots. □

Lemma 5

We will focus on the case with $dL_t = 0$, but the proof can be extended quite naturally. Let $F(\hat{h}, \hat{c}) = (1 + \hat{h}\hat{c}^{-\gamma})^{1-\gamma}$ where $\hat{h} = h/x$, and let $F_t = F(\hat{h}_t, \hat{c}_t)$, and likewise for derivatives, e.g. $F_{\hat{c},t} = \partial_{\hat{c}}F(\hat{h}_t, \hat{c}_t)$. Also, define $\hat{c} = \tilde{c}/x$. Write

$$e^{-\rho t} \left(U_t^{\tilde{c},a} - F(\hat{h}_t, \hat{c}_t) U_t^{c,0} \right) = \mathbb{E}_t^a \left[\int_t^{\tau^n} e^{-\rho u} \frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} du + \int_t^{\tau^n} d(e^{-\rho u} F_u U_u^{c,0}) + e^{-\rho \tau^n} \left(U_{\tau^n}^{\tilde{c},a} - F_{\tau^n} U_{\tau^n}^{c,0} \right) \right]$$

for a localizing sequence $\{\tau^n\}_{n \in \mathbb{N}}$ with $\tau^n \rightarrow \infty$ a.s. We will show that the rhs is non-positive. First write the integral part

$$\mathbb{E}_t^a \left[\int_t^{\tau^n} e^{-\rho u} \frac{\tilde{c}_u^{1-\gamma}}{1-\gamma} du + \int_t^{\tau^n} d(e^{-\rho u} F_u U_u^{c,0}) \right] = \mathbb{E}_t^a \left[\int_t^{\tau^n} e^{-\rho u} (1-\gamma) U_u^{c,0} Y_u du \right] \quad (42)$$

with

$$\begin{aligned} Y_t &= \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{-\rho F_t + \rho F_t - \hat{c}_t^{1-\gamma} F_t}{1-\gamma} \\ &+ \frac{F_{\hat{c},t}}{(1-\gamma)} \hat{c}_t \left(\frac{r-\rho}{\gamma} + \frac{1+\gamma}{2} (\sigma_t^x + \sigma_t^{\hat{c}})^2 - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} (\sigma_t^x)^2 - \sigma_t^x \sigma_t^{\hat{c}} \right) + \frac{F_{\hat{c},t}}{(1-\gamma)} (1-\gamma) \sigma_t^x \sigma_t^{\hat{c}} \hat{c}_t \\ &+ \frac{F_{\hat{h},t}}{(1-\gamma)} \hat{h}_t \left(r + \frac{\hat{c}_t - \hat{c}_t}{\hat{h}_t} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} (\sigma_t^x)^2 + (\sigma_t^x)^2 \right) - \frac{F_{\hat{h},t}}{(1-\gamma)} (1-\gamma) (\sigma_t^x)^2 \hat{h}_t \\ &+ \frac{(\sigma_t^{\hat{c}})^2 \hat{c}_t^2 F_{\hat{c}\hat{c},t} - 2\sigma_t^{\hat{c}} \sigma_t^x \hat{c}_t \hat{h}_t F_{\hat{c}\hat{h},t} + (\sigma_t^x)^2 \hat{h}_t^2 F_{\hat{h}\hat{h},t}}{2(1-\gamma)} \\ &+ \underbrace{\frac{1}{(1-\gamma)} \left(-(1-\gamma) \sigma_t^x F_t - F_{\hat{c},t} \sigma_t^{\hat{c}} \hat{c}_t + (\sigma_t^x \hat{h}_t + \hat{h}_t \phi \beta \hat{c}_t^\gamma) F_{\hat{h},t} \right)}_{A_t} \frac{a_t}{\beta} \end{aligned}$$

where

$$\begin{aligned} F_{\hat{c},t} &= \gamma(\gamma-1) \hat{h}_t \hat{c}_t^{-\gamma-1} (1 + \hat{h}_t \hat{c}_t^{-\gamma})^{-\gamma} \\ F_{\hat{h},t} &= (1-\gamma) \hat{c}_t^{-\gamma} (1 + \hat{h}_t \hat{c}_t^{-\gamma})^{-\gamma} \end{aligned}$$

$$\begin{aligned}
F_{\hat{c}\hat{c},t} &= -\gamma(\gamma-1)\hat{h}_t\hat{c}_t^{-2-\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} - \gamma^2(\gamma-1)\hat{h}_t\hat{c}_t^{-\gamma-2} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1} \left(1 - \hat{h}_t(\gamma-1)\hat{c}_t^{-\gamma}\right) \\
F_{\hat{c}\hat{h}} &= \gamma(\gamma-1)\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} - \gamma^2(\gamma-1)\hat{h}_t\hat{c}_t^{-2\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1} \\
F_{\hat{h}\hat{h},t} &= -\gamma\hat{c}_t^{-2\gamma}(1-\gamma) \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1}
\end{aligned}$$

We know $e^{-rt}(1-\gamma)U_t^{c_i,0} > 0$, and we will show that $Y_t \leq 0$. To do this, we will split the expression into three parts, $Y_t = A_t + B_t + C_t$, and show each one is non-positive. First take the terms multiplying s_t

$$\begin{aligned}
A_t &= \frac{1}{(1-\gamma)} \left(-(1-\gamma)\sigma_t^x \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{1-\gamma} \right. \\
&\quad \left. -\gamma(\gamma-1)\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \sigma_t^{\hat{c}}\hat{c}_t + (\sigma_t^x\hat{h}_t + \hat{k}_t\phi\beta\hat{c}_t^\gamma)(1-\gamma)\hat{c}_t^{-\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \right) \\
A_t &= \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \left(-\sigma_t^x \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right) + \gamma\hat{h}_t\hat{c}_t^{-\gamma-1}\sigma_t^{\hat{c}}\hat{c}_t + (\sigma_t^x\hat{h}_t + \hat{k}_t\phi\beta\hat{c}_t^\gamma)\hat{c}_t^{-\gamma} \right) \\
A_t &\leq \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \gamma\hat{h}_t\hat{c}_t^{-\gamma}\sigma_t^{\hat{c}} \leq 0
\end{aligned}$$

where we have used $\sigma^x \geq \hat{k}_t\phi\beta\hat{c}^{-\gamma}$ in the first inequality, and $\hat{h}_t \geq 0$ and $\sigma_t^{\hat{c}} \leq 0$ in the second. Here is the only place where $\sigma_t^{\hat{c}} \leq 0$ is used.

Second, all the remaining terms that have σ_t^x or $\sigma_t^{\hat{c}}$ are

$$\begin{aligned}
B_t &= \frac{F_{\hat{c},t}}{(1-\gamma)}\hat{c}_t \left(\frac{1+\gamma}{2}(\sigma_t^x + \sigma_t^{\hat{c}})^2 - \frac{\gamma}{2}(\sigma_t^x)^2 - \sigma_t^x\sigma_t^{\hat{c}} \right) + \frac{F_{\hat{c},t}}{(1-\gamma)}(1-\gamma)\sigma_t^x\sigma_t^{\hat{c}}\hat{c}_t \\
&+ \frac{F_{\hat{h},t}}{(1-\gamma)}\hat{h}_t \left(-\frac{\gamma}{2}(\sigma_t^x)^2 + (\sigma_t^x)^2 \right) - \frac{F_{\hat{h},t}}{(1-\gamma)}(1-\gamma)(\sigma_t^x)^2\hat{h}_t + \frac{(\sigma_t^{\hat{c}})^2\hat{c}_t^2 F_{\hat{c}\hat{c},t} - 2\sigma_t^{\hat{c}}\sigma_t^x\hat{c}_t\hat{h}_t F_{\hat{c}\hat{h},t} + (\sigma_t^x)^2\hat{h}_t^2 F_{\hat{h}\hat{h},t}}{2(1-\gamma)} \\
B_t &= \frac{F_{\hat{c},t}}{(1-\gamma)}\hat{c}_t \left(\frac{1+\gamma}{2}(\sigma_t^x + \sigma_t^{\hat{c}})^2 - \frac{\gamma}{2}(\sigma_t^x)^2 - \gamma\sigma_t^x\sigma_t^{\hat{c}} \right) + \frac{F_{\hat{h},t}}{(1-\gamma)}\hat{h}_t \left(-\frac{\gamma}{2}(\sigma_t^x)^2 + \gamma(\sigma_t^x)^2 \right) \\
&\quad + \frac{(\sigma_t^{\hat{c}})^2\hat{c}_t^2 F_{\hat{c}\hat{c},t} - 2\sigma_t^{\hat{c}}\sigma_t^x\hat{c}_t\hat{h}_t F_{\hat{c}\hat{h},t} + (\sigma_t^x)^2\hat{h}_t^2 F_{\hat{h}\hat{h},t}}{2(1-\gamma)} \\
B_t &= \frac{(\sigma_t^x)^2}{2} \left(\frac{F_{\hat{c},t}}{1-\gamma}\hat{c}_t + \frac{F_{\hat{h},t}}{1-\gamma}\hat{h}_t\gamma + \frac{F_{\hat{h}\hat{h},t}}{1-\gamma}\hat{h}_t^2 \right) + \frac{(\sigma_t^{\hat{c}})^2}{2} \left(\frac{F_{\hat{c},t}}{1-\gamma}(1+\gamma)\hat{c}_t + \frac{F_{\hat{c}\hat{c},t}}{1-\gamma}\hat{c}_t^2 \right)
\end{aligned}$$

$$-\sigma_t^x \sigma_t^{\hat{c}} \left(-\frac{F_{\hat{c},t}}{1-\gamma} \hat{c}_t + \frac{F_{\hat{c}\hat{h},t}}{1-\gamma} \hat{c}_t \hat{h}_t \right)$$

which plugging in the F 's gets us

$$\begin{aligned} B_t = & \frac{(\sigma_t^x)^2}{2} \left(\frac{\gamma(\gamma-1)\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma}}{1-\gamma} \hat{c}_t \right. \\ & \left. + \frac{(1-\gamma)\hat{c}_t^{-\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma}}{1-\gamma} \hat{h}_t\gamma + \frac{-\gamma\hat{c}_t^{-2\gamma}(1-\gamma) \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1}}{1-\gamma} \hat{h}_t^2 \right) \\ & + \frac{(\sigma_t^{\hat{c}})^2}{2} \left(\frac{\gamma(\gamma-1)\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma}}{1-\gamma} (1+\gamma)\hat{c}_t + \right. \\ & \left. \frac{-\gamma(\gamma-1)\hat{h}_t\hat{c}_t^{-2-\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} - \gamma^2(\gamma-1)\hat{h}_t\hat{c}_t^{-\gamma-2} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1} \left(1 - \hat{h}_t(\gamma-1)\hat{c}_t^{-\gamma}\right)}{1-\gamma} \hat{c}_t^2 \right) \\ & - \sigma_t^x \sigma_t^{\hat{c}} \left(-\frac{\gamma(\gamma-1)\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma}}{1-\gamma} \hat{c}_t \right. \\ & \left. + \frac{\gamma(\gamma-1)\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} - \gamma^2(\gamma-1)\hat{h}_t\hat{c}_t^{-2\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1}}{1-\gamma} \hat{c}_t \hat{h}_t \right) \end{aligned}$$

which simplifies to:

$$\begin{aligned} B_t = & \frac{(\sigma_t^x)^2}{2} \left(-\gamma\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t + \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{h}_t\gamma - \gamma\hat{c}_t^{-2\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1} \hat{h}_t^2 \right) \\ & + \frac{(\sigma_t^{\hat{c}})^2}{2} \left(-\gamma\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} (1+\gamma)\hat{c}_t + \gamma\hat{h}_t\hat{c}_t^{-2-\gamma} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t^2 \right. \\ & \left. + \gamma^2\hat{h}_t\hat{c}_t^{-\gamma-2} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1} \left(1 - \hat{h}_t(\gamma-1)\hat{c}_t^{-\gamma}\right) \hat{c}_t^2 \right) \\ & - \sigma_t^x \sigma_t^{\hat{c}} \left(\gamma\hat{h}_t\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t - \gamma\hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t \hat{h}_t + \gamma^2\hat{h}_t\hat{c}_t^{-2\gamma-1} \left(1 + \hat{h}_t\hat{c}_t^{-\gamma}\right)^{-\gamma-1} \hat{c}_t \hat{h}_t \right) \end{aligned}$$

and

$$\begin{aligned}
B_t &= -\frac{(\sigma_t^x)^2}{2} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma-1} \hat{c}_t^{-\gamma-1} \gamma \hat{c}_t^{1-\gamma} \hat{h}_t^2 \\
&\quad - \frac{(\hat{\sigma}_t^{\hat{c}})^2}{2} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma-1} \hat{c}_t^{-\gamma-1} \hat{h}_t^2 \gamma^3 \hat{c}_t^{1-\gamma} \\
&\quad - \sigma_t^x \sigma_t^{\hat{c}} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma-1} \hat{c}_t^{-\gamma-1} \gamma^2 \hat{h}_t^2 \hat{c}_t^{1-\gamma} \\
B_t &= -\frac{(\sigma_t^x + \sigma_t^{\hat{c}} \gamma)^2}{2} \gamma \hat{h}_t^2 \hat{c}_t^{1-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma-1} \hat{c}_t^{-\gamma-1} \leq 0
\end{aligned}$$

And finally all the remaining terms that don't involve σ_t^x or $\sigma_t^{\hat{c}}$ are

$$\begin{aligned}
C_t &= \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} - \frac{\hat{c}_t^{1-\gamma} F_t}{(1-\gamma)} + \frac{\gamma(\gamma-1) \hat{h}_t \hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{(1-\gamma)} \hat{c}_t \left(\frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma}\right) \\
&\quad + \frac{(1-\gamma) \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{(1-\gamma)} \hat{h}_t \left(r + \frac{\hat{c}_t - \hat{c}_t}{\hat{h}_t} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma}\right)
\end{aligned}$$

which is maximized for $\hat{\hat{c}}_t = \hat{c}_t + \hat{h}_t \hat{c}_t^{1-\gamma}$. Plugging this in yields

$$\begin{aligned}
C_t &\leq \frac{\hat{c}_t^{1-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{1-\gamma}}{1-\gamma} - \frac{\hat{c}_t^{1-\gamma} F_t}{(1-\gamma)} + \frac{\gamma(\gamma-1) \hat{h}_t \hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{(1-\gamma)} \hat{c}_t \left(\frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma}\right) \\
&\quad + \frac{(1-\gamma) \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma}}{(1-\gamma)} \hat{h}_t \left(r + \frac{\hat{c}_t - \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right) \hat{c}_t}{\hat{h}_t} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma}\right) \\
C_t &\leq -\gamma \hat{h}_t \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \left(\frac{r-\rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma}\right) \\
&\quad + \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{h}_t \left(r + \frac{\hat{c}_t - \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right) \hat{c}_t}{\hat{h}_t} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma}\right) \\
C_t &\leq \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right) \hat{c}_t^{-\gamma} \hat{h}_t \left\{ \rho - r + \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} \gamma + r - \hat{c}_t^{1-\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} \right\} \\
C_t &\leq \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right) \hat{c}_t^{-\gamma} \hat{h}_t \left\{ \rho - \left(\rho - \hat{c}_t^{1-\gamma}\right) - \hat{c}_t^{1-\gamma} \right\} = 0
\end{aligned}$$

Since $A_u, B_u, C_u \leq 0$ we conclude that $Y_t \leq 0$.

Now for the last term, since the agent's strategy (\tilde{c}, a) is feasible, we have that

$$\lim_{n \rightarrow \infty} \mathbb{E}_t^a \left[e^{-\rho\tau^n} U_{\tau^n}^{\tilde{c}, a} \right] = 0$$

For $\gamma < 1$ we use Fatou's lemma to show

$$\lim_{n \rightarrow \infty} \mathbb{E}_t^a \left[e^{-\rho\tau^n} F_{\tau^n} U_{\tau^n}^{c, 0} \right] \geq 0$$

As a result, when we take $n \rightarrow \infty$ we get $\mathbb{E}^a \left[e^{-\rho t} \left(U_t^{\tilde{c}, a} - F(\hat{h}_t, \hat{c}_t) U_t^{c, 0} \right) \right] \leq 0$.

For $\gamma > 1$, if $\lim_{n \rightarrow \infty} \mathbb{E}_t^a \left[e^{-\rho\tau^n} F_{\tau^n} U_{\tau^n}^{c, 0} \right] = 0$ for any feasible strategy (\tilde{c}, a) , then we are done. To show this, let $N_t = X_t + h_t \hat{c}_t^{-\gamma}$, so that $F_t U_t^{c, 0} = \frac{N_t^{1-\gamma}}{1-\gamma}$. The law of motion of N satisfies

$$dN_t \leq (\lambda_1 N_t - \lambda_2 \tilde{c}_t) dt + \sigma_t^N dZ_t^a$$

where $\lambda_2 = \hat{c}_h^{-\gamma} > 0$, and $\lambda_1 = \bar{\mu}^x + r + \hat{c}^{1-\gamma} + \gamma |\bar{\mu}^{\hat{c}}| + \gamma(1+\gamma)(\bar{\sigma}^{\hat{c}})^2$. Here's where we use the assumption that μ^x , $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$ are bounded and $\hat{c} \leq \hat{c}_h$ is uniformly bounded away from 0. Notice that stealing only reduces the drift of N_t , since the change in the drift of x and h cancel out, and it increases the drift of \hat{c} . Since $N_t > 0$ always, Lemma 12 and its corollary (with $\tilde{\sigma}^N = 0$) ensure the desired limit.

Lemma 6

We use the HJB equation (12) with $\mu^{\hat{c}} = \sigma^{\hat{c}} = 0$. Lemma 14 ensures that $\hat{v}(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_*, \hat{c}_h]$. The same argument as in Theorem 2 shows that $\hat{v}_r(\hat{c})$ from (25) is the cost corresponding to the stationary contract with \hat{c} and σ^x given by (22), as long as the contract is indeed admissible and delivers utility u_0 to the agent. We can check that $\mu^x < r$ for the stationary contract if and only if $\hat{c} > \hat{c}_*$, where \hat{c}_* is given by (24). In this case, since $\mu^x < r$ arguing as in the proof of Lemma 4 we can show that the stationary contract is admissible and delivers utility u_0 to the agent if and only if $\hat{c} > \hat{c}_*$. Since the contract satisfies (8), (9), and (10) by construction, Lemma 5 then ensures that it is incentive compatible. This completes the proof.

Lemma 7

The contract generated by the optimal portfolio plan with $\tilde{\phi} = \phi$ is stationary with bounded σ^x . Plugging the consumption and portfolio weight (26) and (27) into the dynamic budget constraint (20) with $\tilde{\phi}_t = \phi$, we obtain the growth rate of net worth (and hence of x)

$$\begin{aligned}\mu^n &= r + \frac{1}{\gamma} \frac{\alpha^2}{(\phi\beta)^2} - \frac{\rho - r(1-\gamma)}{\gamma} + \frac{1-\gamma}{\gamma} \frac{1}{2} \frac{1}{\gamma} \frac{\alpha^2}{(\phi\beta)^2} \\ \mu^n &= r + \frac{1}{\gamma} \frac{\alpha^2}{(\phi\beta)^2} \frac{1+\gamma}{2\gamma} - \frac{\rho - r(1-\gamma)}{\gamma}\end{aligned}$$

Using $\alpha < \bar{\alpha}$ we obtain that

$$\mu^n < r$$

which implies $\mu^x < r$. Since $\sigma^x = \sigma^n$ bounded, we can argue as in the proof of Lemma 4 to show that the contract is indeed admissible and delivers utility $u_0^{op} = \frac{(\omega^{op} n_0)^{1-\gamma}}{1-\gamma}$ to the agent. Since the contract satisfies (8), (9), and (10) by construction, Lemma 5 then ensures that it is incentive compatible. This completes the proof.

Lemma 12. *Assume there are some constants $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\lambda_3 > 0$ such that for any feasible strategy $(\tilde{c}, a, \tilde{\sigma}^h)$ there is a non-negative process N with*

$$dN_t \leq ((\lambda_1 + \lambda_2 \tilde{\sigma}_t^N) N_t - \lambda_3 \tilde{c}_t) dt + \sigma_t^N N_t dZ_t^a + \tilde{\sigma}_t^N N_t d\tilde{Z}_t$$

for some processes σ^N and $\tilde{\sigma}^N$, which can depend on the strategy. Then there is a constant $\lambda_4 > 0$ such that for any $T > 0$ and any feasible strategy $(\tilde{c}, a, \tilde{\sigma}^h)$

$$\mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq \lambda_4 \frac{N_0^{1-\gamma}}{1-\gamma}$$

Proof. First define n_t as the solution to the SDE

$$dn_t = ((\lambda_1 + \lambda_2 \tilde{\sigma}_t^N) n_t - \tilde{c}_t) dt + \sigma_t^N n_t dZ_t^a + \tilde{\sigma}_t^N n_t d\tilde{Z}_t$$

and $n_0 = \frac{N_0}{\lambda_3}$. It follows that $n_t \geq N_t \geq 0$. Now define ζ as

$$\frac{d\zeta_t}{\zeta_t} = -\lambda_1 dt - \lambda_2 d\tilde{Z}_t, \quad \zeta_0 = 1$$

and

$$\tilde{n}_t = \int_0^t \zeta_s \tilde{c}_s ds + \zeta_t n_t$$

We can check that \tilde{n}_t is a local martingale under P^a . Since $\zeta_t > 0$ and $n_t \geq 0$ it follows that

$$\mathbb{E}^a \left[\int_0^{\tau^m \wedge T} \zeta_s \tilde{c}_s ds \right] \leq \mathbb{E}^a \left[\int_0^{\tau^m \wedge T} \zeta_s \tilde{c}_s ds + \zeta_{\tau^m \wedge T} n_{\tau^m \wedge T} \right] = n_0$$

where $\{\tau^m\}$ reduces the stochastic integral and has $\lim_{m \rightarrow \infty} \tau^m = \infty$ a.s. Taking $m \rightarrow \infty$ and using the monotone convergence theorem we obtain

$$\mathbb{E}^a \left[\int_0^T \zeta_s \tilde{c}_s ds \right] \leq n_0$$

Now we want to maximize $\mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right]$ subject to this budget constraint. Notice that a appears both in the budget constraint and objective function, but does not affect ζ , so we can ignore it since we are choosing \tilde{c} . The candidate solution c has

$$e^{-\rho t} c_t^{-\gamma} = \zeta_t \mu$$

where $\mu > 0$ is the Lagrange multiplier and is chosen so that the budget constraint holds with equality. For any \tilde{c} that satisfies the budget constraint we have

$$\begin{aligned} \mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] &< \mathbb{E}^a \left[\int_0^T e^{-\rho t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} + c_t^{-\gamma} (\tilde{c}_t - c_t) \right) dt \right] \\ &= \mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] + \mu \mathbb{E}^a \left[\int_0^T \zeta_t (\tilde{c}_t - c_t) dt \right] \leq \mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \end{aligned}$$

Now since $c_t = (\zeta_t \mu)^{-\frac{1}{\gamma}} e^{-\frac{\rho}{\gamma} t}$ it follows a geometric brownian motion so $\mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right]$ is finite. Because of homothetic preferences, we know that $\mathbb{E}^a \left[\int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] = \tilde{\lambda}_4 \frac{n_0^{1-\gamma}}{1-\gamma} = \lambda_4 \frac{N_0^{1-\gamma}}{1-\gamma}$ for some $\lambda_4 > 0$. \square

Corollary. For $\gamma > 1$, $\lim_{n \rightarrow \infty} \mathbb{E}_t^a \left[e^{-\rho \tau^n} \frac{N_{\tau^n}^{1-\gamma}}{1-\gamma} \right] = 0$ for any feasible strategy $(\tilde{c}, a, \tilde{\sigma}^h)$.

Proof. The continuation utility at any stopping time τ^n has

$$\begin{aligned} U_{\tau^n}^{\tilde{c},a} &= \mathbb{E}_{\tau^n}^a \left[\int_{\tau^n}^{\tau^n+T} e^{-\rho(t-\tau^n)} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho(T-\tau^n)} U_{\tau^n+T}^{\tilde{c},a} \right] \\ &\leq \mathbb{E}_{\tau^n}^a \left[\int_{\tau^n}^{\tau^n+T} e^{-\rho(t-\tau^n)} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq \lambda_4 \frac{N_{\tau^n}^{1-\gamma}}{1-\gamma} \end{aligned}$$

So at $t = 0$ we get

$$U_0^{\tilde{c},a} = \mathbb{E}^a \left[\int_0^{\tau^n} e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho\tau^n} U_{\tau^n}^{\tilde{c},a} \right] \leq \mathbb{E}^a \left[\int_0^{\tau^n} e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho\tau^n} \lambda_4 \frac{N_{\tau^n}^{1-\gamma}}{1-\gamma} \right]$$

Take limits $n \rightarrow \infty$ and use the monotone convergence theorem on the first term on the right hand side to get $0 \geq \lim_{n \rightarrow \infty} \mathbb{E}_t^a \left[e^{-\rho\tau^n} \frac{N_{\tau^n}^{1-\gamma}}{1-\gamma} \right] \geq 0$. \square

Lemma 13. *Let x and \hat{c} satisfy the laws of motion (8) and (9), assume that there is a finite cost function $\hat{v}(\hat{c})$. Assume that, for some $\hat{c}_l \in (0, \hat{c}_h)$ and $\hat{v}_l \leq \hat{v}(\hat{c}_l)$, as $\hat{c} \searrow \hat{c}_l$ we have $\sigma^{\hat{c}}(\hat{c}) \rightarrow 0$ and $A(\hat{c}; \hat{v}_l) \rightarrow 0$. Then $\mu^{\hat{c}}(\hat{c}_l) > 0$.*

Proof. Looking at (9), with $\sigma^{\hat{c}} = 0$ we get for the drift

$$\mu^{\hat{c}} = \frac{r - \rho}{\gamma} + \frac{1}{2}(\sigma^x)^2 - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}$$

So $\mu^{\hat{c}} > 0$ implies

$$\frac{1}{2}(\sigma^x)^2 > \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}$$

Since we also want $A(\hat{c}; \hat{v}) = 0$, we get

$$\begin{aligned} 0 &= \hat{c} - r\hat{v} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2}(\sigma^x)^2 \right) \\ &< \hat{c} - \hat{v}\hat{c}^{1-\gamma} \equiv M \end{aligned}$$

Notice that if $\hat{v} = \hat{c}^\gamma$ we have $M = 0$. If $\hat{v} > \hat{c}^\gamma$ we have $M < 0$ and if $\hat{v} < \hat{c}^\gamma$ we have $M > 0$. So for $A(\hat{c}; \hat{v}) = 0$ and $\mu^{\hat{c}} > 0$ we need $\hat{v} < \hat{c}^\gamma$. In fact, if $\hat{v} = \hat{c}^\gamma$ and in addition

$$\frac{1}{2} \left(\frac{\alpha}{\phi\beta\gamma} \right)^2 = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma} \quad (43)$$

then we have $A = 0$ and $\mu^{\hat{c}} = 0$ (notice that with $\hat{v}' = 0$ and $\hat{v} = \hat{c}^\gamma$ the FOC for σ^x (15) yields $\sigma^x = \frac{\alpha}{\phi\beta\gamma}$). In this case, because we have $\mu^{\hat{c}} = 0$ we therefore have the value of a stationary contract, i.e. $\hat{v} = \hat{v}_r(\hat{c})$. So let (\hat{c}_T, \hat{v}_T) be defined by $\hat{v}_T = \hat{c}_T^\gamma$ and (43). Suppose this point exists for $\hat{c}_T \in [\hat{c}_*, \hat{c}_h]$. It must be the case that $\hat{v}_T \geq \hat{v}_l$ since \hat{v}_T is the cost of a stationary contract (because $\hat{c}_T \geq \hat{c}_*$ so it's an incentive compatible stationary contract) but not optimal, and we assumed $\hat{v}_l \leq \hat{v}(\hat{c}_l)$ (possibly the same).

First we will show that $\mu^{\hat{c}} \geq 0$, and then make the inequality strict. Towards contradiction, suppose $\mu^{\hat{c}} < 0$ at \hat{c}_l . Then it must be the case that $\hat{v}_l > \hat{c}_l^\gamma$ because we have $A(\hat{c}_l, \hat{v}_l) = 0$. We will show that $A(\hat{c}_l, \hat{v}_l) > 0$ and get a contradiction. First take the derivative of A :

$$A'_{\hat{c}}(\hat{c}_l, \hat{v}_l) = 1 - \hat{v}_l \left(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality holds for all $\hat{c} < \hat{v}_l^{\frac{1}{\gamma}}$. So $A(\hat{c}_l, \hat{v}_l) > A(\hat{v}_l^{\frac{1}{\gamma}}, \hat{v}_l)$. Letting $\hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}}$ we get

$$\begin{aligned} A(\hat{c}_l, \hat{v}_l) &> \hat{c}_m - r\hat{v}_l + \hat{v}_l \left(\frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \frac{1}{2} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\gamma} \right) \\ &= \hat{c}_m - r\hat{v}_l + \hat{v}_l \left(\frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \gamma \frac{\rho - \hat{c}_T^{1-\gamma}}{1-\gamma} - (\rho - r) \right) \\ \implies A(\hat{c}_l, \hat{v}_l) &> \hat{c}_m + \hat{v}_l \frac{\gamma \hat{c}_T^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} = \hat{c}_m^\gamma \gamma \frac{\hat{c}_T^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} \geq 0 \end{aligned}$$

where the last equality uses $\hat{v}_l = \hat{c}_m^\gamma$ and the last inequality uses $\hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}} \leq \hat{v}_T^{\frac{1}{\gamma}} = \hat{c}_T$. This is a contradiction, and therefore it must be the case that $\mu^{\hat{c}} \geq 0$ at \hat{c}_l .

It's clear from the previous argument that $\mu^{\hat{c}}(\hat{c}_l) = 0$ only if $(\hat{c}_l, \hat{v}_l) = (\hat{c}_T, \hat{v}_T)$. This can only happen if the optimal contract is indeed stationary. We will show this cannot be the case because $\alpha > 0$. First, note that (\hat{c}_T, \hat{v}_T) is a tangency point where $\hat{v}_r(\hat{c})$ touches the locus $\hat{v}_b(\hat{c})$ defined by $A(\hat{c}; \hat{v}_b(\hat{c})) = 0$. If $(\hat{c}_l, \hat{v}_l) = (\hat{c}_T, \hat{v}_T)$ then this must be the minimum point for $\hat{v}_r(\hat{c})$, so the derivative of both $\hat{v}_r(\hat{c})$ and $\hat{v}_b(\hat{c})$ must be zero. This means that $A'_{\hat{c}}(\hat{c}_l, \hat{v}_l) = 0$. However,

$$1 - \hat{v}_l \left(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality follows from $\hat{v}_l = \hat{v}_T = \hat{c}_l^\gamma$ (note that $\hat{c}_l > 0$ because as Lemma 11 shows $A(\hat{c}, \hat{v}_l)$ is strictly positive for \hat{c} near 0). This can't be a minimum of $\hat{v}_r(\hat{c})$. Therefore $(\hat{c}_l, \hat{v}_l) \neq (\hat{c}_T, \hat{v}_T)$ and $\mu^{\hat{c}}(\hat{c}_l) > 0$.

It only remains to show that (\hat{c}_T, \hat{v}_T) exists. Because there is a finite cost function, we know from Lemma14 that $\alpha < \bar{\alpha}$. Then the lhs of (43) is positive and is bounded by

$$0 < lhs < \frac{\rho - r(1 - \gamma)}{\gamma(1 + \gamma)} > 0$$

The rhs of (43) can be written

$$\begin{aligned} rhs &= \frac{\rho - r(1 - \gamma)}{(1 - \gamma)\gamma} - \frac{\hat{c}^{1-\gamma}}{1 - \gamma} = (\rho - r(1 - \gamma)) \left(\frac{1}{(1 - \gamma)\gamma} - \frac{\hat{c}^{1-\gamma}}{1 - \gamma} \frac{1}{\rho - r(1 - \gamma)} \right) \\ &= \frac{(\rho - r(1 - \gamma))}{\gamma} \left(\frac{1}{1 - \gamma} - \frac{\hat{c}^{1-\gamma}}{1 - \gamma} \frac{\gamma}{\rho - r(1 - \gamma)} \right) \end{aligned}$$

For $\hat{c} = \hat{c}_h$ the rhs is zero. For $\hat{c} = \hat{c}_*$ we get

$$rhs = \frac{\rho - r(1 - \gamma)}{\gamma(1 + \gamma)}$$

Then there is a $\hat{c} \in [\hat{c}_*, \hat{c}_h]$ solving (43) as desired. This completes the proof. \square

Lemma 14. *The cost function of stationary contracts $\hat{v}_r(\hat{c})$ defined by (25) is strictly positive for all $\hat{c} \in (\hat{c}_*, \hat{c}_h]$ if and only if*

$$\alpha < \bar{\alpha} \equiv \frac{\phi\beta\gamma\sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}}$$

Proof. We need to check the numerator in (25), since the denominator is positive for all $\hat{c} \geq \hat{c}_*$:

$$\hat{c} - \frac{\alpha}{\phi\beta} \hat{c}^\gamma \sqrt{2} \sqrt{\frac{\rho - r(1 - \gamma)}{(1 - \gamma)\gamma} \left(1 - \left(\frac{\hat{c}}{\hat{c}_h} \right)^{1-\gamma} \right)}$$

The rest of the proof consists of evaluating this expression at $\hat{c} = \hat{c}_*$ and showing it is

non-positive iff the bound is violated. We get \hat{c} times

$$1 - \frac{\alpha}{\phi\beta} \sqrt{2} \frac{\sqrt{\frac{\rho-r(1-\gamma)}{\gamma(1-\gamma)} \frac{1-\gamma}{1+\gamma}}}{\frac{2\gamma}{1+\gamma} \frac{\rho-r(1-\gamma)}{\gamma}}$$

$$1 - \frac{\alpha}{\phi\beta} \frac{1}{\sqrt{2} \frac{\gamma}{\sqrt{1+\gamma}}} \frac{1}{\sqrt{\frac{\rho-r(1-\gamma)}{\gamma}}}$$

So if $\alpha \geq \bar{\alpha}$ the numerator is non-positive, and if $\alpha < \bar{\alpha}$ then it's strictly positive. This completes the proof. \square

Appendix B: Aggregate risk

We can incorporate aggregate risk into the environment and extend all our results in a natural way. Let \tilde{Z} be an independent Brownian motion that represents aggregate risk, with price π in the market. Let the return on capital be

$$dR_t = (r + \pi\tilde{\beta} + \alpha - a_t) dt + \beta dZ_t + \tilde{\beta} d\tilde{Z}_t$$

The agent can now invest his hidden savings in the market, so they follow¹⁷

$$dh_t = (rh_t + \pi\tilde{\sigma}_t^h h_t + c_t - \tilde{c}_t + \phi k_t a_t) dt + \tilde{\sigma}_t^h h_t d\tilde{Z}_t$$

where $\tilde{\sigma}_t^h$ is the fraction of his hidden savings invested in the risky market, for which he receives a premium π . Since aggregate shocks are observable, the contract (c, k) specifies consumption and capital as functions of the history of not only realized returns R , but also aggregate shocks \tilde{Z} . After signing the contract the agent can choose a strategy $(\tilde{c}, a, \tilde{\sigma}_t^h)$ to maximize his utility, which can also depend on the history of both R and \tilde{Z} . The agent's utility and the principal's objective function are still given by (1) and (2).¹⁸ Since there is now aggregate risk that pays a premium, we need to slightly modify the parameter restrictions

$$\hat{c}_h \equiv \left(\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma) \left(\frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} > 0 \quad (44)$$

¹⁷As before, the agent cannot secretly invest in risky capital.

¹⁸The definitions of feasible strategy, and admissible, incentive compatible, and optimal contracts are unchanged.

and

$$\alpha < \bar{\alpha} \equiv \frac{\phi\beta\gamma\sqrt{2}}{\sqrt{1+\gamma}} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2}(1-\gamma)\left(\frac{\pi}{\gamma}\right)^2} \quad (45)$$

We can check that with $\pi = 0$ we recover the formulas without aggregate risk.

Since the contract can depend on the history of aggregate shocks \tilde{Z} , so can his continuation utility $U^{c,0}$ and his consumption c . However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings in aggregate risk, his Euler equation needs to be modified appropriately. His discounted marginal utility $\exp\left(\int_0^t r - \rho + \pi_s \tilde{\sigma}_s^h - \frac{1}{2}(\tilde{\sigma}_s^h)^2 ds + \int_0^t \tilde{\sigma}_s^h d\tilde{Z}_s\right) c_t^{-\gamma}$ must be a supermartingale for any $\tilde{\sigma}^h$, since otherwise he could save a dollar, investing it in aggregate risk, and consume it later when the marginal utility is expected to be larger. Because aggregate risk pays a premium π , the agent would like to invest in it and expose his consumption to aggregate risk with loading π/γ . An incentive compatible contract must allow this.

Lemma 15. *If $\mathcal{C} = (c, k)$ is an incentive compatible contract, the agent's continuation utility $U^{c,0}$ and consumption c satisfy the laws of motion*

$$dU_t^{c,0} = \left(\rho U_t^{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma} \right) dt + \Delta_t \beta dZ_t + \tilde{\sigma}_t^u d\tilde{Z}_t \quad (46)$$

$$\frac{dc_t}{c_t} = \left(\frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma_t^c)^2 + \frac{1 + \gamma}{2} (\tilde{\sigma}_t^c)^2 \right) dt + \sigma_t^c dZ_t + \tilde{\sigma}_t^c d\tilde{Z}_t + dL_t \quad (47)$$

for some Δ , $\tilde{\sigma}^u$, σ^c , $\tilde{\sigma}^c$, and a weakly increasing processes L , such that

$$\Delta_t \geq c_t^{-\gamma} \phi k_t \quad (48)$$

$$\tilde{\sigma}_t^c = \frac{\pi_t}{\gamma} \quad (49)$$

In light of this, we must modify the laws of motion of the state variables x and \hat{c} . Using Ito's lemma:

$$\frac{dx_t}{x_t} = \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma_t^x)^2 + \frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 \right) dt + \sigma_t^x dZ_t + \tilde{\sigma}_t^x d\tilde{Z}_t \quad (50)$$

$$\begin{aligned} \frac{d\hat{c}_t}{\hat{c}_t} = & \left(\frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{(\sigma_t^x)^2}{2} + \gamma \sigma_t^x \sigma_t^{\hat{c}} + \frac{1 + \gamma}{2} (\sigma_t^{\hat{c}})^2 \right. \\ & \left. + \frac{(\tilde{\sigma}_t^x)^2}{2} + \gamma \tilde{\sigma}_t^x \tilde{\sigma}_t^{\hat{c}} + \frac{1 + \gamma}{2} (\tilde{\sigma}_t^{\hat{c}})^2 \right) dt + \sigma_t^{\hat{c}} dZ_t + \tilde{\sigma}_t^{\hat{c}} d\tilde{Z}_t + dL_t \end{aligned} \quad (51)$$

and the incentive compatibility constraints can be written

$$\sigma_t^x = \hat{c}_t^{-\gamma} \phi \hat{k}_t \beta \quad (52)$$

$$\tilde{\sigma}_t^{\hat{c}} = \frac{\pi}{\gamma} - \tilde{\sigma}_t^x \quad (53)$$

As before, \hat{c} has an upper bound \hat{c}_h , which must be modified to take into account the aggregate risk.

Lemma 16. *For any incentive compatible contract \mathcal{C} , $\hat{c}_t \in (0, \hat{c}_h]$ at all times, where \hat{c}_h is given by (44). If ever $\hat{c}_t = \hat{c}_h$, then the continuation contract satisfies $k_{t+s} = 0$ and $\hat{c}_{t+s} = \hat{c}_h$ for all future times $t + s$, and gives the agent a unique consumption path with growth $(r - \rho)/\gamma + \frac{1+\gamma}{2}(\pi/\gamma)^2$ and exposure to only aggregate risk π/γ . This continuation contract has cost $\hat{v}_h x_t$ to the principal, where $\hat{v}_h \equiv \hat{c}_h^\gamma$.*

The HJB equation As in the case without aggregate risk, the optimal contract can be characterized with the HJB equation

$$\begin{aligned} r\hat{v} = & \min_{\sigma^x, \sigma^{\hat{c}}, \tilde{\sigma}^x} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \beta} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 + \frac{\gamma}{2} (\tilde{\sigma}^x)^2 - \pi \tilde{\sigma}^x \right) \\ & + \hat{v}' \hat{c} \left(\frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \sigma^{\hat{c}} + \frac{1 + \gamma}{2} (\sigma^{\hat{c}})^2 + \frac{(\tilde{\sigma}^x)^2}{2} \right. \\ & \left. + (1 + \gamma) \tilde{\sigma}^x \tilde{\sigma}^{\hat{c}} + \frac{1 + \gamma}{2} (\tilde{\sigma}^{\hat{c}})^2 - \tilde{\sigma}^{\hat{c}} \pi \right) + \frac{\hat{v}''}{2} \hat{c}^2 \left((\sigma^{\hat{c}})^2 + (\tilde{\sigma}^{\hat{c}})^2 \right) \end{aligned} \quad (54)$$

where $\tilde{\sigma}^{\hat{c}} = \frac{\pi}{\gamma} - \tilde{\sigma}^x$. The first order conditions for σ^x and $\sigma^{\hat{c}}$ are the same as before, given by (15) and (16). Now, however, we also have a first order condition for $\tilde{\sigma}^x$

$$\tilde{\sigma}^x = \frac{\pi}{\gamma}$$

which implies $\tilde{\sigma}^{\hat{c}} = \frac{\pi}{\gamma} - \tilde{\sigma}_t^x = 0$. Notice that this is precisely the first-best aggregate risk sharing arrangement. Indeed, there is no conflict between the principal and the agent with

respect to aggregate risk sharing. We can then characterize the optimal contract as follows.

Theorem 3. *The principal's cost function $\hat{v}(\hat{c})$ has a flat portion on $[0, \hat{c}_l]$ and a strictly increasing portion on $[\hat{c}_l, \hat{c}_h]$, for some $\hat{c}_l \in (0, \hat{c}_h)$. The HJB equation (54) holds with equality above \hat{c}_l and with inequality below \hat{c}_l , i.e.*

$$r\hat{v} < \min_{\sigma^x} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\beta} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} \left(\frac{\pi}{\gamma}\right)^2 \right) \quad \forall \hat{c} < \hat{c}_l. \quad (55)$$

Below \hat{c}_l , $\hat{v}(\hat{c}) = \hat{v}(\hat{c}_l)$, and at \hat{c}_l the cost function $\hat{v}(\hat{c})$ satisfies the smooth-pasting condition $\hat{v}'(\hat{c}_l) = 0$, and $\hat{v}''(\hat{c}_l) > 0$.

For all t , aggregate risk sharing is first-best, $\tilde{\sigma}_t^x = \pi/\gamma$ and $\tilde{\sigma}_t^{\hat{c}} = 0$. The optimal contract starts at $\hat{c}_0 = \hat{c}_l$, where σ_0^x is chosen without taking into account its effect on the agent's precautionary motive, to maximize

$$\sigma^x \hat{c}^\gamma \frac{\alpha}{\phi\beta} - \hat{v}(\hat{c}) \frac{\gamma}{2} (\sigma^x)^2 \quad (56)$$

At \hat{c}_l we have $\mu^{\hat{c}}(\hat{c}_l) > 0$ and $\sigma^{\hat{c}}(\hat{c}_l) = 0$. For all $t > 0$, we have $\hat{c}_t > \hat{c}_l$, $\sigma_t^{\hat{c}} \leq 0$ and $\sigma_t^x \geq 0$, but at a level lower than that which maximizes (56).

We also have a verification theorem for the HJB equation.

Theorem 4. *Let $\hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \rightarrow [\hat{v}_l, \hat{v}_h]$ be a strictly increasing C^2 solution to the HJB equation (54) for some $\hat{c}_l \in (0, \hat{c}_h)$, such that $\hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h]$, $\hat{v}'(\hat{c}_l) = 0$, $\hat{v}''(\hat{c}_l) > 0$ and $\hat{v}(\hat{c}_h) = \hat{v}_h$. If $\gamma < \frac{1}{2}$, we also need to check that*

$$1 - \hat{v}_l (\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \alpha^2 (\phi\beta)^{-2} \hat{v}_l^{-2}) \leq 0 \quad (57)$$

Then,

1) For any incentive compatible contract $\mathcal{C} = (c, k)$ that delivers at least utility u_0 to the agent, we have $\hat{v}(\hat{c}_l) ((1-\gamma)u_0)^{\frac{1}{1-\gamma}} \leq J_0(\mathcal{C})$.

2) Let \mathcal{C}^* be a contract generated by the policy functions of the HJB. Specifically, the state variables x^* and \hat{c}^* are solutions to (50) and (51), (with potential absorption at \hat{c}_h), with initial values $x_0^* = ((1-\gamma)u_0)^{\frac{1}{1-\gamma}}$ and $\hat{c}_0^* = \hat{c}_l$. If \mathcal{C}^* is admissible and $\sigma^{\hat{c}^*}$ is bounded, then \mathcal{C}^* is an optimal contract, with cost $J_0(\mathcal{C}^*) = \hat{v}(\hat{c}_l) ((1-\gamma)u_0)^{\frac{1}{1-\gamma}}$.

We can also obtain sufficient conditions for admissibility.

Lemma 17. *If the candidate contract C^* constructed in Theorem 4 has $\mu^{x^*} < r + \pi^2/\gamma$, then C^* is admissible and delivers utility u_0 to the agent.*

Verifying incentive compatibility. As in the case without aggregate shocks, a crucial step in the construction of the optimal contract is the verification of global incentive compatibility. With aggregate shocks, the agent has more options at his disposal. He could potentially find attractive to steal and at the same time save the proceeds by investing them in aggregate risk. However, using the fact that the optimal contract has $\tilde{\sigma}^x = \pi/\gamma$ we can extend the result of Lemma 5 to the case with aggregate risk, as follows.

Lemma 18. *Let $C = (c, k)$ be an admissible contract with associated processes x and \hat{c} satisfying (50) and (51), and (48) and (49), with bounded μ^x , $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$, and with \hat{c} uniformly bounded away from zero and bounded above by \hat{c}_h . Suppose that the contract satisfies the following properties*

$$\begin{aligned}\sigma_t^{\hat{c}} &\leq 0 \\ \tilde{\sigma}_t^x &= \frac{\pi}{\gamma}\end{aligned}$$

Then for any feasible strategy $(\tilde{c}, a, \tilde{\sigma}^h)$, with associated hidden savings h , we have the following upper bound on the utility

$$U_t^{\tilde{c}, a} \leq \left(1 + \frac{h_t}{x_t} \hat{c}_t^{-\gamma}\right)^{1-\gamma} U_t^{c, 0}$$

In particular, since $h_0 = 0$, for any feasible strategy $U_0^{\tilde{c}, a} \leq U_0^{c, 0}$, and the contract C is therefore incentive compatible.

Stationary contracts. We can also define stationary contracts with $\mu^{\hat{c}} = \sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$. We only need to set

$$\frac{1}{2}(\sigma^x)^2 = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma} - \frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 \quad (58)$$

and $\tilde{\sigma}^x = \pi/\gamma$ which implies $\tilde{\sigma}^{\hat{c}} = 0$.

Lemma 19. *Assume $\alpha < \bar{\alpha}$. Take any $\hat{c} \in (\hat{c}_*, \hat{c}_h]$, where*

$$\hat{c}_* \equiv \left(\frac{2\gamma}{1+\gamma}\right)^{\frac{1}{1-\gamma}} \hat{c}_h \in (0, \hat{c}_h) \quad (59)$$

There is an incentive compatible stationary contract for this \hat{c} , where $\sigma_r^x(\hat{c})$ is given by (58), $\tilde{\sigma}_r^x(\hat{c}) = \pi/\gamma$, and the cost $\hat{v}_r(\hat{c})x_0$ is given by

$$\hat{v}_r(\hat{c}) = \frac{\hat{c} - \frac{\alpha}{\phi\beta}\hat{c}^\gamma\sqrt{2}\sqrt{\frac{\hat{c}_h^{1-\gamma}}{1-\gamma}\left(1 - \left(\frac{\hat{c}}{\hat{c}_h}\right)^{1-\gamma}\right)}}{2r - \rho - \frac{1+\gamma}{1-\gamma}\rho + \gamma\left(\frac{\pi}{\gamma}\right)^2 + \frac{\hat{c}^{1-\gamma}}{1-\gamma}(1+\gamma)} \quad (60)$$

For $\hat{c} \leq \hat{c}_*$ the growth rate $\mu_r^x(\hat{c}) > r + \frac{\pi^2}{\gamma}$ and the corresponding stationary contract violates the No-Ponzi condition (3) and is therefore not admissible. Since stationary contracts are not necessarily optimal, we have $\hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c})$.

Proofs

Lemma 15

Since all stochastic processes are now adapted to the filtration generated by both Z and \tilde{Z} , the same argument as in Lemma 1 yields (46). For (47), the argument is similar to Lemma 2. We know that the agent could reduce consumption by a small amount ϵ at any time t and save it to consume later at $t' \geq t$. Since he can invest it in aggregate risk, he obtains $\epsilon \exp(\int_t^{t'} r - \rho + \pi_s \tilde{\sigma}_s^h - \frac{1}{2}(\tilde{\sigma}_s^h)^2 ds + \int_t^{t'} \tilde{\sigma}_s^h d\tilde{Z}_s)$ at time t' . For this to not be attractive, it must be the case that

$$c_t^{-\gamma} \geq \mathbb{E}_t \left[\exp\left(\int_t^{t'} r - \rho + \pi_s \tilde{\sigma}_s^h - \frac{1}{2}(\tilde{\sigma}_s^h)^2 ds + \int_t^{t'} \tilde{\sigma}_s^h d\tilde{Z}_s\right) c_{t'}^{-\gamma} \right]$$

so that the process $Y_t = \exp(\int_0^t r - \rho + \pi_s \tilde{\sigma}_s^h - \frac{1}{2}(\tilde{\sigma}_s^h)^2 ds + \int_0^t \tilde{\sigma}_s^h d\tilde{Z}_s) c_t^{-\gamma}$ is a supermartingale for any $\tilde{\sigma}_s^h$. Using the Doob-Meyer decomposition, the Martingale Representation Theorem, and Ito's lemma as in Lemma 2, we write

$$\frac{dc_t}{c_t} = \mu_t^c dt + \sigma_t^c dZ_t + \tilde{\sigma}_t^c d\tilde{Z}_t + dL_t$$

Since the drift of Y must be weakly negative for any $\tilde{\sigma}^h$, we obtain

$$r - \rho + \pi_t \tilde{\sigma}_t^h - \gamma \mu_t^c + \frac{\gamma}{2}(1+\gamma)(\sigma_t^c)^2 + \frac{\gamma}{2}(1+\gamma)(\tilde{\sigma}_t^c)^2 - \gamma \tilde{\sigma}_t^c \tilde{\sigma}_t^h = 0$$

for any $\tilde{\sigma}_t^h$, and L weakly increasing. Since this must hold for any $\tilde{\sigma}^h$ we get

$$\tilde{\sigma}_t^c = \frac{\pi}{\gamma}$$

$$\mu_t^c = \frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2}(\sigma_t^c)^2 + \frac{1 + \gamma}{2}(\tilde{\sigma}_t^c)^2$$

Finally, the same argument as in Lemma 2 yields (48). This completes the proof.

Lemma 16

The same argument as in Lemma 2 shows that $\hat{c}_t \leq \left(\frac{\rho - r(1 - \gamma)}{\gamma}\right)^{\frac{1}{1 - \gamma}}$, but because we cannot control $\sigma^{\hat{c}}$ independently of σ^x we might have an even lower bound. At this bound, the volatility must be zero (or we would cross it), so $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$, which requires $\tilde{\sigma}^x = \pi/\gamma$. The drift of \hat{c} then simplifies to

$$\mu^{\hat{c}} = \frac{r - \rho}{\gamma} + \frac{1}{2}(\sigma_t^x)^2 + \frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 - \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma}$$

which is greater or equal than $\frac{r - \rho}{\gamma} + \frac{1}{2}\left(\frac{\pi}{\gamma}\right)^2 - \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma}$. This in turn is positive for any $\hat{c} \geq \hat{c}_h$ given by (28). If ever $\hat{c}_t > \hat{c}_h$ there is a positive probability that $\hat{c}_t > \left(\frac{\rho - r(1 - \gamma)}{\gamma}\right)^{\frac{1}{1 - \gamma}}$ which we know cannot happen. We conclude that $\hat{c}_t \leq \hat{c}_h$ always. Since at this point the volatilities and drift are zero, $\hat{c}_{t+s} = \hat{c}_h$ forever. Since $\sigma_{t+s}^x = 0$, (48) implies $k_{t+s} = 0$ for all $s \geq 0$. Since $c_{t+s} = \hat{c}_h x_{t+s}$, and $\tilde{\sigma}_{t+s}^x = \pi/\gamma$, the volatility of consumption is also $\sigma_{t+s}^c = 0$ and $\tilde{\sigma}_t^c = \pi/\gamma$. The growth rate of consumption is then $(r - \rho)/\gamma + \frac{1 + \gamma}{2}(\pi/\gamma)^2$. We can compute the cost of the continuation contract to the principal explicitly and obtain $\hat{v}_h x_t$ where $\hat{v}_h \equiv \hat{c}_h^\gamma$. This completes the proof.

Theorem 9

See Theorem 3.

Theorem 3

The proof is very similar to the case without aggregate shocks in Theorem 1, except we use the definition of $A(\hat{c}, \hat{v})$

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \frac{\left(\frac{\hat{c}^\gamma \alpha}{\phi\beta}\right)^2}{\hat{v}\gamma} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

Also instead of using Lemma 13 to show $\mu^{\hat{c}} > 0$, we use the more general Lemma 21.

Theorem 4

The proof is very similar to the case without aggregate shocks in Lemma 4, except we define

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \frac{\left(\frac{\hat{c}^\gamma \alpha}{\phi\beta}\right)^2}{\hat{v}\gamma} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

and use it to show that the HB holds as an inequality below \hat{c}_l . We also know that for the HJB to have a minimum, it must be the case that $\tilde{\sigma}^x = \pi/\gamma$ and $\tilde{\sigma}^{\hat{c}} = 0$ are the optimal policies. Lemma 13 shows $\mu^{\hat{c}}(\hat{c}_l) > 0$. As a result, we use Lemma 18 to establish the incentive compatibility of the candidate optimal contract.

Lemma 17

We know that $\hat{c}_t^* \in [\hat{c}_l, \hat{c}_h]$ and recall that $\hat{c}_l > 0$. Then an upper bounded $\mu^{x^*} < r + \pi^2/\gamma$ implies a bounded $0 \leq \sigma^{x^*} \leq \bar{\sigma}_X$, and we also know that $\tilde{\sigma}^x = \pi/\gamma$. Then

$$\mathbb{E}^Q \left[\int_0^\infty e^{-rt} (|c_t^*| + |k_t^* \alpha|) dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\bar{\sigma}_X \hat{c}_h^\gamma}{\phi\beta} \alpha \right\} \mathbb{E}^Q \left[\int_0^\infty e^{-rt} x_t^* dt \right] < \infty$$

where the last inequality follows from $\mu^{x^*} < r + \pi^2/\gamma$ (notice the expectations is taken under Q). Let $U^* = \frac{(x^*)^{1-\gamma}}{1-\gamma}$, so using the law of motion of x^* , (8), we get

$$U_0^* = \mathbb{E} \left[\int_0^{\tau^n} e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho\tau^n} U_{\tau^n}^* \right]$$

with $\tau^n \rightarrow \infty$ a.s. Use the monotone convergence theorem and notice that

$$\lim_{n \rightarrow \infty} \mathbb{E} [e^{-\rho\tau^n} U_{\tau^n}^*] = 0$$

because $\rho - (1 - \gamma)(\mu^{x^*} - \frac{\gamma}{2}(\sigma^{x^*})^2 - \frac{\gamma}{2}(\frac{\pi}{\gamma})^2) = \hat{c}^{1-\gamma} \geq \min\{\hat{c}_t^{1-\gamma}, \hat{c}_h^{1-\gamma}\} > 0$. We then get that $U_0^{c^*,0} = U_0^* = u_0$. We conclude that the contract is indeed admissible.

Lemma 18

As in the proof of Lemma 5, we get

$$e^{-\rho t} \left(U_t^{\tilde{c},a} - F(\hat{h}_t, \hat{c}_t) U_t^{c,0} \right) = \mathbb{E}_t^a \left[\int_t^{\tau^n} e^{-\rho u} (1 - \gamma) U_u^{c,0} Y_u d + e^{-\rho\tau^n} \left(U_{\tau^n}^{\tilde{c},a} - F_{\tau^n} U_{\tau^n}^{c,0} \right) \right]$$

where Y_t has the same expression as before, except that a new term \tilde{B}_t appears for the exposure to aggregate risk, $Y_t = A_t + B_t + \tilde{B}_t + C_t$. We have already shown in the proof of Lemma 5 that A_t , B_t , and C_t are non-positive, so we only need to show that the new term $\tilde{B}_t \leq 0$ too. Noting that $\tilde{\sigma}^{\hat{c}} = 0$, we get

$$\begin{aligned} \tilde{B}_t &= \frac{F_{\hat{c},t}}{1-\gamma} \hat{c}_t \left(\frac{1+\gamma}{2} (\tilde{\sigma}_t^x)^2 - \frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 \right) + \frac{F_{\hat{h},t}}{1-\gamma} \hat{h}_t \left(-\frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 + (\tilde{\sigma}_t^x)^2 + \pi \sigma_t^h - \tilde{\sigma}_t^x \sigma_t^h + (1-\gamma) \tilde{\sigma}_t^x (\sigma_t^h - \tilde{\sigma}_t^x) \right) \\ &\quad + \frac{F_{\hat{h}\hat{h},t}}{2(1-\gamma)} \hat{h}_t^2 (\sigma_t^h - \tilde{\sigma}_t^x)^2 \end{aligned}$$

Using $\tilde{\sigma}_t^x = \pi/\gamma$ we simplify to

$$\begin{aligned} \tilde{B}_t &= \frac{F_{\hat{c},t}}{1-\gamma} \hat{c}_t \left(\frac{1+\gamma}{2} (\tilde{\sigma}_t^x)^2 - \frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 \right) + \frac{F_{\hat{h},t}}{1-\gamma} \hat{h}_t \left(-\frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 + (\tilde{\sigma}_t^x)^2 - (1-\gamma) (\tilde{\sigma}_t^x)^2 \right) \\ &\quad + \frac{F_{\hat{h}\hat{h},t}}{2(1-\gamma)} \hat{h}_t^2 (\sigma_t^h - \tilde{\sigma}_t^x)^2 \end{aligned}$$

Use $F_{\hat{h},\hat{h},t} = -\gamma \hat{c}_t^{-2\gamma} (1-\gamma) \left(1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma-1}$ to show the last term is non-positive. We then get

$$\tilde{B}_t \leq \frac{F_{\hat{c},t}}{1-\gamma} \hat{c}_t \left(\frac{1+\gamma}{2} (\tilde{\sigma}_t^x)^2 - \frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 \right) + \frac{F_{\hat{h},t}}{1-\gamma} \hat{h}_t \left(-\frac{\gamma}{2} (\tilde{\sigma}_t^x)^2 + (\tilde{\sigma}_t^x)^2 - (1-\gamma) (\tilde{\sigma}_t^x)^2 \right)$$

$$\begin{aligned}\tilde{B}_t &\leq \frac{(\tilde{\sigma}_t^x)^2}{2} \left(\frac{F_{\hat{c},t}}{1-\gamma} \hat{c}_t + \frac{F_{\hat{h},t}}{1-\gamma} \hat{h}_t \gamma \right) \\ &= \frac{(\tilde{\sigma}_t^x)^2}{2} \left(-\gamma \hat{h}_t \hat{c}_t^{-\gamma-1} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{c}_t + \hat{c}_t^{-\gamma} \left(1 + \hat{h}_t \hat{c}_t^{-\gamma}\right)^{-\gamma} \hat{h}_t \gamma \right) = 0\end{aligned}$$

For the last term we follow the same strategy as in the proof of Lemma 5. Since the agent's strategy $(\tilde{c}, a, \tilde{\sigma}^h)$ is feasible, we have that

$$\lim_{n \rightarrow \infty} \mathbb{E}_t^a \left[e^{-\rho \tau^n} U_{\tau^n}^{\tilde{c}, a} \right] = 0$$

If $\gamma < 1$ we use Fatou's lemma to show

$$\lim_{n \rightarrow \infty} \mathbb{E}_t \left[e^{-\rho \tau^n} F_{\tau^n} U_{\tau^n}^{c, 0} \right] \geq 0$$

If instead $\gamma > 1$, let $N_t = x_t + h_t \hat{c}_t^{-\gamma}$, so that $F_t U_t^{c, 0} = \frac{N_t^{1-\gamma}}{1-\gamma}$. The law of motion of N satisfies the inequality

$$dN_t \leq ((\lambda_1 + \lambda_2 \tilde{\sigma}_t^N) N_t - \lambda_3 \tilde{c}_t) dt + \sigma_t^N N_t dZ_t^a + \tilde{\sigma}_t^N N_t d\tilde{Z}_t$$

where $\lambda_1 = \bar{\mu}^x + r + \hat{c}^{1-\gamma} + \gamma |\bar{\mu}^{\hat{c}}| + \gamma(1+\gamma)(\bar{\sigma}^{\hat{c}})^2$, $\lambda_2 = \pi$, and $\lambda_3 = \hat{c}_h^{-\gamma}$. Here is where we use the assumption that μ^x , $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$ are bounded and that $\hat{c} \leq \hat{c}_h$ and uniformly bounded away from zero, and also $\tilde{\sigma}_t^x = \pi/\gamma$ which implies $\tilde{\sigma}_t^{\hat{c}} = 0$. Stealing only reduces the drift of N_t because the effect on the drift of x and h cancel's out, and it increases the drift of \hat{c} . Since $N_t > 0$ always, Lemma 12 and it's corollary ensure that $\lim_{n \rightarrow \infty} \mathbb{E}_t \left[e^{-\rho \tau^n} F_{\tau^n} U_{\tau^n}^{c, 0} \right] \geq 0$. This completes the proof.

Lemma 19

The proof is similar to the case without aggregate shocks in Lemma 6, using the HJB (54). First, $\hat{v}_r(\hat{c}) > 0$ for all $\hat{c} \in (\hat{c}_*, \hat{c}_h)$ from Lemma 20. We can check that $\mu^x < r + \frac{\pi^2}{\gamma}$ for the stationary contract if and only if $\hat{c} > \hat{c}_*$, where \hat{c}_* is given by (59). Arguing as in the proof of Lemma 17 we can show that the stationary contract is admissible and delivers utility u_0 to the agent if and only if $\hat{c} > \hat{c}_*$. Since the contract satisfies (50), (51), and (52) and (53) by construction, Lemma 18 then ensures that it is incentive compatible. This completes the proof.

Lemma 20. *The cost function of stationary contracts $\hat{v}_r(\hat{c})$ defined by (25) is strictly positive for all $\hat{c} \in (\hat{c}_*, \hat{c}_h]$ if and only if $\alpha < \bar{\alpha}$.*

Proof. We need to check the numerator in (60), since the denominator is positive for all $\hat{c} \geq \hat{c}_*$:

$$\hat{c} \left[1 - \frac{\alpha}{\phi\beta} \sqrt{2} \sqrt{\frac{1}{1-\gamma} \hat{c}^{\gamma-1} \left(\left(\frac{\hat{c}_h}{\hat{c}} \right)^{1-\gamma} - 1 \right)} \right]$$

The rest of the proof consists of evaluating this expression at $\hat{c} = \hat{c}_*$ and showing it is non-positive iff the bound is violated. We get \hat{c} times

$$1 - \frac{\alpha}{\phi\beta} \sqrt{2} \sqrt{\frac{1}{1-\gamma} \frac{1+\gamma}{2\gamma} (\hat{c}_h)^{\gamma-1} \left(\frac{1+\gamma}{2\gamma} - 1 \right)} = 1 - \frac{\alpha}{\phi\beta} \sqrt{2} \sqrt{1+\gamma} \frac{1}{2\gamma} \sqrt{(\hat{c}_h)^{\gamma-1}}$$

So if $\alpha \geq \bar{\alpha}$ the numerator is non-positive, and if $\alpha < \bar{\alpha}$ then it's strictly positive. This completes the proof. \square

Lemma 21. *Let x and \hat{c} satisfy the laws of motion (50) and (51), assume that there is a finite cost function $\hat{v}(\hat{c})$. Assume that, for some $\hat{c}_l \in (0, \hat{c}_h)$ and $\hat{v}_l \leq \hat{v}(\hat{c}_l)$, as $\hat{c} \searrow \hat{c}_l$ we have $\sigma^{\hat{c}}(\hat{c}) \rightarrow 0$ and $A(\hat{c}; \hat{v}_l) \rightarrow 0$ and $\tilde{\sigma}^x(\hat{c}) = \frac{\pi}{\gamma}$ and $\tilde{\sigma}^{\hat{c}}(\hat{c}) = 0$, where*

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \frac{\left(\frac{\hat{c}^\gamma \alpha}{\phi\beta} \right)^2}{\hat{v}^\gamma} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)$$

Then $\mu^{\hat{c}}(\hat{c}_l) > 0$.

Proof. Looking at (51), with $\sigma^{\hat{c}} = \tilde{\sigma}^{\hat{c}} = 0$ we get for the drift

$$\mu^{\hat{c}} = \frac{r - \rho}{\gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\tilde{\sigma}^x)^2 - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}$$

So $\mu^{\hat{c}} > 0$ implies

$$\frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\tilde{\sigma}^x)^2 > \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}$$

Since we also want $A(\hat{c}; \hat{v}) = 0$, we get

$$0 = \hat{c} - r\hat{v} + \hat{v} \left(\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} \left(\frac{\pi}{\gamma} \right)^2 \right)$$

$$< \hat{c} - \hat{v}\hat{c}^{1-\gamma} \equiv M$$

Notice that if $\hat{v} = \hat{c}^\gamma$ we have $M = 0$. If $\hat{v} > \hat{c}^\gamma$ we have $M < 0$ and if $\hat{v} < \hat{c}^\gamma$ we have $M > 0$. So for $A(\hat{c}; \hat{v}) = 0$ and $\mu^{\hat{c}} > 0$ we need $\hat{v} < \hat{c}^\gamma$. In fact, if $\hat{v} = \hat{c}^\gamma$ and in addition

$$\frac{1}{2} \left(\frac{\alpha}{\phi\beta\gamma} \right)^2 + \frac{1}{2} \left(\frac{\pi}{\gamma} \right)^2 = \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma} \quad (61)$$

then we have $A = 0$ and $\mu^{\hat{c}} = 0$ (notice that with $\hat{v}' = 0$ and $\hat{v} = \hat{c}^\gamma$ the FOC for σ^x (15) yields $\sigma^x = \frac{\alpha}{\phi\beta\gamma}$). In this case, because we have $\mu^{\hat{c}} = 0$ we therefore have the value of a stationary contract, i.e. $\hat{v} = \hat{v}_r(\hat{c})$ given by (60). So let (\hat{c}_T, \hat{v}_T) be defined by $\hat{v}_T = \hat{c}_T^\gamma$ and (61). Suppose this point exists for $\hat{c}_T \in [\hat{c}_*, \hat{c}_h]$. It must be the case that $\hat{v}_T \geq \hat{v}_l$ since \hat{v}_T is the cost of a stationary contract (because $\hat{c}_T \geq \hat{c}_*$ so it's an incentive compatible stationary contract) but not optimal, and we assumed $\hat{v}_l \leq \hat{v}(\hat{c}_l)$ (possibly the same).

First we will show that $\mu^{\hat{c}} \geq 0$, and then make the inequality strict. Towards contradiction, suppose $\mu^{\hat{c}} < 0$ at \hat{c}_l . Then it must be the case that $\hat{v}_l > \hat{c}_l^\gamma$ because we have $A(\hat{c}_l, \hat{v}_l) = 0$. We will show that $A(\hat{c}_l, \hat{v}_l) > 0$ and get a contradiction. First take the derivative of A :

$$A'_c(\hat{c}_l, \hat{v}_l) = 1 - \hat{v}_l \left(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality holds for all $\hat{c} < \hat{v}_l^{\frac{1}{\gamma}}$. So $A(\hat{c}_l, \hat{v}_l) > A(\hat{v}_l^{\frac{1}{\gamma}}, \hat{v}_l)$. Letting $\hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}}$ we get

$$\begin{aligned} A(\hat{c}_l, \hat{v}_l) &> \hat{c}_m - r\hat{v}_l + \hat{v}_l \left(\frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \frac{1}{2} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\gamma} - \frac{\gamma}{2} \left(\frac{\pi}{\gamma} \right)^2 \right) \\ &= \hat{c}_m - r\hat{v}_l + \hat{v}_l \left(\frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \gamma \frac{\rho - \hat{c}_T^{1-\gamma}}{1-\gamma} - (\rho - r) \right) \\ \implies A(\hat{c}_l, \hat{v}_l) &> \hat{c}_m + \hat{v}_l \frac{\gamma \hat{c}_T^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} = \hat{c}_m^\gamma \frac{\hat{c}_T^{1-\gamma} - \hat{c}_m^{1-\gamma}}{1-\gamma} \geq 0 \end{aligned}$$

where the last equality uses $\hat{v}_l = \hat{c}_m^\gamma$ and the last inequality uses $\hat{c}_m = \hat{v}_l^{\frac{1}{\gamma}} \leq \hat{v}_T^{\frac{1}{\gamma}} = \hat{c}_T$. This is a contradiction, and therefore it must be the case that $\mu^{\hat{c}} \geq 0$ at \hat{c}_l .

It's clear from the previous argument that $\mu^{\hat{c}}(\hat{c}_l) = 0$ only if $(\hat{c}_l, \hat{v}_l) = (\hat{c}_T, \hat{v}_T)$. This can only happen if the optimal contract is indeed stationary. We will show this cannot be the case because $\alpha > 0$. First, note that (\hat{c}_T, \hat{v}_T) is a tangency point where $\hat{v}_r(\hat{c})$ touches

the locus $\hat{v}_b(\hat{c})$ defined by $A(\hat{c}; \hat{v}_b(\hat{c})) = 0$ (indeed, it is easy to see that $\hat{v}_b(\hat{c}) \leq \hat{v}_r(\hat{c})$ because A is the result of minimizing the HJB with $\hat{v}' = \hat{v}'' = 0$, while \hat{v}_r sets $\sigma^{\hat{c}} = 0$). If $(\hat{c}_l, \hat{v}_l) = (\hat{c}_T, \hat{v}_T)$ then this must be the minimum point for $\hat{v}_r(\hat{c})$, so the derivative of both $\hat{v}_r(\hat{c})$ and $\hat{v}_b(\hat{c})$ must be zero. This means that $A'_{\hat{c}}(\hat{c}_l, \hat{v}_l) = 0$. However,

$$1 - \hat{v}_l \left(\hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \left(\frac{\alpha}{\phi\beta} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0$$

where the inequality follows from $\hat{v}_l = \hat{v}_T = \hat{c}_l^\gamma$ (note that $\hat{c}_l > 0$ because as Lemma 11 shows $A(\hat{c}, \hat{v}_l)$ is strictly positive for \hat{c} near 0). This can't be a minimum of $\hat{v}_r(\hat{c})$. Therefore $(\hat{c}_l, \hat{v}_l) \neq (\hat{c}_T, \hat{v}_T)$ and $\mu^{\hat{c}}(\hat{c}_l) > 0$.

It only remains to show that (\hat{c}_T, \hat{v}_T) exists. Because there is a finite cost function, we know from Lemma 14 that $\alpha < \bar{\alpha}$. Then the lhs of (61) is positive and is bounded by

$$\frac{1}{2} \left(\frac{\pi}{\gamma} \right)^2 < lhs < \frac{1}{1+\gamma} \hat{c}_h^{1-\gamma} + \frac{1}{2} \left(\frac{\pi}{\gamma} \right)^2$$

The rhs of (61) can be written

$$\begin{aligned} rhs &= \frac{\rho - r(1-\gamma)}{(1-\gamma)\gamma} - \frac{\hat{c}^{1-\gamma}}{1-\gamma} = (\rho - r(1-\gamma)) \left(\frac{1}{(1-\gamma)\gamma} - \frac{\hat{c}^{1-\gamma}}{1-\gamma} \frac{1}{\rho - r(1-\gamma)} \right) \\ &= \frac{(\rho - r(1-\gamma))}{\gamma} \left(\frac{1}{1-\gamma} - \frac{\hat{c}^{1-\gamma}}{1-\gamma} \frac{\gamma}{\rho - r(1-\gamma)} \right) \end{aligned}$$

For $\hat{c} = \hat{c}_h$ the rhs is $\frac{1}{2} \left(\frac{\pi}{\gamma} \right)^2$. For $\hat{c} = \hat{c}_*$ we get

$$rhs = \frac{1}{1+\gamma} \hat{c}_h^{1-\gamma} + \frac{1}{2} \left(\frac{\pi}{\gamma} \right)^2$$

Then there is a $\hat{c} \in [\hat{c}_*, \hat{c}_h]$ solving (61) as desired. This completes the proof. \square

Appendix C - numerical algorithm

TO BE COMPLETED