

# A Simple Theory of Asset Pricing under Model Uncertainty

Leonid Kogan and Tan Wang\*

August, 2003

## Abstract

The focus of our paper is on the implications of model uncertainty for the cross-sectional properties of returns. We perform our analysis in a tractable single-period mean-variance framework. We show that there is an uncertainty premium in equilibrium expected returns on financial assets and study how the premium varies across the assets. In particular, the cross-sectional distribution of expected returns can be formally described by a two-factor model, where expected returns are derived as compensation for the asset's marginal contribution to the equilibrium risk and uncertainty of the market portfolio. Thus, the standard result that expected returns are related only to systematic, and not diversifiable risk, carries over to economies with model uncertainty as well. Our two-factor pricing model also illustrates that model uncertainty in financial markets may be distinguished from risk, addressing some of the observational equivalence issues raised in the literature.

---

\*Leonid Kogan is with the Sloan School of Management of the Massachusetts Institute of Technology, Cambridge, MA 02142, USA. lkogan@mit.edu; Tan Wang is with the Sauder School of Business, University of British Columbia, Vancouver, British Columbia, Canada, V6T 1Z2. tan.wang@commerce.ubc.ca. The authors are grateful to Lorenzo Garlappi, Burton Hollifield, Raman Uppal, Stan Zin, and seminar participants at Carnegie Mellon University and MIT for helpful comments. The paper was presented at the 2002 Western Finance Association Meetings. Tan Wang thanks the Social Sciences and Humanities Research Council of Canada for financial support.

# 1 Introduction

The purpose of this paper is to study the implications of model uncertainty for the cross-sectional properties of asset prices in a simple equilibrium setting.

The recent focus on model uncertainty in the literature is driven by the difficulty of reconciling traditional asset pricing theories with the empirical data. Limited success of the standard theories could be in part due to the commonly made assumption that economic agents possess perfect knowledge of the data generating process. For instance, the classical theories of Sharpe (1964), Lucas (1978), Breeden (1979) and Cox, Ingersoll and Ross (1985), assume that, while the payoffs of financial assets are random, agents know the underlying probability law exactly. In reality this is often not the case. Then the natural question is: how are the prices of financial assets affected by investors' lack of knowledge about the probability law, or their uncertainty about what the true model is.

The importance of model uncertainty has long been recognized in finance. While the literature appears under different names, such as parameter uncertainty, Knightian uncertainty, the defining characteristic of that literature is the recognition of the fact that the agents of the economy do not have a perfect knowledge of the probability law that governs the realization of the states of the world. Various issues have been studied. Dow and Werlang (1992) use the uncertainty averse preference model developed by Schmeidler (1989) to study a single period portfolio choice problem. Maenhout (1999) examines a similar problem in a continuous-time economy, but from the point of view of robust portfolio rules. Kandel and Stambaugh (1996), Brennan (1998), Barberis (2000), and Xia (2001) show that parameter uncertainty can affect significantly investors' portfolio choice. Frost and Savarino (1986), Gennotte (1986), Balduzzi and Liu (1999), Pastor (2000) and Uppal and Wang (2001) examine the implication of model uncertainty for portfolio choices when there are multiple risky assets. Detemple (1986), Epstein and Wang (1994), Chen and Epstein (2001), Epstein and Miao (2001), and Brennan and Xia (2001) study the implications for equilibrium asset prices in the representative agent and heterogenous agent economies respectively. Routledge and Zin (2002) examine the connection between model uncertainty and liquidity. There is also a significant literature, for example Lewellen and Shanken (2001) and Brav and Heaton (2002), on the effect of learning about an unknown parameter on the equilibrium asset prices.

The focus of our paper is on the cross-sectional properties of returns. We perform our analysis in a tractable single-period mean-variance framework. We show that there is an uncertainty premium in equilibrium expected returns on financial assets and study how the premium varies across the assets. We find that the cross-sectional distribution of expected returns can be formally described by a two-factor model, where expected returns are derived as compensation for the asset's contribution to the equilibrium risk and uncertainty of the portfolio held by the agent. Thus, the standard result that expected returns are related only to systematic, and not diversifiable risk, carries over to economies with model uncertainty as well.

In light of the large empirical literature on the cross-sectional characteristics of asset returns, understanding the implications of model uncertainty and uncertainty aversion even in such a simple setting is of significant value. While prior research on model uncertainty has been concerned with its implications for the time-series of asset prices, by characterizing the cross-section of returns we are able to address some of the observational equivalence issues raised in the literature. That is, whether model uncertainty in financial markets can be distinguished from risk, and whether uncertainty aversion of the representative agent can be distinguished from risk aversion (Anderson, Hansen and Sargent, 1999).

In the rest of this introduction, we will describe briefly our approach to formalizing model uncertainty and its relation to the literature. The most common way of modelling imperfect knowledge of the model and parameters is in the Bayesian framework (Kandel and Stambaugh (1996), Lewellen and Shanken (2001), Barberis (2000) and Pástor (2000)). The key feature of this approach is that if a parameter of the model is unknown, a prior distribution of the parameter is introduced. The second approach, adopted by Dow and Werlang (1992), Epstein and Wang (1994, 1995), Chen and Epstein (2001), Epstein and Miao (2001), and the third approach, adopted by Anderson, Hansen and Sargent (1999), Maenhout (1999), Uppal and Wang (2001), follow the view of Knight (1921) that model uncertainty, or more precisely, the decision makers' view of model uncertainty, cannot be represented by a probability prior. There is a significant literature in psychology and experimental economics that documents the contrast between the Bayesian and Knightian approaches. The evidence documented there is that, when faced with uncertainty about the true probability law, people's behavior tend to be inconsistent with the prediction of

the Bayesian approach (Ellsberg (1963)). In fact, the behavior is inconsistent with any probabilistically sophisticated preference (Machina and Schmeidler (1992)).

The second and third approaches differ in how uncertainty and uncertainty aversion are modelled. Maenhout (1999), Uppal and Wang (2001), use the preference first introduced by Anderson, Hansen and Sargent (1999) in their study of the implications of preference for robustness for macroeconomic and general asset pricing issues.<sup>1</sup> This class of preferences has been extended in Uppal and Wang (2001), and axiomatized in a static setting in Wang (2001). For this class of preferences, uncertainty is described by a set of priors and the investor's aversion to it is introduced through a penalty function. Dow and Werlang (1992), Epstein and Wang (1994, 1995), Chen and Epstein (2001) and Epstein and Miao (2001) use the multi-prior expected utility developed by Gilboa and Schmeidler (1989).<sup>2</sup> Here both uncertainty and uncertainty aversion are introduced through a set of priors. This paper is based on the multi-prior expected utility preferences with a careful design of the set of priors to distinguish between the uncertainty and uncertainty aversion aspects of the set.

The rest of the paper is organized as follows. Section 2 describes the model. Section 4 presents the main result of this paper, the asset pricing implications of model uncertainty. Section 5 concludes.

## 2 The Model

In this section we formulate the individual choice problem under model uncertainty. We define a new measure of uncertainty and study its properties. We show that such a measure parallels in many respects the notion of variance as a measure of risk. In particular, as with return variance, our measure of uncertainty allows for a meaningful concept of diversification.

---

<sup>1</sup>See Hansen and Sargent (2001) for more on this type of preferences.

<sup>2</sup>Dow and Werlang is based more directly on the Choquet expected utility developed by Schmeidler (1989). However, for the case they studied, Choquet expected utility coincides with multi-prior expected utility.

## 2.1 The Setting

We assume a one-period representative agent economy. Consumption takes place only at the end of the period. The agent is endowed with an initial wealth  $W_0$ . Without loss of generality, we assume  $W_0 = 1$ .

The financial markets consist of  $N$  risky assets in perfectly elastic supply and one risk-free asset in zero net supply. As indicated in the introduction, the investors do not have perfect knowledge of the distribution of the returns of the  $N$  risky assets. More specifically, they know that the returns  $R = (R_1, \dots, R_N)^\top$  follow a joint normal distribution with density function

$$f(R) = (2\pi)^{-n/2} |\Omega|^{-1/2} \exp \left\{ -\frac{1}{2} (R - \mu)^\top \Omega^{-1} (R - \mu) \right\}$$

where

$$\mu = \mathbb{E}[R], \quad \Omega = \mathbb{E}[(R - \mu)(R - \mu)^\top].$$

The risk of returns is summarized by the non-degenerate variance-covariance matrix  $\Omega$ . We assume that investors have precise knowledge of  $\Omega$ . However, they do not know exactly the mean return vector  $\mu$ . This is motivated by the fact that it is much easier to obtain accurate estimates of the variance and covariance of returns than their expected values, e.g., Merton (1992). The imperfect knowledge of the asset return distribution gives rise to model uncertainty.

## 2.2 The Preferences

Each agent in the economy has a state-independent utility function  $u(W)$ . Due to lack of perfect knowledge of the probability law of asset returns, however, the agent's preference is not represented by the standard expected utility, but instead by a multi-prior expected utility

$$U(W, \mathcal{P}(P)) = \min_{Q \in \mathcal{P}(P)} \{ \mathbb{E}^Q[u(W)] \}, \quad (1)$$

where  $\mathbb{E}^Q$  denotes the expectation under the probability measure  $Q$ ,  $\mathcal{P}(P)$  is a set of probability measures that depends on the probability measure  $P$ , called the reference prior. This

multi-prior structure of preferences exhibits uncertainty aversion. The set  $\mathcal{P}(P)$  captures the degree of model uncertainty perceived by the agent. Higher degree of model uncertainty is captured by a larger set  $\mathcal{P}(P)$ . The general nature and the axiomatic foundation of these preferences has been well studied in the literature (Gilboa and Schmeidler (1989)). What is specific to this paper is the structure of  $\mathcal{P}(P)$ , which we now describe.

## A Single Source of Information

We begin with the basic case when the shape of the set  $\mathcal{P}$  is derived from a single source of information about the distribution of stock returns. Specifically, we define

$$\mathcal{P}(P) = \{Q : E[\xi \ln \xi] \leq \eta\},$$

where  $\xi$  is the density of  $Q$  with respect to  $P$  and  $\eta$  is a parameter to be described shortly. Mathematically, the set  $\mathcal{P}$  includes all probability measures that are close to the reference measure  $P$ , where the distance is measured by the relative entropy index  $E[\xi \ln \xi]$ . The idea of defining the set  $\mathcal{P}$  using the relative entropy index is not new and has been used in the robust control literature (see Hansen, Sargent, Turmuhambetova and Williams (2002) for a formal connection between the robust control and multiple prior expected utility formulations.

The above definition of  $\mathcal{P}(P)$  has an intuitive interpretation. Since the investor lacks a perfect knowledge of the probability law of the returns, he may use econometric techniques to estimate a particular model of asset returns. As a result, the investor would come up with a model described by the probability measure  $P$ . However, he is not completely confident that this is the true model, due to not having enough data in the specification analysis and the parameter estimation, or due to simplifying assumptions made for tractability. On the other hand, the econometric analysis does provide more information than just the probability measure  $P$ . The true model can be narrowed down to a set  $\mathcal{P}$  of probability measures. Each element in  $\mathcal{P}$  is a possible alternative to the reference prior  $P$ . Let  $Q$  be an element in  $\mathcal{P}$  and let its density be denoted by  $\xi$ , so that

$$dQ = \xi dP. \tag{2}$$

Knowing that the reference measure  $P$  is subject to misspecification and that the possible alternative is  $Q$ , the problem is how to evaluate the alternative. For this purpose we use the

relative entropy index,  $E[\xi \ln \xi]$ . One interpretation of the index is that it is an approximation to the empirical log-likelihood ratio.<sup>3</sup> To elaborate, suppose that the data set available to the investor has  $T$  observations. Then the empirical log-likelihood ratio of the two models is

$$\frac{1}{T} \sum_{t=1}^T \ln \xi(X_t).$$

Now suppose that  $X_t$ ,  $t = 1, \dots, T$ , takes finitely many values,  $x_1, \dots, x_k$  in the data series. Then

$$\frac{1}{T} \sum_{t=1}^T \ln \xi(X_t) = \frac{1}{T} \sum_{i=1}^k \sum_{X_t=x_i} \ln \xi(X_t) = \sum_{i=1}^k \frac{T_i}{T} \ln \xi(x_i),$$

where  $T_i$  is the number of  $t$  such that  $X_t = x_i$ . By the law of large numbers, under the alternative model  $Q$ ,  $T_i/T$  converges to  $Q(x) = \xi(x)P(x)$  and hence  $\frac{1}{T} \sum_{t=1}^T \ln \xi(X_t)$  converges to  $E[\xi \ln \xi]$ . Thus, if  $Q$  is the true probability law,  $E[\xi \ln \xi]$  is a good approximation to the empirical log-likelihood when  $T$  is large. According to the traditional likelihood ratio theory, if the above sum is large, then the two alternatives,  $Q$  and  $P$ , can be clearly distinguished.<sup>4</sup> Therefore, the set of possible alternative models is given by

$$\mathcal{P}(P) = \{Q : E[\xi \ln \xi] \leq \eta\}$$

where  $\eta$  is the parameter describing how much uncertainty there is about the reference probability  $P$ . For example,  $\eta$  could be chosen to define a rejection region for a test of the reference model  $P$  with a 95% confidence level. The choice of  $\eta$  depends on the investor's aversion to uncertainty. Larger values of  $\eta$  allow for a larger set of alternative models. Thus, more uncertainty averse agents are willing to entertain alternative models that are relatively far from the reference model  $P$ , as measured by their relative entropy. An investor more averse to uncertainty would require a higher confidence level.

For analytical tractability, we assume that stock returns are jointly normally distributed under the alternative models. Furthermore, we assume that the variance-covariance matrix

---

<sup>3</sup>See Anderson, Hansen and Sargent (1999) and Hansen and Sargent (2000) for other interpretations of the index.

<sup>4</sup>It is worth emphasizing that large  $\frac{1}{T} \sum_{t=1}^T \xi(X_t) \ln \xi(X_t)$  should not be interpreted as evidence for *rejecting* the reference model  $P$ , as in the usual likelihood test: as explained above, the very fact that  $P$  is the reference prior implies that the investor has already gone through the preliminary analysis and picked  $P$ . The issue at this stage is only to find an index that summarizes the information available.

of the returns is the same under all measures in  $\mathcal{P}$ , reflecting the fact that the investor knows the variance-covariance matrix  $\Omega$  precisely. Let  $Q$  be a measure in  $\mathcal{P}$  with the density (with respect to  $P$ ) given by

$$(2\pi)^{-n/2} |\Omega|^{-1/2} \exp \left\{ -\frac{1}{2} (R - \hat{\mu})^\top \Omega^{-1} (R - \hat{\mu}) \right\},$$

which can be written as

$$(2\pi)^{-n/2} |\Omega|^{-1/2} \exp \left\{ -\frac{1}{2} (R - \mu)^\top \Omega^{-1} (R - \mu) \right\} \\ \times \exp \left\{ -\frac{1}{2} (\mu - \hat{\mu})^\top \Omega^{-1} (\mu - \hat{\mu}) - (\mu - \hat{\mu})^\top \Omega^{-1} (R - \mu) \right\}.$$

Thus, the likelihood ratio of  $Q$  over  $P$  is given by

$$\xi(R) = \exp \left\{ \frac{1}{2} (\mu - \hat{\mu})^\top \Omega^{-1} (\mu - \hat{\mu}) - (\mu - \hat{\mu})^\top \Omega^{-1} (R - \hat{\mu}) \right\}. \quad (3)$$

Given this particular structure of the set  $\mathcal{P}(P)$ , we can introduce  $v = \mu - \hat{\mu}$  and re-write the representative investor's objective as

$$\min_{v \in \mathcal{V}} E [\xi u(W)], \quad (4)$$

where  $\xi$  is now given by

$$\xi(R) = \exp \left\{ \frac{1}{2} v^\top \Omega^{-1} v - v^\top \Omega^{-1} (R - \mu + v) \right\} \quad (5)$$

and the set  $\mathcal{V}$  corresponds to  $\mathcal{P}$ :

$$\mathcal{V} = \left\{ v : E[\xi \ln \xi] = \frac{1}{2} v^\top \Omega^{-1} v \leq \eta \right\}.$$

## Multiple Sources of Information

In general, the investor's knowledge about the distribution of asset returns may come from different sources and it is often about a subset of the assets, as opposed to the joint distribution of all assets as in the previous subsection. To accommodate this, let  $J_k$ ,  $k = 1$ ,

$\dots, K$ , be subsets of  $\{1, \dots, N\}$ , each set  $J_k$  having  $N_k$  elements,  $J_k = \{j_1, \dots, j_{N_k}\}$ , so that the information is about the distribution of  $R_{J_k} = (R_{j_1}, \dots, R_{j_{N_k}})$ . Sets  $J_k$  are not necessarily disjoint. We assume that  $\cup_k J_k = \{1, \dots, N\}$ , so that the investor has at least some information about each asset. We assume that the reference probability distributions implied by the various sources of information for the corresponding subsets of assets coincide with the marginal distributions of the reference model  $P$ . Consider the density function of the distribution of  $R_{J_k}$ ,

$$(2\pi)^{-1} |\Omega_{J_k}|^{-1/2} \exp \left\{ -\frac{1}{2} (R_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (R_{J_k} - \hat{\mu}_{J_k}) \right\},$$

where  $\hat{\mu}_{J_k} = (\hat{\mu}_{j_1}, \dots, \hat{\mu}_{j_{N_k}})$ , and  $\Omega_{J_k}$  is the variance-covariance matrix of  $R_{J_k}$ , which is a sub-matrix of  $\Omega$ . This density function can be written as

$$\begin{aligned} & \exp \left\{ -\frac{1}{2} (\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (\mu_{J_k} - \hat{\mu}_{J_k}) - (\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (R_{J_k} - \mu_{J_k}) \right\} \\ & \times (2\pi)^{-1} |\Omega_{J_k}|^{-1/2} \exp \left\{ -\frac{1}{2} (R_{J_k} - \mu_{J_k})^\top \Omega_{J_k}^{-1} (R_{J_k} - \mu_{J_k}) \right\}. \end{aligned}$$

Thus, the likelihood ratio of the marginal distribution  $Q_{J_k}$  over  $P_{J_k}$  is

$$\xi_{J_k} = \exp \left\{ \frac{1}{2} (\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (\mu_{J_k} - \hat{\mu}_{J_k}) - (\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (R_{J_k} - \hat{\mu}_{J_k}) \right\}.$$

To relate to the probability measure  $Q$ , suppose its density function is

$$(2\pi)^{-n/2} |\Omega|^{-1/2} \exp \left\{ -\frac{1}{2} (R - \hat{\mu})^\top \Omega^{-1} (R - \hat{\mu}) \right\}.$$

Then

$$\begin{aligned} & (2\pi)^{-1} |\Omega_{J_k}|^{-1/2} \exp \left\{ -\frac{1}{2} (R_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (R_{J_k} - \hat{\mu}_{J_k}) \right\} \\ & = \int (2\pi)^{-n/2} |\Omega|^{-1/2} \exp \left\{ -\frac{1}{2} (R - \hat{\mu})^\top \Omega^{-1} (R - \hat{\mu}) \right\} dR_{J_k-}, \end{aligned}$$

where  $J_k-$  is the complement of the set  $J_k$ :  $J_k- = \{1, \dots, N\} \setminus J_k$ . Thus,  $\xi_{J_k}$  is the likelihood ratio of the marginal distribution of  $Q$  over that of  $P$ .

For notational convenience, let  $\hat{\Omega}_{J_k}^{-1}$  denote the  $N \times N$ -matrix whose element in the  $j_m$ th row and  $j_n$ th column, for  $j_m$  and  $j_n$  in  $J_k$ , is equal to the element in the  $m$ th row and  $n$ th column of the matrix  $\Omega_{J_k}^{-1}$ ; all other elements are zero. Then

$$(\mu_{J_k} - \hat{\mu}_{J_k})^\top \Omega_{J_k}^{-1} (\mu_{J_k} - \hat{\mu}_{J_k}) = (\mu - \hat{\mu})^\top \hat{\Omega}_{J_k}^{-1} (\mu - \hat{\mu}) = v^\top \hat{\Omega}_{J_k}^{-1} v$$

In the case where there are multiple sources of information, the representative investor's preferences are described by

$$\min_{v \in \mathcal{V}} \mathbb{E} [\xi u(W)], \quad (6)$$

where  $\xi$  is given by (5), and similarly to the single source information case,

$$\mathcal{V} = \{v : \mathbb{E}[\xi_{J_k} \ln \xi_{J_k}] = \frac{1}{2} v^\top \hat{\Omega}_{J_k}^{-1} v \leq \eta_k, \quad k = 1, \dots, K\}. \quad (7)$$

## 2.3 A Measure of Uncertainty

To understand how the investor trades off uncertainty and expected return, it is useful to introduce a metric for uncertainty about the distribution of returns. This metric is independent of the utility function  $u(W)$  and is determined only by the set  $\mathcal{P}$ . We show in the next section that our measure of uncertainty shares many properties with the variance as a standard measure of risk of returns.

Let  $x$  be a return on a portfolio  $\theta$ ,  $x = \theta^\top R$ . It's distribution is normal and its variance is the same under  $P$  and all measures  $Q \in \mathcal{P}$ . Define

$$\Delta(x) = \sup_{Q \in \mathcal{P}} \mathbb{E}^Q[x] - \mathbb{E}^P[x] \quad (8)$$

to be the uncertainty of  $x$ . Equivalently,

$$\Delta(\theta) = \sup_v \theta^\top v \quad (9)$$

subject to

$$\mathbb{E}[\xi_{J_k} \ln \xi_{J_k}] = \frac{1}{2} v^\top \hat{\Omega}_{J_k}^{-1} v \leq \eta_k, \quad k = 1, \dots, K. \quad (10)$$

If  $\eta_k$ ,  $k = 1, \dots, K$ , are interpreted as defining confidence sets for the marginal distribution of assets in sets  $J_k$ , then  $[-\Delta(\theta), \Delta(\theta)]$  is the corresponding confidence interval for the expected return on the portfolio  $\theta$ .

The measure of uncertainty  $\Delta(\theta)$  is independent of the utility function  $u(W)$ . Thus, our definition of uncertainty reflects the properties of the set  $\mathcal{P}$  of candidate probability measures, not the preferences of the decision maker.

We will denote a solution of (9) by  $v(\theta)$ . Note that the solution may not be unique in general, with multiple values of  $v$  corresponding to the same value of the objective function. The following lemma shows that when all portfolio weights are non-zero, which is the case for the market portfolio in equilibrium, the solution of (9) is indeed unique.<sup>5</sup>

**Proposition 1** *For  $\theta$  such that all of its components are non-zero, the solution of (9) is unique. There exists nonnegative coefficients  $\phi_k(\theta)$  depending on  $\theta$  such that*

$$v(\theta) = \Omega_{\mathbf{u}}(\theta)\theta, \tag{11}$$

where

$$\Omega_{\mathbf{u}}(\theta) = \left( \sum_{k=1}^K \phi_k(\theta) \hat{\Omega}_{J_k}^{-1} \right)^{-1}.$$

*The coefficient  $\phi_k(\theta)$  is equal to zero if the  $k$ th constraint is not binding, but at least one of the coefficients is strictly positive.*

## 2.4 Diversification of Uncertainty

In this section we summarize some of the properties of our measure of uncertainty, drawing a parallel with the variance as a measure of risk (return variance is the appropriate measure of risk in our model, since asset returns are jointly normally distributed). The key result

---

<sup>5</sup>One of the typical features of the multi-prior expected utility model is that the solution of the maxmin problem is often not unique. The analytical feature of our formulation of the set  $\mathcal{P}(P)$  is that, due to Proposition 1, the minimizer for the equilibrium situation we are considering is always unique. The crucial property of the set  $\mathcal{P}(P)$  that gives rise to this uniqueness is the strict convexity of the relative entropy function, as can be seen in the proof of Proposition 1.

of this section, stated in Proposition 2, is used in the following sections to derive the asset pricing implications of model uncertainty.

The definition of portfolio uncertainty  $\Delta(\theta)$  given in (9) implies that it is a convex and symmetric function of the portfolio composition,  $\Delta(-\theta) = \Delta(\theta)$ , just as the variance of portfolio returns. The function  $\Delta(\theta)$  is homogeneous of degree one, unlike the variance, which is homogeneous of degree two. As an illustration, we plot the portfolio uncertainty as a function of its composition in Figure 1. For comparison, we also plot the variance of portfolio returns in Figure 2. As expected, the two functions look qualitatively similar.

As with the standard measure of risk, variance, one can draw a distinction between the total uncertainty of an asset (or a portfolio) and its systematic uncertainty. The systematic uncertainty of the asset  $i$  with respect to a portfolio  $\theta$  is defined as its marginal contribution to the total portfolio uncertainty, in analogy with the definition of systematic risk:

$$\beta_{\mathbf{u},i}(\theta) = \frac{\partial \ln \Delta(\theta)}{\partial \theta_i}.$$

The following proposition shows that  $\beta_{\mathbf{u},i}(\theta)$  is well defined, as long as all components of the portfolio  $\theta$  are non-zero and characterizes the sensitivity of the portfolio uncertainty to its composition.

**Proposition 2** *Assuming that all components of the portfolio weights vector  $\theta$  are non-zero, the sensitivity of the uncertainty of a portfolio to a change in its composition is given by*

$$\frac{\partial \ln \Delta(\theta)}{\partial \theta} = \frac{1}{\Delta(\theta)} v(\theta) = \frac{\Omega_{\mathbf{u}}(\theta)\theta}{\theta^\top \Omega_{\mathbf{u}}(\theta)\theta}. \quad (12)$$

This proposition implies in particular that systematic uncertainty of the market portfolio is equal to its total uncertainty. Also, since  $v(\theta) \in \mathcal{V}$ , it is immediate that the total uncertainty of an asset exceeds its systematic uncertainty, i.e.,

$$\Delta(e_i) = \max_{v \in \mathcal{V}(\phi)} e_i^\top v \geq \beta_{\mathbf{u},i}(\theta)\Delta(\theta) = e_i^\top v(\theta),$$

where  $e_i = (0, \dots, 0, 1, 0, \dots, 0)^\top$  with the  $i$ th component of the vector equal to 1.

In the above, we have considered the sensitivity of portfolio uncertainty to a change in the composition of the portfolio when the portfolio weights are non-zero. This corresponds to the case when the portfolio already has a loading of all the assets. The other interesting case is when an asset is not in the portfolio to begin with, but is to be added to the portfolio. As the following proposition shows, this case is not as simple as the other case and the reason is that  $\Delta(\theta)$  is in general no longer differentiable.

**Proposition 3** *Let  $\theta$  be a portfolio with  $\theta_j = 0$ . Let  $\mathcal{K} = \{k : j \in J_k\}$ . If there exists a solution  $\bar{v}$  of (9) such that for all  $k \in \mathcal{K}$ ,*

$$\frac{1}{2} \bar{v}^\top \hat{\Omega}_{J_k}^{-1} \bar{v} = \frac{1}{2} \bar{v}_{J_k}^\top \Omega_{J_k}^{-1} \bar{v}_{J_k} < \eta_k, \quad (13)$$

*then  $\Delta(\theta)$  is not differentiable in  $\theta_j$  at  $\theta_j = 0$ . Otherwise  $\Delta(\theta)$  is differentiable in  $\theta_j$  at  $\theta_j = 0$  and  $\partial\Delta(\theta)/\partial\theta_j = \bar{v}_j$ , where  $\bar{v}$  is any solution of (9).*

The intuition of this proposition can be illustrated by the following example. There are two assets and two sources of information, one for each asset,

$$\frac{1}{2} v_j^2 \sigma_j^2 \leq \eta_j, \quad j = 1, 2.$$

Let  $\theta = (\theta_1, \theta_2)$  be a portfolio where  $\theta_1 > 0$  and  $\theta_2 = 0$ . In this case,

$$\Delta(\theta) = \sqrt{2\eta_1} \sigma_1 \theta_1.$$

and the solutions of (8) are of the form,  $\bar{v} = (\sqrt{2\eta_1} \sigma_1, \bar{v}_2)$  where  $\bar{v}_2$  is arbitrary as long as it satisfies the constraint above. According to the proposition, the derivative  $\partial\Delta(\theta)/\partial\theta_2$  at  $\theta_2 = 0$  does not exist.

While this example is special, the insight revealed applies more generally. Notice that, when  $\theta_2 = 0$ , the source of information about the second asset is irrelevant for the uncertainty of the portfolio. In other words, the source of information is not reflected in the portfolio uncertainty when  $\theta_2 = 0$ . The moment when  $\theta_2$  becomes positive, this source of information starts to contribute to the uncertainty of the portfolio. The rate at which it adds to the

uncertainty of the portfolio is given by  $\sqrt{2\eta_2}\sigma_2$ . This rate is  $-\sqrt{2\eta_2}\sigma_2$  when  $\theta_2$  becomes negative. As a result,  $\Delta(\theta)$  is not differentiable in  $\theta_2$ . More generally, when the information about a particular asset has not been fully reflected, which is what (13) characterizes, the rates at which an asset contributes to the uncertainty of the portfolio differ, depending on whether the asset is added in a long or short position, and non-differentiability arises. This potential non-differentiability can play an important role in modelling of the bid-ask spread of asset prices (see Routledge and Zin (2002)). In this paper such complications do not arise, since we assume that all assets are in positive supply.

### 3 Portfolio Choice

In this section we re-formulate the agent's portfolio choice problem in a form that is particularly convenient for deriving the asset pricing results below.

Under the preferences introduced above, the investor's utility maximization problem is

$$\sup_{\theta} \min_{v \in \mathcal{V}} \mathbb{E} [\xi u(W)], \quad (14)$$

subject to (5), (7), and the wealth constraint

$$W = [\theta^\top (R - r1) + 1 + r]$$

The following proposition shows that the solution of the minimization is given by the solution  $v(\theta)$  of (9).

**Proposition 4** *Problem (14) is equivalent to*

$$\max_{\theta} \min_{|y| \leq \Delta(\theta)} \{ \mathbb{E} [\xi(\theta, y) u(W)] \}, \quad (15)$$

where

$$\xi(\theta, y) = \exp \left\{ -\frac{y^2}{2\theta^\top \Omega \theta} - \frac{y(\theta^\top R - \theta^\top \mu + y)}{\theta^\top \Omega \theta} \right\}.$$

is the density of the return on the portfolio  $\theta$  with respect to the reference measure  $P$ . Furthermore, if  $(\theta, v)$  is the solution of (14) and  $\theta$  is such that all of its components are non-zero,

then

$$v = v(\theta), \tag{16}$$

where  $v(\theta)$  is a solution of (9). Moreover, the optimal portfolio policy  $\theta$  satisfies

$$E [u'(W - \Delta(\theta)) (R - r1 - v(\theta))] = 0. \tag{17}$$

Proposition 4 has an important implication. It shows that for a given portfolio composition  $\theta$ , the agent would evaluate his/her expected utility under the measure indexed by  $v$ , which is independent of the agent's utility function  $U(W)$  and can be determined as a solution of (9). This greatly simplifies the analysis of equilibrium asset prices, since knowing the composition of the market portfolio is sufficient for computing the adjustment of expected returns,  $v(\theta)$ , due to model uncertainty. Then, a restriction on expected returns follows directly from (17).

Second, Proposition 4 shows how our measure of uncertainty  $\Delta(\theta)$  can be used to understand the agent's portfolio choice. The objective (15) no longer involves the entire set of probability measures  $\mathcal{P}(P)$ , as does the original formulation (14). Instead, it uses a scalar summary of model uncertainty,  $\Delta(\theta)$ . Specifically, for each portfolio composition  $\theta$ , the agent reduces the expected return on the portfolio by  $\Delta(\theta)$  and then evaluates the expected utility function in a standard manner.

## 4 The Asset Pricing Model

In this section we derive a pricing model, which relates the cross-sectional distribution of expected returns on the  $N$  risky assets to their covariances with the market portfolio. These securities are assumed to be available in perfectly elastic supply and their return distribution is defined exogenously. The market portfolio, however, is endogenously determined, giving rise to a non-trivial pricing model. Note that this formulation is isomorphic to an exchange economy in which the supply of the risky assets is exogenously fixed, while their prices (and returns) are determined endogenously.

## 4.1 Risk Premium and Uncertainty Premium, CARA Utility

We first consider the case of the representative investor with a constant absolute risk aversion (CARA) utility function,

$$u(W) = -e^{-\gamma W}$$

We extend our results to the case of a general utility function below.

Let  $\theta_m$  denote the composition of the market portfolio and define  $\Delta_m = \Delta(\theta_m)$ . Also, define a pricing kernel

$$\zeta = \frac{e^{-\gamma W}}{\mathbb{E}[e^{-\gamma W}]}$$

Then, according to Proposition 4, returns on the risky assets satisfy

$$\mathbb{E}[\zeta R] = r1 + v(\theta_m), \quad (18)$$

and therefore the market return satisfies

$$\mathbb{E}[\zeta R_m] = r + \Delta_m. \quad (19)$$

From these relations, we find that the expected return premia on the risky assets and on the market portfolio are given by

$$\mu - r1 = \gamma \text{cov}(R_m, R) + v(\theta_m) \quad (20)$$

$$\mu_m - r1 = \underbrace{\gamma \sigma_m^2}_{\lambda, \text{ risk premium}} + \underbrace{\Delta_m}_{\lambda_u, \text{ uncertainty premium}} \quad (21)$$

The first term in (21) may be viewed as the market risk premium, being proportional to the variance of the market portfolio. The proportionality coefficient depends on the preferences of the representative agent, i.e., the absolute risk aversion coefficient of the agent,  $\gamma$ . We will denote the first term by  $\lambda$ . The second term,  $\Delta_m$ , has a natural interpretation of the market uncertainty premium, and depends on the degree of uncertainty of the market portfolio. We denote it by  $\lambda_u$ .

## 4.2 Risk Premium and Uncertainty Premium, General Utility

We now generalize the results of the previous section to non-CARA utility functions. Again, we start with the relation,

$$\mathbb{E}[\zeta R] = r1 + v(\theta_m), \quad (22)$$

where

$$\zeta = \frac{U'(W - \Delta_m)}{\mathbb{E}[U'(W - \Delta_m)]}$$

Thus, as in the case of the CARA utility function,

$$\mathbb{E}[\zeta R_m] = r + \Delta_m, \quad (23)$$

By applying Stein's Lemma to (22) and (23), we find that the expected return premia on the individual stocks and on the market are given by

$$\mu - r1 = \frac{\mathbb{E}[U''(W - \Delta_m)]}{\mathbb{E}[U'(W - \Delta_m)]} \text{cov}(R_m, R) + v(\theta_m) \quad (24)$$

$$\mu_m - r1 = \frac{\mathbb{E}[U''(W - \Delta_m)]}{\mathbb{E}[U'(W - \Delta_m)]} \sigma_m^2(\theta) + \Delta_m \quad (25)$$

The first term in (25) may be viewed as the market risk premium, being proportional to the variance of the market portfolio. The proportionality coefficient depends on the preferences of the representative agent. For a special case of the CARA utility function,  $U(W) = -\exp(-\gamma W)$ , it equals the absolute risk aversion coefficient of the agent,  $\gamma$  (see equation (18)). In general, however, this term is affected by the agent's uncertainty aversion as well, since it depends on  $\Delta_m$ . The only exception is the case of constant absolute risk aversion, when  $U''(W)/U'(W)$  is independent of the level of  $W$ . With this reservation in mind, we will denote the entire first term by  $\lambda$ . The second term,  $\Delta_m$ , has a natural interpretation of the market uncertainty premium. We will denote it by  $\lambda_u$ .

### 4.3 Asset Pricing Model

According to (24), the expected excess return on an individual risky asset is determined by two terms. The first term is standard, given by the product of the representative investor's risk aversion coefficient and the covariance of the individual asset return with the market portfolio. The second term captures the effect of model uncertainty. Since  $v(\theta_m)$  is a solution of (9), it depends not just on the uncertainty about the returns on an individual asset, but also on the nature of the information about the joint distribution of returns on all the assets. Equations (24) and (21) imply a relation between expected excess returns on individual assets, which we state as the following proposition.

**Proposition 5** *The equilibrium vector of expected excess returns is given by*

$$\mu - r1 = \lambda\beta + \lambda_u\beta_u, \quad (26)$$

where  $\lambda$  and  $\lambda_u$  are the market risk and uncertainty premia and  $\beta$  and  $\beta_u$  are the risk and uncertainty betas with respect to the market portfolio:

$$\begin{aligned} \lambda &= \frac{\mathbb{E}[U''(W - \Delta_m)]}{\mathbb{E}[U'(W - \Delta_m)]} \sigma_m^2(\theta), \\ \lambda_u &= \Delta_m, \\ \beta &= \frac{\partial \ln \sigma^2(\theta_m)}{\partial \theta_m} = \frac{1}{\sigma_m^2} \Omega \theta_m, \\ \beta_u &= \frac{\partial \ln \Delta(\theta_m)}{\partial \theta_m} = \frac{1}{\Delta_m} \Omega_u(\theta_m) \theta_m. \end{aligned}$$

In the proposition,  $\beta$  defines the vector of market risk betas of stocks, i.e., their betas with respect to the market portfolio. As stated in the proposition, an equivalent definition of the market risk beta is as sensitivity of the total risk of the market portfolio to a change in its composition, i.e.,  $\beta = \partial \ln \sigma^2(\theta_m) / \partial \theta_m$ . The definition of the market uncertainty betas  $\beta_u$  is analogous. According to Proposition 2,  $\beta_u$  defines the sensitivity of the uncertainty of the market portfolio to a change in its composition. We also find that, like risk, uncertainty is partially “diversifiable” in a sense that for a particular asset only its contribution the total market uncertainty is compensated in equilibrium by higher expected return.

In equilibrium, the investor is compensated for bearing both risk and uncertainty. Thus, two assets with the same beta with respect to the market risk may have different equilibrium expected returns. This would set our model apart conceptually from the standard CAPM, which would require the expected excess returns on all assets in our setting to be proportional to their market beta.

#### 4.4 A Single Source of Information

We first consider the simplest case when the representative investor uses a single source of information about asset returns, solving the problem given by (4). In that case, Proposition 1 implies that the matrix  $\Omega_u$  is proportional to the variance-covariance matrix of returns. In this case it is easy to show that

$$\Omega_u = \frac{\sqrt{2\eta}}{\sigma_m} \Omega,$$

and hence

$$\mu - r1 = \frac{\mathbb{E}[U''(W - \Delta_m)]}{\mathbb{E}[U'(W - \Delta_m)]} \Omega \theta_m + \frac{\sqrt{2\eta}}{\sigma_m} \Omega \theta_m = \left( \frac{\mathbb{E}[U''(W - \Delta_m)]}{\mathbb{E}[U'(W - \Delta_m)]} + \frac{\sqrt{2\eta}}{\sigma_m} \right) \sigma_m^2 \beta.$$

Since the utility-dependent coefficient

$$\frac{\mathbb{E}[U''(W - \Delta_m)]}{\mathbb{E}[U'(W - \Delta_m)]}$$

is not observable, the cross-sectional distribution of expected asset returns in a world with a single source of information will be observationally indistinguishable from that in a world where there is no model uncertainty. In fact, it follows from the equation above that in an economy with a single source of information the standard CAPM holds:

$$\mu - r1 = \beta(\mu_m - r).$$

#### 4.5 Two Sources of Information

The reason that in the case of a single source of information the uncertainty premium is observationally indistinguishable from the risk premium is that the two are proportional to

each other in the cross-section. When there is more than one source of information, this is no longer the case and hence the observational equivalence no longer holds.

In this section we consider an important special case where in addition to the information about the joint distribution of all  $N$  assets, the representative agent has an additional source of information about the joint distribution of the first  $J$  assets. For instance, it may be that the historical sample of returns on the first  $J$  assets is longer than the overall sample and therefore there is less uncertainty about the expected returns on these assets.

In such case, the uncertainty of the market portfolio  $\theta_m$  is given by

$$\Delta_m = \sup_v \theta_m^\top v, \quad (27)$$

subject to

$$\frac{1}{2}v^\top \Omega^{-1}v \leq \eta \quad (28)$$

$$\frac{1}{2}v^\top \hat{\Omega}_J^{-1}v \leq \eta_J \quad (29)$$

Let  $\phi$  and  $\phi_J$  denote the Lagrange multipliers on the constraints (28) and (29) respectively. The first constraint is always binding and hence  $\phi > 0$ . Assuming that  $\eta_J$  is sufficiently small, the second constraint is binding as well and  $\phi_J > 0$ . The matrix  $\Omega_u$  in Proposition 1 is given by

$$\Omega_u = \frac{1}{\phi} \left( \Omega^{-1} + \frac{\phi_J}{\phi} \hat{\Omega}_J^{-1} \right)^{-1}$$

It is straightforward to verify that

$$\Omega_u = \frac{1}{\phi} \left( \Omega - \frac{\phi_J}{\phi + \phi_J} \Omega \hat{\Omega}_J^{-1} \Omega \right)$$

Given the explicit form of the uncertainty matrix, the uncertainty beta  $\beta_u$  in Proposition 5 is given by

$$\beta_u = \frac{1}{\Delta_m} \left( \Omega \theta_m - \frac{\phi_J}{\phi + \phi_J} \Omega \hat{\Omega}_J^{-1} \Omega \theta_m \right)$$

The first term,  $\Omega\theta_m$  is a vector of covariances of returns with the market portfolio and is proportional to the vector of standard risk betas. The second term is proportional to the vector of covariances of asset returns with the return on a particular portfolio, with weights  $\theta_p = (1^\top \hat{\Omega}_J^{-1} \Omega \theta_m)^{-1} \hat{\Omega}_J^{-1} \Omega \theta_m$ . Such a portfolio has a simple intuitive interpretation. It's return is a linear projection of the market return on the space of the first  $J$  assets. Thus, expected excess returns on the risky assets satisfy

$$\mu_j - r = \lambda \frac{1}{\sigma_m^2} \text{cov}(R_j, R_m) + \lambda_u \frac{1}{\Delta_m} \text{cov}(R_j, R_m) - \lambda_u \frac{\phi_J (1^\top \hat{\Omega}_J^{-1} \Omega \theta_m)}{\phi + \phi_J} \text{cov}(R_j, R_p), \quad j = 1, \dots, N$$

where  $R_p$  denotes the return on portfolio  $\theta_p$ .

Thus, in presence of an additional source of information about the first  $J$  risky assets, the equilibrium vector of expected excess returns can be described by a two-factor model, in which the first factor is the market portfolio and the second factor is a projection of the market return on the subset of the first  $J$  assets. This implication of uncertainty aversion is distinct from the pricing model implied by standard preferences. Since returns in our model are normal, a standard model of preferences would imply the familiar CAPM relation, regardless of the functional form of the utility function.

While the pricing relation above is distinct from that of any static model with standard expected utility preferences, it appears observationally equivalent to a particular intertemporal capital asset pricing model (ICAPM, see Merton 1973, Section 15). In particular, a similar two-factor pricing formula would be obtained in a frictionless dynamic continuous-time economy with diffusion information structure in which the investment opportunity set is driven by a single state variable, such that the portfolio  $R_p$  introduced above is the corresponding hedging portfolio (see Merton (1973)). One could therefore argue that model uncertainty has implications distinct from traditional static models, but indistinguishable from standard dynamic pricing models, such as Merton's ICAPM. Such an argument has an obvious limitation. While it is true that one could always create a dynamic economy supporting a particular factor structure of returns, to argue that such a model explains the empirical observations it is necessary to show that the factor portfolios provide a hedge against changes in the investment opportunity set. As our analysis shows, in an economy with model uncertainty the cross-section of expected returns may have a multi-factor struc-

ture even when the returns are identically distributed over time (which is highlighted by our static formulation). This establishes a conceptual distinction between an economy with model uncertainty and a dynamic economy with standard expected-utility preferences. Thus, to test the implications of model uncertainty empirically, it is crucial to take a stand on the structure of model uncertainty in the economy. This could allow one to identify the nature of the factor portfolios, which otherwise may appear unrelated to the state variables describing changes in the investment opportunity set.

## 5 Conclusion

We have developed a single-period equilibrium model incorporating not only risk, but also model uncertainty. We have introduced a notion of a measure of uncertainty and characterized an uncertainty premium in equilibrium expected returns on financial assets.

We have shown that the cross-sectional distribution of expected returns can be formally described by a two-factor model, where expected returns are derived as compensation for the asset's contribution to the risk and uncertainty of the portfolio held by the agent in equilibrium. Thus, the standard result that expected returns are related only to systematic, and not diversifiable risk, carries over to economies with model uncertainty as well.

While prior research on model uncertainty has been concerned with its implications for the time-series of asset prices, by characterizing the cross-section of returns we were able to address some of the observational equivalence issues raised in the literature. In particular, we demonstrated that the effect of model uncertainty in our framework is distinct from risk aversion and cannot be captured by any specification of the risk aversion parameter.

# Appendix

## Proof of Proposition 1

Suppose to the contrary that  $\bar{v}$  and  $\underline{v}$  are two distinct solutions. Let  $v(a) = a\bar{v} + (1-a)\underline{v}$ . The strict convexity of all the functions defining the choice set implies that for  $a \in (0, 1)$ ,

$$\frac{1}{2}v(a)^\top \hat{\Omega}_{J_k}^{-1}v(a) \leq \eta_k, \quad k = 1, \dots, K.$$

Now let  $k$ , if exists, be such that

$$\frac{1}{2}v(a)^\top \hat{\Omega}_{J_k}^{-1}v(a) = \eta_k$$

holds for  $a = 0$ ,  $a = 1$ , and for some  $a \in (0, 1)$ . Then it must be the case that  $\bar{v}_{J_k} = \underline{v}_{J_k}$ .

Denote by  $A$  the set of such  $k$ . If

$$J_A = \cup_{k \in A} J_k = \{1, \dots, n\},$$

then  $\bar{v} = \underline{v}$ , a contradiction to assumption. So,  $J_A \neq \{1, \dots, n\}$ . Without loss of generality, we assume that  $J_A = \{2, \dots, n\}$ . Then for all  $v$  of the form  $v = (v_1, \bar{v}_2, \dots, \bar{v}_n)$  with  $v_1 \in R$ ,

$$\frac{1}{2}v^\top \hat{\Omega}_{J_k}^{-1}v = \eta_k, \quad k \in A.$$

Note that  $v(a)$  is of the form  $(a\bar{v}_1 + (1-a)\underline{v}_1, \bar{v}_2, \dots, \bar{v}_n)$ . Thus for  $v = (0.5\bar{v}_1 + 0.5\underline{v}_1, \bar{v}_2, \dots, \bar{v}_n)$ ,

$$\frac{1}{2}v^\top \hat{\Omega}_{J_k}^{-1}v < \eta_k, \quad k \notin A.$$

Combining the two cases,  $k \in A$  and  $k \notin A$ , together, by continuity, there is a  $\epsilon > 0$  such that for all  $v = (v_1, \bar{v}_2, \dots, \bar{v}_n)$  with  $v_1 \in (0.5\bar{v}_1 + 0.5\underline{v}_1 - \epsilon, 0.5\bar{v}_1 + 0.5\underline{v}_1 + \epsilon)$ ,

$$\frac{1}{2}v^\top \hat{\Omega}_{J_k}^{-1}v \leq \eta_k, \quad k = 1, \dots, K.$$

But, given the linearity of the objective function, this means  $\bar{v}$  and  $\underline{v}$  cannot be the solution of (9). This is a contradiction.

The second statement of the proposition is a straightforward application of the Lagrangian duality approach.

## Proof of Proposition 2

Since the constraint set  $\mathcal{P}$  is convex and compact,  $\Delta(\theta)$  is a convex function. Optimality conditions imply that  $v(\theta)$  is a subgradient of the value function  $\Delta(\theta)$  at  $\theta$ . The solution  $v$  of is unique, according to lemma 1. Thus, the function  $\Delta(\theta)$  has a unique subgradient, therefore it is in fact differentiable, and  $v$  is equal to the gradient of  $\Delta(\theta)$ . This establishes the statement of the lemma.

## Proof of Proposition 3

Observe that  $\partial\Delta(\theta)/\partial\theta_j$  exists if and only if all solutions of (9) have the same  $j$ th component. For the first claim of the lemma, assume without loss of generality that  $j = 1$ . If the condition of the lemma is satisfied, there exists a  $\epsilon > 0$  such that for any  $|x| < \epsilon$ ,  $v_x = \bar{v} + (x, 0, \dots, 0)$  satisfies all constraints of (9). Since  $\theta_1 = 0$ ,  $v_x$  is also a solution of (9). The claim follows.

For the second part, let  $\bar{v}$  be a solution of (9). If it is the unique solution of (9), then  $\partial\Delta(\theta)/\partial\theta_j$  exists. Suppose  $\bar{v}$  and  $\underline{v}$  are two distinct solutions of (9). Let  $v(a) = a\bar{v} + (1-a)\underline{v}$ . We claim that there exists a  $k \in \mathcal{K}$  such that

$$\frac{1}{2}v(a)_{J_k}^\top \Omega_{J_k}^{-1}v(a)_{J_k} = \eta_k$$

holds for  $a = 0$ ,  $a = 1$  and some  $a \in (0, 1)$ . Suppose the contrary. By strict convexity,

$$\frac{1}{2}v(a)_{J_k}^\top \Omega_{J_k}^{-1}v(a)_{J_k} < \eta_k, \quad k \in \mathcal{K}$$

for  $a \in (0, 1)$ . Also the convexity of all the functions defining the choice set implies that for  $a \in (0, 1)$ ,

$$\frac{1}{2}v(a)^\top \hat{\Omega}_{J_k}^{-1}v(a) = \frac{1}{2}v(a)_{J_k}^\top \Omega_{J_k}^{-1}v(a)_{J_k} \leq \eta_k, \quad k = 1, \dots, K.$$

Since the objective function of (9) is linear,  $v(a)$  is a solution of (9) for all  $a \in (0, 1)$ . But this is a contradiction to assumption of the lemma. Thus the claim is shown. It then follows from the claim that  $\bar{v}_{J_k} = \underline{v}_{J_k}$  and hence  $\bar{v}_j = \underline{v}_j$ . Since  $\bar{v}$  and  $\underline{v}$  are arbitrary, we have  $\bar{v}_j = \underline{v}_j$  for all solutions of (9). The differentiability follows.  $\blacksquare$

### Proof of Proposition 4

Since the distribution of  $W$  depends only on the distribution of  $\theta^\top R$ , for each fixed  $\theta$ ,  $E[\xi u(W)]$  depends only on  $y = \theta^\top v$ , and it is given by

$$E[\xi u(W)] = E[\xi(\theta, y)u(W)],$$

where

$$\xi(\theta, y) = \exp \left\{ -\frac{y^2}{2\theta^\top \Omega \theta} - \frac{y(R_\theta - \theta^\top \mu + y)}{\theta^\top \Omega \theta} \right\}.$$

Thus the original utility function can be written as

$$\max_{\theta} \min_{|y| \leq \phi \Delta(\theta)} (E[\xi(\theta, y)u(W)]) \tag{A1}$$

which is (15). The characterization for  $v$  follows immediately.

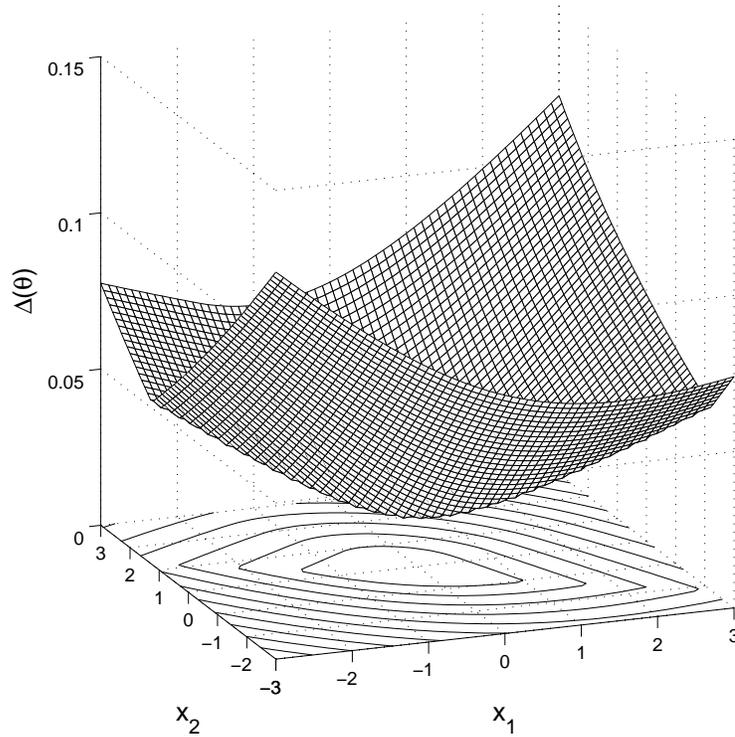


Figure 1: Portfolio uncertainty

The portfolio uncertainty measure,  $\Delta(\theta)$  is plotted as a function of the portfolio composition,  $(x_1, x_2)$ . The portfolio consists of three risky assets and the moments of their returns are given by

$$\mu = \begin{bmatrix} 0.06 \\ 0.08 \\ 0.12 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0.090 & 0.045 & 0.036 \\ 0.045 & 0.090 & 0.009 \\ 0.036 & 0.009 & 0.090 \end{bmatrix}$$

The portfolio composition is parameterized by  $(x_1, x_2)$ :

$$\theta = \frac{1}{\sqrt{3}}e_0 + x_1e_1 + x_2e_2, \quad e_0 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad e_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

so that the vectors  $e_0, e_1$  and  $e_2$  form an orthonormal basis and  $1^\top \theta = 1, \forall (x_1, x_2)$ . We assume that there exist two sources of information, the second being about the first two assets:  $J_1 = \{1, 2, 3\}, J = \{1, 2\}$ . The feasible set  $\mathcal{V}$  in (10) is given by

$$v^\top \Omega^{-1} v \leq 0.0025, \quad v^\top \hat{\Omega}_{J_2}^{-1} v \leq 0.005.$$

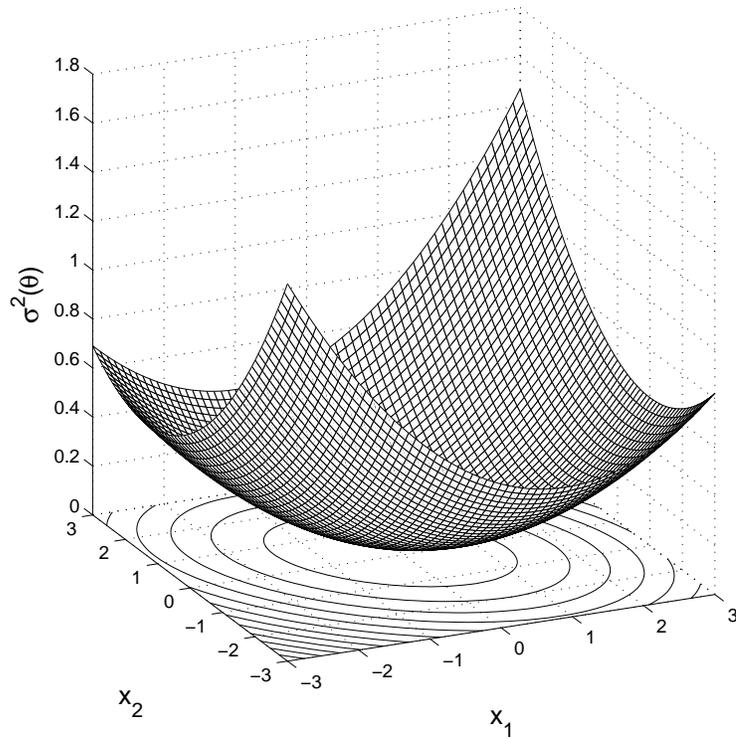


Figure 2: Portfolio variance

The portfolio variance,  $\sigma^2(\theta)$  is plotted as a function of the portfolio composition,  $(x_1, x_2)$ . The model parameters are given in the caption to Figure 1.

## References

- [1] Anderson, E., L. Hansen and T. Sargent (1999): “Robustness, Detection and the Price of Risk,” working paper, University of Chicago.
- [2] Banz, R. (1981): “The Relation between Return and Market Value of Common Stocks,” *Journal of Financial Economics*, 9, 3-18.
- [3] Barberis, N. (2000): “Investing for the long run when returns are predictable,” *Journal of Finance* 55, 225-264.
- [4] Balduzzi, P., and L. Liu (1999): “Estimation risk, learning, and international investment,” Working Paper, Boston College.
- [5] Brav, A. and J.B. Heaton (2002): “Competing Theories of Financial Anomalies,” *Review of Financial Studies*, 15, 575-606.
- [6] Breeden, D. (1979): “An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities,” *Journal of Financial Economics*, 7, 265-296.
- [7] Brennan, M. (1998): “The role of learning in dynamic portfolio decisions,” *European Finance Review* 1, 295 -306.
- [8] Brennan, M. and Y. Xia (2001): “Stock Return Volatility and Equity Premium,” *Journal of Monetary Economics*, 47, 249-283.
- [9] Chen, Z. and L. Epstein (2001): “Ambiguity, risk and asset returns in continuous time,” *Econometrica*, 70, 1403-1443.
- [10] Cox, J., J. Ingersoll, and S. Ross (1985): “An Intertemporal General Equilibrium Model of Asset Prices,” *Econometrica*, 53, 363-384.
- [11] Detemple, J. (1986): “Asset pricing in an economy with incomplete information,” *Journal of Finance* 61, 383-392.
- [12] Dow, J. and S. Werlang (1992): “Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio”, *Econometrica*, 60, pp.197-204.
- [13] Ellsberg, D. (1961): “Risk, Ambiguity, and the Savage Axioms,” *Quarterly Journal of Economics*, 75, pp.643-669.
- [14] Epstein, L. and J. Miao (2001): “A two-person dynamic equilibrium under ambiguity,” forthcoming in *Journal of Economic Dynamics and Control*.
- [15] Epstein, L. and T. Wang (1994): “Intertemporal Asset Pricing under Knightian Uncertainty,” *Econometrica*, 62, pp.283-322.
- [16] Epstein, L. and T. Wang (1995): “Uncertainty, Risk-Neutral Measures and Security Price Booms and Crashes,” *Journal of Economic Theory*, 67, pp.40-80.

- [17] Frost, P. and J. Savarino, (1986): “An empirical Bayes approach to efficient portfolio selection,” *Journal of Financial and Quantitative Analysis* 21, 293 -305.
- [18] Gennotte, G. (1986): “Optimal portfolio choice under incomplete information,” *Journal of Finance* 41, 733-746.
- [19] Gilboa, I. and D. Schmeidler (1989): “Maxmin Expected Utility Theory with Non-Unique Prior,” *Journal of Mathematical Economics*, 18, pp.141-153.
- [20] Hansen, L. and T. Sargent (2001): “Robust Control and Model Uncertainty,” *American Economic Review*, Vol. 91, pp.60-66.
- [21] Hansen, L., T. Sargent, G. Turmuhambetova, and N. Williams (2002): “Robustness and Uncertainty Aversion,” working paper.
- [22] Kandel, S., and R. Stambaugh (1996): “On the predictability of stock returns: An asset allocation perspective,” *Journal of Finance* 51, 385-424.
- [23] Knight, F. (1921): *Risk, Uncertainty and Profit*, Houghton, Mifflin, Boston.
- [24] Lowellen, J. and J. Shanken (2001): “Learning, Asset-Pricing Tests, and Market Efficiency,” *Journal of Finance*, 57, 1113-1145.
- [25] Lucas, R. (1978): “Asset prices in an exchange economy,” *Econometrica* 46, 1429-1445.
- [26] Machina, M. and D. Schmeidler (1992): “A More Robust Definition of Subjective Probability,” *Econometrica*, 60, pp.745-780.
- [27] Maenhout, P. (1999): “Robust portfolio rules and asset pricing,” Working paper, Harvard University.
- [28] Merton, R. (1992): *Continuous-Time Finance*. Blackwell Publishers, Cambridge MA.
- [29] Pástor, L. (2000): “Portfolio selection and asset pricing models,” *Journal of Finance* 55, 179-223.
- [30] Routledge, B. and S. Zin (2002): “Model Uncertainty and Liquidity,” working paper, Carnegie Mellon University.
- [31] Schmeidler, D. (1989): “Subjective Probability and Expected Utility without Additivity,” *Econometrica*, 57, pp.571-587.
- [32] Sharpe, W. (1964): “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk,” *Journal of Finance*, 19, 425-442.
- [33] Uppal, R. and T. Wang (2001): “Model Misspecification and Under-Diversification,” forthcoming in *Journal of Finance*.
- [34] Xia, Y. (2001): “Learning about Predictability: the Effect of Parameter Uncertainty on Dynamic Asset Allocation,” *Journal of Finance* 56, 205-246.