Valuations and Dynamics of Negotiations

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Abstract

This paper analyzes three-party negotiations in the presence of externalities, deriving a close form solution for the stationary subgame perfect Nash equilibrium of a standard non-cooperative bargaining model. Players' values are monotonically increasing (or decreasing) in the amount of negative (or positive) externalities that they impose on others. Moreover, players' values are continuous and piecewise linear on the worth of bilateral coalitions, and are inextricably related to their negotiation strategies: the equilibrium value is the Nash bargaining solution when no bilateral coalitions form; the Shapley value when all bilateral coalitions form; or the nucleolus, when either one bilateral coalition among 'natural partners' or two bilateral coalitions including a 'pivotal player' form.

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1 Introduction

This paper studies multilateral negotiations in the presence of externalities. These problems are important in economics, appearing in such diverse areas as mergers and acquisitions, bankruptcy and international treaty negotiations, as well as the formation of labor unions and coalitional governments. What is the sequencing of negotiations and how do the parties involved form valuations? Our goal in this paper is to analyze a standard strategic model of negotiations and to derive a close form solution answering the question above.

While bilateral negotiations have been extensively studied, much less is known about the more complex problem of negotiations involving three or more parties, where, for example, coalition formation is an important aspect of the negotiations. We restrict our attention here to three-player games and develop an explicit solution for all three-player games with externalities. Having a simple off-the-shelf solution for three-player games is helpful in extending our understanding one step beyond bilateral interactions, and is useful in applications where coalition formation and externalities play an important role.

The bargaining model analyzed herein is a standard non-cooperative model (see literature review below). Our main contribution is to characterize its stationary subgame perfect Nash equilibrium. In our model there are three players who can form bilateral coalitions and the grand coalition. A set of parameters describe the worth of the grand coalition and of all bilateral coalitions, including the amount of externalities they impose on excluded players. The bargaining game evolves over time with players making offers followed by responses every period.

We show that the equilibrium value, referred to as the coalitional bargaining value (CBV), is a continuous and piecewise linear function of the parameters of the game. Specifically, the space of all games is divided into four convex regions (eight including all permutations), and in each region the CBV is a linear function of the parameters of the game. For three-player games without externalities, in one of the regions the CBV coincides with the Nash bargaining solution, in another region with the Shapley value, and in the other regions with the nucleolus. For three player games with externalities, a similar treatment applies as long as an adjusted measure for the worth of pairwise coalitions is used to generalize the Shapley value and the nucleolus. This adjustment involves measuring the worth of a pairwise coalition adding the amount of negative externalities (or subtracting the amount of positive externalities) that it creates for the excluded player. Therefore, this paper proposes a way to select a specific cooperative solution concept for all three-player games without externalities, and a generalization to situations with externalities.

The strategies employed by players in each region have an intuitive economic interpretation in terms of credible outside options (see also Sutton (1986)). This interpretation serves to enhance our understanding of the related cooperative solution concepts and shows how valuations are inextricably related to the equilibrium negotiation strategies. First, the CBV is equal to the Nash bargaining solution (equal split of the surplus) and no bilateral coalitions forms, if the (adjusted) worth of all bilateral coalition is less than a third of the grand coalition value. In this region no player is able to demand more than an equal share of surplus because the outside option of forming a bilateral coalition is not credible. Second, the CBV coincides with the (generalized) Shapley value and all bilateral coalitions can form in equilibrium, if the sum of the (adjusted) values of all bilateral coalitions is greater than the grand coalition value. In these games, there is an advantage from being the proposer (first mover advantage) and a disadvantage from being excluded from a bilateral coalition.

Finally, there are two novel cases in which the CBV coincides with the (generalized) nucleolus: games where only the ‘natural coalition’ among two ‘natural partners’ creates significant value, and games where only the two pairwise coalitions including a ‘pivotal player’ create significant value. In the first case, the player excluded from the natural coalition agrees with a payoff lower than an equal split of the surplus, and the natural partners equally split the gains from forming the natural coalition–an outcome that is driven by the fact that only the natural coalition can credibly form in equilibrium. In the second case, both non-pivotal players agree to form a coalition with the pivotal player, receiving a payoff lower than an equal split of the surplus–an outcome that is driven by the fact that only the pairwise coalitions including the pivotal player can credibly form.

There is now an extensive literature studying non-cooperative coalitional bargaining games–we refer below to the most closely related literature. Earlier papers in the area analyzed the properties of games without externalities: Gul (1989), Chatterjee et al. (1993), Moldovanu and Winter (1995), Okada (1996), Hart and Mas-Colell (1996), Krishna and Serrano (1996), Seidmann and Winter (1998), among others. Later studies considered the extension to coalitional bargaining games with externalities. For example, Jehiel and Moldovanu (1995ab), Bloch (1996), Yi (1997), Ray and Vohra (1999), Montero (1999), and Gomes (2001) showed that a much broader range of applications can be analyzed when allowing for externalities. Moreover, they addressed several general properties (such as efficiency, bargaining delays, existence, stability, uniqueness, and convergence) of the equilibrium for games with an arbitrary number of players.

Similarly to Moldovanu (1990), Serrano (1993), and Cornet (2003), the contribution of this paper to the literature is its in depth analysis of 3-player games. Moldovanu (1990) studies the coalition-proof Nash equilibria of 3-player games without side payments. Among other properties, he shows that if the game is balanced then the equilibrium payoff is in the core. Serrano (1993) studies 3-player bargaining games without externalities in which responders may exit an have endogenous outside options. When the order of proposers corresponds to the power players have in the underlying coalition function, the unique Markov perfect equilibrium outcome of the game is the prenucleolus. Cornet (2003) studies 3-player negotiations with externalities using a model in which bargaining takes place using the demand-making framework originally proposed by Binmore (1985). In this framework players sequentially pose demands and accept or reject standing demands from other player(s). In Cornet's framework acceptance leads to the formation of a two (or three) player coalition and the game terminates, while in our framework the game goes on even when a two-player coalition has already been formed. Cornet (2003) analyzes the (stationary) subgame perfect equilibrium and shows that it is not unique and may be inefficient. His main conclusion is that creation of positive or negative externalities by a coalition is irrelevant to the equilibrium value. The properties of our solution are different because we show that the players' values are increasing (decreasing) on the amount of negative (positive) externality that they can impose on others.

Croson, et al. (2004) experimentally compare the new equilibrium value proposed in this paper to that of competing concepts in situations with and without externalities, and they also examine empirically the dynamics of coalition formation. Their experimental results indicate that the CBV performs significantly better than other leading solution concepts such as the Shapley value and the nucleolus. Moreover, Croson, et al. (2004) show that the dynamics of coalition formation is as predicted by our model in over 75% of the experiments conducted. Overall these results indicate that the CBV is an attractive off-the-shelf solution concept for three-player games with and without externalities.

The remainder of the paper is organized as follows: Section 2 presents the negotiation model, Section 3 characterizes the equilibrium and derives the explicit formula for the CBV, Section 4 studies the relationship of the CBV with cooperative game theory concepts, and Section 5 concludes. The Appendix contains the proofs of all propositions.

2 The Bargaining Model

The game has three players $N = \{1, 2, 3\}$. Each player owns an indivisible tradeable resource or right, and they can buy or sell resources in exchange of a transfer of utility. Players that acquire resources continue trading, and players that sell their resources leave the game. As a result of trading, different ownership or *coalition structures (c.s.)* may arise, starting from the initial c.s. $[1|2|3]$ in which all resources are owned by different players: the c.s. $[ij|k]$, where one player, either i or j, owns both resources $\{i, j\}$, or the c.s. $[N]$ where one player owns all resources. We assume that players owning the same resources play identical strategies, and thus to simplify notation we do not keep track of who owns resources C, instead referring to this player as *coalition* C.[1]

Players are expected utility maximizers and have a common per period discount factor equal to $\delta \in [0, 1)$. A set of parameters describes the flow of utility generated by the resources for all the possible c.s. Accordingly, if the c.s. is $[1|2|3]$, the flow of utility to player i is $(1-\delta)\, u_i$; if the c.s. is $[ij|k]$, the flow of utility to coalition $\{i, j\}$ and player k are, respectively, equal to $(1-\delta)\, U_{ij}$ and $(1-\delta)\, U_k$; and finally if the c.s. is $[N]$, the flow of utility to the grand coalition N is $(1-\delta)\, U$. Note that the specification above can capture any positive or negative externalities that the coalition $\{j, k\}$ creates for player i, whenever $u_i < U_i$ or $u_i > U_i$, respectively. The set of parameters u is also known as a *partition function form* (see Thrall and Lucas (1963) and Ray and Vohra (1999)). A *characteristic function form* corresponds to a special partition function form where $u_i = U_i$, and thus there are no externalities. Without any loss of generality, we consider only 0-normalized partition functions, that is $u_i = 0$. Furthermore, all partition functions considered here are such that a three-player agreement is efficient, i.e., $U \geq U_i + U_{jk}$ for all distinct i, j, and k and $U \geq 0$.

We model negotiations as an infinite horizon non-cooperative game with complete information, utilizing the partition function as the basic underlying structure. The negotiation game evolves with players making offers (to acquire the resources of other players) followed by players that have received offers accepting or rejecting them, as in Rubinstein (1982). Specifically, the game is defined recursively by the following extensive form, starting from

[1] Alternatively, we could have interpreted the formation of a coalition as a binding agreement (i.e., not necessarily an ownership agreement), in which the coalition acts as an agent maximizing the aggregate utility of the coalition members.

the c.s. [1|2|3]: *(Random proposer's choice)* At the beginning of each period one of the players belonging to the current c.s. is randomly chosen to be the proposer. In order to capture the role of the players' opportunity to propose, if the c.s. is [1|2|3], player i is proposer with arbitrary probability $p_i = p_i[1|2|3]$, and if the c.s. is $[ij|k]$, coalition ij and player k are proposers with probabilities $p_{ij}[ij|k]$ and $p_k[ij|k]$. *(Proposal stage)* The proposer then makes one of the following choices: an offer to buy the resources of another player, say j, at price t_j; an offer to buy the resources of both players, say j and k, at prices t_j and t_k; or makes no offers (i.e., pass his opportunity to propose). *(Response stage)* The player(s) receiving the offer respond sequentially either accepting or rejecting the offer (the order of response turns out to be irrelevant). An exchange of ownership then takes place if *all* player(s) receiving the offer accept(s) it. After trading a new c.s. arises (or the c.s. remains the same if any player rejected the offer), with those players selling their resources leaving the game and the proposer remaining in the game. Flow payoffs occur at the end of each period according to the partition function described in the previous paragraph.[2] The game is repeated, after a lapse of one period of time, with a new proposer being randomly chosen as described above, until the game terminates when the c.s. $[N]$ forms.

Our notion of equilibrium is *stationary subgame perfect Nash equilibrium or Markov perfect equilibrium (MPE)*. A strategy profile σ is MPE if it is a subgame perfect Nash equilibrium and the strategies are Markovian, i.e., the strategies at each stage of the game do not depend on the history of the game nor on calendar time. Formally, at the proposal stage a Markovian strategy depends only on the current c.s. and who is the proposer; at the response stage a Markovian strategy depends only on the current c.s., the proposer, the offer made by the proposer, and (if a player is the second responding) the response of the first responder.

[2]For example, the situation in which an offer by player i is accepted by j leads to a flow payoff equal to $(1-\delta)\,U_{ij} - t_j$ to player i, $(1-\delta)\,U_k$ to player k, and player j leaves the game with a final payoff equal to t_j.

3 The Coalition Bargaining Value

The interesting action happens at the initial stage. After any bilateral agreement, the analysis reduces to a standard two-player bargaining game. It is a well-known result that the random proposer bilateral bargaining game has a unique (stationary) subgame perfect equilibrium (this game is an unessential variation of Rubinstein's (1982) alternating offer bargaining game). In the unique equilibrium, the continuation values of the bilateral game with c.s. $[ij|k]$ are equal to

$$V_{ij} = U_{ij} + p_{ij}([ij|k])\left(U - U_k - U_{ij}\right) \text{ and } V_k = U_k + p_k([ij|k])\left(U - U_k - U_{ij}\right), \tag{1}$$

respectively, for coalition ij and player k. Note that this is the value at the subgame $[ij|k]$ before the proposer has been chosen. It is also useful to define the continuation values at the end of subgame $[ij|k]$, after offers have been rejected, which are equal to $V_{ij}^{\delta} = \delta V_{ij} + (1-\delta)U_{ij}$ and $V_k^{\delta} = \delta V_k + (1-\delta)U_k$.

Consider any MPE σ and let v_i be the equilibrium continuation value of player i when the c.s. is $[1|2|3]$, before the proposer has been chosen. Any player responding to an offer can, by rejecting it, maintain the negotiations on the same state. Therefore, when faced with any offer, responders accept it only if the offer price is above or equal to $v_i^{\delta} := \delta v_i$. On the other hand, proposers choose whom to extend an offer to based on which deal produces the largest gains–always offering the minimum prices that are acceptable. Specifically, say that player i's strategy puts probability $\sigma_i^{\delta}(S)$ on making an offer to form coalition S (where $i \in S$), and let the gain associated with the formation of coalition S be e_S^{δ}, which is equal to $V_{ij}^{\delta} - v_i^{\delta} - v_j^{\delta}$ if $S = \{i,j\}$ (symmetric for $S = \{i,k\}$), $U - v_i^{\delta} - v_j^{\delta} - v_k^{\delta}$ if $S = N$, and zero if $S = \{i\}$ (no coalition forms). Note that this implies that in equilibrium coalition S forms with probability $\mu_S^{\delta} = \sum_{i \in N} p_i \sigma_i^{\delta}(S)$. Also note that the value v_i is endogenously given, and must be equal to $\left(1 - \mu_{jk}^{\delta}\right) v_i^{\delta} + \mu_{jk}^{\delta} V_i^{\delta} + p_i \max_{i \in S} e_S^{\delta}$. This is because, when player i proposes, his value is $v_i^{\delta} + \max_{i \in S \subset N} e_S^{\delta}$, and when another player proposes, i's value is V_i^{δ}, if i is excluded from the offer, and v_i^δ otherwise.

We obtain the equilibrium for all $\delta > \bar{\delta}$, where $\bar{\delta} < 1$, by constructing strategies σ_i^δ satisfying

$$\sigma_i^\delta(S) = 0 \text{ if } e_S^\delta < \max_{i \in S \subset N} e_S^\delta \text{ for all } i, \tag{2}$$

and satisfying

$$v_i^\delta = \delta\left(\mu_{jk}^\delta V_i^\delta + \left(1 - \mu_{jk}^\delta\right) v_i^\delta + p_i \max_{i \in S \subset N} e_S^\delta\right) \text{ for all } i. \tag{3}$$

We are particularly interested in the limit equilibrium value when δ converges to one (which corresponds to an arbitrarily small interval between offers). The limit equilibrium value is referred to as the *coalition bargaining value* (CBV) of the game.

Proposition 1 *Consider any 0-normalized three-player game where the grand coalition is efficient (i.e., $U \geq 0$ and $U \geq U_{ij} + U_k$) and the opportunities to propose satisfy $p_i < \frac{1}{2}$. Moreover, let V_{ij} and V_i be equal to $V_{ij} = U_{ij} + p_{ij}([ij|k])(U - U_k - U_{ij})$ and $V_k = U_k + p_k([ij|k])(U - U_k - U_{ij})$. There exists a MPE σ^δ for all $\delta > \bar{\delta}$, where $\bar{\delta} < 1$, such that:*

Part A. *The equilibrium values v_i^δ converge to v_i (the coalition bargaining value) when δ converges to one:*

Case i : If $V_{12} \leq (p_1 + p_2)U$, $V_{13} \leq (p_1 + p_3)U$, and $V_{23} \leq (p_2 + p_3)U$, then

$$v_i = p_i U \quad \textit{for all } i \in N,$$

Case ii : If $V_{12} \geq (p_1 + p_2)U$, $V_{13} + \frac{p_2}{p_1+p_2} V_{12} \leq U$, and $V_{23} + \frac{p_1}{p_1+p_2} V_{12} \leq U$, then

$$v_1 = \frac{p_1}{p_1 + p_2} V_{12},\ v_2 = \frac{p_2}{p_1 + p_2} V_{12},\ \textit{and } v_3 = U - V_{12},$$

Case iii : If $V_{12} + \frac{p_3}{p_1+p_3} V_{13} \geq U$, $V_{13} + \frac{p_2}{p_1+p_2} V_{12} \geq U$, and $V_{12} + V_{13} + V_{23} \leq 2U$, then

$$v_1 = U - V_2 - V_3,\ v_2 = V_2,\ \textit{and } v_3 = V_3,$$

Case iv : If $V_{12} + V_{13} + V_{23} \geq 2U$, then

$$v_i = \frac{1}{3}(U - 2V_{jk} + V_{ij} + V_{ik}) \quad \textit{for all } i \in N.$$

Part B: *The following coalitions form in each of the cases above* ($\mu_S^\delta = \sum_{i \in N} p_i \sigma_i^\delta(S)$ *is the probability of coalition S forming):*

Case i: $\mu_N^\delta = 1$ *(all* $\mu_{ij}^\delta = 0$*),*
Case ii: $\mu_{12}^\delta + \mu_N^\delta = 1$ *(* $\mu_{12}^\delta > 0$ *and* $\mu_{13}^\delta = \mu_{23}^\delta = 0$*),*
Case iii: $\mu_{12}^\delta + \mu_{13}^\delta + \mu_N^\delta = 1$ *(*$\mu_{12}^\delta, \mu_{13}^\delta > 0$ *and* $\mu_{23}^\delta = 0$*),*
Case iv: $\mu_{12}^\delta + \mu_{13}^\delta + \mu_{23}^\delta = 1$ *(all* $\mu_{ij}^\delta > 0$ *and* $\mu_N^\delta = 0$*).*

In the proof (see appendix) we consider a partition of the parameter space into regions defined by the inequalities *i*, *ii*, *iii*, and *iv*. In each region, we compute the values v_i^δ and the strategies σ_i^δ satisfying the system of equations and inequalities (2)-(3). We then take the limit of the solution in each region and show that the limit satisfies part A and B of proposition 1 (explicit formulas for μ_S^δ's are also provided in the appendix).[3]

Note that each case above defines a convex region, and there are a total of eight regions when all permutations are included (three of types *ii* and *iii*). Moreover, all games belong to one of the eight regions and the (interior) of the regions are disjoint.

The sequencing of negotiations is different in each region. Intuitively, region *i* corresponds to the case where no pairwise coalitions create much value, so the threat of forming a pairwise coalition is not credible and would only benefit the excluded player (accordingly the strategies are such that $\mu_{ij}^\delta = 0$). Region *ii* corresponds to the case where $\{1, 2\}$ is the only bilateral coalition that creates significant value and is the only one that should arise in equilibrium ($\mu_{13}^\delta = \mu_{23}^\delta = 0$). In region *iii* only pairwise coalitions with player 1 create significant value, so they are the only ones that should arise in equilibrium ($\mu_{23}^\delta = 0$). In region *iv*, the sum of values created by pairwise coalitions surpass the grand coalition value, and the solution predicts that all pairwise coalitions arise, but not the grand coalition ($\mu_N^\delta = 0$). Croson,

[3]The equilibrium we obtain in proposition 1 is the unique equilibrium of the game. The uniqueness result is based on Gomes (2001, theorem 6 and corollary 3), who studies local and global uniqueness properties of the equilibria of n-player coalitional bargaining games with externalities.

et al. (2004) consider games in each of the four regions. Their experimental results for the transition probabilities are in line with proposition 1, part B.[4]

The close form solution exhibited in proposition 1 allows for the evaluation of *comparative statics* effects associated with changes in the coalitions' worth *and* in the proposers' probabilities. We discuss below how valuations are intrinsically linked to the negotiation strategies, and the intuition for the comparative statics effects.

In region i, the CBV is the split of the surplus according to players's opportunities to propose. The intuition behind this result is that the threat of any pair of players i and j to form coalition $\{i,j\}$ is not credible because the most the coalition $\{i,j\}$ can get by alienating player k is V_{ij}, and $V_{ij} \leq (p_i+p_j)U$, which is the amount they can get by conforming to the equilibrium strategies. In other words, the ability of players to demand more than a proportional split of the surplus by threatening to form a pairwise coalition is an outside option that is not credible (see Sutton (1986)). The CBV prediction has the following comparative statics implication in this region: the expected outcome of players should be insensitive to local changes in the coalition's worth and is increasing in the proposer probability, as long as the conditions for belonging to region i are maintained.

In region iv, the strategy of proposer i is to choose a player randomly, say j, and offer him the value δv_j to form the pairwise coalition $\{i,j\}$. Conditional on player i been the proposer and making an offer to player j, the value of left out player k is equal to V_k, which is smaller than v_k because

$$V_k - v_k = \frac{2U - (V_{12} + V_{13} + V_{23})}{3} \leq 0,$$

(obtained after taking into account that $V_k = U - V_{ij}$). Therefore in this region there is an advantage from being the proposer and a disadvantage from being excluded from a pairwise coalition.

[4]Croson et al. (2004, table VII) experimental results for the transition probabilities from [1|2|3] to [N], [12|3], [13|2], [23|1], and [1|2|3] are, respectively: case i, 84%, 11%, 4%, 0, and 2%; case ii, 60%, 26%, 0, 6%, and 7%; case iii, 46%, 20%, 23%, 11%, and 0; case iv, 3%, 42%, 40%, 14%, and 0.

The comparative statics with respect to the proposer probability yields a surprising result. Based on the discussion above one would expect that in region *iv* the value of a player would be increasing in his opportunity to propose, but this is not the case (note that the limit equilibrium values are not a function of the proposer probabilities). The explanation comes from the analysis of the equilibrium strategies. In the appendix, we show that coalition jk forms in equilibrium with probability $\mu_{jk} = p_i$ (in the limit when $\delta \to 1$). Thus increases in player i proposer probability p_i are offset by players j and k who are more likely to form coalition jk (and conditional on coalition jk forming, the value of player i is $V_i \leq v_i$).

It is interesting to note that in region *iv*, if there are no externalities, the value of players are exactly equal to Shapley value. Recall that the Shapley value is the concept Shapley (1953) derived from axioms, and, in particular, its value for 3-player games is equal to $Sh_i = \frac{1}{6}(2U - 2U_{jk} + U_{ij} + U_{ik})$. Thus the Shapley value arises as the equilibrium in situations where players rush to form any pairwise coalitions and there are significant first mover advantages (see also Gul (1989)).[5]

In the next section, we further discuss the connection with cooperative solution as well as the results in regions *ii* and *iii*.

4 Further Properties of the Solution

A situation of special interest is the one in which all players have an equal opportunity to propose (i.e., $p_i([1|2|3]) = \frac{1}{3}$ and $p_{ij}([ij|k]) = p_k([ij|k]) = \frac{1}{2}$). The CBV in this case simplifies to the following expression.

Proposition 2 *Consider any three-player game where all players have equal probability to propose, and let* $\overline{U}_{ij} = U_{ij} - U_k$*. The CBV is:*

Case i : *If* $\overline{U}_{12} \leq \frac{U}{3}$, $\overline{U}_{13} \leq \frac{U}{3}$, *and* $\overline{U}_{23} \leq \frac{U}{3}$ *then*

$$v_i = \frac{U}{3} \quad \textit{for all } i;$$

[5]Gul (1989) also obtains the Shapley value as the solution under a similar condition.

Case ii : *If* $\overline{U}_{12} \geq \frac{U}{3}$, $2\overline{U}_{13} + \overline{U}_{12} \leq U$, *and* $2\overline{U}_{23} + \overline{U}_{12} \leq U$ *then*

$$v_1 = v_2 = \frac{1}{4}\left(U + \overline{U}_{12}\right), \text{ and } v_3 = \frac{1}{2}\left(U - \overline{U}_{12}\right);$$

Case iii : *If* $\overline{U}_{12} + \overline{U}_{13} + \overline{U}_{23} \leq U$, $2\overline{U}_{13} + \overline{U}_{12} \geq U$, *and* $2\overline{U}_{12} + \overline{U}_{13} \geq U$ *then*

$$v_1 = \frac{1}{2}\left(\overline{U}_{12} + \overline{U}_{13}\right), \ v_2 = \frac{1}{2}\left(U - \overline{U}_{13}\right), \text{ and } v_3 = \frac{1}{2}\left(U - \overline{U}_{12}\right);$$

Case iv : *If* $\overline{U}_{12} + \overline{U}_{13} + \overline{U}_{23} \geq U$ *then*

$$v_i = \frac{1}{6}\left(2U - 2\overline{U}_{jk} + \overline{U}_{ij} + \overline{U}_{ik}\right) \text{ for all } i.$$

Note that the CBV depends only on an adjusted measure $\overline{U}_{ij}$ of the coalition's worth, where $\overline{U}_{ij} = U_{ij} - U_k$ is the value that coalition $\{i, j\}$ creates plus (minus) the amount of negative (positive) externalities that it creates for the excluded player k. Therefore, players' equilibrium values increase or decrease with the amount of negative or positive externalities they impose on others. Interestingly, any game with externalities has similar value and dynamics compared to a game without externalities (a characteristic function game) once coalitions' values are adjusted, $\overline{U}_{ij}$, to take externalities into account.

4.1 Examples

A better understanding of the negotiation strategies can be grasped by analyzing a few specific examples illustrating the cases previously discussed (in all examples we assume that players have equal opportunities to propose).

Example 1 *Three firms compete in an industry in which there are the following merger gains:* $u_i = U_i = 0$, $U = 1$, $U_{12} = v_H$, $U_{13} = v_{L_1}$, $U_{23} = v_{L_2}$ *where* $v_H \in \left[\frac{1}{3}, 1\right]$ *and* $v_{L_1} \leq v_{L_2} \leq \frac{1-v_H}{2} \leq v_H$.

What are the prices at which firms merge? Are there any natural merger partners in this industry? The bargaining value and strategies provide a direct answer to the questions above, as one can easily verify that this game belongs to region *ii* and thus the bargaining value is

$$v_1 = \frac{1+v_H}{4},\ v_2 = \frac{1+v_H}{4},\ \text{and}\ v_3 = \frac{1-v_H}{2},$$

where $v_1 = v_2 \geq v_3$. In this situation, we only expect to see the bilateral merger between firms 1 and 2. On the contrary, say that firms 1 and 3 merge. Their profitability increases by v_{L_1}, and there are still gains from further consolidation with firm 2. Firm 2 and conglomerate $\{1,3\}$ split the merger gains in a Nash bargaining way, each getting, respectively, $\frac{1}{2}(1-v_{L_1})$ and $\frac{1}{2}(1+v_{L_1})$. Note that the value of the conglomerate $\{1,3\}$ is $\frac{1}{2}(1+v_{L_1}) \leq \frac{1}{4}(3-v_H) = v_1 + v_3$. Therefore, one can predict that firms 1 and 3 are not going to merge, and by the same reasoning, one can also rule out a merger between firms 2 and 3.

Consider now a merger between firms 1 and 2. How should the value of the conglomerate $\{1,2\}$ be split among firms 1 and 2? Firm 2 has an apparent stronger bargaining position than firm 1 because $v_{L_1} \leq v_{L_2}$ and thus it seems reasonable that firm 2 should receive a higher share of the value than firm 1. However, this intuitive idea is wrong: Firm 2 does not have any credible outside options other than to merge with firm 1, and thus the Nash bargaining solution is an equal split of the value of the conglomerate $\{1,2\}$.

Example 2 *Formation of labor unions: In this game* $u_i = U_i = 0$, $U_{12} = v$, $U_{13} = v$, $U_{23} \leq 1-2v$ *where* $v \in \left[\frac{1}{3}, 1\right]$, *and* $U = 1$. *The firm is player 1, and players 2 and 3 are two unions.*

What are the firm's profits and the employee wages? Are the workers better off forming only one union to collectively bargain for wages or bargaining separately as two unions? This game belongs to region *iii*, and thus the bargaining value is equal to

$$v_1 = v,\ v_2 = \frac{1-v}{2},\ \text{and}\ v_3 = \frac{1-v}{2},$$

where $v_1 \geq v_2 = v_3$. Note that the unions are not willing to agree to a wage lower than $\frac{1-v}{2}$. Otherwise, the union could just wait until the firm signs a contract with the other union and bargain with the firm for a wage equal to half of the extra profits that the union could create, which results in a wage equal to $\frac{1}{2}(1-v)$. Interestingly, the threat of forming only one union to bargain for higher wages is not credible. The larger union can bargain for a total wage package equal to half of the surplus that it creates, which is equal to $\frac{1}{2}(1+U_{23}) \leq 1-v$.[6]

Example 3 *One-seller and two-buyer market game: Consider the negotiation game where* $u_i = U_i = 0$, $U_{12} = v_H = 1$, $U_{13} = v_L$, $U_{23} = 0$, *and* $U = v_H = 1$, *with* $v_L < v_H = 1$. *In this game player 1 is the seller, player 2 is the high valuation buyer, and player 3 is the low valuation buyer.*

By proposition 1 we have that the bargaining value is

$$v_1 = \frac{v_H}{2} + \frac{v_L}{6},\ v_2 = \frac{v_H - v_L}{2} + \frac{v_L}{6} = \frac{1}{2}v_H - \frac{1}{3}v_L,\ \text{and } v_3 = \frac{v_L}{6},$$

because $\overline{U}_{12}+\overline{U}_{13}+\overline{U}_{23} = v_H+v_L = 1+v_L > 1$.[7] Note that if the valuation of the buyers are the same $v_H = v_L = 1$ then the only point in the core of the market game is $(1,0,0)$, where the seller extracts all the surplus from the two buyers. In this case the CBV, which is equal to $\left(\frac{4}{6},\frac{1}{6},\frac{1}{6}\right)$, does not belong to the core.

Are the CBV predictions reasonable? Shouldn't we expect competition between the two buyers to drive the good's price to 1, as the core predicts? The main reason why the seller can't extract the entire surplus from the buyers is that both buyers have the option of forming a cartel to bid for the good and then buy it at a very low price (0.5), rather than initiating a bidding war. The seller knows about that all too well, and, rather than

[6]Kovenock and Widdows (1989) also analyze negotiations between a firm a two unions. They show that the sequencing of negotiations is related to what contingent contracts are available to unions.

[7]This solution generalizes the solution of the one-seller two-buyer market game in Osborne and Rubinstein (1990) when players are allowed to use contracts and resell the resource.

auctioning the good, the seller prefers to negotiate an intermediate price (between 0.5 and 1) with one buyer, leaving the second buyer with nothing. Because all agreements are binding after a deal is sealed (i.e., either a buyers' cartel is formed or the good is sold) there is no way for the excluded player to undo the deal by enticing one of the players with a slightly better offer.

4.2 Cooperative Solutions and the CBV

We now discuss the CBV's relationship with cooperative solution concepts. The CBV is closely related to classic cooperative game theory solutions for characteristic function games (i.e., partition function games without externalities, where $\overline{U}_{ij} = U_{ij}$).

Our next result shows that the CBV in regions *i*, *ii*, and *iii* coincides with the nucleolus. We recall that the nucleolus is the concept introduced by Schmeidler (1969), who proved that the nucleolus always exists and is a unique point belonging to the core of the game, whenever the core is non-empty. Kohlberg (1971) then showed that the nucleolus is a piecewise linear function of the characteristic function of the game, and Brune (1983) computed the nucleolus with its regions of linearity for three-person games (see appendix). We have seen in section 3 that the CBV coincides with the Shapley value in region *iv*.[8]

Proposition 3 *The CBV of any* 0*-normalized superadditive characteristic function game is the nucleolus, if* $U_{12}+U_{13}+U_{23} \leq U$*, or the Shapley value, if* $U_{12}+U_{13}+U_{23} \geq U$*.*

Note first that belonging to regions *i*, *ii*, or *iii* is indeed equivalent to the constraint $U_{12} + U_{13} + U_{23} \leq U$. Comparing the formula for the nucleolus derived by Brune (1983) with the formula for the CBV yields the above result (see appendix). While the nucleolus is a concept that is mathematically

[8] It is also worth pointing out the relationship between the CBV and the core. It is straightforward that the core of a three-player superadditive characteristic function game is non-empty if and only if $U_{12}+U_{13}+U_{23} \leq 2U$. Therefore, we conclude that the Shapley value is the CBV of all games with an empty core (because whenever the core is empty the game belongs to region *iv*).

very attractive and simple, economists have had difficulties in developing a motivation for it. The strategies employed by players in region *ii* and *iii*, where the nucleolus arise,[9] have an intuitive economic interpretation in terms of credible outside options which we now discuss.

For games that satisfy the conditions of case *ii*, there exists a pair of players $\{i, j\}$ (*natural partners*) that are willing to form a pairwise coalition. According to proposition 2, the outcome of negotiations when i and j are natural partners is $v_k = V_k$ and $v_i = v_j = \frac{V_{ij}}{2}$ whenever case *ii* holds, which one can easily see is equivalent to $V_k \leq \frac{V}{3}, V_{ik} \leq v_i + v_k$, and $V_{jk} \leq v_j + v_k$ (these inequalities can be verified by substituting expression (1) for V_i and V_{jk}). Here is the intuition for the result. Note first that the proposed solution is consistent with coalition $\{i, j\}$ being the only pairwise coalition forming; consider the alternative and suppose that the coalition $\{i, k\}$ forms; then the payoff for the coalition is V_{ik} and the payoff of the player left out is V_j. But since $V_{ik} \leq v_i + v_k$, then the coalition $\{i, k\}$ is worse off (with respect to the proposed equilibrium). The payoffs of the players i and j, $v_i = v_j = \frac{V_{jk}}{2}$, are also consistent with the fact that only the pairwise coalition $\{i, j\}$ may form: players i and j bargain over V_{ij} using as disagreement points their zero status quo values.

For games that satisfy the conditions of case *iii*, a *pivotal* player is included in all pairwise coalitions that are proposed, and the pairwise coalition between the non-pivotal players is never proposed. According to proposition 2, the outcome of negotiations when player i is pivotal is $v_i = V - V_j - V_k$, $v_j = V_j$, and $v_k = V_k$ whenever case *iii* holds, which one can easily see is equivalent to $V_{jk} \leq V_j + V_k, V_j \leq \frac{V_{ij}}{2}$, and $V_k \leq \frac{V_{ik}}{2}$. The intuition for this result is that players j and k cannot demand a higher payoff than V_j and V_k from player i by threatening to form the coalition $\{j, k\}$, since they would be worse off pursuing this strategy ($V_{jk} \leq V_j + V_k$). Also, note that players j and k are not willing to accept any offer lower than V_j and V_k because they can guarantee this amount by credibly holding out. This is so because if j holds out then i would successfully bargain with k to form a coalition;

[9] See section 3 for a discussion of the equilibrium in region *i*.

k's gains are $\frac{V_{ik}}{2} \geq V_k$, and thus k does not want to hold out when j holds out.

5 Conclusion

This paper explicitly derives the solution of a standard model for three-player negotiations with externalities. The analysis shows how the players' values are inextricably related to the equilibrium sequencing of negotiations. A simple way to deal with externalities is developed: add to the worth of a bilateral coalition the amount of negative externalities (or subtract the amount of positive externalities) that it creates for the excluded player. Players' equilibrium values are monotonically increasing or decreasing in the amount of negative or positive externalities they impose on others.

We show that the equilibrium value can be any of the following: the Nash bargaining solution, in the case where the value of all (adjusted) pairwise coalitions are less than a third of the grand coalition value; the Shapley value, in the case where the sum of the (adjusted) values created by all pairwise coalitions is greater than the grand coalition value; or the nucleolus, in the case where only the 'natural coalition' among two 'natural partners' creates significant value, and in the case where only the two pairwise coalitions including a 'pivotal player' create significant value.

We believe that the solution is economically intuitive and the experimental results of Croson, et al. (2004) indicate that it is an interesting candidate to be an off-the-shelf solution for applications, filling a gap in the literature. A natural (difficult) next step for future research is to derive close form solutions for games with an arbitrary number of players and externalities. Moreover, it would be interesting to establish links between the solution and existing or novel cooperative solution concepts (see for example Maskin (2003)). The results in this paper suggest that any plausible solution concept that applies to all games is likely to be a piecewise linear function, and it would be important to explicitly indicate the negotiation strategies associated with each region of linearity.

Appendix

Proof of Proposition 1: First note that cases *i*-*iv* form a partition of the parameter space: the union of cases *i*, *ii*, and *iii* is the half-space $V_{12}+V_{13}+V_{23} \leq 2U$ (and so it follows that *i*, *ii*, and *iii* is a partition of the half-space) and case *iv* corresponds to the complementary half-space $V_{12}+V_{13}+V_{23} \geq 2U$.

Denote limits when δ converges to one without the superscript δ: $v_i = \lim_{\delta\to 1} v_i^\delta$, $V_{ij} = \lim_{\delta\to 1} V_{ij}^\delta$, and $e_S^{'} = \lim_{\delta\to 1} \frac{de_S^\delta}{d\delta}$, etc. We have that $\frac{dV_{ij}^\delta}{d\delta} + \frac{dV_k^\delta}{d\delta} = (U - W_{ij})$ where $W_{ij} = U_{ij} + U_k$ and $V_k^\delta + V_{ij}^\delta = \delta U + (1-\delta) W_{ij}$.

We consider below a partition of the parameter space into open regions defined by strict inequalities *i*, *ii*.a-b, *iii*.a-d (subdivisions of cases *ii* and *iii* in which the strategies are slightly different but the limit solutions turn out to be the same), and *iv*. Consider the analysis of each of the regions separately in the remainder of the proof, and assume momentarily that $U > 0$ and $U - U_{ij} + U_k > 0$ (strict superadditivity).

$$\begin{aligned} V_{12} &< (p_1+p_2)\,U \\ V_{13} &< (p_1+p_3)\,U \\ V_{23} &< (p_2+p_3)\,U \end{aligned} \qquad \textbf{(i)}$$

Whenever (i) holds let $\sigma_i^\delta(N) = 1$ and $v_i^\delta = \delta p_i U$ (so $\mu_N = 1$). Eqs. (3) hold because $v_i^\delta = \delta p_i e_N^\delta + \delta v_i^\delta$, $e_N^\delta = U - \sum_{i\in N} v_i^\delta$ (note that $v_i = p_i U$). Ineqs. (2) hold because $e_N^\delta \geq e_S^\delta$ for all $S \subset N$: the excesses are equal to $e_N^\delta = (1-\delta)\,U$, $e_{ij}^\delta = V_{ij}^\delta - \delta\,(p_i+p_j)\,U$, $e_i^\delta = 0$ and the inequalities hold because $V_{ij} < (p_i+p_j)\,U$ for all $i, j \in N$.

Consider separately subregions *ii*.a and *ii*.b of region *ii*.

$$\begin{aligned} V_{ij} &> (p_i+p_j)\,U \\ (p_i+p_j)\,V_{ik} + p_j V_{ij} &< (p_i+p_j)\,U \\ (p_i+p_j)\,V_{jk} + p_i V_{ij} &< (p_i+p_j)\,U \\ V_{ij} - (p_i+p_j)^2\,(U - W_{ij}) &> (p_i+p_j)\,U \end{aligned} \qquad \textbf{(ii.a)}$$

(There are a total of three symmetric cases): Whenever (*ii*.a) holds let ${\sigma_i}^\delta(\{i,j\}) = 1$, ${\sigma_j}^\delta(\{i,j\}) = 1$ and ${\sigma_k}^\delta(N) = 1$ (so $\mu_N = p_k$ and $\mu_{ij} = p_i + p_j$), and v^δ be the solution of the system of linear eqs. (3):

$$\begin{aligned} v_i^\delta &= \delta p_i e_{ij}^\delta + \delta v_i^\delta \\ v_j^\delta &= \delta p_j e_{ij}^\delta + \delta v_j^\delta \\ v_k^\delta &= \delta p_k e_N^\delta + \delta\left((p_i+p_j)\,V_k^\delta + p_k v_k^\delta\right) \\ e_N^\delta &= U - \sum_{l\in N} v_l^\delta \\ e_{ij}^\delta &= V_{ij}^\delta - v_i^\delta - v_j^\delta \end{aligned}$$

Eqs. (3) have only one solution for all δ (that can be obtained applying, for example, Cramer's rule) and this solution converges to $v_i = \frac{p_i V_{ij}}{p_i+p_j}$ and $v_j = \frac{p_j V_{ij}}{p_i+p_j}$, $v_k = V_k = U - V_{ij}$. Ineqs. (2) holds if there exists $\overline{\delta} < 1$ such that for all $\delta \in [\overline{\delta}, 1)$, $e_N^\delta \geq \max\{e_{ik}^\delta, e_{jk}^\delta, 0\}$ and $e_{ij}^\delta \geq \max\{e_N^\delta, e_{ik}^\delta, e_{jk}^\delta, 0\}$: both excesses e_N^δ and e_{ij}^δ converge to zero when δ converges to one. To show that $e_{ij}^\delta \geq e_N^\delta$, for all $\delta \in [\overline{\delta}, 1)$, when both excesses converge to the same value, we prove that $e_{ij}^{'} < e_N^{'}$. Applying L'Hôpital's rule to the expression $v_i^\delta = \frac{e_{ij}^\delta}{p_i(1-\delta)}$, derived from the first eq. in the system, yields $e_{ij}^{'} = -\frac{v_i}{p_i}$, a result that will be used repeatedly in the remainder of the proof. From the solution to the system of equations, $e_N^\delta = (1-\delta)\left(\delta(p_i+p_j)(U - W_{ij}) + U\right)$, and thus $\frac{de_N^\delta}{d\delta} = -\left((p_i+p_j)(U-W_{ij}) + U\right)$. The inequality above holds because it is equivalent to $\frac{V_{ij}}{p_i+p_j} > (p_i+p_j)(U-W_{ij})+U$, (last inequality in *ii*.a). The condition $e_N^\delta \geq e_{ik}^\delta$ also holds as long as $\lim_{\delta\to 1} e_N^\delta = 0 > \lim_{\delta \to 1} e_{ik}^\delta$, which corresponds to $V_{ik} - \frac{p_i V_{ij}}{p_i+p_j} - V_k < 0 \Leftrightarrow (p_i+p_j)\, V_{ik} + p_j V_{ij} < (p_i+p_j)\, U$. By symmetry, $e_N^\delta \geq e_{jk}^\delta$ holds whenever, $(p_i+p_j)\, V_{jk} + p_i V_{ij} < (p_i+p_j)\, U$.

$$\begin{array}{rcl} V_{ij} & > & (p_i+p_j)\, U \\ (p_i+p_j)\, V_{ik} + p_j V_{ij} & < & (p_i+p_j)\, U \\ (p_i+p_j)\, V_{jk} + p_i V_{ij} & < & (p_i+p_j)\, U \\ V_{ij} - (p_i+p_j)^2 (U - W_{ij}) & < & (p_i+p_j)\, U \end{array} \qquad \textbf{(ii.b)}$$

(There are a total of three symmetric cases) *ii*.b strategy: players i and j randomize over the choices $\{i, j\}$ and N, and player k chooses N (let the associated transition probability be μ^δ). The transition probability is such that $\mu_{ij}^\delta \geq 0$, $\mu_N^\delta \geq 0$, and $\mu_N^\delta + \mu_{ij}^\delta = 1$, and, in addition, $\mu_N^\delta \in [p_k, 1]$, because player k's only choice is N, and k proposes with probability p_k. Moreover, given any transition probability μ^δ satisfying the conditions above, we can always find a strategy profile σ^δ with associated transition probability μ^δ. Let μ^δ and v^δ be a solution of the (non-linear) system of eqs. (3):

$$\begin{array}{rcl} v_i^\delta & = & \delta p_i e^\delta + \delta v_i^\delta \\ v_j^\delta & = & \delta p_j e^\delta + \delta v_j^\delta \\ v_k^\delta & = & \delta p_k e^\delta + \delta \left(\mu_{ij}^\delta V_k^\delta + \mu_N^\delta v_k^\delta\right) \\ e^\delta & = & U - \sum_{l \in N} v_l^\delta \\ e^\delta & = & V_{ij}^\delta - v_i^\delta - v_j^\delta \\ 1 & = & \mu_N^\delta + \mu_{ij}^\delta \end{array}$$

The system of equations excluding the third and last eqs. and the variables μ_{ij}^δ and μ_{ij}^δ, is linear and can be solved using Cramer's rule. The solution converges to $v_i = \frac{p_i V_{ij}}{p_i+p_j}$, and $v_k = V_k = U - V_{ij}$. After the expressions for v_k^δ and e^δ have

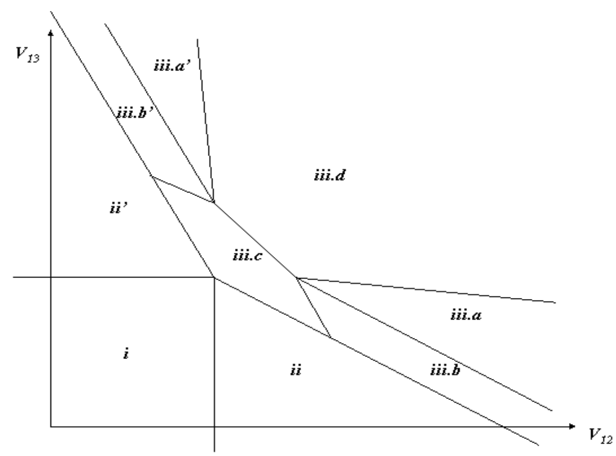

Figure 1: Case *iii* subcases.

been obtained we can solve for μ_{ij}^{δ} and μ_N^{δ} considering the third and last eqs. (The solution also converges to $\mu_N = \frac{(p_i+p_j)(2U-W_{ij})-V_{ij}}{(p_i+p_j)(U-W_{ij})}$ and $\mu_{ij} = \frac{V_{ij}-(p_i+p_j)U}{(p_i+p_j)(U-W_{ij})}$). The restriction $\mu_N > p_k = (1-p_i-p_j)$ on the transition probability corresponds to $V_{ij} < (p_i+p_j)\,U + (p_i+p_j)^2\,(U-W_{ij})$, and the restriction $\mu_{ij} > 0$, corresponds to, $V_{ij} > (p_i+p_j)\,U$, and both holds. Moreover, ineqs. (2) hold: (observe that $\lim_{\delta\to 1} e^{\delta} = 0$ and $e^{\delta} \geq 0$ for $\delta < 1$) $e^{\delta} \geq e_{ik}^{\delta}$ holds, for δ close enough to one, because $\lim_{\delta\to 1} e^{\delta} = 0 > \lim_{\delta\to 1} e_{ik}^{\delta} = V_{ik} - \frac{p_i V_{ij}}{p_i+p_j} - V_k < 0 \Leftrightarrow (p_i+p_j)\,V_{ik} + p_j V_{ij} < (p_i+p_j)\,U$. Symmetrically, $e^{\delta} \geq e_{jk}^{\delta}$ also holds, because $\lim_{\delta\to 1} e_{jk}^{\delta} = V_{jk} - \frac{p_j V_{ij}}{p_i+p_j} - V_k < 0 \Leftrightarrow (p_i+p_j)\,V_{jk} + p_i V_{ij} < (p_i+p_j)\,U$.

Consider the decomposition of case *iii* into four subcases *iii*.a-*iii*.d. Figure 1 illustrates each of the subcases (projected in the V_{12}-V_{13} space). Note that, as figure 3 illustrates, all four subcases have a common intersection point.

$$\begin{aligned}
(p_2-p_3)\,V_{12} + (1-2p_3)\,V_{13} &< (1-2p_3)\,U + p_1p_3\,(-U + (1-p_3)\,(W_{12}-W_{13})) \\
(p_1+p_2)\,V_{13} + p_2V_{12} &> (p_1+p_2)\,U + p_1p_3\,(U-W_{13}) \qquad \textbf{(iii.a)} \\
V_{12}+V_{13}+V_{23} &< 2U
\end{aligned}$$

(there are a total of six symmetric cases corresponding to all permutations of the players). *iii*.a strategy: players 1 and 2 choose $\{1,2\}$ and player 3 chooses $\{1,3\}$, and v^δ's are the (unique) solution of the system of linear eqs. (3)

$$\begin{aligned} v_1^\delta &= \delta p_1 e_{12}^\delta + \delta v_1^\delta \\ v_2^\delta &= \delta p_2 e_{12}^\delta + \delta\left((p_1+p_2)v_2^\delta + p_3 V_2^\delta\right) \\ v_3^\delta &= \delta p_3 e_{13}^\delta + \delta\left((p_1+p_2)V_3^\delta + p_3 v_3^\delta\right) \\ e_{12}^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\ e_{13}^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta \end{aligned}$$

The limit solution is $v_1 = U - V_2 - V_3$, and $v_j = V_j$ for $j = 2,3$. Ineqs. (2) are $e_{12}^\delta \geq e_{13}^\delta$, $e_{13}^\delta \geq e_N^\delta$, and $e_{13}^\delta \geq e_{23}^\delta$ for all $\delta \in [\overline{\delta}, 1)$: all the excesses e_{12}^δ, e_{13}^δ, and e_N^δ converge to zero as δ converges to one. So we analyze the derivatives of the excesses evaluated at $\delta = 1$ (see also case *ii*.a) which are equal to $e_{12}^{'} = -(p_1)^{-1}(V_{12} + V_{13} - U)$, $e_{13}^{'} = -(p_1 p_3)^{-1}((p_1+p_2)(V_{13}-U) + p_2 V_{12}) + (-U + (1-p_3)(W_{12} - W_{13}))$, $e_N^{'} = p_3 W_{13} + (1-p_3) W_{12} - 2U$. The ineq. $e_{12}^\delta \geq e_{13}^\delta$ holds for $\delta \in [\overline{\delta}, 1)$, because $p_1 + p_2 + p_3 = 1$ and first inequality in *iii*.a, imply $e_{13}^{'} - e_{12}^{'} = -(p_1 p_3)^{-1}\begin{pmatrix} (p_2 - p_3) V_{12} + (1-2p_3)(V_{13} - U) \\ -p_1 p_3 (-U + (1-p_3)(W_{12} - W_{13})) \end{pmatrix} > 0$, and the inequality $e^\delta(13) \geq e_N^\delta$ holds for $\delta \in [\overline{\delta}, 1)$, because the inequality in *iii*.a imply $e_N^{'} - e_{13}^{'} = (p_1 p_3)^{-1}((p_1+p_2)(V_{13}-U) + p_2 V_{12} - p_1 p_3 (U - W_{13})) > 0$, and finally $e_{13}^\delta \geq e_{23}^\delta$ holds because $\lim_{\delta\to1} e_{13}^\delta = 0 > \lim_{\delta\to1} e_{23}^\delta$: $\lim_{\delta\to1} e_{23}^\delta = V_{23} - v_2 - v_3 = V_{23} - (U - V_{13}) - (U - V_{12}) = V_{12} + V_{13} + V_{23} - 2U < 0$

$$\begin{aligned} (p_1+p_2)V_{13} + p_2 V_{12} &> (p_1+p_2)U \\ (p_1+p_2)V_{13} + p_2 V_{12} &< (p_1+p_2)U + p_1 p_3 (U - W_{13}) \qquad \textbf{(iii.b)} \\ (p_1+p_3)V_{12} + p_3 V_{13} &> (p_1+p_3)U + p_1(p_1+p_2)(U - W_{12}) \\ V_{12} + V_{13} + V_{23} &< 2U \end{aligned}$$

(there are six symmetric cases corresponding to all permutations of the players). *iii*.b strategy: players 1 and 2 choose $\{1,2\}$, and player 3 randomizes over the choice of $\{1,3\}$ or N. The transition prob. must satisfy $\mu_N^\delta + \mu_{13}^\delta = p_3$, because player 3 is the only player choosing $\{1,3\}$ and N (and 3 is proposer with prob. p_3), and $\mu_N^\delta \geq 0$ and $\mu_{13}^\delta \geq 0$ (reciprocally, given any μ^δ satisfying the restrictions above, a strategy profile with transition prob. equal to μ^δ can be constructed). Let μ^δ and v^δ's be a solution of the (non-linear) system of eqs. (3),

$$\begin{aligned} v_1^\delta &= \delta p_1 e_1^\delta + \delta v_1^\delta \\ v_2^\delta &= \delta p_2 e_2^\delta + \delta\left(\left(1-\mu_{13}^\delta\right) v_2^\delta + \mu_{13}^\delta V_2^\delta\right) \\ v_3^\delta &= \delta p_3 e_2^\delta + \delta\left(p_3 v_3^\delta + (p_1+p_2) V_3^\delta\right) \\ e_1^\delta &= V_{12}^\delta - v_1^\delta - v_2^\delta \\ e_2^\delta &= V_{13}^\delta - v_1^\delta - v_3^\delta \\ e_2^\delta &= U - v_1^\delta - v_2^\delta - v_3^\delta \end{aligned}$$

The system of equations excluding the second and last eqs. and the variables μ_{13}^{δ} and μ_{N}^{δ}, is a linear system of equations that can be solved using Cramer's rule. The solutions converges to $v_1 = U - V_2 - V_3$ and $v_j = V_j$ for $j = 2, 3$. After the expressions for v_2^{δ} and e_2^{δ} have been obtained we can solve for μ_{13}^{δ} and μ_{N}^{δ} considering the third and last eqs. The solution converges to $\mu_{13} = \frac{(p_1+p_2)V_{13}+p_2V_{12}-(p_1+p_2)U}{p_1(U-W_{13})}$, and the restrictions that $\mu_{13} \geq 0$ and $\mu_N \geq 0$ corresponds to inequalities one and two in iii.b. Ineqs. (2) $e_1^{\delta} \geq e_2^{\delta}$ and $e_2^{\delta} \geq e_{23}^{\delta}$, for all $\delta \in [\overline{\delta}, 1)$: Observe that $\lim_{\delta \to 1} e_1^{\delta} = \lim_{\delta \to 1} e_2^{\delta} = 0$, and so we analyze the derivatives of e_1^{δ} and e_2^{δ} at $\delta = 1$. Differentiating with respect to δ the expressions for e_1^{δ} and e_2^{δ} obtained from the solutions of the system of equations above yield $e_1^{'} = -p_1^{-1} v_1 = -p_1^{-1}(V_{12} + V_{13} - U)$ and $e_2^{'} = -p_1^{-1}((p_1+p_2)V_{13} + p_2 V_{12} - p_2 U + p_1(p_1+p_2)(U - W_{12}))$. The ineq. $e_2^{'} - e_1^{'} = p_1^{-1}((p_1+p_3)(V_{12} - U) + p_3 V_{13} - p_1(p_1+p_2)(U - W_{12})) > 0$ holds (because $p_1 + p_2 + p_3 = 1$ and the third inequality in *iii*.b), which implies that $e_1^{\delta} \geq e_2^{\delta}$ for all $\delta \in [\overline{\delta}, 1)$ for some $\overline{\delta} < 1$ close enough to one. As we have already argued (see iii.a), $e_2^{\delta} \geq e_{23}^{\delta}$ holds whenever $V_{12} + V_{13} + V_{23} < 2U$.

$$
\begin{aligned}
&(p_1+p_3)V_{12} + p_3 V_{13} > (p_1+p_3)U \\
&(p_1+p_2)V_{13} + p_2 V_{12} > (p_1+p_2)U \\
&\frac{(p_1+p_3)V_{12} + p_3 V_{13} - (p_1+p_3)U}{p_1(U - W_{12})} + \frac{(p_1+p_2)V_{13} + p_2 V_{12} - (p_1+p_2)U}{p_1(U - W_{13})} < 1 \\
&(p_1+p_3)V_{12} + p_3 V_{13} < (p_1+p_3)U + p_1(p_1+p_2)(U - W_{12}) \qquad \textbf{(iii.c)} \\
&(p_1+p_2)V_{13} + p_2 V_{12} < (p_1+p_2)U + p_1(p_1+p_3)(U - W_{13}) \\
&V_{12} + V_{13} + V_{23} < 2U
\end{aligned}
$$

(there are total of three such symmetric cases): all players randomize over the choices of $\{1,2\}$, $\{1,3\}$ and N. The transition probabilities μ_{12}^{δ} and μ_{13}^{δ} (and $\mu_N^{\delta} = 1 - \mu_{13}^{\delta} - \mu_{12}^{\delta}$) must satisfy $\mu_{13}^{\delta} \geq 0$, $\mu_{12}^{\delta} \geq 0$ and $\mu_N^{\delta} = 1 - \mu_{13}^{\delta} - \mu_{12}^{\delta} \geq 0$, and in addition, because players 1 and 2 are the only players that can choose $\{1,2\}$, the weight assigned to μ_{12} must satisfy, $\mu_{12}^{\delta} \leq p_1 + p_2$. Also, the same considerations applies to $\{1,3\}$, and thus $\mu_{13}^{\delta} \leq p_1 + p_3$. Let μ^{δ} and v^{δ}'s be a solution of the non-linear system of eqs. (3):

$$
\begin{aligned}
v_1^{\delta} &= \delta p_1 e^{\delta} + \delta v_1^{\delta} \\
v_2^{\delta} &= \delta p_2 e^{\delta} + \delta\left(\left(1 - \mu_{13}^{\delta}\right) v_2^{\delta} + \mu_{13}^{\delta} V_2^{\delta}\right) \\
v_3^{\delta} &= \delta p_3 e^{\delta} + \delta\left(\left(1 - \mu_{12}^{\delta}\right) v_3^{\delta} + \mu_{12}^{\delta} V_3^{\delta}\right) \\
e^{\delta} &= V_{12}^{\delta} - v_1^{\delta} - v_2^{\delta} \\
e^{\delta} &= V_{13}^{\delta} - v_1^{\delta} - v_3^{\delta} \\
e^{\delta} &= U - v_1^{\delta} - v_2^{\delta} - v_3^{\delta} \\
\mu_N^{\delta} &= 1 - \mu_{13}^{\delta} - \mu_{12}^{\delta}
\end{aligned}
$$

The system of four equations (eqs. 1, 4, 5, and 6 above) and four variables (v_1^{δ}, v_2^{δ}, v_3^{δ}, and e^{δ}) is a linear system that has a unique solution that converges to

$v_1 = U - V_2 - V_3$, $v_j = V_j$ for $j = 2, 3$. The solution for μ_{12}^{δ} and μ_{13}^{δ} (and $\mu_N^{\delta} = 1 - \mu_{13}^{\delta} - \mu_{12}^{\delta}$) can be directly obtained from eqs. 2 and 3 above and converge to, $\mu_{12} = \frac{(p_1+p_3)V_{12}+p_3V_{13}-(p_1+p_3)U}{p_1(U-W_{12})}$ and $\mu_{13} = \frac{(p_1+p_2)V_{13}+p_2V_{12}-(p_1+p_2)U}{p_1(U-W_{13})}$. Note that the conditions $\mu_{13} > 0$, $\mu_{12} > 0$ and $1 - \mu_{13} - \mu_{12} > 0$ correspond to the first three inequalities in *iii*.c. Moreover the restrictions $\mu_{12} < p_1 + p_2$ and $\mu_{13} < p_1 + p_3$, correspond respectively to the fourth and fifth inequalities in *iii*.c. The last inequality guarantees that $e_2^{\delta} \geq e_{23}^{\delta}$.

$$
\begin{aligned}
&(p_2 - p_3) V_{12} + (1 - 2p_3) V_{13} > (1 - 2p_3) U + p_3 p_1 (-U + (1 - p_3)(W_{12} - W_{13})) \\
&(p_3 - p_2) V_{13} + (1 - 2p_2) V_{12} > (1 - 2p_2) U + p_2 p_1 (-U + (1 - p_2)(W_{13} - W_{12})) \\
&\frac{(p_1 + p_2) V_{13} + p_2 V_{12} - (p_1 + p_2) U}{p_1 (U - W_{13})} + \frac{(p_1 + p_3) V_{12} + p_3 V_{13} - (p_1 + p_3) U}{p_1 (U - W_{12})} > 1 \\
&V_{12} + V_{13} + V_{23} < 2U \qquad \textbf{(iii.d)}
\end{aligned}
$$

(there are total of three such symmetric cases): player 1 randomizes over the choices $\{1, 2\}$ and $\{1, 3\}$, and players 2 and 3 choose $\{1, 2\}$ and $\{1, 3\}$, respectively. The transition probabilities associated with the strategy profile are such that $\mu_{1j}^{\delta} \geq p_j$ for $j = 2, 3$ because player j's only choice is coalition $\{1, j\}$ and player j proposes with probability p_j. Consider the non-linear system of eqs. (3):

$$
\begin{aligned}
v_1^{\delta} &= \delta p_1 e^{\delta} + \delta v_1^{\delta} \\
v_2^{\delta} &= \delta p_2 e^{\delta} + \delta \left(\mu^{\delta} v_2^{\delta} + \left(1 - \mu^{\delta}\right) V_2^{\delta}\right) \\
v_3^{\delta} &= \delta p_3 e^{\delta} + \delta \left(\mu^{\delta} V_3^{\delta} + \left(1 - \mu^{\delta}\right) v_3^{\delta}\right) \\
e^{\delta} &= V_{12}^{\delta} - v_1^{\delta} - v_2^{\delta} \\
e^{\delta} &= V_{13}^{\delta} - v_1^{\delta} - v_3^{\delta}
\end{aligned}
$$

(where $\mu^{\delta} = \mu_{12}^{\delta}$ and $\mu_{13}^{\delta} = 1 - \mu^{\delta}$). We first prove that there exist a solution of the system with $\mu^{\delta} \in (p_2, 1 - p_3)$ for all $\delta \in [\overline{\delta}, 1)$, for some $\overline{\delta} < 1$. The first step is to solve the system of (linear) eqs. composed of eqs. 1, 2, 4, and 5 for the variables v_1^{δ}, v_2^{δ}, v_3^{δ}, and e^{δ} as a function of μ^{δ}. Now, replacing the expressions for v_3^{δ} and e^{δ} into eq. 3, yields a quadratic equation in μ, $q^{\delta}(\mu) = 0$. The quadratic expression $q^{\delta}(\mu)$ evaluated at $\delta = 1$ yields, $q(\mu) = p_1 (W_{12} - W_{13}) \mu^2 + (V_{13} + V_{12} - U - p_1 (U + W_{12} - W_{13})) \mu - ((p_1 + p_3)(V_{12} - U) + p_3 V_{13})$. Developing the expressions for $q(p_2)$ and $q(1 - p_3)$ yields $q(p_2) = -(p_3 - p_2) V_{13} - (1 - 2p_2)(V_{12} - U) + p_2 p_1 (-U + (1 - p_2)(W_{13} - W_{12})) < 0$, and $q(1 - p_3) = (p_2 - p_3) V_{12} + (1 - 2p_3)(V_{13} - U) - p_3 p_1 (-U + (1 - p_3)(W_{12} - W_{13})) > 0$. Therefore, by the continuity of $q(\mu)$ with respect to μ, the quadratic equation $q(\mu) = 0$ has one solution in the interval $(p_2, 1 - p_3)$. Moreover, by continuity with respect to δ, there exists $\overline{\delta} < 1$, such that all equations $q^{\delta}(\mu) = 0$ (for all $\delta \in [\overline{\delta}, 1)$) also have one solution $\mu^{\delta} \in (p_2, 1 - p_3)$. The solutions v^{δ} and e^{δ} obtained are such that $\lim_{\delta \to 1} e^{\delta} = 0$ (this can be obtained directly from the first equation), and $v_j = \lim_{\delta \to 1} v_j^{\delta} = V_j$ for $j = 2, 3$, and $v_1 = \lim_{\delta \to 1} v_1^{\delta} = U - V_2 - V_3$, (the second equation, in the limit, is $v_2 = \mu_{12} v_2 + \mu_{13} V_2$, which combined with $\mu_{12} + \mu_{13} = 1$ and the third equation yields the expressions above for the limit equilibrium payoffs).

Ineqs. (2): $e^{\delta} \geq e_N^{\delta}$, for all $\delta \in [\overline{\delta}, 1)$, where $e_N^{\delta} = U - \sum_{i=1}^{3} v_i^{\delta}$. Because $\lim_{\delta \to 1} e^{\delta} = \lim_{\delta \to 1} e_N^{\delta} = 0$, it is sufficient to show that $e' < e_N'$. Adding up the first three eqs. yields $\sum_{i=1}^{3} v_i^{\delta} = \delta \mu_{12}^{\delta} \left(V_{12}^{\delta} + V_3^{\delta}\right) + \delta \mu_{13}^{\delta} \left(V_{13}^{\delta} + V_2^{\delta}\right)$, and the derivative with respect to δ is $\sum_{i=1}^{3} v_i^{'} = 2U - \mu_{12} W_{12} - \mu_{13} W_{13}$, which implies that $g \triangleq e_N' - e' = \frac{v_1}{p_1} - 2U + \mu_{12} W_{12} + \mu_{13} W_{13}$. Differentiating the second eq. in the system with respect to δ yields (at $\delta = 1$), $-p_2 \frac{v_1}{p_1} = \mu_{13} \left(v_2^{'} - V_2^{'}\right) - V_2$. But note that $\frac{dv_2^{\delta}}{d\delta} = \sum_{i=1}^{3} \frac{dv_i^{\delta}}{d\delta} - \left(\frac{dv_1^{\delta}}{d\delta} + \frac{dv_3^{\delta}}{d\delta}\right)$ and from the fifth equation in the system, $v_2^{'} - V_2^{'} = -\left(g + (U - W_{13})\right)$. Therefore, $\mu_{13} = \frac{\frac{p_2 v_1}{p_1} - V_2}{g + (U - W_{13})} = \frac{(p_1+p_2)V_{13} + p_2 V_{12} - (p_1+p_2)U}{p_1 g + p_1 (U - W_{13})}$ and a symmetric equation holds for μ_{12}. But g is the solution of $\frac{(p_1+p_2)V_{13} + p_2 V_{12} - (p_1+p_2)U}{g + p_1(U - W_{13})} + \frac{(p_1+p_3)V_{12} + p_3 V_{13} - (p_1+p_3)U}{g + p_1(U - W_{12})} = 1$, because $\mu_{13} + \mu_{12} = 1$, thus $g > 0$ (because of the third inequality in *iii*.d) which shows $e^{\delta} \geq e_N^{\delta}$ and concludes this case.

$$V_{12} + V_{13} + V_{23} > 2U \qquad \textbf{(iv)}$$

Strategy: all players randomize over pairwise coalitions $\{i, j\}$ (including the proposers). The associated transition probabilities satisfy $\mu_{12}^{\delta} + \mu_{13}^{\delta} + \mu_{23}^{\delta} = 1$, and moreover, the transition probabilities must satisfy the inequalities $\mu_{12}^{\delta} + \mu_{13}^{\delta} \geq p_1$ (because 1's only choices are $\{1, 2\}$ and $\{1, 3\}$, and 1 is proposer with probability p_1) and, similarly, $\mu_{12}^{\delta} + \mu_{23}^{\delta} \geq p_2$ and $\mu_{13}^{\delta} + \mu_{23}^{\delta} \geq p_3$. Note also that given any transition probability μ satisfying the conditions above, we can always find a strategy profile σ with associated transition probability μ^{δ}. Let μ^{δ} and v^{δ} be a solution of the system of non-linear eqs. (3):

$$\begin{aligned}
v_1^{\delta} &= \delta p_1 e^{\delta} + \delta \left(\left(\mu_{12}^{\delta} + \mu_{13}^{\delta}\right) v_1^{\delta} + \mu_{23}^{\delta} V_1^{\delta} \right) \\
v_2^{\delta} &= \delta p_2 e^{\delta} + \delta \left(\left(\mu_{12}^{\delta} + \mu_{23}^{\delta}\right) v_2^{\delta} + \mu_{13}^{\delta} V_2^{\delta} \right) \\
v_3^{\delta} &= \delta p_3 e^{\delta} + \delta \left(\left(\mu_{13}^{\delta} + \mu_{23}^{\delta}\right) v_3^{\delta} + \mu_{12}^{\delta} V_3^{\delta} \right) \\
e^{\delta} &= V_{12}^{\delta} - v_1^{\delta} - v_2^{\delta} \\
e^{\delta} &= V_{13}^{\delta} - v_1^{\delta} - v_3^{\delta} \\
e^{\delta} &= V_{23}^{\delta} - v_2^{\delta} - v_3^{\delta} \\
1 &= \mu_{12}^{\delta} + \mu_{13}^{\delta} + \mu_{23}^{\delta}
\end{aligned}$$

Substituting $v_i = \frac{1}{3}\left(U + V_{ij} + V_{ik} - 2V_{jk}\right)$, $\mu_{ij} = p_k$, and $e = \frac{1}{3}\left(V_{12} + V_{13} + V_{23} - 2U\right)$ into eqs. show that it is a solution for $\delta = 1$. By the implicit function theorem (IFT) a solution of the system for all $\delta \in [\overline{\delta}, 1)$, for some $\overline{\delta} < 1$, is also guaranteed because the Jacobian evaluated at the solution point and $\delta = 1$ (where the Jacobian is the natural one associated the system of the equations) is a non-singular matrix. Thus the problem of finding solutions for δ in a neighborhood of $\delta = 1$ satisfies all conditions of the IFT.[10] Moreover, for $\overline{\delta}$ close enough to one,

[10] Note that the Jacobian associated with all other cases considered before, evaluated at $\delta = 1$, are singular, and thus we cannot apply the IFT to the previous cases.

the solution also satisfies the inequalities such as $\mu_{12}^{\delta} + \mu_{13}^{\delta} > p_1$ (the inequality $\mu_{12} + \mu_{13} = p_3 + p_2 > p_1$ is strict because $p_1 < \frac{1}{2}$ and $p_1 + p_2 + p_3 = 1$) and $e^{\delta} > 0$ (because $e^1 = \frac{1}{3}(V_{12} + V_{13} + V_{23} - 2U) > 0$ is strict). Ineq. (2) is $e^{\delta} \geq e_N^{\delta}$. Since $\lim_{\delta \to 1} e_N^{\delta} = 0$ and $\lim_{\delta \to 1} e^{\delta} > 0$ we can guarantee that there exists $\overline{\delta} < 1$ such that $e^{\delta} \geq e_N^{\delta}$ for all $\delta \in [\overline{\delta}, 1)$.

Consider now any game in the frontier of any of the regions we considered above (i.e., assume that some of the strict inequalities are binding). Note that such game can be approximated by a sequence interior games. Because the results holds for all games in the interior, and the MPE correspondence is an upper hemi-continuous correspondence of the parameters of the game, it implies that the results also hold for all games in the frontier. Q.E.D.

Proof of Proposition 3: Let v and η denote the CBV and the nucleolus, respectively. According to Brune (1983), the nucleolus of a three-person 0-normalized superadditive game satisfying $U_{12} \geq U_{13} \geq U_{23}$:
If $U_{12} \leq \frac{U}{3}$ then $\eta = (\frac{U}{3}, \frac{U}{3}, \frac{U}{3})$,
If $U_{12} \geq \frac{U}{3}$ and $U_{12} + 2U_{13} \leq U$ then $\eta = (\frac{U+U_{12}}{4}, \frac{U+U_{12}}{4}, \frac{U-U_{12}}{2})$,
If $U_{12} + 2U_{23} \leq U$ and $U_{12} + 2U_{13} \geq U$ then $\eta = (\frac{U_{12}+U_{13}}{2}, \frac{U-U_{13}}{2}, \frac{U-U_{12}}{2})$,
If $2(U_{13}+U_{23})-U_{12} \geq U$ then $\eta = (\frac{U+U_{12}+U_{13}-2U_{23}}{3}, \frac{U+U_{12}+U_{23}-2U_{13}}{3}, \frac{U+U_{13}+U_{23}-2U_{12}}{3})$,
If $U_{12}+2U_{23} \geq U$, $2(U_{13}+U_{23})-U_{12} \leq U$ then $\eta = (\frac{U+2U_{13}+U_{12}-2U_{23}}{4}, \frac{U+2U_{23}+U_{12}-2U_{13}}{4}, \frac{U-U_{12}}{2})$.

We have argued before that the union of cases *i, ii,* and *iii* is equal to the half-space $V_{12} + V_{13} + V_{23} \leq 2U$, which, after using expression (1), is equal to $U_{12} + U_{13} + U_{23} \leq U$.

First note that the CBV and the nucleolus coincide for games in regions *i*-*iii*: if case *i* holds then obviously $v = \eta$; if case *ii* holds, which is equivalent to $U_{12} \geq \frac{U}{3}$, $2U_{13} + U_{12} \leq U$, and $2U_{23} + U_{12} \leq U$, then $v = (\frac{U+U_{12}}{4}, \frac{U+U_{12}}{4}, \frac{U-U_{12}}{2})$. Note that the inequalities imply that $U_{12} \geq U_{13}$ and $U_{12} \geq U_{23}$. Now if $U_{13} \geq U_{23}$ then $v = \eta$, and similarly, by the symmetry in the nucleolus formula, if $U_{23} \geq U_{13}$ then also $v = \eta$; if case *iii* holds, which corresponds to *(a)* $2U_{13} + U_{12} \geq U$, *(b)* $U_{12} + U_{13} + U_{23} \leq U$, and $U_{12} + U_{13} \geq U$ then $v = (\frac{U_{12}+U_{13}}{2}, \frac{U-U_{13}}{2}, \frac{U-U_{12}}{2})$. Note that the inequalities imply that $U_{12} \geq U_{13}$ and $U_{12} \geq U_{23}$. Now suppose that $U_{13} \geq U_{23}$. Combining inequalities a and b above (more precisely consider $(a)+2(b) \geq 0$) yields $U_{12}+2U_{23} \leq U$, which implies that $v = \eta$. A similar argument applies to the symmetric case where $U_{23} \geq U_{13}$. Thus the CBV coincides with the nucleolus whenever $U_{12} + U_{13} + U_{23} \leq U$.

Now, if condition $U_{12}+U_{13}+U_{23} \geq U$ holds, which is equivalent to $V_{12}+V_{13}+V_{23} \geq 2U$, then the CBV is equal to $v = \frac{1}{6}(2(U - U_{jk}) + U_{ij} + U_{ik})$, which is equal to the Shapley value. Q.E.D.

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Coalition Bargaining Games: Local and Global Uniqueness of Equilibria

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Abstract

This paper studies generic properties of Markov Perfect equilibrium of coalitional bargaining games. We show that in almost all games (except in a set of measure zero of the parameter space) the equilibrium is locally unique and stable, and comparative statics analysis are well-defined and can be performed using standard calculus tools. Global uniqueness does not hold in general, but the number of equilibria is finite and odd. In addition, a sufficient condition for global uniqueness is derived, and using this sufficient condition we show that there is a globally unique equilibrium in three-player superadditive games with externalities.

JEL: C71, C72, C78, D62

KEYWORDS: Coalitional bargaining, contracts, externalities, renegotiation

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1 Introduction

In this paper we study the properties of the equilibrium of coalitional bargaining games with externalities. The externalities present in the environment are described by a set of exogenous parameters, conveniently expressed using a partition function form. The partition function form assigns a worth to each coalition depending on the coalition structure (or collection of coalitions) formed by the remaining players. This general formulation is valuable because it can address situations in which the formation of coalitions impose positive or negative externalities (see also Ray and Vohra (1999), Bloch (1996), and Jehiel and Moldovanu (1995)).

Our coalitional bargaining game is based on the non-cooperative game proposed in Gomes (2005), where at each stage a player is randomly chosen to make an offer to form coalitions, followed by players who have received offers making their response whether or not to accept the offer. Like in Gul (1989) and Seidmann and Winter (1998), coalitions after forming do not leave the game and may continue negotiating the formation of further coalitions.

We focus on the properties of the Markov perfect equilibrium of the game (*MPE*) where the set of states are all possible coalition structures. The MPE solutions characterize, jointly, both the expected equilibrium value of coalitions, and the Markov state transition probability that describes the path of coalition formation.

The goal of this paper is to develop a thorough analysis of the uniqueness properties of the equilibria–the existence and efficiency properties of the equilibria have been developed in Gomes (2005). We prove that for almost all games (except in a closed set of measure zero) the MPE are locally unique and locally stable, i.e., local uniqueness is robust to small perturbations of the exogenous parameters of the game. Moreover, the number of equilibrium solutions is finite and odd for almost all games. Therefore, we extend to coalitional bargaining games similar results that hold for other well-known economic models such as Walrasian equilibrium of competitive economies

(Debreu (1970)), Nash equilibrium of n-person strategic form games (Wilson (1971) and Harsanyi (1973)), and the Markov perfect equilibrim of stochastic games (Lagunoff and Holler (2000)).

Although, the equilibria is not global unique, we provide a sufficient condition for global uniqueness. This sufficient condition holds for three-player coalitional bargaining games whenever the grand coalition is efficient (which includes the class of superadditive games), and thus there is a globally unique equilibrium for the class of three-player games (see also Gomes (2004)).

How do the equilibrium value of players and the path of coalition formation change as a result of changes in exogenous parameters such as the partition function form and the probability of being the proposer? Knowing how to address these questions is of considerable practical interest to negotiators, as they, for example, may be able to invest in changing the likelihood of being proposers in negotiations. We show how to answer these questions using standard calculus results (the implicit function theorem), which provides a powerful tool for quickly answering comparative statics questions by simply evaluating Jacobian matrices at the solution. We illustrate the applications of the technique using the apex and quota games (see Shapley (1953), Davis and Maschler (1965), and Maschler (1992)), and some interesting insights emerge. Surprisingly, a player sometimes may not benefit by investing in obtaining more initiative to propose in negotiations, because other players may adjust their strategies in such a way that lead the proposer to be worse off. The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase as well.

The paper also proposes a method to compute the equilibria. We show that the problem of finding equilibria is equivalent to finding solutions of a mixed nonlinear complementarity problem (*MNCP*). Such problems have been extensively studied in the mathematical programing literature (see Harker and Pang (1990) and Cottle, Pang, and Stone (1992)), and several numerical algorithms have been developed to solve them. Hence, the computation of equilibria is a task that can be undertaken using several proven numerical algorithms.

The remainder of the paper is organized as follows: section 2 presents the negotiation model; section 3 addresses the existence and efficiency of equilibria; section 4 shows how to compute the equilibria; section 5 develops the local uniqueness, stability, and genericity properties of the equilibria; section 6 includes the examples; section 7 addresses the number of equilibrium solutions; and section 8 concludes.

2 The Model

Coalition formation in this paper is modeled as an infinite horizon complete information game, and our goal in this paper is to analyze the properties of the equilibrium outcome. In a nutshell, the coalition formation process is such that during any period a player is chosen at random to propose a coalition, and a payment to all other coalition members. All prospective members respond in turn to the offer, and the coalition is formed only if all its members agree to the contract.

Formally, let $N = \{1, 2, \cdots, n\}$ be a set of n agents. A *coalition* is a subset of agents and a *coalition structure (c.s.)* $\pi = \{B_1, ..., B_K\}$ is a *partition* of the set of agents into disjoint coalitions. The game starts at the state where all agents are in solo coalitions (the c.s. is $\{\{1\}, ..., \{n\}\}$). The *coalition bargaining game* describes the extensive form of the coalition formation process. Consider that at the beginning of a certain period of the game the c.s. is π. One of the coalitions $i \in \pi$ is randomly chosen with probability $p_i(\pi) > 0$ to be the proposer. Coalition i then makes an offer (S, t) where $S \subset \pi$ is a set of coalition in π and t is a vector of transfers satisfying $\sum_{j \in S} t_j = 0$. All coalitions in S respond in a fixed sequential order whether they accept or not the offer. If they all accept the offer a new coalition $\mathcal{S} = \cup_{j \in S}\{j\}$ is formed under the control of the proposing coalition i and the coalitions $j \in S \backslash i$ ceding control receive the lump-sum payment t_j

(and the coalition structure evolves to $\pi S = \mathcal{S} \cup (\pi \backslash S)$).[1] Otherwise, if any one of the coalitions receiving the offer rejects it, no new coalition is formed and the coalition structure remains equal to π. After a lapse of one period of time, the game is repeated starting with the prevailing c.s. with a new proposer being randomly chosen as just described.

Remark: The model above is similar to the coalitional bargaining model introduced in Gomes (2005). In this study, coalition formation is also alternatively modelled as a process where agents are allowed to offer contracts specifying monetary transfers among signatories contingent on who contracted with whom. One advantage of this later approach is that the original cast of agents act as decision makers, rather than delegating the decision making to an agent (or third-party). However, it is shown that the equilibrium outcome of both models are equivalent, and thus the model in this paper can be seen as reduced form for the process of coalition formation. Since the focus here is exactly on the study of the equilibrium outcome properties, we prefer to use a reduced form model of coalition formation.

All agents have the same expected intertemporal utility function and are risk-neutral and have common discount factor $\delta \in (0,1)$.[2] When coalitions form they may impose externalities on other coalitions. This possibility is captured by a *partition function form* $v = (v_i(\pi))_{\substack{\pi \in \Pi \\ i \in \pi}}$, where coalition i's payoff flow (during a period of time), when the coalition structure is π, is equal to $(1-\delta)\, v_i(\pi)$ (so if the game stays at c.s. π forever, the value of coalition i is $v_i(\pi)$). The payoffs are distributed at the end of each period, after the coalition formation stage, with coalitions ceding control receiving a final lump-sum transfer payoff and the coalition acquiring control receiving, in addition to lump-sum transfers, the payoff given by the partition function form.

Our notion of equilibrium is *Markov perfect equilibrium (MPE)*, where the set of states is all the coalition structures (this is just referred at the

[1] So, for example, if $\pi = \{\{1,2\},\{3,4\},\{5\}\}$ and coalition $i = \{1,2\}$ proposes to form coalition $S = \{\{1,2\},\{3,4\}\}$, then $\pi S = \{\{1,2,3,4\},\{5\}\})$ and $\mathcal{S} = \{1,2,3,4\}$. Note that the coalition structure becomes coarser as time elapses.

[2] Thus their utility over a stream of random payoffs $(x_t)_{t=0}^{\infty}$ is $\sum_{t=0}^{\infty} \delta^t E(x_t)$.

equilibria throughout the paper). So players use strategies that only depends on the current state (and for responders also on the proposal and the response of previous players), but neither on the history of the game nor on calendar time.

3 Generic Local Uniqueness and Stability

In this section we show that almost all games (except in a set of measure zero of the parameter space) have equilibria that are locally unique and stable. These results imply that almost all games have only a finite number of equilibria, and provide tools for performing comparative statics analysis in coalitional bargaining games.

3.1 Characterization of Equilibrium

Let us be given a MPE σ and let $\phi_i(\pi|\sigma)$ represent the *equilibrium continuation value* (at the beginning of a period before a proposer is chosen) of coalition $i \in \pi$ when the c.s. is π (the value is derived from the stochastic process induced by σ). The equilibrium continuation value at the end of a period (after the coalition formation stage) is equal to (gross of lump-sum transfers)

$$x_i(\pi|\sigma) = \delta\phi_i(\pi|\sigma) + (1-\delta)\, v_i(\pi), \tag{1}$$

because coalition i receives the flow of payoff $(1-\delta)\, v_i(\pi)$ during the current period and, after a delay of one period, at the beginning of the next period, coalition i 's value is $\delta\phi_i(\pi|\sigma)$.

An equilibrium σ is characterized by several properties which we now summarize. The minimum offer that coalition j receiving offer (S,t) is willing to accept is one where $t_j \geq x_j(\pi|\sigma)$ (because upon rejection no transfers are made and the state remains at π). In turn, coalition i proposes offers (S,t) that maximizes the value $x_{\mathcal{S}}(\pi S|\sigma) - \sum_{j\in S\setminus i} t_j$ subject to the constraint that $t_j \geq x_j(\pi|\sigma)$ (because a new coalition $\mathcal{S}$ is formed whose value is $x_{\mathcal{S}}(\pi S|\sigma)$ if the offer is accepted by all players). So the *excess* of proposer

i is equal to $\max_{S \ni i} \{e(\pi)(S)(x)\}$ where,

$$e(\pi)(S)(x) = x_{\mathcal{S}}(\pi S|\sigma) - \sum_{j \in S} x_j(\pi|\sigma). \tag{2}$$

Proposers may randomize across coalitions S that maximizes the excess and $\sigma_i(\pi)(S)$ represents the probability that coalition S is chosen by player i (note that when an offer (S, t) is made the transfers t are uniquely determined and can be inferred from S). Let the support structure of a strategy profile σ be denoted by $\mathcal{C} = (\Sigma_i(\pi))$ where $\Sigma_i(\pi) = \mathit{supp}(\sigma_i(\pi))$ (i.e., $S \in \Sigma_i(\pi)$ if and only if $\sigma_i(\pi)(S) > 0$ and $\sum_{S \in \Sigma_i(\pi)} \sigma_i(\pi)(S) = 1$).

A Markovian strategy σ induces a *transition probability* $\mu = \mu(\sigma)$ on the state space which is defined by

$$\mu(\sigma)(\pi)(S) = \sum_{j \in \pi} p_j \sigma_j(\pi)(S). \tag{3}$$

and represents the probability that coalition $\mathcal{S}$ forms from state π (and a transition to state πS takes place). The *equilibrium point (or outcome)* associated with a MPE σ is the pair $(x(\sigma), \mu(\sigma))$ composed of the equilibrium value and the transition probabilities.

A key characterization result is that σ is a MPE of the coalitional bargaining game if and only if there exists a payoff structure $x = (x_i(\pi))$ and a probability distribution $(\sigma_i(\pi)(S))$ satisfying the following (see also Gomes (2005)):

- the support of $\sigma_i(\pi)$ satisfies

$$\mathit{supp}(\sigma_i(\pi)) \subset \arg\max_{S \ni i} \{e(\pi)(S)(x)\}, \tag{4}$$

- the following system of equations holds

$$\begin{aligned} x_i(\pi) &= \delta p_i(\pi) \max_{S \ni i} \{e(\pi)(S)(x)\} + (1-\delta) v_i(\pi) \\ &\quad + \delta \sum_{S \subset \pi} \mu(\sigma)(\pi)(S) \left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S) \right), \end{aligned} \tag{5}$$

and the equilibrium continuation value of σ satisfies $x_i(\sigma|\pi) = x_i(\pi)$.[3]

Note that equation (5) holds because the value of coalition $j \neq i$ conditional on offer S being made is either $x_j(\pi)$ if $j \in S$, or $x_j(\pi S)$ if $j \notin S$, and the proposer's i value is $\max_{S \ni i} \{e(\pi)(S)(x)\} + x_i(\pi)$.

An important difficulty in the analysis is that the equilibrium is not locally unique in the equilibrium strategies σ. There exist a continuum of equilibrium strategies whenever there is one equilibrium in which coalition S is the best response choice of more than one player, say i and j: a continuum of strategies can be constructed by having player i reducing the weight it chooses S, with player j increasing the weight on S, so that the overall transition probability μ is unchanged. Uniqueness with respect to the strategy profile does not hold in general, and the best we can hope for is to have uniqueness with respect to the equilibrium outcome.

To derive our main results about generic local uniqueness we introduce some notation to represent the equations that characterize the equilibrium outcome in a form that is amenable to the use of differential calculus tools.

Consider an MPE σ with support structure $\mathcal{C} = (\Sigma_i(\pi))$. Based on our characterizations and definitions, if $i_1, i_2 \in \pi$ satisfy $\Sigma_{i_1}(\pi) \cap \Sigma_{i_2}(\pi) \neq \emptyset$ then $e(\pi)(S)(x) = e(\pi)(T)(x)$ for all $S, T \in \Sigma_{i_1}(\pi) \cup \Sigma_{i_2}(\pi)$. More generally, if for any $i_1, i_m \in \pi$ there exists a sequence $i_1, i_2, ..., i_m \in \pi$ such that $\Sigma_{i_k}(\pi) \cap \Sigma_{i_{k+1}}(\pi) \neq \emptyset$, then $e(\pi)(S)(x) = e(\pi)(T)(x)$ for all $S, T \in \cup_{k=1}^{m} \Sigma_{i_k}(\pi)$. The following definition will soon prove useful.

Definition 1 *(Connection) Consider an arbitrary support structure* $\mathcal{C} = (\Sigma_i(\pi))$. *Let us define two coalitions* i_1,$i_m \in \pi$ *to be connected if there exists a sequence* $i_1, i_2, ..., i_m$ *such that* $\Sigma_{i_k}(\pi) \cap \Sigma_{i_{k+1}}(\pi) \neq \emptyset$. *Let the maximal connected components be denoted by* $P_r(\pi) \subset \pi$ *for* $r = 1, ..., q(\pi)$. *Define also* $C_r(\pi)$ *as* $C_r(\pi) = \cup_{i \in P_r(\pi)} \Sigma_i(\pi)$.

[3] The MPE σ is: proposers' strategies are $\sigma_i(\pi)(S,t) = \sigma_i(\pi)(S)$, if $t_j = x_j(\pi)$ for all $j \in S \backslash i$, and $t_i = -\sum_{j \in S \backslash i} t_j$ and $\sigma_i(\pi)(S,t) = 0$ otherwise; the strategies of responders' $j \in S \backslash i$ are to accept any offer (S,t) proposed by coalition i at state π if $t_j \geq x_j(\pi)$ and to reject it otherwise.

Note that connection is an equivalence relation (transitivity, symmetry, and reflexivity hold). So the maximal connected components, that is, the set which includes *all* players whose supports are connected, denoted by $P_r(\pi)$, are the equivalent classes of the connection relation.

Certainly by our discussion above it must be the case that all coalitions in the same connected component have the same excess

$$e_r(\pi) = x_S(\pi S) - \sum_{i \in S} x_i(\pi), \tag{6}$$

for all $S \in C_r(\pi)$, where r represents any of the $q(\pi)$ maximal connected components (so the excess is a function $e(x)$ of x). The payoffs x that are candidates for equilibrium outcome with support $\mathcal{C}$ must also satisfy inequalities

$$e(\pi)(S)(x) \geq e(\pi)(T)(x) \text{ for all } S \in \Sigma_i(\pi) \text{ and } T \notin \Sigma_i(\pi) \text{ with } i \in T, \tag{7}$$

and we let the set of payoffs consistent with an equilibrium with support $\mathcal{C}$ to be defined by

$$\mathcal{E}_{\mathcal{C}} = \left\{ x \in R^d : \text{ such that all inequalities (7) hold} \right\}. \tag{8}$$

Moreover, the associated Markov transition probability $\mu = \mu(\sigma)$ (besides satisfying $\sum_S \mu(\pi)(S) = 1$) must also satisfy the following equations

$$\sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi).$$

This is so because, by definition 1, $\Sigma_j(\pi) \subset C_r(\pi)$ for all $j \in P_r(\pi)$ and, in addition, $\Sigma_k(\pi) \cap C_r(\pi) = \emptyset$ for all $k \notin P_r(\pi)$. In addition, any transition probability $\mu = \mu(\sigma)$ that is consistent with an equilibrium σ with support $\mathcal{C}$ must belong to the set $\mathcal{M}_{\mathcal{C}}$ where

$$\mathcal{M}_{\mathcal{C}} = \{\mu = \mu(\sigma) : \text{ where } \sigma \in \Delta_{\mathcal{C}}\}, \tag{9}$$

and $\Delta_{\mathcal{C}}$ is the set of all strategies with support $\mathcal{C}$.

We are now ready to introduce a system of equations that will play an important role throughout the rest of the proof: For each support structure $\mathcal{C}$ define a mapping $F_{\mathcal{C}} : R^d \times R^m \times R^q \to R^d \times R^m \times R^q$ by

$$F_{\mathcal{C}}(x, \mu, e) = \begin{pmatrix} f_{\mathcal{C}}(x, \mu, e) \\ E_{\mathcal{C}}(x, e) \\ M_{\mathcal{C}}(\mu) \end{pmatrix} \tag{10}$$

where the maps $f_{\mathcal{C}}(x, \mu, e)$, $E_{\mathcal{C}}(x, e)$, and $M_{\mathcal{C}}(\mu)$ associated with support $\mathcal{C}$ are defined by

$$\begin{aligned} (f_{\mathcal{C}})_i(\pi)(x, \mu, e) &= x_i(\pi) - \delta p_i(\pi) e_r(\pi) - (1-\delta) v_i(\pi) \\ &\quad - \delta \left(\sum_S \mu(\pi)(S) \left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S) \right) \right), \\ E_{\mathcal{C}}(\pi)(S)(x, e) &= \sum_{i \in S} x_i(\pi) + e_r(\pi) - x_S(\pi S), \\ M_{\mathcal{C}}(\pi)(r)(\mu) &= \sum_{j \in P_r(\pi)} p_j(\pi) - \sum_{S \in C_r(\pi)} \mu(\pi)(S), \end{aligned} \tag{11}$$

for all r, i, and S satisfying $r = 1, ..., q(\pi)$, $i \in P_r(\pi)$, $S \in C_r(\pi)$, and all $\pi \in \Pi$, where $P_r(\pi)$ and $C_r(\pi)$ are as in definition 1.

The characterization of the equilibrium can be restated as the equivalent problem of finding solutions of problem $F_{\mathcal{C}}$, $F_{\mathcal{C}}(z) = 0$, where $z = (x, \mu, e)$ represents an equilibrium point (note that the excess e at a solution satisfies expression (6) and thus is given by a function of $e(x)$ of x). The next proposition follows directly from the arguments and definitions presented previously.

Proposition 1 *If σ is an MPE with support $\mathcal{C}$ then the equilibrium point $z = (\mu(\sigma), x(\sigma), e(\sigma))$ is a solution of problem $F_{\mathcal{C}}$, where $\mu(\sigma) \in \mathcal{M}_{\mathcal{C}}$, $x(\sigma) \in \mathcal{E}_{\mathcal{C}}$. Reciprocally, if $z = (x, \mu, e)$ is a solution of problem $F_{\mathcal{C}}$ satisfying $\mu \in \mathcal{M}_{\mathcal{C}}$ and $x \in \mathcal{E}_{\mathcal{C}}$ then there exists an MPE σ with equilibrium point z and support $\mathcal{C}$.*

So any equilibrium σ with support $\mathcal{C}$ has an equilibrium point $z = (x, \mu, e)$ that solves problem $F_{\mathcal{C}}$, $F_{\mathcal{C}}(z) = 0$. The reciprocal result also holds if a solution of $F_{\mathcal{C}}$ also satisfy a system of inequalities (represented by $\mu \in \mathcal{M}_{\mathcal{C}}$ and $x \in \mathcal{E}_{\mathcal{C}}$).

3.2 Regular and Nondegenerate Games

We seek to determine in this section conditions under which the equilibrium outcome is locally unique and stable. When global uniqueness is not achievable, as is the case in general for the coalitional bargaining games (see example 2), the next best property is local uniqueness. An equilibrium outcome is locally unique if we cannot obtain another equilibrium outcome arbitrarily close to it. On the other hand, stability is a property that ensures that comparative statics exercises are well defined. Roughly speaking an equilibrium point is stable if it changes smoothly for any small changes of the parameters of the game.

According to the previous section the equilibrium outcome are solutions of $F_{\mathcal{C}}(z) = 0$. Intuitively, in order to obtain (local) uniqueness it is necessary that the problem have the same number of equations and of unknowns. Indeed, there are $d = \sum_{\pi \in \Pi} |\pi|$ equations $f_{\mathcal{C}}$, $m = \sum_{\pi \in \Pi} \sum_{r=1}^{q(\pi)} m_r(\pi)$, where $m_r(\pi) = |C_r(\pi)|$, equations $E_{\mathcal{C}}$, and $q = \sum_{\pi \in \Pi} q(\pi)$ equations $M_{\mathcal{C}}$ (a total of $d+m+q$ equations). Moreover, the unknowns are the d dimensional variable x, the m dimensional variable μ, and the q dimensional variable e (a total of $d+m+q$ unknowns). So the number of equations and unknowns coincide.

We now introduce the concepts of regularity and nondegeneracy. Games that satisfy both of these technical conditions are shown to have equilibrium outcome that are locally unique and stable.

Definition 2 *(Regular game) A solution z of problem $F_{\mathcal{C}}(z) = 0$ is regular if the Jacobian $d_z F_{\mathcal{C}}$ is nonsingular. A support $\mathcal{C}$ is regular if all the solutions of problem $F_{\mathcal{C}}$ are regular.*[4] *Finally, a game is regular if all supports are regular.*

[4] By definition, if $F_{\mathcal{C}}$ have no solutions then the support $\mathcal{C}$ is regular.

The Jacobian $d_z F_{\mathcal{C}}$ is a matrix of order $d+m+q$, and is nonsingular if and only if it has full rank (equal to $d+m+q$). It has the following special structure

$$d_z F_{\mathcal{C}} = \begin{bmatrix} d_{(x,e)} f_{\mathcal{C}} & d_\mu f_{\mathcal{C}} \\ d_{(x,e)} E_{\mathcal{C}} & 0 \\ 0 & d_\mu M_{\mathcal{C}} \end{bmatrix}, \tag{12}$$

where the matrix$d_{(x,e)} E_{\mathcal{C}}$ has m rows and $d+q$ columns, and matrix $d_\mu M_{\mathcal{C}}(\mu)$ has q rows and m columns. This special structure of the Jacobian matrix will be explored later on to show that it is nonsingular almost everywhere.[5]

Remark: The solution at a given c.s. π only depends on the variables evaluated at coalition structures that are coarser than π. Thus the Jacobian matrix $d_z F_{\mathcal{C}}$ can be partitioned into an upper block triangular structure with diagonal blocks equal to $d_{z(\pi)} F_{\mathcal{C}}(\pi)$ where $z(\pi) = (x(\pi), \mu(\pi), e(\pi))$ for all $\pi \in \Pi$, where all entries to the left of the diagonal blocks are zero. Therefore, the Jacobian matrix $d_z F_{\mathcal{C}}$ is nonsingular if and only if all the diagonal blocks $d_{z(\pi)} F_{\mathcal{C}}(\pi)$ are nonsingular.

Consider now the nondegeneracy technical condition, which is a property of the support that roughly means that all choices outside the support are not best response strategies (Harsanyi (1973) and Kojima et al (1985) refer to a similar property in the context of n-person non-cooperative games as quasi-strong property). This condition is used in the next proposition to show that nearby games have equilibrium with the same support.

Definition 3 *(Nondegenerate game) A support $\mathcal{C} = (\Sigma_i(\pi))$ is a nondegenerate if all solutions of $F_{\mathcal{C}}$ satisfy*

$$e(\pi)(S)(x) \neq e(\pi)(T)(x) \text{ for all } S \in \Sigma_i(\pi) \text{ and } T \notin \Sigma_i(\pi) \text{ with } i \in T. \tag{13}$$

A game is nondegenerate if all supports are nondegenerate.

[5] Note that if Jacobian matrix is nonsingular then the $m \times (d+q)$-matrix $dE_{\mathcal{C}}$ must have rank m. So, for example, support structures with more than $d+q$ coalitions in the support ($m > d+q$) are not candidates for a regular equilibrium point. Reciprocally, we show in the next section that if matrix $dE_{\mathcal{C}}$ has rank m then for almost all partition functions the Jacobian matrix is nonsingular.

We now show that the implicit function theorem implies that regular and nondegenerate equilibrium points are locally unique and stable. Formally, we show that for any game (v^*, p^*) and regular and nondegenerate equilibrium point (x^*, μ^*) associated with an equilibrium with support $\mathcal{C}$, there exists an open neighborhood $B \subset R^d \times \Delta^d$ of (v^*, p^*), an open neighborhood $W \subset R^d \times R^m$ of (x^*, μ^*) and a local mapping $(x(v,p), \mu(v,p)) \in R^d \times R^m$ such that $(x(v,p), \mu(v,p))$ is the *only* equilibrium point in the neighborhood W for all games $(v,p) \in B$.

Proposition 2 *(Local uniqueness and stability) Regular and nondegenerate coalitional bargaining games have equilibrium points that are locally unique and stable.*

Proof: The implicit function theorem immediately implies that, for any game (v^*, p^*) and regular solution $z^* = (x^*, \mu^*, e^*)$ there exists an open neighborhood $B \subset R^d \times R^d$ of (v^*, p^*), an open neighborhood $\widetilde{W} \subset R^d \times R^m \times R^q$ of (x^*, μ^*, e^*), and a mapping $z(v,p) = (x(v,p), \mu(v,p), e(v,p)) \in R^n \times R^m \times R^q$ such that $z(v,p)$ is the only solution of problem $F_{\mathcal{C}}$ in $\widetilde{W}$ for all games $(v,p) \in B$. Note that since $e(v,p)$ can be expressed as a function of $x(v,p)$ (see equation (6)) then $z(v,p)$ is the only solution in a cylinder $W \times R^q$ for W an open neighborhood of (x^*, μ^*).

It remains to show that $z(v,p)$ is indeed an equilibrium point. By proposition 1, $z(v,p)$ is an equilibrium point if (i) $x(v,p) \in \mathcal{E}_{\mathcal{C}}$ and (ii) $\mu(v,p) \in \mathcal{M}_{\mathcal{C}}$. We show below that (i) and (ii) hold:

(i) Because $\mathcal{C}$ has full support (nondegeneracy condition), all the inequalities in (7) are strict for x^* and thus, by continuity, the inequalities also hold for all $x(v,p)$ in an open neighborhood Q of x^*, which is equivalent to $x(v,p) \in \mathcal{E}_{\mathcal{C}}$.

(ii) Key for the result that for all μ close to μ^* then $\mu \in \mathcal{M}_{\mathcal{C}}$ (there exists an equilibrium strategy $\sigma \in \Sigma_{\mathcal{C}}$ that yields the transition probability μ) is that the support structure is constructed so that each collection of sets C_r are composed of supports that are all connected (see definition 1). The following lemma establishes this result.

Lemma 1 *For any support $\mathcal{C}$ and $\mu^* \in \mathcal{M}_{\mathcal{C}}$ there exists an open neighborhood U around μ^* such that for all μ in U that satisfies all equations $\sum_{S \in C_r(\pi)} \mu(\pi)(S) = \sum_{j \in P_r(\pi)} p_j(\pi)$ there exists σ with support $\mathcal{C}$ such that $\mu(\sigma) = \mu$.*

Therefore, since (i) and (ii) hold in the neighborhood $W = Q \times U$, the pair $(x(v,p), \mu(v,p))$ is an equilibrium point, which completes the proof. Q.E.D.

3.3 Genericity of Equilibria

Regularity and nondegeneracy are here shown to be generic properties. The parameter space used to establish the result is the set of all games in $R^d \times \Delta^d$. Formally, a *generic* property is one that holds for all games, except possibly those in a subset of Lesbegue measure zero on $R^d \times \Delta^d$ (i.e., the property holds for *almost all* games).[6] Combining with the results of the previous section, we prove that local uniqueness and stability hold for almost all games.

This genericity result is established using the well-known transversality theorem from differential calculus (see Guillemin and Pollack (1974) and Hirsch (1976) and the appendix for a restatement of the theorem). The key result of this section is the following.

Proposition 3 *(Genericity) Almost all coalitional bargaining games (v,p) in $R^d \times \Delta^d$ are regular and nondegenerate. Therefore almost all games are locally unique and stable.*

Key for the proof is to show that for all supports $\mathcal{C}$ the Jacobian $d_z F_{\mathcal{C}}$ at any solutions of problem $F_{\mathcal{C}}(z) = 0$ is nonsingular, for almost every parameter (v,p) in $R^d \times \Delta^d$; that is all supports are regular almost everywhere.

In order to illustrate the arguments involved to prove regularity, consider first that $\mathcal{C}$ is a support where matrix $dE_{\mathcal{C}}$ has rank m. The solutions

[6] We have just argued that the set of regular and nondegenerate games is an open set. Therefore the set of games that are not regular nor degenerate is a closed set.

of problem $F_{\mathcal{C}}$ can be represented as the zeros of the augmented problem $F_{\mathcal{C}}(z,v)=0$, where we take into account the dependency with respect to the game. The Jacobian of this mapping is

$$d_{(z,v)}F_{\mathcal{C}}=\left[\begin{array}{ccc} * & * & -(1-\delta)I \\ d_{(x,e)}E_{\mathcal{C}} & 0 & 0 \\ 0 & d_{\mu}M_{\mathcal{C}} & 0 \end{array}\right],$$

where $d_zF_{\mathcal{C}}=\left[\begin{array}{cc} * & * \\ d_{(x,e)}E_{\mathcal{C}} & 0 \\ 0 & d_{\mu}M_{\mathcal{C}} \end{array}\right]$ and $d_vF_C=\left[\begin{array}{c} -(1-\delta)I \\ 0 \\ 0 \end{array}\right]$, and $*$ denotes arbitrary coeficients. The augmented Jacobian is a surjective matrix (with rank equal to the number of rows) because all blocks $dE_{\mathcal{C}}$, $dM_{\mathcal{C}}$, and $-(1-\delta)I$ have rank equal to the number of rows, and because of the disposition of zeros. Thus $F_{\mathcal{C}}$ is transversal to zero (i.e., $F_{\mathcal{C}}\overline{\pitchfork}0$) or 0 is a regular value of the augmented problem. By the transversality theorem, for almost every v, $F_{\mathcal{C}}(v)$ is also transversal to zero, $F_{\mathcal{C}}(v)\overline{\pitchfork}0$. Thus the square Jacobian matrix $d_zF_{\mathcal{C}}(v)$ is surjective at all solutions of $F_{\mathcal{C}}$, and thus nonsingular at all solutions for almost all $v\in R^d$ and all $p\in\Delta^d$.

Note that when the support $\mathcal{C}$ is such that matrix $dE_{\mathcal{C}}$ has rank smaller than m the Jacobian $d_zF_{\mathcal{C}}$ is singular for all parameters. However, in the appendix, we show that the further augmented problem $F_{\mathcal{C}}(z,v,p)=0$ is such that $d_{(z,v,p)}F_{\mathcal{C}}(z,v,p)=0$ in surjective, and thus, by the transversality theorem, $d_zF_{\mathcal{C}}$ is nonsingular almost everywhere in the parameter space $R^d\times\Delta^d$.

The argument to prove that almost all games are nondegenerate is as follows. Given any support $\mathcal{C}$ consider an hyperplane H in the space R^{d+m+q} defined by equality (13), $e\left(\pi\right)\left(S\right)\left(x\right)=e\left(\pi\right)\left(T\right)\left(x\right)$ for some pair S,T (so $\mathcal{C}$ is nondegenerate if there is no solution of $F_{\mathcal{C}}(z)=0$ such that $z\in H$). Applying the transversality theorem again (see Guillemin and Pollack (1974)) it follows that, for almost no parameters, there are solutions $F_{\mathcal{C}}(z)=0$ such that $z\in H$: because the codimension of H in the space R^{d+m+q} is 1, the transversality theorem applied to the surjective problem $F_{\mathcal{C}}(z,v,p)=0$ restricted to the domain $H\times R^d\times\Delta^d$ implies that for almost no parameters

(v,p) there are no solutions of $F_{\mathcal{C}}(z)=0$ such that $z\in H$. Using the fact that a finite union (there are only a finite number of pairs S,T) of sets of measure zero is a set of measure zero, we conclude that there exists a set of parameters, with complement of measure zero, where $F_{\mathcal{C}}$ is regular and nondegenerate.

3.4 Comparative Statics Analysis

Understanding how the value of coalitions and the path of coalition formation changes in response to changes in the exogenous parameters of the game v and p is a relevant comparative statics exercise. Regular and nondegenerate games are very convenient because they allow us to perform comparative statics analysis using standard calculus tools.

The following corollary is an immediate application of the implicit function theorem and proposition 2. The sensitivity matrix $\mathcal{S}_{\mathcal{C}}$ allow us to evaluate how the equilibrium point changes $\Delta z=\mathcal{S}_{\mathcal{C}}(\Delta v,\Delta p)$ in response to local changes of the game.

Corollary 1 *(Comparative Statics) Let (v,p) be a regular and nondegenerate game and $z=(x,\mu,e)$ be an equilibrium with support $\mathcal{C}$. The first-order effects of a change in the exogenous parameters (v,p) on the solution z is given by the sensitivity matrix $\mathcal{S}_{\mathcal{C}}=-[d_zF_{\mathcal{C}}]^{-1}d_{(v,p)}F_{\mathcal{C}}$ (i.e., $\Delta z=\mathcal{S}_{\mathcal{C}}(\Delta v,\Delta p)$). In particular, the effect of a local change Δv of coalitional values are given by $\Delta z=\left([d_zF_{\mathcal{C}}]^{-1}\right)_{\cdot x}(1-\delta)\Delta v$, where $\left([d_zF_{\mathcal{C}}]^{-1}\right)_{\cdot x}$ denotes the submatrix with the first d columns of the inverse Jacobian.*

The first-order effects with respect to changes in value Δv are given by the sensitivity matrix $-[d_zF_{\mathcal{C}}]^{-1}d_vF_{\mathcal{C}}$. But since

$$d_vF_{\mathcal{C}}=\begin{pmatrix} d_vf_{\mathcal{C}}\\ d_vE_{\mathcal{C}}\\ d_vM_{\mathcal{C}}\end{pmatrix}=-\begin{pmatrix}(1-\delta)I\\ 0\\ 0\end{pmatrix}, \tag{14}$$

the sensitivity matrix $-[d_zF_{\mathcal{C}}]^{-1}d_vF_{\mathcal{C}}$ simplifies to $\left([d_zF_{\mathcal{C}}]^{-1}\right)_{\cdot x}(1-\delta)$.So evaluating the inverse of the Jacobian matrix at the solution yields the first-order effects of changes in value.

We illustrate with the next example the comparative statics properties of quota games, first introduced by Shapley (1953) (see also Maschler (1992)).

Example 1: Quota Games

Consider a four-player quota game, where each pairwise coalition gets $v_{\{i,j\}} = \omega_i + \omega_j$ for all distinct pairs $i, j \in N$, where the quotas of the four players are $(\omega_1, \omega_2, \omega_3, \omega_4) = (10, 20, 30, 40)$, and all remaining coalitions get $v_S = 0$ for all $S \subset N$, $S \neq \{i, j\}$. Players are very patient (i.e., we are interested in the limit when δ converges to 1), and they all have an equal chance to be proposers.

The equilibrium point is depicted in Figure 1. Interestingly, the equilibrium strategy of player 1 is to wait for a pairwise coalition to form, an strategy that allows player 1 to get significantly more than his quota. The solution thus makes predictions that are consistent with experimental results reported in Maschler (1992), where player 1 realized that he was weak and that his condition would improve if he waited until a pairwise coalition formed, and captures an important strategic element of the game. Indeed, player 1 is better off if the coalition $\{2, 3\}$ forms, rather than $\{2, 4\}$ or $\{3, 4\}$, because in the ensuing pairwise bargaining with 4, player 1 can get a payoff equal to 25.[7]

How do players' value change with changes in quotas and proposers' probabilities? Evaluating the value-sensitivity matrix with respect to changes in quotas, as we have seen in section 3.4, yields

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \end{bmatrix} = \begin{bmatrix} 0.366 & 0.549 & 0.062 & 0.022 \\ 0.211 & 0.816 & -0.020 & -0.007 \\ 0.211 & -0.183 & 0.979 & -0.007 \\ 0.211 & -0.183 & -0.020 & 0.992 \end{bmatrix} \begin{bmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \\ \Delta\omega_4 \end{bmatrix},$$

and the coalition formation sensitivity matrix satisfies $\frac{\partial\mu(\{i,j\})}{\partial\omega_i} > 0$ and $\frac{\partial\mu(\{j,k\})}{\partial\omega_i} < 0$ for all distinct i, j, and k in {2,3,4} (for the sake of space we report only the signs of entries).

[7] However, strategies considered in this paper rule out the possibility that player 1 makes side payments to players 2 and/or 3 in order to encourage them to form coalition $\{2, 3\}$.

The information contained in the value sensitivity matrix provides the following results: the value of all players increases when their quotas increase, but increases in the quota of player 1 are shared by all players, while increases in the quotas of either player 2, 3 or 4 are almost completely appropriated by them (in fact, the other two players distinct from player 1 suffer a loss). Moreover, when a player's quota goes up, all coalitions including this player become more likely to form (and coalitions not including this player are less likely to form).

The comparative statics with respect to changes in proposers' probability is described by the value-sensitivity matrix

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \Delta\phi_3 \\ \Delta\phi_4 \end{bmatrix} = \begin{bmatrix} 0 & -5.42 & 1.96 & 3.45 \\ 0 & 1.80 & -0.65 & -1.15 \\ 0 & 1.80 & -0.65 & -1.15 \\ 0 & 1.80 & -0.65 & -1.15 \end{bmatrix} \begin{bmatrix} \Delta p'_1 \\ \Delta p'_2 \\ \Delta p'_3 \\ \Delta p'_4 \end{bmatrix},$$

where, in order to preserve the sum of probabilities equal to one, we consider $p_i = p'_i/(\sum_{j=1}^4 p'_j)$, and the coalition formation sensitivity matrix satisfies $\frac{\partial\mu(\{i,j\})}{\partial p'_i} < 0$ and $\frac{\partial\mu(\{j,k\})}{\partial p'_i} > 0$ for all distinct i, j, and k in $\{2, 3, 4\}$.

This comparative statics analysis reveals a surprising result: When player 2 has more initiative to propose, he benefits and player 1 loses from it. Interestingly, though, the opposite happens when players 3 and 4 have more initiative. Their equilibrium payoffs decrease when they have more initiative to propose![8]

4 Uniqueness and the Global Number of Equilibria

We show in this section that even though there can be multiple equilibrium points, as illlustrated by the next example, almost all games have a finite and

[8] This result can be rationalized as follows: when p_2 increases, coalitions {2,3} and {2,4} are less likely to form and coalition {3,4} more likely; since player 1's gains are lowest when coalition {3,4} forms he indirectly suffers when p_2 increases. By similar reasoning, when p_4 increases, coalitions {2,4} and {3,4} are less likely to form and coalition {2,3} more likely, which benefits player 1 and hurts the other players.

odd number of MPE equilibria. Moreover, we derive a sufficient condition for the global uniqueness of equilibria. The result states that if the index of each equilibrium solution is non-negative, where the index is equal to the sign of the determinant of the Jacobian matrix $d_z F_{\mathcal{C}}$, then there is a globally unique equilibrium. We prove that the sufficient condition holds for three-player superadditive games with externalities, and thus there is only one equilibrium in this class of games.

Let us start by showing that coalitional bargaining games may have multiple equilibrium points.

Example 2: War of Attrition (Multiple Equilibria)

The following three-player symmetric example have seven MPE solutions. The partition function that describes this game is $v_i(\{\{1\},\{2\},\{3\}\} = 0$, $v_{\{i,j\}}(\{\{i,j\},\{k\}\}) = 1$, and $v_{\{k\}}(\{\{i,j\},\{k\}\}\}) = 3$. The three-player coalition are not allowed (or has a very low value) and we assume that proposers are chosen with equal probabilities and $\delta \in (0.5, 1)$. We describe below all the equilibria (results are derived solving equations (1), but we omit the details).

There is an equilibrium in which the expected equilibrium value is $x = (0.5, 0.5, 0.5)$; the transition probabilities are $\mu(\{i,j\}) = \frac{1-\delta}{5\delta}$, for all pairs $\{i,j\}$, and $\mu(\emptyset) = \frac{8\delta-3}{5\delta}$, where $\emptyset$ represents no proposal (or remaining at the initial state). In this equilibrium, each of the three players refrains from proposing with high probability, and only proposes with a small probability to the other two players. They all reject any proposals below 0.5, and thus players are indifferent between proposing or not.

There are three other equilibria (they are all symmetric so we just focus on one of them), in which the expected equilibrium values are $x = (0, 1, 1)$; the transition probabilities are $\mu(\{1,2\}) = \mu(\{1,3\}) = \frac{1-\delta}{2\delta}$, and $\mu(\emptyset) = \frac{2\delta-1}{\delta}$. In this equilibrium, players 2 and 3 reject any proposals lower than 1, make no proposal with high probability, and, when proposing, choose to form a coalition with player 1. Player 1 cannot afford to pay more than 1 to form a coalition and thus it makes no proposals with probability one.

Finally, there are three additional equilibria (they are also symmetric), in which the expected equilibrium values are $x = \left(\frac{6\delta}{3-\delta}, \frac{\delta}{3-\delta}, \frac{\delta}{3-\delta}\right)$, which converges to $x = (3, 0.5, 0.5)$ when $\delta \to 1$; the transition probabilities are $\mu(\emptyset) = \frac{1}{3}$, $\mu(\{2,3\}) = \frac{2}{3}$. In this equilibrium, player 1's strategy is to refrain from proposing and reject any proposal worth less than 3, and player 2's and player 3's strategies are to always "give in" and propose to form the coalition $\{2,3\}$. ■

The main result of this section establishes a formula for counting the number of equilibrium points; the formula is based on the Index theorem (see for example Mas-Colell et al. (1995)). The index is a number that is assigned to each equilibrium point of a regular game. Say that z is an equilibrium point with support $\mathcal{C}$; the index of z is defined as the sign of the determinant of the Jacobian $d_z F_{\mathcal{C}}$ evaluated at z (and is either $+1$ or -1). We denote, index $z = sign \det (d_z F_{\mathcal{C}})$.

The result is useful because it implies that there is an odd number of equilibria. In particular, the number of equilibrium points is not zero, so there exists at least one equilibrium. We will see next that the result can be used to obtain global uniqueness for specific classes of games.

While the Index theorem has been applied to establish similar results for-competitive economies (Debreu (1970), and Mas-Colell (*)) and normal form games (Wilson (1971) and Harsanyi (1973)), the application to coalitional bargaining games is a bit more involved due to the fact that the equilibrium is not locally unique in terms of the equilibrium strategies, but only in terms of the equilibrium points (see section 3.1). To prove the result we use a stronger version of the Index theorem for correspondences developed in McLennan (1989)-see the appendix for the restatement of the Lefschetz fixed point theorem (LFPT) which is used in the proof. On the other hand, for both competitive economies and normal form games a standard version of the Index theorem developed in differential calculus textbooks suffices to develop the formula for the number of equilibria.

Proposition 4 *Almost all games (all regular and nondegenerate games) have a finite and odd number of equilibrium points. Moreover,*

$$\sum_{\mathcal{C}} \sum_{z:\in MPE_{\mathcal{C}}} sign \det \left(d_z F_{\mathcal{C}}\right) = +1,$$

where the summation is over all supports $\mathcal{C}$ and equilibrium points z with support $\mathcal{C}$.

PROOF: Consider the correspondence $\mathcal{F} : X \to X$ where, $\mathcal{F}(x) \subset R^d$ is defined by

$$\left\{ y \in R^d : \begin{array}{l} \text{where } y_i(\pi) = \delta p_i(\pi) \max\limits_{S \ni i} \{e(\pi)(S)(x)\} + (1-\delta) v_i(\pi) \\ +\delta \left(\sum_{S \subset \pi} \sum_{j \in \pi} p_j(\pi) \sigma_j(\pi)(S) \left(\mathbb{I}_{[i \in S]} x_i(\pi) + \mathbb{I}_{[i \notin S]} x_i(\pi S) \right) \right), \\ \text{and } supp\left(\sigma_i(\pi)\right) \subset \arg\max\limits_{S \ni i} \{e(\pi)(S)(x)\} \end{array} \right\}. \tag{15}$$

The set X is the convex and compact set $X \subset R^d$ defined by $X = \underset{\pi \in \Pi}{\times} X(\pi)$, where

$$X(\pi) = \{x(\pi) \in R^{|\pi|} \text{ such that } \sum_{i \in \pi} x_i(\pi) \leq \overline{v} \text{ and } x_i(\pi) \geq \underline{v}_i\},$$

and $\underline{v}_i = \min\limits_{\pi \ni i} \{v_i(\pi)\}$ and $\overline{v} = \max\limits_{\pi \in \Pi} \left\{\sum_{i \in \pi} v_i(\pi)\right\}$. By the definition of X it is easy to verify that indeed $\mathcal{F}(X) \subset X$.

The set of fixed points of $\mathcal{F}$, $\mathcal{F}^* = \{x \in X : x \in \mathcal{F}(x)\}$, corresponds to the equilibrium points of the game (see section 3.1). The set $\mathcal{F}^*$ is finite for all regular and nondegenerate games v: all the equilibrium points are, by proposition 1, solutions of $F_{\mathcal{C}}(x, \mu, e) = 0$ for some some support $\mathcal{C}$. But since the game is regular the solutions are locally isolated (proposition 2), and since the solution belongs to the compact X then there is only a finite number of solutions.

Moreover, it can be easily shown that $\mathcal{F} : X \to X$ is an upper hemi-continuous convex-valued correspondence (thus $\mathcal{F}(x)$ is contractible for all $x \in X$). The set $X \subset R^d$, Cartesian product of simplexes, is a simplicial complex and thus $\mathcal{F}$ satisfies the conditions of the *LFPT*.

Let U_{x^*} be an open neighborhood around each $x^* \in \mathcal{F}^*$, so that x^* is the only fixed point in $\overline{U}_{x^*}$. The Additivity Axiom of the Lefschetz index implies

$$\Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}). \tag{16}$$

In addition, the Lefschetz index is

$$\Lambda(\mathcal{F}, X) = 1. \tag{17}$$

But $\mathcal{F}$ can be approximated by a continuous map $f' : X \to X$ such that $\Lambda(\mathcal{F}, X) = \Lambda(f', X)$ (Continuity Axiom), and X is a contractible set, and thus there is an homotopy $\varphi : X \times [0,1] \to X$ where $\varphi_1 = I_X$ and $\varphi_0 = z_0 \in X$. Therefore, any continuous map $f' : X \to X$ is homotopic to the constant map so, by the Weak Normalization and Homotopy Axioms, $\Lambda(\mathcal{F}, X) = \Lambda(f', X) = 1$. Therefore, equations (16) and (17) imply,

$$\Lambda(\mathcal{F}, X) = \sum_{x^* \in \mathcal{F}^*} \Lambda(\mathcal{F}, U_{x^*}) = 1.$$

The next lemma, proved in the appendix, establishes a formula for computing the Lefschetz index $\Lambda(\mathcal{F}, U_{x^*})$, which shows that it is equal to $sign \det(d_{z^*} F_{\mathcal{C}})$ where $z^* = (x^*, \mu^*, e^*)$ is the equilibrium point.

Lemma 2 *The Lefschetz index of a regular and nondegenerate equilirbrium point $z^* = (x^*, \mu^*, e^*)$ is equal to $\Lambda(\mathcal{F}, U_{x^*}) = sign \det(d_{z^*} F_{\mathcal{C}})$, and is equal to either $+1$ or -1.*

Q.E.D.

Proposition 4 implies a sufficient condition for global uniqueness of equilibria.

Corollary 2 *All regular and nondegenerate coalitional bargaining games have a globally unique equilibrium if $\det(d_z F) \geq 0$ where the Jacobian is evaluated at any solution of problem $F_{\mathcal{C}}$ for all supports $\mathcal{C}$.*

Because of the special structure of the Jacobian $d_z F_{\mathcal{C}}$, we conjecture that a sufficient condition for $\det(dF_{\mathcal{C}}) \geq 0$ at all solutions of problem $F_{\mathcal{C}}$ is that the inequalities $x_i(\pi) - x_i(\pi S) \geq 0$, for all $S \in C_r(\pi)$ and $i \notin S$, hold. Direct computation of determinants reveals, for all games with three players and all *CDS*s $\mathcal{C}$, that the inequalities imply that $\det(dF_{\mathcal{C}}) \geq 0$ (see corollary 5).

The inequality $x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, is a weak condition that has a natural economic interpretation: player i is excluded from the offer S if and only if moving from c.s. π to πS imposes a negative externality on player i (i.e., $x_i(\pi S) \leq x_i(\pi)$). We show in the next corollary that these inequalities hold for all three-player games where the grand coalition is efficient.

Therefore, the equilibrium point computed explicitly in Gomes (2004) is the unique equilibrium point for almost all games.

Proposition 5 *Almost all three-player games with externalities where the grand coalition is efficient,* $v(\{N\}) \geq \sum_{S \in \pi} v_S(\pi)$ *for all* π*, in particular superadditive games, have a globally unique equilibrium.*

Proof: We focus on the c.s. $\pi = \{\{1\},\{2\},\{3\}\}$ because we already know that two-player games have a unique equilibrium (Rubinstein (1982)).

We first show that $X(i,S) = x_i(\pi) - x_i(\pi S) \geq 0$, where $i \notin S$, if there is a positive probability that S is chosen in equilibrium. Say that $S = \{j,k\}$ (if $S = \emptyset$ (no proposal case) then $x_i(\pi S) = x_i$ and if $S = N = \{1,2,3\}$ then there are no elements $i \notin S$). In order to simplify the notation, let $x_i = x_i(\pi)$, $x_i(\pi S) = x_i(jk)$, $x_S(\pi S) = x_{jk}(jk)$, and $V = v_N(\{N\})$. Suppose that S is chosen in equilibrium with positive probability. Then $e(S)(x) \geq e(N)(x)$, which is equivalent to,

$$x_{jk}(jk) - x_j - x_k \geq V - x_i - x_j - x_k, \tag{18}$$

and

$$x_{jk}(jk) + x_i(jk) + x_i - x_i(jk) \geq V. \tag{19}$$

But since there is no delay in the formation of the grand coalition when the game is at the c.s. $\{\{jk\}, \{i\}\}$, we have that

$$x_{jk}(jk) + x_i(jk) = \delta V + (1-\delta)(v_{jk}(jk) + v_i(jk)).$$

Replacing this expression into (19) yields

$$X(i, jk) = x_i - x_i(jk) \geq (1-\delta)(V - (v_{jk}(jk) + v_i(jk))) \geq 0.$$

We now compute $\det(dF_{\mathcal{C}})$ for all admissible *CDS* $\mathcal{C} = (C, P)$, and show that $\det(dF_{\mathcal{C}}) \geq 0$. From the definition of *CDS*s it follows that $P = (P_1, ..., P_q)$ is a partition of N and $C = (C_1, ..., C_q)$ is an ordered disjoint collection of subsets $S \subset N$ satisfying: for all $S \in C_r$ then $S \cap P_r \neq \emptyset$ and $S \subset \cup_{s=1}^{r} P_s$, and also $\cup_{S \in C_r} S \supset P_r$. Moreover, proposition **??** implies that there is no $S = \{i\}$ that is chosen in equilibrium, and thus $C_r \subset \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. A list of all admissible *CDS*s (except for permutations of the players) follows with the corresponding value for $\det(dF_{\mathcal{C}})$ (i, j, and k are distinct elements of N, and $d_i = 1 - \delta \sum_S \mu(S) \mathbb{I}_{[i \in S]}$, $z(i, jk) = \delta X(i, jk)$, and $w_i = \delta p_i$):

For example, the Jacobian $dF_{\mathcal{C}}$ for $\mathcal{C} = (\{\{1,2\}, \{1,3\}, \{2,3\}\})$ in a three-player game is,

$$\begin{bmatrix} 1-\delta\left(\mu_{\{1,2\}}+\mu_{\{1,3\}}\right) & 0 & 0 & -\delta\, p_1 & 0 & 0 & \delta\, X_{1,\{2,3\}} \\ 0 & 1-\delta\left(\mu_{\{1,2\}}+\mu_{\{2,3\}}\right) & 0 & -\delta\, p_2 & 0 & \delta\, X_{2,\{1,3\}} & 0 \\ 0 & 0 & 1-\delta\left(\mu_{\{1,3\}}+\mu_{\{2,3\}}\right) & -\delta\, p_3 & \delta\, X_{3,\{1,2\}} & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix},$$

$CDS\ \mathcal{C}$	$\det(dF_{\mathcal{C}})$
$(\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\})$	$z(2,13)z(1,23)z(3,12)$
$(\{\{1,2\},\{1,3\},\{1,2,3\}\})$	$z(3,12)z(2,13)(w_1+d_1)$
$(\{\{1,2\},\{1,2,3\}\})$	$z(3,12)(d_2w_1+d_1d_2+w_2d_1)$
$(\{\{1,2,3\}\})$	$d_1w_3d_2+d_1d_3d_2+d_1d_3w_2+d_2d_3w_1$
$(\{\{1,2\},\{2,3\},\{1,3\}\})$	$\sum_{i,j,k}(d_i+2w_i)\,z(k,ij)z(j,ik)$
$(\{\{1,2\},\{1,3\}\})$	$\sum_{i,j\neq 1}(d_iw_1+d_id_1+w_id_1)z(j,1i)$
$(\{\{1,2\}\},\{\{1,2,3\},\{1,3\},\{2,3\}\})$	$z(2,13)z(1,23)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,2,3\},\{1,3\}\})$	$z(2,13)(w_1+d_1)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,2,3\}\})$	$(d_2w_1+d_2d_1+w_2d_1)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,3\},\{2,3\}\})$	$\left(\sum_{i,j\neq 3}(2w_i+d_i)z(j,i3)\right)(w_3+d_3)$
$(\{\{1,2\}\},\{\{1,3\}\})$	$(d_2w_1+d_2d_1+w_2d_1)(w_3+d_3)$

Note that the first 6 entries of the table corresponds to *CDS*s with $P = (\{1,2,3\})$ and the last entries to *CDS*s with $P = (\{1,2\},\{3\})$.

The determinant for all *CDS*s are nonnegative because it is a sum of nonnegative terms. Corollary 2 implies that there is a unique global MPE solution. Q.E.D.

5 Conclusion

This paper studied the equilibrium properties of n-player coalitional bargaining games in an environment with widespread externalities (where the exogenous parameters are expressed in a partition function form). The coalitional bargaining problem is modeled as a dynamic non-cooperative game in which contracts forming coalitions may be renegotiated. The equilibrium concept used is stationary subgame or Markov perfect equilibrium, where the set of states is all possible coalition structures.

A comprehensive analysis of the equilibrium properties is developed. We show that for almost all games (except in a closed set of measure zero) the equilibrium is locally unique and stable to small perturbations of the exogenous parameters, and the number of equilibria is finite and odd. Global uniqueness does not hold in general, but a sufficient condition for global uniqueness is derived, and this sufficient condition is shown to prevail in three-player superadditive games.

Comparative statics analysis can be easily performed using standard calculus tools, allowing us to understand how the value of players and the path of coalition formation changes in response to changes in the exogenous parameters. Being able to answer comparative statics questions is valuable to negotiators, because they may be able, for example, to invest in changing the likelihood of being proposers. Applications of the technique are illustrated using the apex and quota games, and some interesting insights emerge: surprisingly, a player may not benefit from having more initiative to propose (other players may adjust their strategies in such a way that lead the proposer to be worse off). The analysis also suggests several interesting regularities: when the exogenous value of a coalition increases, both the equilibrium value of the coalitional members and the likelihood that the coalition forms increase.

Appendix

Proof of lemma 1: The same steps of the proof applies to each c.s. π separately, so to simplify notation we eliminate explicit references to π below. The following claim implies the lemma, as shown below.

Claim: *Given any* $\mu = (\mu(S))_{S \in C_r}$ *close to zero satisfying* $\sum_{S \in C_r} \mu(S) = 0$ *there exists* $\sigma = (\sigma_i(S))_{\substack{S \in \Sigma_i \\ i \in P_r}}$ *close to zero satisfying* $\sum_{S \in \Sigma_i} \sigma_i(S) = 0$ *for all* $i \in P_r$ *such that* $\mu(\sigma) = \mu$.

There exists $\sigma^* \in \Delta_{\mathcal{C}}$ such that $\mu(\sigma^*) = \mu^*$ (as $\mu^* \in \mathcal{M}_{\mathcal{C}}$). Given any μ close to μ^* define $\Delta\mu = \mu - \mu^*$ (which is close to zero). Consider a $\Delta\sigma$ given by the claim (related to $\Delta\mu$) and let $\sigma = \sigma^* + \Delta\sigma$. Such σ satisfies $\sigma_i(S) > 0$ for all $S \in \Sigma_i$ (because $\sigma_i^*(S) > 0$ for all $S \in \Sigma_i$ and $\Delta\sigma(S)$ are close to zero) and $\sum_{S \in \Sigma_i} \sigma(S) = \sum_{S \in \Sigma_i} \sigma^*(S) + \sum_{S \in \Sigma_i} \Delta\sigma_i(S) = 1$. So $\sigma \in \Delta_{\mathcal{C}}$ and, by linearity of $\mu(\cdot)$, $\mu(\sigma) = \mu$. Therefore, it is sufficient to prove the claim.

Proof of claim: It is enough to analyze each component $r = 1, ..., q$ separately so we drop the subscript r (so $P = P_r$ and $C = C_r$, and say that $\#P = p$). By construction (definition of P and C), all supports Σ_i for $i \in P$ are connected, so there exists an ordering $(i_1, ..., i_k, ..., i_p)$ of P where $P^k = \{i_1, ..., i_k\}$ and $C^k = \cup_{i \in P^k} \Sigma_i$ satisfy $C^k \cap \Sigma_{i_{k+1}} \neq \varnothing$ for all $k = 1, ..., p-1$ (starting from any element in P, each new element in the order is chosen so that it has a support connected to some of the previous elements chosen).

The proof now proceeds by induction. The induction hypothesis is: Suppose that the following statement holds for P^k and C^k: Given any $\mu^k = \left(\mu^k(S)\right)_{S \in C^k}$ close to zero satisfying $\sum_{S \in C^k} \mu^k(S) = 0$ there exists $\sigma^k = \left(\sigma_i^k(S)\right)_{\substack{S \in \Sigma_i \\ i \in P^k}}$ close to zero satisfying $\sum_{S \in \Sigma_i} \sigma_i^k(S) = 0$ for all $i \in P^k$ such that $\mu\left(\sigma^k\right) = \mu^k$. The statement also holds for $k+1$: Consider any $\mu = (\mu(S))_{S \in C^{k+1}}$ close to zero satisfying $\sum_{S \in C^{k+1}} \mu(S) = 0$. Let $\hat{S} \in C^k \cap \Sigma_{i_{k+1}}$. Define $\sigma_{i_{k+1}}(S) = \mu(S)$ for all $S \in \Sigma_{i_{k+1}} \backslash \{\hat{S}\}$ and let $\sigma_{i_{k+1}}\left(\hat{S}\right) = -\sum_{S \in \Sigma_{i_{k+1}} \backslash \{\hat{S}\}} \mu(S)$. Also define $\mu^k(S) = \mu(S)$ for all $S \in C^k \backslash \Sigma_{i_{k+1}}$, $\mu^k(S) = 0$ for all $S \in C^k \cap \Sigma_{i_{k+1}} \backslash \{\hat{S}\}$, and let $\mu^k\left(\hat{S}\right) = -\sum_{S \in C^k \backslash \{\hat{S}\}} \mu^k(S)$. Thus, using the induction hypothesis, there exists $\sigma = (\sigma_i(S))_{\substack{S \in \Sigma_i \\ i \in P^{k+1}}}$ which is close to zero, satisfies $\sum_{S \in \Sigma_i} \sigma_i(S) = 0$ for all $i \in P^{k+1}$ and is such that $\mu(\sigma) = \mu$. Since the statement is true for $k = 1$ (just let $\sigma = \mu$) and, by induction, the statement is also true for $k = p$, which is exactly the claim (as $P^k = P$ and $C^k = C$), this completes the proof. Q.E.D.

Transversality theorem: Suppose $f : U \times V \to R^n$ is continuously differentiable where $U \subset R^n$ and $V \subset R^m$ are open sets. If the $n \times (n+m)$ Jacobian $d_{(x,q)}f$ has rank n whenever $f(x,q) = 0$ (i.e., $f \pitchfork 0$) then the system of n equation and n unknowns $f(\cdot, q^*) = 0$ is regular for almost every $q^* \in V$.

Proof of Theorem 3: The proof is by induction on the number of players and the induction hypothesis is: for games with less than n players, almost all parameters (v,p) in $R^d \times \Delta^d$ are regular and nondegenerate *and* all such games have local solution mappings $x(v,p)$ that are surjective.

The hypothesis holds for one player games: the only support is $\mathcal{C} = \{\{1\}\}$ and the Jacobian matrix of problem $F_{\mathcal{C}}$ is obviously nonsingular. Now, let π be a c.s. with n players, and let us represent by a subscript 0 the references to the c.s. π and by the subscript -0 the references to all its proper subgames. Let $V_0 \times \Delta_0$ represent the set of all $(v_i(\pi), p_i(\pi))$ and $V_{-0} \times \Delta_{-0}$ the set of all $(v_i(\pi'), p_i(\pi'))_{\substack{\pi' \in \Pi \\ \pi' \neq \pi}}$.

Let $R_{-0} \subset V_{-0} \times \Delta_{-0}$ be the set of games that are regular and nondegenerate and the local mappings $x_{-0}(v,p)$ are surjective. According to the induction hypothesis almost all games of $V_{-0} \times \Delta_{-0}$ belong to R_{-0}. Consider the solutions of the augmented problem $F_{\mathcal{C}_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) = 0$ where $z_0 = (x_0, \mu_0, e_0)$ and we consider that $x_{-0}(v_{-0,}p_{-0})$ changes with $v_{-0,}p_{-0}$ (even though expressions $\sum_{i \in S} x_i(\pi) + e_r(\pi) - x_S(\pi S)$ do not depend directly on the parameters v, p, the term $x_S(\pi S)$ is a function of v_{-0}, p_{-0}). The Jacobian matrix at the solution, $d_{(z_0, v_0, p_0, v_{-0}, p_{-0})} F_{\mathcal{C}_0}$, is

$$\begin{bmatrix} * & * & -(1-\delta) I_0 & 0 & * \\ * & 0 & 0 & 0 & -d_{(v_{-0}, p_{-0})} g \circ x_{-0}(v_{-0}, p_{-0}) \\ 0 & d_\mu M_{\mathcal{C}_0} & 0 & d_{p_0} M_{\mathcal{C}_0} & 0 \end{bmatrix}, \quad (20)$$

where $g : V_{-0} \to R^m$ is the linear map $g(x_{-0})(S) = x_S(\pi S)$ for all the sets in the support $\mathcal{C}_0 = (\Sigma_i(\pi))$, and $*$ denotes arbitrary coefficients. Note that the linear map g is surjective, and thus the composition $g \circ x_{-0}(v_{-0}, p_{-0})$ is surjective (the composition of surjective maps is surjective). But then we have that $F_{\mathcal{C}_0}(z_0, v_0, p_0, v_{-0}, p_{-0}) \pitchfork 0$ because all blocks $-(1-\delta) I_0$, $d_\mu M_{\mathcal{C}_0}$, and $-d_{(v_{-0}, p_{-0})} g \circ x_{-0}(v_{-0}, p_{-0})$ are surjective. Therefore, by the transversality theorem, for almost every $(v,p) \in R^d \times \Delta^d$, $F_{\mathcal{C}_0}(z_0) \pitchfork 0$. Because of the block triangular structure of the Jacobian matrix $d_z F_{\mathcal{C}}(z)$ (see remark on section 3.2) this shows that $d_z F_{\mathcal{C}}(z)$ is nonsingular ($\mathcal{C}$ regular) almost everywhere. The argument to show that $\mathcal{C}$ is nondegenerate almost everywhere is the same one discussed in section 3.3.

To complete the proof, it still remains to show that the local solution mappings $x\,(v,p)$ of problem $F_{\mathcal{C}}$ are surjective. But

$$d_{(v,p)}x = \begin{bmatrix} d_{(v_0,p_0)}x_0 & d_{(v_{-0},p_{-0})}x_0 \\ 0 & d_{(v_{-0},p_{-0})}x_{-0} \end{bmatrix},$$

because x_{-0} does not depend on (v_{-0},p_{-0}), and it is thus enough to prove that $x_0\,(v,p)$ is surjective (by the induction hypothesis $d_{(v_{-0},p_{-0})}x_{-0}$ is surjective). The implicit function theorem gives us the expression of the derivative of the local mappings (refer to (11)) as, $d_{(v,p)}x_0\,(v,p) = -\left[d_{z_0}F_{\mathcal{C}_0}\right]_n^{-1} d_{(v,p)}F_{\mathcal{C}_0}$, where $\left[d_{z_0}F_{\mathcal{C}_0}\right]_n^{-1}$ is the submatrix of $\left[d_{z_0}F_{\mathcal{C}_0}\right]^{-1}$ restricted to the first n rows, and $d_{(v,p)}F_{\mathcal{C}_0}$ is given by (20). But both $\left[d_{z_0}F_{\mathcal{C}_0}\right]_n^{-1}$ and $d_{(v,p)}F_{\mathcal{C}_0}$ are surjective so $d_{(v,p)}x_0\,(v,p)$ is surjective. Thus we conclude that $x_0\,(v,p)$ is surjective. Q.E.D.

Lefschetz Fixed Point Theorem (LFPT) (McLennan 1989): Let $\mathcal{T}$ be the collection of admissible triples (X,F,U) where $X \subset R^m$ is a finite simplicial complex, $F : X \to X$ is a upper hemicontinuous contractible valued correspondence (u.h.c.c.v.), $U \subset X$ is open, and there are no fixed points of F in $\overline{U} - U$. Then there is a unique Lefschetz fixed point index $\Lambda(X,F,U)$ that satisfying the following axioms (when X is implicitly given we just say $\Lambda(F,U)$):
(Localization axiom): If $F_0, F_1 : X \to X$ are u.h.c.c.v. correspondences that agree on $\overline{U}$, and (X,F_1,U), $(X,F_0,U) \in \mathcal{T}$, then $\Lambda(X,F_1,U) = \Lambda(X,F_0,U)$.
(Continuity axiom): If $(X,F,U) \in \mathcal{T}$, then there is a neighborhood W of $Gr(F)$ such that $\Lambda(X,F',U) = \Lambda(X,F,U)$ for all u.h.c.c.v. correspondences $F' : X \to X$ with $Gr(F') \in W$.
(Homotopy axiom): If $h : [0,1]\times X \to X$ is a homotopy with $(X,h_t,U) \in \mathcal{T}$, for all t, then $\Lambda(X,h_0,U) = \Lambda(X,h_1,U)$.
(Additivity axiom): If $(X,F,U) \in \mathcal{T}$ and $U_1,...,U_r$ is a collection of pairwise disjoint open subsets of U such that there are no fixed points of F in $U - (\cup_{k=1}^r U_k)$ then $\Lambda(X,F,U) = \sum_{k=1}^r \Lambda(X,F,U_k)$.
(Weak Normalization axiom): For $y \in X$, let c_y be the constant correspondence $c_y(x) = \{y\}$. If $y \in U$ then $\Lambda(X,c_y,U) = 1$.
(Commutativity axiom): If $X \subset R^m$ and $Y \subset R^n$ are finite simplicial complexes, $f : X \to Y$ and $g : Y \to X$ are continuous functions, and $\Lambda(X, g\circ f, U) = \Lambda(X, f\circ g, g^{-1}(U))$.

PROOF OF LEMMA 2: Define the correspondence $\mathbf{F}(x) = x - \mathcal{F}(x)$, where $\mathcal{F}(x)$ is the correspondence defined in (15). The Lefschetz index of $\mathcal{F}$ and the degree of $\mathbf{F}$ are related by $\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}, U, 0)$ (see McLennan (1989)), and, for convenience, we work in the remainder of the proof with the concept of degree.

For each point x consider the mixed linear complementarity problem, or $MLCP(0)$

$$\begin{aligned} h(\sigma) &= 0, \\ g(e,x) &\geq 0, \\ e \text{ free variable, } \sigma &\geq 0 \text{ and } \sigma^T g(e,x) = 0, \end{aligned} \tag{21}$$

where the functions h and g were defined in (**??**). Let $z(x) = (e(x), \sigma(x))$ be a solution of the $MLCP(0)$ (there can be multiple solutions). Note that $\mathbf{F}(x) = \{f(x, z(x)) : \ z(x) \text{ is a solution of } MLCP(0)\}$, where the function f has also been defined in (**??**).

Let (x^*, e^*, μ^*) be any regular and nondegenerate MPE with an associated *CDS* $\mathcal{C} = (C, P)$, with $C = (C_1, ..., C_q)$ and $P = (P_1, ..., P_q)$. By lemma 1, there exists $\sigma^* \in \Sigma_C$ such that $\mu^* = \mu^*(\sigma^*)$, and (x^*, σ^*) is *MPE*. Furthermore, because all points in P_r are connected, we can choose a strategy profile σ^* satisfying $supp(\sigma_i^*) = \mathcal{C}_r \cap \{S \subset \pi : i \in S\}$ for all $i \in P_r$.

Consider now the perturbed mixed linear complementarity problem, or $MLCP(\varepsilon)$

$$\begin{gathered} h(\varepsilon)(e,\sigma) = h(\sigma) + \varepsilon(e - e^*) = 0, \\ g(\varepsilon)(x,\sigma,e) = g(x,e) + \varepsilon(\sigma - \sigma^*) \geq 0, \\ e \text{ free variable, } \sigma \geq 0, \ \sigma^T g(\varepsilon) = 0, \end{gathered} \tag{22}$$

where $\varepsilon > 0$. The Jacobian matrix $M(\varepsilon)$ of $MLCP(\varepsilon)$ is a P-matrix (i.e., a matrix with all its principal minors positive). This is so because (see Cottle et al. (1992, pg. 154)), $M(\varepsilon) = M + \varepsilon I$, where M is the Jacobian of $MLCP(0)$, is a P_0-matrix (i.e., a matrix with all its principal minors nonnegative). Let us prove that M is a P_0-matrix: Consider the principal matrix $M_{\beta\beta}$ associated with a subset β of lines (or columns).[9] We now show that either $\det(M_{\beta\beta})$ is equal to zero or one. Note first that $\det(M_{\beta\beta}) = \prod_{i\in\pi}\det\left(M_{\beta_i\beta_i}\right)$ where $\beta = \cup_i\beta_i$ and β_i are the elements of β with entry i (either e_i or $\sigma_i(S)$ for some $S \ni i$). But $\det\left(M_{\beta_i\beta_i}\right) = 1$ if $\beta_i = \{e_i, \sigma_i(S)\}$

[9] We refer to the lines corresponding to ∂h_i and $\partial g_i(S)$ as lines λ_i and $\sigma_i(S)$, and the columns corresponding to $\frac{\partial \cdot}{\partial \lambda_i}$ and $\frac{\partial \cdot}{\partial \sigma_i(S)}$ as columns λ_i and $\sigma_i(S)$. Also, we use the standard notation that $A_{\alpha\alpha}$, $A_{\cdot\alpha}$, and $A_{\alpha\cdot}$ represent the submatrix of A with, respectively, rows and columns, columns, and rows extracted from the index set α. Also, $\overline{\alpha}$ denotes the complementary set of α.

and is zero otherwise. Therefore, we conclude that all principal minors of M are nonnegative, and thus M is a P_0-matrix.

Given that $MLCP(\varepsilon)$ has a P-matrix then there is a unique solution $z_\varepsilon(x)$ (Cottle et al. (1992, pg. 150)) for all x: $MLCP(\varepsilon)$ can be transformed into a standard LCP eliminating the variable e and the equation $h(\varepsilon) = 0$ (this is possible because $M_{ee}(\varepsilon) = \varepsilon I$ is nonsingular), and the transformed LCP also has a P-matrix (the Schur complement of $M_{ee}(\varepsilon)$ in $M(\varepsilon)$). Note that, in addition, we have that $z_\varepsilon(x^*) = (e^*, \sigma^*)$, and that $z_\varepsilon(x)$ converge to a solution of $MLCP(0)$ when $\varepsilon \to 0$ (Cottle et al. (1992, pg. 442)), and that $z_\varepsilon(x)$ is piecewise linear in x.

We now show that, because x^* is a strong solution, there exists an $\overline{\varepsilon} > 0$ such that for every $0 < \varepsilon < \overline{\varepsilon}$ there exists an open neighborhood U_ε of x^* such that $z_\varepsilon(x)$ is smooth in U_ε. Moreover, if we let α represent the index set

$$\alpha = \{\sigma_i(S) : \text{for all } S \in \mathcal{C}_r \text{ and } i \in S\}, \tag{23}$$

then all $\sigma_i(S)$-coordinates of the solution $z_\varepsilon(x)$ that do not belong to α are zero, and $z_\varepsilon(x)$ are explicitly given by $(M_{\alpha\alpha}(\varepsilon))^{-1} q_\alpha(x)$, where $M_{\alpha\alpha}(\varepsilon)$ is

$$M_{\alpha\alpha}(\varepsilon) = \begin{bmatrix} \varepsilon I_{\alpha\alpha} & (d_e g)_\alpha \\ d_\alpha h & \varepsilon I_{ee} \end{bmatrix},$$

and the vector $q_\alpha(x)$ has e_i-coordinate equal to $(\varepsilon e_i^* - 1)$, and $\sigma_i(S)$-coordinate in α equal to $\varepsilon\sigma_i(S)^* + e(S)(x)$, for all $0 < \varepsilon < \overline{\varepsilon}$ and $x \in U_\varepsilon$.

In order to prove the above claim consider the function

$$\varphi(x) = \min \cup_{r=1}^{q} \{e(S)(x) - e(T)(x) : S \in C_r, T \cap P_r \neq \emptyset, \text{ and } T \notin C_r\}.$$

Naturally, the function φ is continuous in x and, because x^* is a strong solution, $\varphi(x^*) > 0$. Therefore, there exists an $\overline{\varepsilon} > 0$ and an open neighborhood $U \subset U_{x^*}$ of x^*, such that all $x \in U$ satisfy $\varphi(x) > 2\overline{\varepsilon}$. Now suppose that the solution $z_\varepsilon(x)$ for $x \in U$ is such that a $\sigma_i(T)$-coordinate is non-zero for $T \notin C_r$ and $i \in P_r$. Then $g_i(\varepsilon)(T) = 0$ which is equivalent to $e_i + \varepsilon(\sigma_i(T) - \sigma_i^*(T)) - e(T)(x) = 0$, and implies $e(T)(x) \geq e_i - \varepsilon$. Also, $g_i(\varepsilon)(S) \geq 0$ for all S, and thus $e_i + \varepsilon(\sigma_i(S) - \sigma_i^*(S)) - e(S)(x) \geq 0$, which implies that $e(S)(x) \leq e_i + \varepsilon$. Therefore, $e(S)(x) - e(T)(x) \leq 2\varepsilon \leq 2\overline{\varepsilon}$ for $x \in U$, in contradiction with $\varphi(x) > 2\overline{\varepsilon}$ for all $x \in U$. Now, since $z_\varepsilon(x^*) = (e^*, \sigma^*)$, and $supp(\sigma_i^*) = \mathcal{C}_r \cap \{S \subset \pi : i \in S\}$, and $z_\varepsilon(x)$ is continuous, then there exists an open neighborhood $U_\varepsilon \subset U_{x^*}$ of x^* where all $\sigma_i(S)$-coordinates of the solution belonging to α are non-zero. This implies that $g_i(\varepsilon)(S) = 0$ holds for all $\sigma_i(S)$ in α, and thus $z_\varepsilon(x) = (M_{\alpha\alpha}(\varepsilon))^{-1} q_\alpha(x)$.

Define the mapping $\mathbf{F}_{\varepsilon}(x) = f(x, z_{\varepsilon}(x))$ (this mapping is well-defined due to the uniqueness of $z_{\varepsilon}(x)$), where $\mathbf{F}_{\varepsilon}(x^*) = 0$. Since f is smooth and $z_{\varepsilon}(x) \to z(x)$ then $\mathbf{F}_{\varepsilon}(x) \to \mathbf{F}(x)$. Therefore, for every $\delta > 0$ there exists $\overline{\varepsilon}$ such that $\underset{x \in \overline{U}}{dist}(\mathbf{F}_{\varepsilon}(x), \mathbf{F}(x)) < \delta$, for all $0 < \varepsilon \leq \overline{\varepsilon}$. But since $\mathbf{F}(x)$ has no zeros in the boundary of ∂U then $\mathbf{F}_{\varepsilon}(x)$ also does not have any zeros in ∂U. By the homotopy and continuity property of the degree, $\deg(\mathbf{F}, U, 0) = \deg(\mathbf{F}_{\varepsilon}, U, 0)$, for ε close to zero.

Therefore, it only remains to show that $\deg(\mathbf{F}_{\varepsilon}, U, 0) = sgn(\det(dF_{\mathcal{C}}(z^*)))$ for ε close to zero, where $z^* = (x^*, e^*, \mu^*)$. This result follows from $sgn(\det(d_x\mathbf{F}_{\varepsilon}(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*))) \neq 0$, as we will show. Indeed, this implies that $\mathbf{F}_{\varepsilon}$ is nonsingular at x^*, and thus there exists an open neighborhood $V \subset U$ of x^* where x^* is the only zero of $\mathbf{F}_{\varepsilon}$. But since the point x^* is the only zero of $\mathbf{F}(x)$ in $U \subset U_{x^*}$, and $\mathbf{F}_{\varepsilon}(x) \to \mathbf{F}(x)$ then there are no zeros of $\mathbf{F}_{\varepsilon}$ in the compact region $\overline{U} \backslash V$, for ε small enough, and thus x^* is the only zero of $\mathbf{F}_{\varepsilon}$ in U. A well-known property of the degree then implies that $\Lambda(\mathcal{F}, U) = \deg(\mathbf{F}_{\varepsilon}, U, 0) = sgn(\det(d_x\mathbf{F}_{\varepsilon}(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$.

We now show that $sgn(\det(d_x\mathbf{F}_{\varepsilon}(x^*))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$, for ε small enough. Consider $F(x, \sigma, e)(\varepsilon)$,

$$F(x,\sigma,e)(\varepsilon) = \begin{pmatrix} f(x,\sigma,e) \\ h(\sigma) + \varepsilon(e - e^*) \\ g(e,x) + \varepsilon(\sigma - \sigma^*) \end{pmatrix}.$$

Simple linear algebra shows that the Jacobian $d_x\mathbf{F}_{\varepsilon}(x^*)$ is the Schur complement of $M_{\alpha\alpha}(\varepsilon)$ in $dF_{\alpha\alpha}(\varepsilon)$ ($d_x\mathbf{F}_{\varepsilon}(x^*) = dF_{\alpha\alpha}(\varepsilon)/M_{\alpha\alpha}$), where

$$dF_{\alpha\alpha}(\varepsilon) = \begin{bmatrix} (d_x f) & d_\alpha f & d_e f \\ (d_x g)_\alpha & \varepsilon I_{\alpha\alpha} & (d_e g)_\alpha \\ 0 & d_\alpha h & \varepsilon I_{ee} \end{bmatrix}, \tag{24}$$

is evaluated at point (x^*, e^*, σ^*). Therefore, $\det(d_x\mathbf{F}_{\varepsilon}(x^*)) = \det(dF_{\alpha\alpha}(\varepsilon)) / \det(M_{\alpha\alpha})$ (see Cottle et al. (1992, pg. 75)). But since $\det(M_{\alpha\alpha}) > 0$ (M is a P-matrix) then $sgn(\det(d_x\mathbf{F}_{\varepsilon}(x^*))) = sgn(\det(dF_{\alpha\alpha}(\varepsilon)))$.

We claim that $sgn(\det(dF_{\alpha\alpha}(\varepsilon))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$. In order to prove the claim we use the following formula for the determinant (Cottle et al. (1992), pg. 60): for an arbitrary diagonal matrix D, $\det(A + D) = \sum_{\gamma} \det D_{\overline{\gamma\gamma}} \det A_{\gamma\gamma}$ where the summation ranges over all subsets γ of lines. Observe that matrix $dF_{\alpha\alpha}(\varepsilon) = A + D$, where $A = dF_{\alpha\alpha}(0)$ and D is the diagonal matrix,

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon I_{\alpha\alpha} & 0 \\ 0 & 0 & \varepsilon I_{ee} \end{bmatrix}.$$

Developing the expression for $\det(dF_{\alpha\alpha}(\varepsilon))$ using the formula above we get a polynomial in ε ($\det D_{\overline{\gamma\gamma}}$ is a power of ε). We are only interested in the non-zero coefficient with lowest order because, when ε converges to zero, this is the coefficient that determines the sign of $\det(dF_{\alpha\alpha}(\varepsilon))$.

The rows and columns of matrix $A = dF_{\alpha\alpha}(0)$ corresponding to $\sigma_i(S)$ and e_i are

$$\begin{aligned} R(\sigma_i(S)) &= \sum_j \mathbb{I}_{[j\in S]} e(x_j) + e(e_i), \\ R(e_i) &= -\sum_{S\in C_r} e(\sigma_i(S)), \\ C(\sigma_i(S)) &= \sum_j \mathbb{I}_{[j\notin S]} x_j(S) e(x_j) - e(e_i), \\ C(e_i) &= -p_i e(x_i) + \sum_{S\in C_r} e(\sigma_i(S)), \end{aligned}$$

where vectors $e(x_i)$, $e(e_i)$, and $e(\sigma_i(S))$ are the unit vectors at, respectively, coordinates x_i, e_i, and $\sigma_i(S)$.

Consider $A_{\alpha\cdot}$ the submatrix of A corresponding to the rows α of A. Let β be a maximal subset of α such that $rank(A_{\beta\cdot})$ is different from zero ($|\beta| = rank(A_{\alpha\cdot})$ and $rank(A_{\beta\cdot}) = rank(A_{\alpha\cdot})$). Note that $A_{\gamma\gamma}$ where γ is the set of lines $\gamma = \beta \cup \{e_i : i \in \pi\} \cup \{x_i : i \in \pi\}$ is equal to $A_{\gamma\gamma} = dF_{\beta\beta}(0)$, according to the definition (24). Also, $\det A_{\gamma'\gamma'} = 0$ for set of lines γ' that strictly contains γ because β is a maximal subset of α such that $rank(A_{\beta\cdot}) \neq 0$. We now show that $\det(dF_{\beta\beta}(0)) = \det(dF_{\mathcal{C}}(z^*)) \neq 0$, which proves that the lowest-order non-zero coefficient is equal to a positive integer (the number of maximal subsets $\beta \subset \alpha$) multiplied by $\det(dF_{\mathcal{C}}(z^*))$, and thus $sgn(\det(dF_{\alpha\alpha}(\varepsilon))) = sgn(\det(dF_{\mathcal{C}}(z^*)))$, for ε small enough.

We now propose an algorithm replaces all rows and columns $\sigma_i(S)$'s with the same S by only one row and column $\sigma_i(S)$ for all $S \in C_r$, and also replaces all rows and columns e_i for all $i \in P_r$ by only one row and column e_r for each $r = 1, ..., q$.

Algorithm: Start with matrix $A = dF_{\beta\beta}(0)$.

Step 1: Choose an element r, that have not yet been chosen, from the set $\{1, 2, ..., q\}$ and proceed to the next step, or else, stop if the choice is not possible.

Step 2: Choose two distinct rows $\sigma_i(S)$ and $\sigma_j(S)$ of A with $j \neq i$ and $S \in C_r$ and proceed to the next step, or else return to step 1 if the choice is not possible.

Step 3: Subtract row $\sigma_i(S)$ from row $\sigma_j(S)$ (i.e., $R(\sigma_j(S)) = R(\sigma_j(S)) - R(\sigma_i(S))$), and add column e_j to column e_i (i.e., $C(e_i) = C(e_i) + C(e_j)$). The matrix that is obtained after the two operations have the same determinant as matrix A. Let this matrix be the new matrix A. After these two operations, row $\sigma_j(S)$ of A has only one non-zero entry at column e_j, with a value equal to 1. The determinant of A can be computed by a co-factor expansion along row $\sigma_j(S)$, and $|A| = (-1)^{(\#\sigma_j(S)+\#e_j)}|A'|$, where A' is the submatrix obtained after deleting row $\sigma_j(S)$ and column e_j of matrix A.

Now, perform the following symmetric transformations on the submatrix A': Subtract column $\sigma_i(S)$ from column $\sigma_j(S)$ (i.e., $C(\sigma_j(S)) = C(\sigma_j(S)) - C(\sigma_i(S))$) and add row e_j to row e_i (i.e., $R(e_i) = R(e_i) + R(e_j)$). The matrix that is obtained after the two operations have the same determinant as A'. Let this matrix be the new matrix A'. After these two operations, column $\sigma_j(S)$ of A' has only one non-zero entry at row e_j, with a value equal to -1. The determinant of A' can be computed by a co-factor expansion along column $\sigma_j(S)$, and $|A'| = (-1) \times (-1)^{(\#\sigma_j(S)+\#e_j-1)}|A''|$, where A'' is the submatrix of A' obtained after deleting column $\sigma_j(S)$ and row e_j: observe that the column $\sigma_j(S)$ of A' is in the same location as row $\sigma_j(S)$ of A', but row e_j appears one entry before column e_j of A (because the row $\sigma_j(S)$ that has been removed appears before row e_j). Putting together the expressions for the determinant yields $|A| = |A''|$. Let matrix A'' be the new matrix A, and return to step 2.

Because β is a maximal subset of α with $rank(A_{\beta \cdot}) \neq 0$ and $rank(E_C) \neq 0$, the algorithm starts with matrix $A = dF_{\beta\beta}(0)$ and ends with matrix $A = \det(dF_{\mathcal{C}}(z^*))$ (maintaining the same determinant in all steps).

Therefore, $\det(dF_{\beta\beta}(0)) = \det(dF_{\mathcal{C}}(z^*))$, as we claimed. Q.E.D.

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