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**Folk Theorems on Transmission Access:  
Proofs and Counter Examples**

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- [11] S. Stoft, "On the meaning of transmission prices." Private Communication, 18 October 1994.
- [12] J. Walsh, "Comments of San Diego Gas & Electric Company (U 902-E) On Competitive Wholesale Electric Markets and Market Institutions in the Restructured Electric Industry." CPUC Submission, San Francisco, CA, July 26, 1994.

a very stringent regulation. Thus, the efficient equilibrium—the economic dispatch—is realized in a way that resembles how centralized economies were expected to realize efficient allocations. The difficulties with this centralized approach, in terms of information collection and incentive incompatibilities, have yet to be addressed in the specific context of transmission access.

Most proposals on transmission access have been focussed on short term efficiency and the problems of transition from the current organization of regulated utilities to a regime of open access. While these short and medium term considerations are important, they should not form the basis for the design of the institutions that will implement open access transmission. The success of open access will be far more dependent on how those institutions will shape the longer term outcomes: how much innovation and investment are encouraged, how easy is it for new agents to overcome entry barriers, how new commodity packages will be introduced, etc. We expect to return to a mathematical formulation of these issues.

## References

- [1] W. Hogan, “Contract networks for electric power transmission,” *Journal of Regulatory Economics*, vol. 4, pp. 211–242, September 1992.
- [2] E. Kahn and R. Baldick, “Reactive power is a cheap constraint,” *The Energy Journal*, vol. 15, no. 4, pp. 191–201, 1994.
- [3] A. R. Bergen, *Power Systems Analysis*. Prentice-Hall, 1986.
- [4] S. S. Oren, S. A. Smith, R. B. Wilson, and H. P. Chao, “Priority service: Unbundling the quality attributes of electric power,” in *EPRI Interim Report, EA-4851*, Electric Power Research Institute, Palo Alto, CA, 1986.
- [5] C.-W. Tan and P. Varaiya, “Interruptible electric power service contracts,” *J. Economic Dynamics and Control*, vol. 17, pp. 495–517, May 1993.
- [6] R. Kaye, F. Wu, and F. Wu, “Optimizing demand side contributions to power system security,” in *Service Opportunities for Electric Utilities: Creating Differentiated Products* (S. Oren and S. Smith, eds.), pp. 147–168, Kluwer, 1993.
- [7] R. E. Bohn, M. C. Caramanis, and F. C. Schweppe, “Optimal pricing in electrical networks over space and time,” *Rand Journal of Economics*, vol. 15, pp. 360–376, Autumn 1984.
- [8] F. C. Schweppe, M. C. Caramanis, R. E. Tabors, and R. E. Bohn, *Spot Pricing of Electricity*. Kluwer Academic Publishers, 1988.
- [9] D. Baron and R. Myerson, “Regulating a monopolist with unknown costs,” *Econometrica*, vol. 50, pp. 911–930, 1982.
- [10] W. Hogan and L. Ruff, “Reshaping the electricity industry: Competitive market structure and regulatory policy.” Prepared for Wisconsin Electric Power Company, November 1, 1994.

$$\begin{aligned} \text{subject to } q_i^r + \bar{q}_i &= \sum_{j=1}^n Y_{ij} \sin(\theta_i - \theta_j), \quad i = 1, \dots, n; \\ q_{ij}^o &= Y_{ij} \sin(\theta_i - \theta_j) \leq C_{ij}, \quad 1 \leq i, j \leq n. \end{aligned}$$

In comparison to the original economic dispatch problem (6)-(8), we may call the limited economic dispatch a “second best” solution. It is easy to show that if Minop selects the second best dispatch and if it sets the nodal prices to be the corresponding Lagrange variables, then the merchandizing surplus is positive.

The larger is the set of allocated rights, the smaller is Minop’s discretion. As a result, it may be impossible to achieve economic efficiency. Consider the three-node network of Figure 6. Suppose that ACR for  $T_{13}^a = 150$  MW have been issued. If that ACR is exercised the dispatch in the left of the figure obtains, and Minop cannot select the dispatch shown in the figure on the right, even if that dispatch is economically efficient. In order to achieve efficiency, it will be necessary to develop a mechanism that permits withdrawal of existing rights and issuing new rights that better match the new demand.

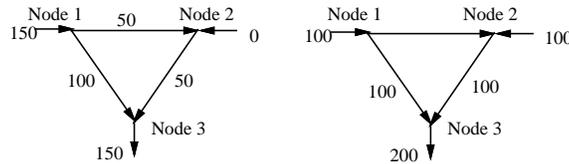


Figure 6: Each link’s capacity is 100 MW. An ACR to transfer 150 MW from 1 to 3 will lead to the dispatch on the left. Those rights will prevent the dispatch on the right.

## 6 Conclusion

We have attempted to clarify some concepts used to formulate claims for various proposals on transmission access. By formulating these concepts within a mathematical model it is possible to prove some of those claims and to show that others are erroneous. The unwary may make two errors, perhaps by drawing an incorrect analogy between power transmission and goods transportation. The first error is to say that just as the price of a good sold at two locations will differ by the cost of transporting that good between those two locations, so will the nodal price difference for power equal the cost of transporting it from one node to the other. The second, and more serious, error is to say that competition will drive the difference in nodal prices to the cost of transporting power, just as competition drives the difference in locational prices of goods to the cost of transporting that good.

Sophisticated proponents of transmission access schemes based on nodal prices and transmission charges or rights, do not make these errors. However, implementation of these schemes requires lodging great monopoly power in a centralized system operator who combines the roles of market maker and dispatcher. Thus, “open” access is gained at the cost of instituting a system operator who, in effect, determines the prices and quantities of power transactions that are permitted. In order to prevent abuse, the proponents have to impose

constraints, and because  $q^a \in FI$  by (19),

$$-p^T q \geq -p^T q^a,$$

as claimed.  $\square$

Thus, if Maxop selects the economic dispatch, it will retain a positive balance from the merchandizing surplus, and achieve an efficient allocation. However, the only requirement on Maxop is to select a market equilibrium. In order to ensure that this equilibrium is the economic dispatch, Maxop's monopoly will have to be severely regulated. We have already discussed the objections to such regulation.

A different way to compensate the former  $k$  grid owners is to give them shares  $\alpha_1, \dots, \alpha_k$ , such that  $\alpha_j \geq 0$  and  $\sum \alpha_j = 1$ . The  $j$ th erstwhile grid owner receives the fraction  $\alpha_j$  of Maxop's nodal market operations, and Maxop retains no portion of the merchandizing surplus.

## 5.2 Actual capacity rights

We define actual capacity rights or ACR as transmission rights that meet characteristics 1 and 2. However, we allow a rights holder to exercise that right or not, as in footnote 15. We strengthen (19) accordingly to

$$[q^a] \subset FI. \tag{21}$$

If an ACR holder of  $T_{ij}$  exercises that right, the system is obliged to transfer  $T_{ij}$  MW of power from  $i$  to  $j$ . To make ACR operational, we introduce a system operator, Minop, with the following attributes. Minop selects a set of nodal prices  $p_i, 1 \leq i \leq n$ , and a dispatch  $q^o$ . Suppose the ACR rights-holders select  $\bar{q} \in [q^a]$ . Then Minop uses the remaining injections

$$q^r = q^o - \bar{q}$$

to support some or all the transactions requested by other agents that don't hold ACR. All power transfers from  $i$  to  $j$  (including those exercised by rights-holders) pay Minop a per MW "transmission charge" of  $p_j - p_i$ . This charge may be positive or negative. Minop receives the merchandizing surplus

$$\frac{1}{2} \sum_i \sum_j (p_j - p_i) q_{ij}^o,$$

where  $\{q_{ij}^o\}$  are the line flows corresponding to the dispatch  $q^o$ .

In this arrangement, Minop's discretion is limited to selecting the transmission charges and an injection  $q^r \in FI - \bar{q}$  that non ACR-holders wish to transact at the announced transmission charges. Minop must also maintain a positive merchandizing surplus. It can do this in several ways. One natural way is to solve the limited economic dispatch problem:

$$\text{minimize}_{q_i^r, \theta_i} \sum_i^n C_i(q_i^r)$$

Thus, owning  $T_{ij}$  units of VCR is equivalent to a claim on an income stream. When the prevailing nodal prices are  $p(t)$  for period  $t$ , Maxop pays the  $T_{ij}$ -owner  $(p_j(t) - p_i(t))T_{ij}$ . The value of this VCR is the present value of this stream, appropriately discounted. The VCR may be sold in a secondary market, like any other financial asset. The fact that this owner may also be a supplier, consumer or trader is irrelevant.

In essence, then, transmission rights in the form of VCR means that some agents are initially given claims to portions of Maxop's merchandizing surplus. Those claims may subsequently be sold to others. Actual power transactions are dealt with separately. They are carried out between Maxop and other agents (including, possibly, VCR-holders) at the nodal prices that Maxop sets. We now analyze these arrangements more closely.

First, how are the initial VCR distributed? It is suggested in [12] that "Initially, such rights will be allocated to those who have paid or are paying the fixed costs of the initial grid or of grid expansions, so that they will be protected from the economic effects of increased grid congestion in the future. Over time, perhaps under the direction of regulators, these rights may be sold or freely traded in secondary markets, all without affecting system operations at any time." The proposal to compensate grid owners through VCR is attractive at first sight, but the proposal has two defects.

First, some VCR-holders may receive a *negative* income stream. In the network of Figure 5, the holder of VCR  $T_{32}$  will receive the rent of  $(p_2 - p_3)T_{32} < 0$ . This seems unreasonable.<sup>15</sup>

The second defect is that, with rare exceptions, the merchandizing surplus  $-p^T q$  will not equal the rents  $-p^T q^a$  distributed to VCR holders. Thus Maxop will retain the balance,  $-p^T(q - q^a)$ . This balance may be positive or negative. Hogan suggests the following result:

**Theorem 2** Suppose the injection  $q$  selected by Maxop is the economic dispatch, and suppose the spot prices  $p$  equal the dual variables of economic dispatch. Then, Maxop will retain a positive balance, i.e.,

$$-p^T q \geq -p^T q^a.$$

**Proof** Because the economic dispatch problem has a convex cost function with linear

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<sup>15</sup>Hogan is aware of this possibility which he calls "displacement" [1, p.236]. He seems to suggest that other users would pay for this negative rent, but he does not explain how they would pay or why.

A more natural way to prevent negative rents is to extend the notion of VCR to permit a  $T_{ij}$ -holder to exercise that right and receive  $(p_j - p_i)T_{ij}$  or not to exercise it and receive 0 rent. Let  $x_{ij} \in [0, 1]$  be the fraction of  $T_{ij}^a$  that is exercised. Let

$$[q^a] = \left\{ \sum_j (x_{ij}T_{ij}^a - x_{ji}T_{ji}^a) \mid i = 1, \dots, n; x_{ij}, x_{ji} \in [0, 1] \right\}$$

be the set of injections corresponding to all possible ways in which the allocated rights can be exercised. Now modify characteristic 2 so that (19) is strengthened to

$$[q^a] \subset FI.$$

All VCR-holders will receive non-negative rents, since they are free not to exercise their rights. The proof of Theorem 2 shows that Maxop will still retain a positive balance if it operates at the economic dispatch. The condition  $[q^a] \subset FI$  may be much stronger than (19). For instance, in the case of Figure 5, the injection vector  $(30, 5, -35) \in FI$ , but  $[(30, 5, -35)] \not\subset FI$ .

**Characteristic 1** A TCR is any triple  $(i, j, T_{ij})$  with  $T_{ij} \geq 0$ . This is abbreviated to  $T_{ij}$  when  $(i, j)$  is understood. We add up all allocated rights of the same type  $(i, j)$  and call

$$T^a = \{T_{ij}^a \geq 0, i, j = 1, 2, \dots, n\}$$

the matrix of allocated rights.

**Characteristic 2** The requirement that “the simultaneous use of all allocated rights is feasible,” should mean that it is feasible to operate the power system to meet all the transactions  $\{T_{ij}^a\}$ . Let

$$q^a = \{q_i^a = \sum_j (T_{ij}^a - T_{ji}^a), i = 1, \dots, n\}, \quad (18)$$

be the injections corresponding to the allocated rights  $T^a$ . Let  $FI$  be the set of all feasible injections, i.e., the set of  $q$  satisfying (7), (8). Then characteristic 2 is interpreted as the requirement

$$q^a \in FI. \quad (19)$$

This says that it is possible to operate the system so as to support the allocated rights  $T^a$ . However, it does not require the system to be operated that way. That will depend on whether a weak or strong right attaches to TCR, as defined below.

**Characteristic 3** To specify this characteristic it is necessary to introduce a system operator, Maxop, with the following attributes. First, Maxop manages a nodal spot market, i.e., all purchases and sales of power at every node are transacted with Maxop at the equilibrium prices  $p = (p_1, \dots, p_n)$ . Second, Maxop selects a dispatch, i.e., the injections  $q = (q_1, \dots, q_n)$  so that  $(q, p)$  is a market equilibrium, as in Definition 2. Third, Maxop collects the merchandizing surplus

$$MS = - \sum p^T q = - \sum_i p_i q_i.$$

Fourth, Maxop pays to the  $T_{ij}^a$  rights holders a “rent” equal to  $(p_j - p_i)T_{ij}^a$ . Using (18), the total rent paid to the rights holders is

$$- p^T q^a = \sum_i \sum_j (p_j - p_i) T_{ij}^a. \quad (20)$$

## 5.1 Virtual capacity rights

We define virtual capacity rights or VCR as transmission rights that meet characteristics 2 and 3 above. An agent (supplier, consumer or trader) who holds VCR of  $T_{ij}$  MW is just indifferent between: (i) purchasing  $T_{ij}$  MW of power at  $i$  at price  $p_i$  and having Maxop transfer that power to  $j$ , and (ii) collecting a rent of  $(p_j - p_i)T_{ij}$  and purchasing  $T_{ij}$  MW of power at  $j$  at price  $p_j$ . Therefore VCR seems to capture Hogan’s notion of rights in a contract network: it “will make a  $[T_{ij}]$  capacity-right holder indifferent between (i) [collecting the rent and] purchasing power at bus  $j$  or (ii) specific performance in actually shipping the additional power from bus  $i$  to bus  $j$  at the current transmission prices  $[p_j - p_i]$ ” [1, p.234].

those corresponding to the economic dispatch. To our knowledge, no one has proposed such an adjustment procedure. In a future publication, we will exhibit an adjustment process between bilateral transactors and the system operator that does the job, *without* calculation of transmission charges.

## 5 Transmission capacity rights

Some proposals for transmission access are framed in terms of transmission capacity rights or TCR. The formulations of TCR vary in terms of the entitlements they give to TCR holders. Different formulations also imply different distribution of decision-making authority among the suppliers, consumers and system operators. Lastly, the formulations vary in terms of how well they meet the criterion of efficiency. We attempt to compare these proposals through a mathematical specification of two versions—one weak and one strong—of TCR. Together, the two versions “span” the range of proposals. The weak version, which we call *virtual capacity rights* or VCR, gives rights in Hogan’s “contract network.” The strong version, which we call *actual capacity rights* or ACR, gives transmission rights in the actual power network. Both versions have shortcomings.

In essence, VCR gives rights holders a claim on an income stream. It requires a central authority, Maxop, that organizes a nodal spot market for power, selects the dispatch, and settles the income claims of rights holders.<sup>13</sup> Maxop can achieve economic efficiency. Maxop can also abuse its authority, favor some holders, disadvantage others, accumulate merchandizing surplus, or operate inefficiently.

By contrast, ACR holders constrain a central authority, Minop, to fulfill the rights holders dispatch decisions.<sup>14</sup> Minop also imposes certain transmission charges. The dispatch constraints imposed on Minop may prevent economic efficiency.

Our mathematical specification is based on Hogan [1]. In his formulation [1, pp 234] a TCR has three characteristics:

1. “A TCR is defined as the right to put power in one bus and take out the same amount of power at another bus in the network.”
2. “We assume that the simultaneous use of all allocated rights is feasible.”
3. “However, in the contract network, we amend the definition of a capacity right to allow for either specific performance or receipt of an equivalent rental payment.”

Characteristics 1 and 2 define the strong version of rights. Characteristics 2 and 3, which dilutes characteristic 1, define the weak version.

We now develop a mathematical formulation of these characteristics.

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<sup>13</sup>It will be seen that Maxop resembles what has been called Poolco.

<sup>14</sup>It will be seen that Minop resembles what has been called Opco.

## 4 Bilateral contracts

We show that bilateral contracts, defined in the obvious way as a transaction involving one buyer and one seller, may not sustain an economic dispatch.

We define a bilateral contract as the right to inject a certain amount of power at node  $i$  and to remove it at node  $j$  at a specified price  $P_{ij}$ . Consider the economic dispatch of Figure 5. The nodal (shadow) prices satisfy (17). Since suppliers at node 3 have a marginal cost of  $p_3$ , they will enter into a bilateral contract with demanders at node 2 only if  $P_{32} \geq p_3$ . But since the demanders' marginal benefit is  $p_2$ , they will be willing to enter the contract only if  $P_{32} \leq p_2$ . Since  $p_3 > p_2$ , the transaction of 5 MW from node 3 to node 2, which is part of the economic dispatch, cannot be sustained as a bilateral contract.

More generally, let us say that a bilateral contract for transfer of power from  $i$  to  $j$  is (commercially) *viable* only if  $p_i \leq p_j$ . Since an economic dispatch need not satisfy this condition as in the example of Figure 5, we conclude that an efficient allocation may not be reached using only viable bilateral contracts. There are at least two ways to achieve efficiency. We explore these within the context of the example of Figure 5.

As we have seen, the bilateral transaction from node 3 to node 2 cannot be sustained, so no power will be injected at node 3.<sup>12</sup> The reduction in injection at node 3 implies that the 30 MW injection at node 1 will lead to a flow on line 13 larger than the 5 MW capacity. Hence the injection at node 1 must be curtailed, reducing the profit of the suppliers at node 1. Realizing this, node 1 suppliers may contract to purchase power from node 3 at price  $p_3$  for resale to node 2 at price  $p_2$ . This transaction on its own yields a loss of  $p_3 - p_2$  per MW. However, generation at node 3 now enables an increase in transmission from node 1 to node 2, and for a per MW gain of  $p_2 - p_1$ . In this way, a node 1 supplier can “bundle” expensive power purchased from node 3 with cheaper power generated at node 1. This example indicates that *trilateral* contracts may provide profitable opportunities that cannot be attained by bilateral contracts alone. In general networks multilateral contracts that “cross subsidize” bilateral transactions may be needed to achieve efficiency. Formation of such multilateral contracts may be facilitated by a centralized coordination mechanism.

The second alternative is to combine the operation of bilateral contracts with a system operator who imposes a per MW “transmission charge” of  $p_i - p_j$  on any (bilateral) contract to transfer power from  $j$  to  $i$ . (The charge may be positive or negative.) Here the  $\{p_i\}$  are the economic dispatch nodal (shadow) prices. It is then obvious that bilateral contracts from any  $i$  to any  $j$  are sustained if and only if the delivery price at  $j$  is  $p_j$ . In other words, the bilateral contracts sustain the economic dispatch. The objection to this alternative is that the system operator can figure out the correct transmission charges only by solving the economic dispatch problem. We have already indicated the difficulties involved in obtaining the cost and demand schedules needed to formulate the economic dispatch problem and in ensuring that the system operator makes the correct calculations. Furthermore, effective competition in a bilateral transactions market requires knowledge of transmission charges, obtained a priori or via an adjustment process in which transmission charges converge to

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<sup>12</sup>A bilateral transaction of less than 5 MW may be sustainable, depending on the marginal cost curve at node 3 and the marginal benefit curve at node 2. The argument below does not change.

Third, if  $\delta_i \neq \delta_j$  then  $p_i \neq p_j$ , even if line  $ij$  is not congested (hence,  $\mu_{ij} = 0$ ). So *nodal price differences do not equate to congestion prices*. This is a distinction from transportation models in which nodal price differences, in competitive equilibrium, equal the marginal cost of transportation between those nodes.

Fourth, if  $\delta_i > \delta_j$ , then  $p_i < p_j$ . This price relation holds whether the line flow is from node  $i$  to node  $j$  or from  $j$  to  $i$ . So, in *any network (which is not radial), one can construct an economic dispatch involving a line flow from a higher priced node to a lower priced node*.<sup>9</sup> This means that price differences are not indicative of merchandizing opportunities.

The three-node example of [1, Figure 3] also exhibits the third and fourth conclusions above. Theorem 1 implies that these phenomena will occur in every three-node network that is not linear.

Fifth, *the strengthening of a line (increasing its admittance) may lead to a larger minimum cost*. To see this, consider the three-node example of Figure 5. Nodes 1 and 3 generate 30

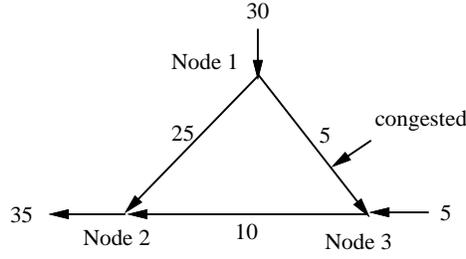


Figure 5: Link 13 is congested. If this is an economic dispatch then  $p_3 > p_2 > p_1$ .

MW and 5 MW respectively, and node 2 consumes 35 MW. Line 13 is congested. Suppose this is an economic dispatch. By Theorem 1, the nodal prices must satisfy  $p_3 > \max\{p_1, p_2\}$ . Because MS is positive by Fact 4, we must in fact have the price relationship

$$p_1 < p_2 < p_3. \quad (17)$$

Now suppose line 23 is strengthened, i.e.,  $Y_{23}$  is increased. If the same power injections are maintained, the flow  $q_{13}$  will *increase* beyond 5 MW.<sup>10</sup> This violates the capacity constraint, and the economic dispatch will reduce the injection in node 1, leading to a larger minimum cost. Of course, strengthening line 12 (increasing  $Y_{12}$ ), even though it is not congested, or strengthening line 13 (increasing  $C_{13}$ ), which is congested, will both relieve line 13 and increase the network capacity. The point of the example is that strengthening an arbitrary line may decrease capacity.<sup>11</sup> A result of Steve Stoft [11] shows that a proportionate increase in  $Y_{ij}$  and  $C_{ij}$  changes the minimum cost in proportion with  $(p_j - p_i)$ .

<sup>9</sup>This will always occur if the node on the receiving end of the congested line is not a sink, but it could also happen in other parallel flow configurations.

<sup>10</sup>Intuitively, the flow from node 1 to node 2 via node 3 now sees a larger admittance and so a larger proportion of the 30 MW injected in node 1 will flow through line 13.

<sup>11</sup>If line flows change during the day or over the year, it is not obvious which line should be strengthened.

Then (12) is satisfied. Equation (11), in matrix form, reads

$$\begin{bmatrix} \sum Y_{1i} & -Y_{12} & \cdot & -Y_{1n} \\ -Y_{21} & \sum Y_{2i} & \cdot & -Y_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -Y_{n1} & -Y_{n2} & \cdot & \sum Y_{ni} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ p_n \end{bmatrix} = \begin{bmatrix} -Y_{1n} \\ 0 \\ 0 \\ Y_{n1} \end{bmatrix} \mu_{1n}. \quad (13)$$

These equations are linearly dependent. We choose to eliminate the last equation. Define  $\pi_i = p_i/p_n, i = 1, \dots, n-1$ , and  $m = \mu_{1n}/p_n$ . Dividing throughout by  $p_n$  we get

$$\begin{bmatrix} \sum Y_{1i} & -Y_{12} & \cdot & -Y_{1n-1} \\ -Y_{21} & \sum Y_{2i} & \cdot & -Y_{2n-1} \\ \cdot & \cdot & \cdot & \cdot \\ -Y_{n-11} & -Y_{n-12} & \cdot & \sum Y_{n-1i} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \cdot \\ \pi_{n-1} \end{bmatrix} = \begin{bmatrix} Y_{1n} \\ Y_{2n} \\ \cdot \\ Y_{n-1n} \end{bmatrix} + \begin{bmatrix} -Y_{1n} \\ 0 \\ \cdot \\ 0 \end{bmatrix} m. \quad (14)$$

Write (14) as

$$Y\pi = h + gm,$$

so

$$\pi = Y^{-1}h + Y^{-1}gm = Y^{-1}h - \delta m, \quad (15)$$

where

$$\delta := -Y^{-1}g.$$

**Fact 6**  $Y, h, \delta$  depend only on the network and not on the power injections. All entries of  $Y^{-1}$  and  $\delta$  are positive. All entries of  $Y^{-1}h$  equal 1.

**Proof** The first assertion follows by checking that the entries of  $Y, h, g$  consist of terms involving line admittances only. The second assertion holds because  $Y$  is diagonally dominant, and because the entries of  $g$  are negative. The last assertion follows by verifying that  $Ye = h$  where  $e$  is the vector all of whose entries are 1.  $\square$

Rewrite (15) as

$$\pi_i = 1 - \delta_i m, \quad i = 1, \dots, n-1.$$

In terms of the nodal prices this equation is

$$p_i = p_n - \delta_i \mu_{1n}, \quad i = 1, \dots, n-1. \quad (16)$$

**Theorem 1** In any economic dispatch in which line  $1n$  is the only congested line, the nodal prices are related by (16). Here the  $\delta_i$  depend only on line admittances, and  $\mu_{1n} \geq 0$  is the congestion price for line  $1n$ .

### 3.1 Some consequences of Theorem 1

First, if  $\mu_{1n} = 0$ , all prices are equal. This is the uncongested case. For the rest of this discussion assume  $\mu_{1n} > 0$ .

Second, the price  $p_n$  at node  $n$  is the largest price. This is not unexpected. The pattern of congestion says it is difficult to deliver power to node  $n$ . This does *not* imply that  $n$  is a net demander.

mal capacity: if  $C_{min}(C_{ij})$  is the minimum value of (6), parametrized by  $C_{ij}$ , then

$$\frac{\partial C_{min}}{\partial C_{ij}} = -\mu_{ij}.$$

**Fact 5** Suppose  $(q, \theta)$  is the economic dispatch and  $p_i, \mu_{ij}$  are the associated Lagrange variables. Then the merchandizing surplus equals the congestion rent, i.e.,

$$-\sum_i p_i q_i = \sum_i \sum_j \mu_{ij} C_{ij} = \frac{1}{2} \sum_i \sum_j (p_j - p_i) q_{ij}.$$

**Proof** This is the duality result. To prove it, multiply (11) by  $\theta_i$  and add to get, after some manipulation,

$$0 = \sum_i \sum_j p_i Y_{ij} (\theta_i - \theta_j) + \sum_i \sum_j \mu_{ij} Y_{ij} (\theta_i - \theta_j).$$

Using (7) in the first term on the right and (12) in the second term gives

$$0 = \sum_i p_i q_i + \sum_i \sum_j \mu_{ij} C_{ij},$$

as claimed. □

In particular, if at the economic dispatch no flow is constrained, the  $\mu_{ij} = 0$  and so MS = 0, as is the case for the network in Figure 3.

### 3 A more interesting example

We now analyze conditions (7)–(12) for an  $n$ -node network in which exactly one line is congested. The optimality conditions take a particularly simple form, and are used to draw several conclusions, some of them counter-intuitive.

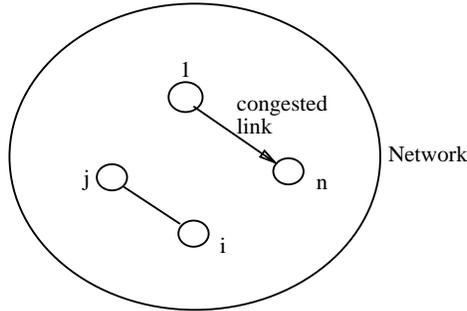


Figure 4: In this  $n$ -node network line  $1n$  is congested. All other lines are not congested.

For any  $n$ -node network (see Figure 4), consider any economic dispatch such that: (1) the flow on link  $1n$ ,  $q_{1n} > 0$ , and it is binding, i.e.,  $q_{1n} = C_{1n}$ ; (2)  $\mu_{1n} > 0$ ; (3) all other flows are not binding so  $\mu_{ij} = 0$ , for  $ij \neq 1n$ .

To obtain the second equality, use  $q_i = \sum_j q_{ij}$  and  $q_{ij} = -q_{ji}$ , to get

$$-\sum_i p_i q_i = -\sum_i \sum_j p_i q_{ij} = -\sum_j \sum_i p_j q_{ji} = \sum_j \sum_i p_j q_{ij}.$$

The term “merchandizing surplus” is not standard. If we imagine a “market maker” who is a party to every purchase and sale of power, MS is the profit resulting from this merchandizing activity.

**Fact 2** At a market equilibrium, MS can be positive or negative.

**Fact 3** A market equilibrium need not be an economic dispatch. Hence, a market equilibrium need not be efficient.

The proof of these “negative” assertions can be given with the 2-node example of Figure 3.

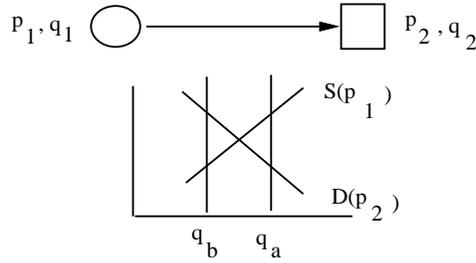


Figure 3: Node 1 is a net generator, 2 is a net consumer. The capacity constraint is not binding at  $q_a$ . At  $q_a$  the market equilibrium has negative MS, at  $q_b$  it has positive MS.

In Figure 3,  $S(p_1)$  is the supply curve for generation at node 1, and  $D(p_2)$  is the demand curve for consumption at node 2. Define  $p_1^a, p_2^a$  by  $S(p_1^a) = D(p_2^a) = q_a$  and  $p_1^b, p_2^b$  by  $S(p_1^b) = D(p_2^b) = q_b$ , where  $q_a, q_b$  are as shown. Check that  $p_1^a > p_2^a$  and  $p_1^b < p_2^b$ . Both  $(p^a, q_a)$  and  $(p^b, q_b)$  are market equilibria. MS is negative at “a” and positive at “b”. Neither equilibrium is the economic dispatch, which is at the point where the two curves intersect. In “b” someone collects the positive MS, in “a” someone pays for the negative MS.

**Fact 4** If the market equilibrium is also an economic dispatch, then MS is positive.

**Proof** Observe that  $(q, \theta) = (0, 0)$  is a feasible dispatch. Hence if  $(q, \theta)$  is an economic dispatch, and if  $p$  satisfies (10), we must have  $-\text{MS} = \sum_i p_i q_i \leq 0$ . This can be written as

$$\sum_{q_i > 0} p_i q_i \leq \sum_{q_i < 0} -p_i q_i.$$

Recalling our sign convention, the term on the right is the revenue from consumption, the term on the left is the payment for generation. Thus MS is positive.  $\square$

**Definition 4** Suppose  $(q, \theta)$  is an economic dispatch. Let  $p_i, \mu_{ij}$  be the associated Lagrange variables. Then  $\mu_{ij}$  is the (shadow) *congestion price* for the constraint  $Y_{ij}(\theta_i - \theta_j) \leq C_{ij}$ , and  $\sum_i \sum_j \mu_{ij} C_{ij}$  is the (shadow) *congestion rent*.

The congestion price has a standard interpretation as the marginal value of increased ther-

To preserve its competitive advantage, a generating company may be unwilling to reveal its costs to its competitors or its customers. For the same reason, an energy-consuming company may be unwilling to reveal to its competitors and suppliers how much it is willing to pay for power. For game-theoretic reasons, suppliers and demanders may have an incentive not to be truthful about their cost or demand data.

Implementation of optimal price regulation may therefore need enforcement procedures to ensure that the required information is forthcoming. As is well-known from the modern theory of regulation, such a revelation problem is seldom “first best,” and requires output and price distortions (see Baron and Myerson [9]).

Even if that were not the case, given the needed data, the system operator may select an operating point that is different from the economic dispatch, perhaps to favor some suppliers or consumers. To prevent this abuse, the regulations must contain audit procedures that can check whether the operating point is the economic dispatch. The ideal implementation of optimal price regulation will replace the system operator by a computer program that calculates and selects the economic dispatch. In practice, this is not so simple. The selection of the thermal constraints and the contingency constraints (which we have ignored) involve a considerable degree of judgement on the part of the system operator. By appropriately selecting these constraints, the operator can choose a dispatch that favors some suppliers or consumers.

In summary, implementing optimal spot price regulation requires a central authority—the system operator—who combines the roles of “dispatcher and market maker” and determines “market-clearing quantities and prices for each half-hour, based on buy and sell offers from the market participants” [10, p. 52]. To ensure economic dispatch, one then has to restrain the choice of this central authority to a single outcome. Under this implementation of optimal price regulation, the system operator’s information and authority are much greater than utility operators currently have (e.g., the latter don’t know demand schedules, and they cannot set prices); the regulation of the operator’s authority is correspondingly much greater than the regulation of utility operators (e.g., regulatory bodies do not prescribe or police utility dispatching rules).

The optimal price regulation equilibrium, more commonly known in industry as “economic dispatch,” is sometimes called the “competitive equilibrium” [10, p.50]. This term, attributed to Schweppe et al [7, 8], is a carryover from location theory. The term is misleading and should be avoided. If “competitive” means market-clearing prices for power at each node that preclude arbitrage opportunities from buying power at one node and selling at another, then *any* market equilibrium satisfying Definition 2 is competitive. A competitive equilibrium may, by happenstance, coincide with the economic dispatch.

We now explore non-uniqueness of the market equilibrium. Recall that by our sign convention,  $q_i > 0$  if there is net generation at  $i$ , and  $q_i < 0$  if there is net consumption at  $i$ .

**Definition 3** The *merchandizing surplus* (MS) at a market equilibrium  $(q, \theta, p)$  is

$$\text{MS} = - \sum_i p_i q_i = \frac{1}{2} \sum_i \sum_j (p_j - p_i) q_{ij}.$$

The optimal solution  $(q, \theta)$  is characterized by the existence of Lagrange multipliers such that (7)–(12) hold:

$$\frac{\partial C_i}{\partial q_i}(q_i) = p_i, \quad i = 1, \dots, n; \quad (10)$$

$$\sum_j Y_{ij}[p_i - p_j + \mu_{ij} - \mu_{ji}] = 0, \quad i = 1, \dots, n; \quad (11)$$

$$\mu_{ij}[Y_{ij}(\theta_i - \theta_j) - C_{ij}] = 0, \quad i, j = 1, \dots, n. \quad (12)$$

Equations (10), (11) follow respectively from the first-order conditions  $\partial\Phi/\partial q_i = 0$  and  $\partial\Phi/\partial\theta_i = 0$ . Equation (12) is the complementary slackness condition.

## 2.4 Some concepts

We introduce some concepts.

**Definition 1** A pair  $(q, \theta)$  is an (optimal) *economic dispatch* if it solves (6)–(8).

**Fact 1**  $(q, \theta)$  is an economic dispatch if and only if there exist  $p_i, \mu_{ij}$  such that (7)–(12) hold.

**Proof** This follows because (6)–(8) is a convex program with linear constraints.  $\square$

**Definition 2** A triple  $(q, \theta, p)$  is a *market equilibrium* if (7), (8), (10) hold.

Every market equilibrium will be sustained by competitive demand and supply. Relation (10) is equivalent to consumer and supplier equilibrium at every node  $i$ , because if node  $i$  is a net demander,  $p_i$  equals the marginal benefit, and if  $i$  is a net supplier,  $p_i$  equals the marginal cost of generation. (Hence, at an equilibrium there is no opportunity for profit from buying power at one node and selling at another.) Relation (7) says that every market clears, since supply equals demand for power at each node. Relation (8) says that transmission constraints are met. Since the  $C_i$  are increasing functions, we must have  $p_i > 0$ .

As will be seen below the market equilibrium is non-unique. The non-uniqueness is an essential feature of power systems, stemming from the wide latitude with which the “system operator” can select the system operating point  $(q, \theta)$ . The more stringent the regulation imposed on the system operator’s decision, the more constrained will be the market equilibrium. We discuss one regulation.

The regulation’s aim is to make the system operator select the operating point that satisfies (11) and (12). Following [7, 8, 1] we call this *optimal price regulation*, because its purpose is to fix the nodal prices  $p_i$  at the Lagrange multipliers corresponding to the economic dispatch. The attractive feature of optimal price regulation is that it forces the system to operate at the unique, efficient equilibrium, namely the economic dispatch.

Of course, for optimal price regulation to work, the various players must be well-behaved. Suppliers and consumers must truthfully reveal to the system operator their cost and demand schedules. The system operator must then correctly calculate the economic dispatch and implement it. Unfortunately, there are strong incentives for the players not to behave in this way.

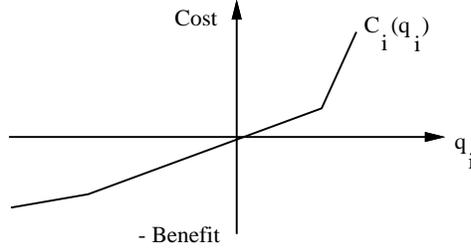


Figure 2: The cost and benefit of  $q_i$  is measured by the convex function  $C_i(q_i)$ .

The economic dispatch problem is the following:

$$\begin{aligned}
& \text{minimize}_{q_i, \theta_i} && \sum_{i=1}^n C_i(q_i) && (5) \\
& \text{subject to } q_i &= & \sum_{j=1}^n Y_{ij} \sin(\theta_i - \theta_j), \quad i = 1, \dots, n; \\
& & & q_{ij} = Y_{ij} \sin(\theta_i - \theta_j) \leq C_{ij}, \quad 1 \leq i, j \leq n.
\end{aligned}$$

This is a nonlinear programming problem. It is difficult to solve in practice because the network size  $n$  is large, and the functions  $C_i$  may not be readily available.<sup>7</sup> We now make one more simplifying assumption: the phase angle differences,  $|\theta_i - \theta_j|$ , are small.<sup>8</sup> We then make the approximation  $\sin(\theta_i - \theta_j) \approx (\theta_i - \theta_j)$  and end up with a convex programming problem with linear constraints:

$$\begin{aligned}
& \text{minimize}_{q_i, \theta_i} && \sum_i^n C_i(q_i) && (6) \\
& \text{subject to } q_i &= & \sum_{j=1}^n Y_{ij}(\theta_i - \theta_j), \quad i = 1, \dots, n; && (7) \\
& & & q_{ij} = Y_{ij}(\theta_i - \theta_j) \leq C_{ij}, \quad 1 \leq i, j \leq n. && (8)
\end{aligned}$$

Associate Lagrange multipliers  $p_i$  with the  $n$  constraints (7) and  $\mu_{ij} \geq 0$  with the  $n^2$  constraints (8), and form the Lagrangian

$$\Phi = \sum_i C_i(q_i) + \sum_i p_i [\sum_j Y_{ij}(\theta_i - \theta_j) - q_i] + \sum_i \sum_j \mu_{ij} [Y_{ij}(\theta_i - \theta_j) - C_{ij}]. \quad (9)$$

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of consuming  $h_i$  and  $q_i$  is the net injection into the network. Alternatively, one can first carry out a per node optimization using  $G_i, B_i$  and then consider only the net effect in terms of  $C_i$ , as we have done. Since the per node optimization problems are independent, this procedure is sound.

<sup>7</sup>In utility dispatch, the problem is a little simpler. One assumes that at nodes  $i$  with net consumption, the demand is fixed or inelastic, i.e.,  $q_i \equiv \bar{q}_i$ . The economic dispatch problem is to meet this demand at minimum generation cost, subject to the line flow constraints. The solution for this fixed demand case is called optimal power flow or OPF.

<sup>8</sup>The assumption that the  $|\theta_i - \theta_j|$  are small means that the line admittances are large relatively to the power flow. If that is not the case, one can use a “linearized” power flow in the neighborhood of a nominal (base case) operating point  $\theta_i = \theta_i^* + \delta\theta_i$  using  $Y_{ij} \sin(\theta_i - \theta_j) \approx Y_{ij} \sin(\theta_i^* - \theta_j^*) + Y_{ij} \cos(\theta_i^* - \theta_j^*)(\delta\theta_i - \delta\theta_j) = q_{ij}^* + \delta q_{ij}$ . One now works with the perturbed quantities  $\delta q_i, \delta\theta_i$  and the “admittances”  $Y_{ij} \cos(\theta_i^* - \theta_j^*)$ . If that is still insufficient, one has to work with the nonlinear problem which, however, may not be convex.

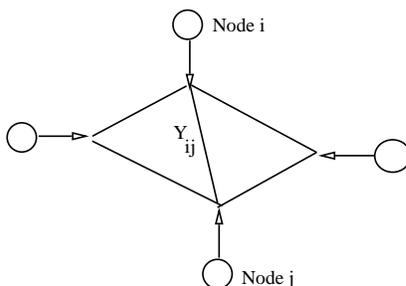


Figure 1: The transmission network has  $n$  nodes. The voltage at node  $i$  is  $v_i(t) = V_i \sin(\omega t + \theta_i)$ .

determined, so are all the line flows. Second, from (3) we see that the maximum value of  $|q_{ij}|$  is  $Y_{ij}$ , since the sine function is bounded by one. More importantly, although we have assumed lossless lines, there is always some energy loss, which is converted into heat. Depending upon how easily that heat is dissipated, there is a thermal limit  $C_{ij} = C_{ji} > 0$ , beyond which the flow should not be increased. Flows beyond  $C_{ij}$  can cause physical damage to the transmission line, with subsequent high probability of power failure. Usually  $C_{ij} \leq Y_{ij}$ , and so we have the “thermal” constraints on each line:

$$q_{ij} = Y_{ij} \sin(\theta_i - \theta_j) \leq C_{ij}, \quad 1 \leq i, j \leq n. \quad (4)$$

The third set of constraints concerns security of operations. It may happen that because of a fault, one of the lines or generators gets disconnected by the safety switches connected to this equipment. As a result, in the post-fault network, the  $Y_{ij}$  or the  $q_i$  change. In turn, this leads to a new set of line flows which may no longer meet the thermal limits (4). If that is the case, additional limits are imposed on the pre-fault line flows so that the post-fault network flows also satisfy the thermal constraints. Note that these constraints depend on the operating point, i.e., the pre-fault power flows. In what follows we ignore these contingency constraints.<sup>5</sup>

The feasible flows and injections of the network are determined by (2), (3), (4).

### 2.3 Economic dispatch

We measure the cost and benefit of the net injection  $q_i$  by the increasing convex function  $C_i(q_i)$ . Generally,  $C_i(0) = 0$ ,  $C_i(q_i) > 0$  for  $q_i > 0$ , and  $C_i(q_i) < 0$  for  $q_i < 0$ . This means that if  $i$  is a net supplier,  $q_i > 0$ ,  $C_i(q_i)$  is the variable cost of generation, and the marginal cost curve is increasing. And if  $i$  is a net demander,  $q_i < 0$ ,  $-C_i(q_i)$  is the consumer benefit, and the marginal benefit curve is decreasing.<sup>6</sup> See Figure 2.

<sup>5</sup>A proper framework for dealing with contingency constraints will, in our opinion, require differentiation of power by reliability, see, e.g., [4, 5, 6].

<sup>6</sup>By formulating the problem in this way we are “netting” out simultaneous generation and consumption at the same node. If one wishes to include these explicitly one should replace the objective function in (5) by  $\sum_i [G_i(g_i) - B_i(h_i)]$  and define  $q_i = g_i - h_i$ , where  $G_i(g_i)$  is the cost of generating  $g_i$ ,  $B_i(h_i)$  is the benefit

power and customers that consume power. The voltage at bus  $i$  is a sinusoidal waveform whose instantaneous value at time  $t$  is

$$v_i(t) = V_i \sin(\omega t + \theta_i).$$

Here  $V_i$  is the magnitude (amplitude) of the sinusoidal waveform,  $\omega = 2\pi \times 60$  is the waveform's frequency in radians per second, and  $\theta_i$  is its phase angle.

Some pairs of nodes are connected by transmission lines through which power can flow. A line connecting  $i$  and  $j$  is characterized by its electrical admittance, denoted  $Y_{ij} > 0$ .<sup>2</sup> We have  $Y_{ij} = Y_{ji}$ .

The (real) power flow over the line from  $i$  to  $j$  is equal to

$$q_{ij} = V_i V_j Y_{ij} \sin(\theta_i - \theta_j),$$

measured in kW or MW.<sup>3</sup> By our sign convention,  $q_{ij} = -q_{ji}$  is positive if the power flows from  $i$  to  $j$ . So the net power  $q_i$ <sup>4</sup> injected into the network at bus  $i$  is the algebraic sum of the  $q_{ij}$ ,

$$q_i = \sum_{j=1}^n q_{ij} = \sum_{j=1}^n V_i V_j Y_{ij} \sin(\theta_i - \theta_j), \quad i = 1, \dots, n. \quad (1)$$

We assume that the voltage magnitude  $V_i$  at bus  $i$  is constant. With no loss of generality, we can then set  $V_i \equiv 1$ , all  $i$ . Then (1) takes a simpler form:

$$q_i = \sum_{j=1}^n Y_{ij} \sin(\theta_i - \theta_j), \quad i = 1, \dots, n. \quad (2)$$

Equations (2) are the “real power flow equations.” Only  $n - 1$  of these equations are independent, since adding these equations gives  $q_1 + \dots + q_n \equiv 0$ . Also, all that matters are the phase angle differences. Thus, one can set, say,  $\theta_n \equiv 0$ , and eliminate the  $n$ th equation. This gives  $n - 1$  equations. Given (exogenous) power injections  $q_1, \dots, q_{n-1}$ , we can solve for the  $n - 1$  “unknown” phase angles  $\theta_1, \dots, \theta_{n-1}$ . We can then obtain the individual line flows from

$$q_{ij} = Y_{ij} \sin(\theta_i - \theta_j). \quad (3)$$

## 2.2 Line flow constraints

There are  $n(n - 1)/2$  line flows—the  $q_{ij}$ . Three sets of constraints limit the values they may take. First, these values are not independent. Unlike standard transportation problems where flows over links can be independently assigned (subject to link capacity constraints), here there are only  $n - 1$  degrees of freedom: once the  $n - 1$  phase angle differences are

<sup>2</sup>If there are several lines joining  $i$  and  $j$ ,  $Y_{ij}$  is the sum of the admittances of the individual lines. If there is no such line,  $Y_{ij} = 0$ .

<sup>3</sup>Only the derivation of this equation and the next requires familiarity with electrical circuits, see [3].

<sup>4</sup>Power engineers use  $p$  for power. We use  $q$  for power, reserving  $p$  for price.

The counterexamples are interesting because they negate plausible propositions, including: (1) uncongested lines do not receive congestion rents (defined through node price differences); (2) nodal prices clear markets for power only if the allocation is efficient; (3) in an efficient allocation power can only flow from nodes with lower prices to nodes with higher prices; (4) strengthening transmission lines or building additional lines increases transmission capacity; (5) transmission capacity rights are compatible with any economically efficient dispatch.

The counterexamples to propositions (1), (3) and (4) above highlight basic differences between power transmission and transportation of goods and expose the incorrect analogies that the unwary may draw. To show these differences we employ a simplified approximation to the economic dispatch problem, ignoring nonlinearities, line losses and reactive power. However, the differences persist under more realistic assumptions.

The counterexample to proposition (2) may serve to clarify a misconception that has been perpetuated by the use of misleading terminology. This misconception is also rooted in an inappropriate analogy between transmission and transportation.

Finally, item (5) comes out of the attempt to provide a mathematical formulation of Hogan's characterizations of transmission rights and contract networks [1].

The paper is organized as follows. Section 1 introduces the equations governing the power flows in a lossless transmission network and the economic dispatch problem. Section 2 considers in detail a network in which one line is congested, i.e., the power flow in the line has reached its capacity limit. Section 3 gives a brief discussion of bilateral contracts and Section 4 considers various formulations of transmission rights.

## 2 Network transmission

In this section we first provide a short tutorial to the power flow and economic dispatch problem in AC power systems. Since our objective is to highlight basic technical issues and prevailing misconceptions, we employ a simplified power system model, leaving out some important aspects such as reactive power and line losses. A more complete model will make it more difficult to focus on the technical issues we raise.<sup>1</sup>

The optimality conditions for the economic dispatch problem are used to introduce the concepts of market equilibrium, nodal prices and merchandizing surplus, congestion price and rent.

### 2.1 Real power flow

The power transmission network is modeled as follows. The network consists of  $n$  nodes or buses, indexed  $i, j = 1, \dots, n$ . See Figure 1. To each node are attached generators that supply

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<sup>1</sup>Losses are typically between two and five percent of total power flow. We consider them in a forthcoming paper. The supply of reactive power does not require energy but it does require fixed equipment. The capacity of that equipment places a limit on the amount of reactive power that can be supplied. This issue is discussed by Kahn and Baldick [2]

# Folk theorems on transmission access: proofs and counterexamples

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## Abstract

Nodal prices, congestion revenues, transmission capacity rights, and compensation for wire ownership are key concepts used to formulate claims about proposals to organize competitive and open transmission access. Underlying those claims are implicit assertions (folk theorems) concerning the regulation of transmission access, the determination of power flows, properties of economic dispatch, and the operations of competitive nodal markets for power. The paper has two objectives. We first formulate these folk theorems as explicit mathematical assertions. We then prove that some of these assertions are true, and we present counterexamples to other assertions.

The counterexamples are interesting because they negate plausible propositions, including: (1) uncongested lines do not receive congestion rents (defined through node price differences); (2) nodal prices clear markets for power only if the allocation is efficient; (3) in an efficient allocation power can only flow from nodes with lower prices to nodes with higher prices; (4) strengthening transmission lines or building additional lines increases transmission capacity; (5) transmission capacity rights are compatible with any economically efficient dispatch.

## 1 Introduction

Nodal prices, congestion revenues, transmission capacity rights, and compensation for wire ownership are key concepts used to formulate claims about proposals to organize competitive and open transmission access. Those claims are put forward by advocates of specific proposals for transmission access such as Poolco or bilateral transactions. The claims are sometimes stated as if they are self-evident or widely accepted. However, they are not self-evident, and furthermore, as we show here, many are false.

Upon closer examination one finds that the claims rest on implicit assertions (folk theorems) concerning the regulation of transmission access, the determination of power flows, properties of economic dispatch, and the operations of nodal markets for power. Our objective is to conduct this examination and to reveal the underlying assertions. To achieve that goal we formulate the folk theorems as explicit mathematical assertions. We then prove that some of these assertions are true, and we present counterexamples to other assertions.