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Optimal Power Flow, Node Prices, and Transmission Toll in a Number of Instructive Examples

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ABSTRACT

A theory of optimal real and reactive power flow, node prices, and transmission tolls suited to decentralized operation of a power grid is explored in the context of various numerical examples. A simple OPF program REX is described. With a view toward achieving an understanding of what an economic network would be, values of various grid components are calculated and compared to the corresponding toll revenues generated. Further REX-addressable questions are invited.

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This paper extends an earlier study[1] of the feasibility of decentralized price-quantity coordination schemes for electric power grids. Like the earlier paper, this one is somewhat idiosyncratic. Certain important and much studied aspects of real systems are ignored in order to focus on some features that we happen to think deserve more attention. In the former category are strategic decisions, market power, intertemporal production costs, and arbitrary regulatory constraints. In the latter category: decentralization, transmission externalities, the economic role of reactive power, and the possibilities for coordinating such simplified systems by means of non-exotic market designs. The hope, perhaps vain, is that the results will carry back to the more complicated contexts.

[1] McGuire (1996)

Both the economic theory and the power engineering theory employed here are completely traditional.[2] But where qualitative results are difficult to extract from theory, or difficult to grasp, we (like others) rely heavily on the posing and computation of simple numerical examples. For the latter purpose we make use of a homemade OPF (optimal power flow) routine called REX. The program is easily understood and used and readily obtainable by the reader to verify examples of ours or others' and to investigate his own.[3]

Most of the problems posed envision a system made up completely of bilateral contractors. This is mainly a matter of expositional choice; more of the workings of the system are revealed. In every case translation to an essentially equivalent pooled system is straightforward.

1. **How might a decentralized bilateral system work?** Suppose the grid operator alone possesses knowledge about the characteristics of the transmission hardware: network topology, link admittances, and thermal capacities. Suppose analogously that the producers and consumers of electric power alone possess knowledge about their own cost and value functions. Moreover, all of this information is new: another day, or hour, has just begun; the dice have been recast and coordination of new actions must be carried out by means of some kind of adjustment process.

One such process is as follows. The grid operator makes a public announcement of tentative complex node voltages (which is exactly equivalent to announcing point-to-point loss ratios). The producers and consumers make deals that maximize their profits in the context of the announced voltages (We assume perfectly competitive behavior), and communicate the resulting nodal power imports and exports (i.e., power injections) to the grid operator. Ordinarily, the grid operator will find that link power flows consistent with the proposed nodal power injections either exceed or fall short of the capabilities of the grid.

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[2] The power engineering theory is intended to be straight Bergen (1986).

[3] A recent exposition, more general and more formal, of a similar analysis is Baughmann *et al* (1997) which— exaggerating only a bit— settles for announcement of necessary conditions for a centrally managed optimum.

More specifically, the link voltage drops caused by the proposed injections are different from those in the tentative voltage announcement that elicited the those same injections.

Seeking improved coordination, the grid operator begins a second round of the adjustment process by announcing the new voltages. Buyers and sellers reformulate their tentative contracts and again inform the grid operator. The process continues until the set of node voltages converges to a fixed set, that is to a set of voltages that are both economically and technically consistent with proposed power flows.

That such a fixed-point voltage vector always exists is easily shown, but finding it (and finding it quickly!) by means of an adjustment process can be tricky. Our concern here is less with the efficiency or feasibility of the adjustment process but rather more with the nature of the resulting equilibrium at the fixed-point voltages. Two important questions to ask are: Does this equilibrium maximize social welfare? and How, precisely, do the actions of buyers and sellers follow from the announced node voltages?

We address the second question first. Let $\mathbf{v}_i = v_i e^{i\theta_i}$ be the given complex voltage at Node $i, (i = 1, .., n)$. Suppose each buyer-seller pair is held responsible for the real and reactive power losses directly associated with its contracted power flow. A seller at Node i considering a contract with a buyer at Node j then knows that he must inject $(\mathbf{v}_i/\mathbf{v}_j)\mathbf{x}$ power at i in order for $\mathbf{x}$ power to be (legitimately) withdrawn at j. Define the complex loss ratio $\mathbf{r}_{ij} = \mathbf{v}_j/\mathbf{v}_i = r_{ij} + i\hat{r}_{ij}$. If $\mathbf{x}_i$ and $\mathbf{x}_j$ denote the complex sent and delivered power quantities in the contract we have

$$\mathbf{x}_j = \mathbf{r}_{ij}\mathbf{x}_i = (r_{ij} + i\hat{r}_{ij})(x_i + i\hat{x}_i)$$

or

$$(x_j + i\hat{x}_j) = (r_{ij}x_i - \hat{r}_{ij}\hat{x}_i) + i(\hat{r}_{ij}x_i + r_{ij}\hat{x}_i) \tag{1}$$

so that

$$\frac{\partial x_j}{\partial x_i} = r_{ij}, \qquad \frac{\partial x_j}{\partial \hat{x}_i} = -\hat{r}_{ij}, \qquad \frac{\partial \hat{x}_j}{\partial x_i} = \hat{r}_{ij}, \qquad \text{and} \qquad \frac{\partial \hat{x}_j}{\partial \hat{x}_i} = r_{ij}. \tag{2}$$

A non-strategic profit-maximizing deal between the seller at i and the buyer at j will maximize total value $f_j(x_j, \hat{x}_j)$ at j less total cost $f_i(x_i, \hat{x}_i)$ at i, so

$$\frac{\partial}{\partial x_i}(f_j(x_j, \hat{x}_j) - f_i(x_i, \hat{x}_i)) = \frac{\partial f_j}{\partial x_j}\frac{\partial x_j}{\partial x_i} + \frac{\partial f_j}{\partial \hat{x}_j}\frac{\partial \hat{x}_j}{\partial x_i} - \frac{\partial f_i}{\partial x_i} = 0$$

and

$$\frac{\partial}{\partial \hat{x}_i}(f_j(x_j, \hat{x}_j) - f_i(x_i, \hat{x}_i)) = \frac{\partial f_j}{\partial x_j}\frac{\partial x_j}{\partial \hat{x}_i} + \frac{\partial f_j}{\partial \hat{x}_j}\frac{\partial \hat{x}_j}{\partial \hat{x}_i} - \frac{\partial f_i}{\partial \hat{x}_i} = 0.$$

Substituting (2) we have finally the relation among marginal costs and values of real and reactive power at origin and destination that characterizes profit-maximizing decisions in the absence of any monetary transmission tolls.

$$\frac{\partial f_i}{\partial x_i} = r_{ij}\frac{\partial f_j}{\partial x_j} + \hat{r}_{ij}\frac{\partial f_j}{\partial \hat{x}_j} \tag{3}$$

and

$$\frac{\partial f_i}{\partial \hat{x}_i} = -\hat{r}_{ij}\frac{\partial f_j}{\partial x_j} + r_{ij}\frac{\partial f_j}{\partial \hat{x}_j}. \tag{4}$$

Return now to the earlier question. Does the own-profit maximization lead to a social-welfare-maximizing equilibrium? There are both obvious and more subtle reasons that the answer is no. First of all, if there are thermal limits on links affected by the transmission in question, then a share of the congestion shadow price must enter the marginal cost equations stated above. That is well recognized. Less attention has been given to other less obvious congestion externalities. A transmission of power from origin Node i to destination Node j imposes— because of loop flow— voltage drops throughout the network. These changes, in turn in "spillover" fashion, affect power losses on other bilateral transmissions.[4] These theoretical reasons for believing that the own-profit-maximizing equilibrium is not a social optimum are borne out by numerical examples. To take the simplest possible case, consider a two-node network with a connecting transmission line of admittance 8. Imagine a producer at Node 1 with a supply curve $p_1 = .10x_1$ and a consumer at Node

[4] McGuire (1996) proposes a calculation for this "spillover" congestion for the strict-DC case. The "second form" of congestion discussed in Hogan (1995) is related to but not strictly the same as our spillover congestion.

2 with demand curve $p_2 = 100 - .10x_2$. The own-profit maximum of 10,000 occurs at $x = (400, 200)$ and $p = (80, 40)$. Marginal costs and values here satisfy profit-maximizing conditions (3) and (4) as is easily verified.[5] Yet, making use of the OPF program REX described in the next section, we find that the *correct* social optimum— total value minus total cost— is found at $x = (269, 178)$ and $p = (26.9, 82.2)$ for a "surplus" of 12,642.

The use of transmission tolls to— in effect— force recognition of these externalities is discussed below. For now we only remark that if transmission tolls t_{ij} and $\hat{t}_{ij}$ *are* imposed respectively on real and reactive O–D power flows, profit maximizing decisions are then characterized by

$$\frac{\partial f_i}{\partial x_i} = r_{ij}\frac{\partial f_j}{\partial x_j} + \hat{r}_{ij}\frac{\partial f_j}{\partial \hat{x}_j} - t_{ij} \tag{5}$$

and

$$\frac{\partial f_i}{\partial \hat{x}_i} = -\hat{r}_{ij}\frac{\partial f_j}{\partial x_j} + r_{ij}\frac{\partial f_j}{\partial \hat{x}_j} - \hat{t}_{ij}. \tag{6}$$

A geometric illustration of the (3)–(4) conditions and the (5)–(6) conditions and their respective solutions is given in Figure 1.

2. **The numerical analysis in this paper** is based on an optimal power flow model REX developed specifically to investigate the economic questions we regard as relevant to coordination. Of course, other optimal power flow models, much more powerful and more comprehensive than REX, can be obtained. Our object has not been to reinvent the wheel, but rather to construct an easily used simple program tailored to our own specific interests. We have also aimed to develop a program that the reader of this paper can use to reproduce our numerical results and to investigate simple examples of his own design. The analytical literature concerned with power-grid coordination questions has been remarkably dependent upon examination of very simple numerical examples– indeed, seldom extending beyond three nodes.[6] Much can be learned from such simple examples,

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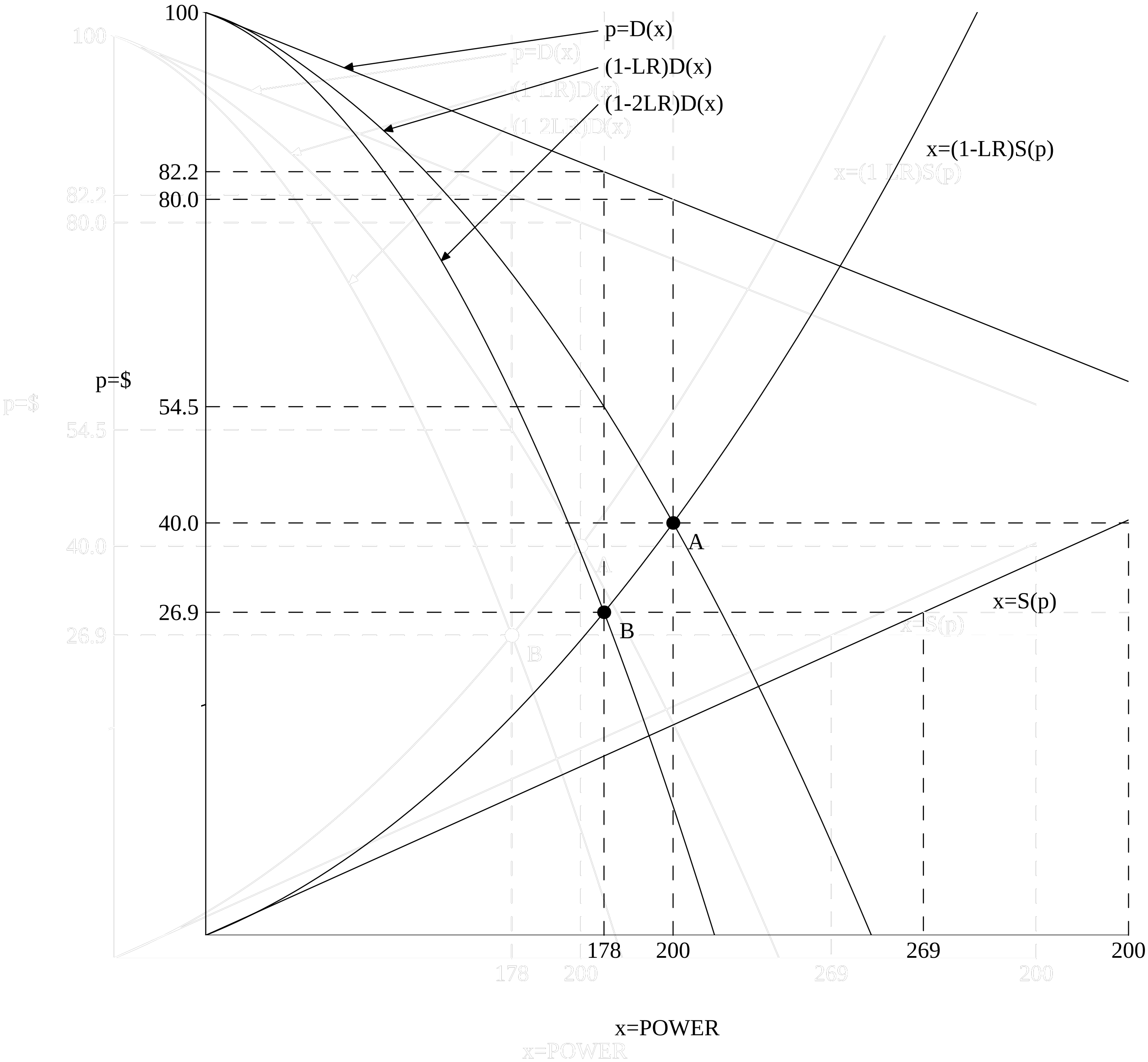
[5] The mistaken WEPEX proposal not to "over-collect" for losses is tantamount to an assertion that this own-profit-maximizing equilibrium is indeed a social optimum. See Stoft (1997).

[6] For economic purposes three nodes are sufficient to study two of the most important features of power transmission: loop flow and dispatch interaction.



$$D(x) = 100 - .10x$$

$$S(p) = 10p$$

$$LR = PowerLoss/PowerDispatched$$

Figure 1. The own-profit maximum (A) and the social optimum (B)

but most often the analysis is *ad hoc* arithmetic difficult to compare from one example to another.[7] Sometimes, where the example is more complicated,[8] policy recommendations are claimed to be supported by numerical outcomes that the reader cannot verify for lack of access to some computer program. Even current textbooks on power engineering are not very helpful.[9]

For most analysts interested in gaining qualitative insight into the working of power grids, the mathematics of optimal power flow is sufficiently difficult to make reliance on simple numerical examples essential. The program REX is a beginning attempt to meet this need in a disciplined fashion.

REX is programmed in the algebraic modeling language GAMS[10] which is a handy front-end for accessing a number of powerful mathematical programming routines such as MINOS, which in turn deal efficiently with discrete and non-linearly constrained extremization problems. The optimal power flow problem is just such a problem; it exceeds the capabilities of the built-in routines of popular PC languages such as Mathematica and MATLAB.

GAMS was designed with the laudable intention of achieving readability through standardization, simple notation, and easy documentation.[11] Although our OPF model is described in traditional form in the next section, the reader may nevertheless find it helpful to examined the REX program listing in the Appendix. The part of the program dealing directly with inputs and with the optimization is short; a longer and no less important part deals with manipulation of the optimization output to find such things as marginal

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[7] Schweppe *et al* (1985) and (1988), Chao and Peck (1996), Hogan (1992), Oren *et al* (1995), Wu *et al* (1996)

[8] e.g. Hogan(1995)

[9] Neither Bergen(1986) nor Wood & Wollenberg(1996) present worked examples of optimal power flow.

[10] All of the computations in this paper employ only the student version of GAMS, available from Boyd & Fraser Publishing for about $60.

[11] See the Preface in Brooke *et al* (1988)

costs of generation, marginal values of consumption, marginal values of link conductance and susceptance, grid revenues, *etc.*

3. **The general form of the model** and the nature of the inputs should be understood prior to examining the program itself. The transmission grid is a network of n nodes and the links among them. Each link is characterized by its admittance (*i.e.,* conductance and susceptance) and its thermal capacity which (at the users option, setting $PSI = 0$ or 1, resp.) may refer to total real power injected into the link or to real power loss *on* the link. Nodes not connected have zero admittance. The complex admittance matrix is $\mathbf{y} = y + i\hat{y}$ with $\mathbf{y}_{ii} = -\sum_j \mathbf{y}_{ij}, \quad y_{ii} > 0, \quad \hat{y}_{ii} < 0$. These sign conventions are adopted to cause node power injections to be positively signed when they represent true power inflows and negatively signed when they represent true power outflows. In summary, the grid inputs are the three $n \times n$ real matrices y, $\hat{y}$, and c, the thermal capacity matrix.

Each node is treated as a single real-power producer, a single real-power consumer, or a passive node. Producers are constrained to non-negative real power injections, consumers to non-positive real power injections, and passive nodes to zero net injections of both real and reactive power. The roles of nodes as real-power producers, consumers, or passives are specified in an input vector ($ID = 1, -1, 0$, resp.). Their roles as reactive-power producers and consumers are left to be determined endogenously.

The motivation for this asymmetric treatment is as follows. In order to allow for the interesting case of complete exclusion from trade of a high-priced producer (or a low-valuing consumer) and in the same model allow him to become a consumer if his price is high enough, one needs to model his excess supply curve with a discontinuity at zero. For the modest goals of this paper such complexity is more confusing than helpful. With respect to reactive power we simply assume that excess supply curves are continuous at zero.

The economic inputs to the model are, then, for each producer (consumer) node an intercept and slope of a real-power supply (demand) curve, an intercept and slope of a reactive power supply curve, and finally a coefficient specifying the interaction between real and reactive supply or demand. Five numbers, then, for each node. In this discussion the terms supply curve and marginal cost function are to be taken as interchangeable; similarly demand curve and marginal value function. Total social welfare (SURPLUS in the REX terminology) is the total value to consumers of real and reactive power received minus total cost to producers of real and reactive power generated, a quadratic function of power injections.

It is most convenient to think of complex node voltages as the ultimate decision variables in an optimal power flow problem. In a case where thermal constraints are not binding, voltages are constrained only by common upper and lower bounds: magnitudes may not exceed some constant ($VOLTTOP$ in REX) and phase angles must for stability lie within some safety interval (given by $ANGLELIM$ in REX). Everything else follows from voltage choices.

To state the model some additional notation is required. A consistent style has been attempted: boldface for complex quantities, the same letter for the real coefficients of that quantity with a hat on the coefficient of i. No subscripts will mean a vector over n nodes or a matrix over n^2 network links. Thus $\mathbf{x}_{ij} = (x_{ij} + i\hat{x}_{ij})$ is complex power injection into link ij. Summarizing both old and new, we have $\mathbf{x}$ is power injection, $\mathbf{z}$ is current injection, $\mathbf{r}$ is a voltage ratio, $\mathbf{y}$ is admittance. The demand and supply curve parameters will be written similarly, $\mathbf{a}$ for the intercepts and $\mathbf{b}$ for the linear slopes (Just a notational convenience: $\mathbf{a}$ and $\mathbf{b}$ are not treated as complex numbers.) Node voltages, unlike all the other variables, will always be written in polar form: $\mathbf{v} = ve^{i\theta}$ with θ a real vector of node voltage phase angles (in radians). Now the model.

The objective is to maximize the sum S of consumers' surpluses and the negative of producer costs :

$$S = -\sum_i (a_i x_i + \tfrac{1}{2} b_i x_i^2) + (\hat{a}_i \hat{x}_i + \tfrac{1}{2} \hat{b}_i^2). \tag{7}$$

The next step is to constrain the node flows to conform to the laws of circuit theory. The complex current injected into link ij is, by Ohm's Law, link admittance times the voltage drop across the link: $\mathbf{z}_{ij} = \mathbf{y}_{ij}(\mathbf{v}_j - \mathbf{v}_i)$. Total current injection at Node i is then

$$\mathbf{z}_i = \sum_j \mathbf{z}_{ij} = \sum_j \mathbf{y}_{ij}\mathbf{v}_j$$

because the second term sums to zero. Node power injection is the product of node voltage and the *conjugate* of node current injection:[12]

$$\mathbf{x}_i = \mathbf{v}_i \sum_i \mathbf{y}^*_{ij}\mathbf{v}^*_j.$$

Expanding this we have, writing θ_{ij} for $\theta_i - \theta_j$,

$$x_i = v_i \sum_j v_j(y_{ij}\cos(\theta_{ij}) + \hat{y}_{ij}\sin(\theta_{ij})) \tag{8}$$

and

$$\hat{x}_i = v_i \sum_j v_j(y_{ij}\sin(\theta_{ij}) + \hat{y}_{ij}\cos(\theta_{ij})). \tag{9}$$

Statements (8) and (9) would serve as they stand for properly constraining node power injections were it not for our interest in the effects on payoff of individual link admittances. Toward that end we rephrase them as follows. Defining $u_{ij} = y_{ij}\cos(\theta_{ij})$, $\hat{u}_{ij} = \hat{y}_{ij}\sin(\theta_{ij})$, $w_{ij} = y_{ij}\sin(\theta_{ij})$, and $\hat{w}_{ij} = \hat{y}_{ij}\cos(\theta_{ij})$ we may rewrite (8) and (9) as

$$x_i = v_i \sum_j v_j(u_{ij} + \hat{u}_{ij}) \tag{10}$$

and

$$\hat{x}_i = v_i \sum_j v_j(w_{ij} - \hat{w}_{ij}). \tag{11}$$

Next, we turn the **u** and **w** definitions into four constraints directly related to admittance (= conductance $+i\times$ susceptance):

$$u_{ij}/\cos(\theta_{ij}) = y_{ij} \tag{12}$$

[12] Beyond his own intuition, this rule and Ohm's Law are just about all that the reader needs to know about circuit theory.

$$\hat{u}_{ij}/\sin(\theta_{ij}) = \hat{y}_{ij} \tag{13}$$

$$w_{ij}/\sin(\theta_{ij}) = y_{ij} \tag{14}$$

$$\hat{w}_{ij}/\cos(\theta_{ij}) = \hat{y}_{ij} \tag{15}$$

Finally, link power injection must not exceed link thermal capacity. Link power injection is $\mathbf{x}_{ij} = \mathbf{v}_i \mathbf{z}^*_{ij}$ which, expanded, yields the thermal constraint on the real component [13]

$$v_i v_j (\cos(\theta_{ij}) y_{ij}) + \sin(\theta_{ij})\hat{y}_{ij}) - v_i^2 y_{ij} \leq c_{ij}. \tag{16}$$

Three additional constraints are obviously required: lids and floors on the components of voltage,

$$0 \leq v_i \leq VOLTTOP, \tag{17}$$

$$-ANGLELIM \leq \theta_i \leq ANGLELIM, \tag{18}$$

and role behavior of the nodes,[14]

$$x_i ID(i) \geq 0 \tag{19}$$

In summary, the REX model is: Choose voltages $\mathbf{v}$ and the related variables to maximize (7) subject to the constraints (10)-(19).

4. **Admittance and capacity values are assembled from an array of shadow prices.** Shadow prices on constraints (12)-(16) reflect the contributions to social payoff of thermal capacities and the components of link admittances that are effected *through various channels*. Because of the symmetry and zero row-sum design of the admittance matrix a change in link ij admittance changes not only $\mathbf{y}_{ij}$ but also $\mathbf{y}_{ji}, \mathbf{y}_{ii}$, and $\mathbf{y}_{jj}$. And

[13] If one wishes to constrain power *loss* instead of power injection (After all, what bends the lines?) one need only subtract the ji injection from the left side of (16). In the REX program this is controlled by the parameter PSI.

[14] When this constraint doesn't work for passive nodes, one has to fiddle with the cost and value parameters.

each of these in turn may affect both real and reactive power flows in different ways. Let the Lagrange multipliers associated with constaints (12)-(16) be denoted $L(u_{ij}, L(\hat{u}_{ij}), L(w_{ij}),$ $L(\hat{w}_{ij})$, and$L(c_{ij})$. (All of these are returned in the REX output.) The total effect on payoff of, for instance, changing link ij conductance is

$$\frac{\partial S}{\partial y_{ij}} + \frac{\partial S}{\partial y_{ii}} + \frac{\partial S}{\partial y_{ji}} + \frac{\partial S}{\partial y_{jj}}.$$

Expanding only the first two terms we get[15]

$$\frac{\partial S}{\partial x_i}\frac{\partial x_i}{\partial u_{ij}}\frac{\partial u_{ij}}{\partial y_{ij}} + \frac{\partial S}{\partial \hat{x}_i}\frac{\partial \hat{x}_i}{\partial w_{ij}}\frac{\partial w_{ij}}{\partial y_{ij}} + \frac{\partial S}{\partial x_i}\frac{\partial x_i}{\partial u_{ii}}\frac{\partial u_{ii}}{\partial y_{ii}} = L(u_{ij}) + L(w_{ij}) + L(u_{ii}).$$

Obviously then, the total effect— the marginal value of link ij conductance— is

$$MV(y_{ij}) = L(u_{ij}) + L(w_{ij}) + L(u_{ii}) + L(u_{ji}) + L(w_{ji}) + L(u_{jj}).$$

An exactly analogous derivation yields the formula for the marginal value of link ij susceptance:

$$MV(\hat{y}_{ij}) = L(\hat{u}_{ij}) + L(\hat{w}_{ij}) + L(\hat{w}_{ii}) + L(\hat{u}_{ji}) + L(\hat{w}_{ji}) + L(\hat{u}_{jj}).$$

The marginal value of thermal capacity is explicitly returned in the Lagrange multiplier $L(c_{ij})$. One complexity, however, remains to be explored. The marginal value of conductance described above is precisely that when thermal capacity is not binding anywhere. When thermal capacity *is* tight on some links the values of conductances and thermal capacities are confounded. A procedure that appears to work in identifying the separate effects is this. One subtracts the "total value" of thermal capacity on a link from the "total value" of conductance on that link. Then divide by conductance to obtain the "unpolluted" marginal value of conductance:

$$\text{True } MV(y_{ij}) = (MV(y_{ij}) \times y_{ij} - MV(c_{ij}) \times c_{ij})/y_{ij}.$$

[15] Equality in the following statement is not to be taken literally. The left-hand side is meant only to guide one in the assembly of Lagrange multipliers. Computation of the left-hand side derivatives does *not* give us proper estimates of marginal values under optimization of all the variables.

The validity of this procedure can be confirmed by direct (and tedious) optimal power flow computations, varying constraint levels one at a time by one unit and re-solving.

All the the marginal value outputs in the REX program are calculated according to the procedures just described.

5. **A classification of examples** discussed here, and those in the literature, into types according to the way they treat real and reactive power is useful.[16] We shall call a model Type GH-PQ if the conductance matrix G is non-zero, the susceptance matrix H is non-zero, the cost and value parameters P for real power are non-zero, and the cost and value parameters Q for reactive power are non-zero. That's the general case. If any one of these four items is zero, we drop the corresponding letter from the type code. Thus a model of Type G-P is a strict DC (lossy) example as in McGuire(1996). A model of Type H-P is the simplest of the popular pseudo-DC lossless models, as in Varaiya & Wu (1995). The numbering of examples will follow this classification: the hundreds digit will indicate the number of nodes in the example, and the tens digit will indicate the type, and the last digit will simply differentiate one example from another in the same class. Table 1 summarizes.

TABLE 1.Model Types

Type	Example No.
G-P	$n10$ series
G-Q	$n20$
G-PQ	$n30$
H-P	$n40$
H-Q	$n50$
H-PQ	$n60$
GH-P	$n70$
GH-Q	$n80$
GH-PQ	$n90$

n =number of nodes

[16] My attempt at notational economy fails me here. In this section the symbols $y, \hat{y}, (a, b), \text{and}(\hat{a}, \hat{b})$ are replaced, respectively, by $G, H, P, andQ$.

Not all of these types merit individual attention, but the classification is nevertheless useful as will be seen.

We start our examination of examples with EX310. This one will serve to show how examples are to be read. The inputs to and the outputs from the REX run are displayed in Table 2. This G-P model is from McGuire (1996)[17] The thermal limits are set so high as to be non-binding. All nodes are connected and Nodes 1 and 2 are given "roles" as producers and Node 3 as a consumer. As in the other examples we shall examine, marginal costs and marginal values are linear; thus (refer to the input list in Table 3) marginal cost of real power produced at Node 1 is $A(1)+B(1)X(1) = .05X(1)$ and marginal value of real power consumed at Node 3 is $A(3) + B(3)X(3) = 100 + .10X(3)$. (Recall that "injection" $X(3)$ is non-positive.) Marginal costs and values of reactive power are similarly given by the parameters AR and BR. The important parameter setting $VOLTTOP = 10$ specifies the upper bound on node voltage magnitude.

TABLE 2. Example 310

```
TABLE 2.  EXAMPLE   310
  Type G-P
=================================================
INPUTS  INPUTS  INPUTS  INPUTS
==================================================
 0.5 < Voltage Magnitude <   10.0

- 1.5 < Phase Angles <  1.5

Thermal limit on total flow or loss? 0 or 1?
  PSI =        1.00

Real Value Parameters: Intercept, Slope, and Role
N1            0.00        0.05        1.00
N2            0.00        0.15        1.00
N3          100.00        0.10       -1.00

Reactive Value Parameters: Intercept & Slope (Role=0)
N1            0.00        0.00        0.00
N2            0.00        0.00        0.00
N3            0.00        0.00        0.00

Conductance, Susceptance, & Thermal Capacity--  PSI= 1
N1    N1             27.000           0.00          0.000
N1    N2            -15.000           0.00       1000.000
N1    N3            -12.000           0.00       1000.000
```

[17] There called Ex 34a.

```
N2    N1             -15.000           0.00       1000.000
N2    N2              23.000           0.00          0.000
N2    N3              -8.000           0.00       1000.000
N3    N1             -12.000           0.00       1000.000
N3    N2              -8.000           0.00       1000.000
N3    N3              20.000           0.00          0.000

=================================================
OUTPUTS  OUTPUTS  OUTPUTS  OUTPUTS
===================================================
Real & Reactive Power Flows & Real & Reactive Power Losses
N1    N1               0.000          0.000          0.000          0.000
N1    N2              44.366          0.000          1.312          0.000
N1    N3             374.096          0.000        116.623          0.000
N2    N1             -43.054          0.000          1.312          0.000
N2    N2               0.000          0.000          0.000          0.000
N2    N3             219.059          0.000         63.696          0.000
N3    N1            -257.473          0.000        116.623          0.000
N3    N2            -155.363          0.000         63.696          0.000
N3    N3               0.000          0.000          0.000          0.000

Voltages & Phase Angles
N1              10.000         -0.001
N2               9.704         -0.001
N3               6.883         -0.001

Real & Reactive Node Injections
N1             418.462          0.000
N2             176.005          0.000
N3            -412.836          0.000

Real & Reactive Node Prices and Real & Reactive Mgl Costs/Values
N1              20.923          0.000        20.923         0.000
N2              26.401          0.000        26.401         0.000
N3              58.716          0.000        58.716         0.000

Real & Reactive Tolls
N1    N1           0.000        0.000
N1    N2           4.697        0.000
N1    N3          19.489        0.000
N2    N1          -4.840        0.000
N2    N2           0.000        0.000
N2    N3          15.243        0.000
N3    N1         -28.316        0.000
N3    N2         -21.492        0.000
N3    N3           0.000        0.000

MV of Conductance, Susceptance, & Thermal Capacity
N1    N1            0.00        -0.42        0.00
N1    N2           13.89        -0.47        0.00
N1    N3          607.55        -0.61        0.00
N2    N1           13.89        -0.47        0.00
N2    N2            0.00        -0.51        0.00
N2    N3          417.38        -0.66        0.00
N3    N1          607.55        -0.61        0.00
N3    N2          417.38        -0.66        0.00
N3    N3            0.00        -0.81        0.00

Real & Reactive Toll Revenue
N1    N1            0.00         0.00        0.00
N1    N2          208.38         0.00      208.38
N1    N3         7290.64         0.00     7290.64
N2    N1          208.38         0.00      208.38
N2    N2            0.00         0.00        0.00
N2    N3         3339.05         0.00     3339.05
N3    N1         7290.64         0.00     7290.64
```

```
N3    N2         3339.05        0.00     3339.05
N3    N3            0.00        0.00        0.00

                10838.07        0.00    10838.07

Values of Conductance, Susceptance, Thermal Capacity, & Total
N1    N1            0.00        0.00        0.00        0.00
N1    N2          208.38        0.00        0.00      208.38
N1    N3         7290.64        0.00        0.00     7290.64
N2    N1          208.38        0.00        0.00      208.38
N2    N2            0.00        0.00        0.00        0.00
N2    N3         3339.05        0.00        0.00     3339.05
N3    N1         7290.64        0.00        0.00     7290.64
N3    N2         3339.05        0.00        0.00     3339.05
N3    N3            0.00        0.00        0.00        0.00

Surplus              26060.828
```

6. **How does the operation of a pool system compare with that of a bilateral system?** Example 310 in Table 2 shows optimal power flow in a very simple strict-DC grid. Node prices for real power are shown at line 58. A properly guided pool operator would simply pay Node 1 and 2 prices for real power purchased at those points and demand Node 3 price for real power sold there. The amounts bought and sold are shown as real power injections at Line 53. Sufficient power is bought to cover line losses. Market participants need know nothing about power losses. Each will find his profit maximized at the posted prices by offering and demanding exactly the listed power injections.

In a properly designed bilateral system buyers and sellers of real power will make their bilateral deals in response to announcements from the grid manager of loss ratios and transmission tolls on origin-destination transactions. Loss ratios are the same as voltage ratios. At the social optimum portrayed in Example 310 the loss ratio for the 1-3 transmission is $v_3/v_1 = 6.883/10$ so only .6883 of power dispatched from 1 to 3 is received at 3. A profit-maximizing deal between Nodes 1 and 3 takes this loss and the toll into account in the following fashion (Let c_i denote marginal cost/value and t_{ij} the ij toll):

$$c_1 = (v_3/v_1)c_3 - t_{13}$$

or

$$20.923 = .6883 \times 58.716 - 19.489.$$

With the data displayed in Example 310, the reader can verify that these marginal costs correspond to the optimal power injections. Notice that the pool system and the bilateral system lead to identical results.

Still another, perhaps more natural, bilateral scheme can be imagined. Let the grid operator announce transmission tolls and (instead of loss ratios) a set of node prices at which he will buy to cover losses. To achieve efficiency these prices (generally different at each node!) must elicit competitive quantity offers from contracting pairs that just match the losses on the corresponding O–D paths. The selling pairs need not know that their sales just cover the losses of their own transmissions. A misguided pool operator could assign loss coverage in a different way but the OPF results tell us that social losses would inevitably follow.

In their pioneering DOE study of "wheeling" costs Schweppe *et al* (1985) treated *one* distinguished pair of contractors (the "wheelers") in the fashion of this last scheme. The cost of this single wheeling contraction was asserted (properly, *ceteris paribus*) to be the cost imposed in terms of losses upon all the participants in the network. The lost power was generated, presumably, by the non-wheelers. Unfortunately and obviously, this non-simultaneous style of cost computation falls apart when *all* power transmitters are treated as wheelers.[18]

7. **Link flows and Origin–Destination (O–D) flows.** Although triangle grids serve to display many of the important coordination problems that arise generally, they sometimes mask the critical difference between link flows and O–D flows. EX610 in Table 3 is a six-node DC grid with producers at Nodes 1 and 2 and consumers at Nodes 5 and 6. Not all nodes are connected, as seen in Figure 2. Thermal limits are non-binding. Optimal link power flows and losses, node injections, and voltages are given in the table and displayed in Figure 3. One (of course, non-unique) set of optimal O–D transactions is shown in Figure 4, and optimal node prices and the relevant O–D tolls are displayed in Figure 5.

The main motive for presenting this example is to provide a context for posing some instructive problems.

Exercise: Verify that the O–D flows and losses of Figure 4 are consistent with the link flows of Figure 3 the prices and tolls of Figure 5, *and* the profit maximizing rules outlined in Section 6.

---

[18] Our plea for a standard procedure for computation of expository examples is lent some force by the capstone numerical example they present to demonstate wheeling costs. The arithmetic is wrong and repeated without correction in the (1988) publication. The margins of the UC library copy of their book are covered with the notes of readers trying vainly to replicate their computation. And in this exposition numerical examples are needed— the notation is ornate and the algebra is dense.

```
TABLE 3.  EXAMPLE   610
   Type G-P
=================================================
INPUTS  INPUTS  INPUTS  INPUTS
=================================================
 2.0 < Voltage Magnitude <   10.0

Thermal limit on total flow or loss? 0 or 1?
  PSI =        1.00

Real Value Parameters: Intercept, Slope, and Role
N1             0.00        0.10                  1.00
N2             0.00        0.05                  1.00
N3             0.00        0.00                  0.00
N4             0.00        0.00                  0.00
N5            80.00        0.05                 -1.00
N6           100.00        0.10                 -1.00

Conductance & Thermal Capacity--  PSI= 1
N1    N1             20.000          0.000
N1    N2            -10.000       1000.000
N1    N3            -10.000       1000.000
N1    N4              0.000          0.000
N1    N5              0.000          0.000
N1    N6              0.000          0.000
N2    N1            -10.000       1000.000
N2    N2             30.000          0.000
N2    N3            -10.000       1000.000
N2    N4            -10.000       1000.000
N2    N5              0.000          0.000
N2    N6              0.000          0.000
N3    N1            -10.000       1000.000
N3    N2            -10.000       1000.000
N3    N3             40.000          0.000
N3    N4            -10.000       1000.000
N3    N5            -10.000       1000.000
N3    N6              0.000          0.000
N4    N1              0.000          0.000
N4    N2            -10.000       1000.000
N4    N3            -10.000       1000.000
N4    N4             40.000          0.000
N4    N5            -10.000       1000.000
N4    N6            -10.000       1000.000
N5    N1              0.000          0.000
N5    N2              0.000          0.000
N5    N3            -10.000       1000.000
N5    N4            -10.000       1000.000
N5    N5             30.000          0.000
N5    N6            -10.000       1000.000
N6    N1              0.000          0.000
N6    N2              0.000          0.000
N6    N3              0.000          0.000
N6    N4            -10.000       1000.000
N6    N5            -10.000       1000.000
N6    N6             20.000          0.000

=================================================
OUTPUTS  OUTPUTS  OUTPUTS  OUTPUTS
===================================================
Real Power Flows & Real Power Losses
N1    N1              0.000          0.000
N1    N2              0.000          0.000
N1    N3            147.485         21.752
N1    N4              0.000          0.000
N1    N5              0.000          0.000
N1    N6              0.000          0.000
```

```
N2    N1              0.000          0.000
N2    N2              0.000          0.000
N2    N3            147.485         21.752
N2    N4            229.934         52.870
N2    N5              0.000          0.000
N2    N6              0.000          0.000
N3    N1           -125.733         21.752
N3    N2           -125.733         21.752
N3    N3              0.000          0.000
N3    N4             70.289          6.798
N3    N5            181.177         45.165
N3    N6              0.000          0.000
N4    N1              0.000          0.000
N4    N2           -177.064         52.870
N4    N3            -63.491          6.798
N4    N4              0.000          0.000
N4    N5            100.164         16.919
N4    N6            140.392         33.237
N5    N1              0.000          0.000
N5    N2              0.000          0.000
N5    N3           -136.012         45.165
N5    N4            -83.245         16.919
N5    N5              0.000          0.000
N5    N6             33.433          2.729
N6    N1              0.000          0.000
N6    N2              0.000          0.000
N6    N3              0.000          0.000
N6    N4           -107.154         33.237
N6    N5            -30.704          2.729
N6    N6              0.000          0.000

Voltages & Real Node Injections
N1             10.000        147.485
N2             10.000        377.418
N3              8.525          0.000
N4              7.701          0.000
N5              6.400       -185.824
N6              5.878       -137.859

Real Node Prices and Real Mgl Costs/Values
N1           14.748      14.748
N2           18.871      18.871
N3           33.648       0.000
N4           46.581       0.000
N5           70.709      70.709
N6           86.214      86.214

Real Tolls & Real Toll Revenue
N1    N1           0.00        0.00
N1    N2           4.12        0.00
N1    N3          13.94     2055.55
N1    N4          21.12        0.00
N1    N5          30.50        0.00
N1    N6          35.92        0.00
N2    N1          -4.12        0.00
N2    N2           0.00        0.00
N2    N3           9.81     1447.55
N2    N4          17.00     3908.82
N2    N5          26.38        0.00
N2    N6          31.80        0.00
N3    N1         -16.35     2055.55
N3    N2         -11.51     1447.55
N3    N3           0.00        0.00
N3    N4           8.43      592.39
N3    N5          19.43     3520.92
N3    N6          25.79        0.00
```

N4	N1	-27.43	0.00
N4	N2	-22.08	3908.82
N4	N3	-9.33	592.39
N4	N4	0.00	0.00
N4	N5	12.18	1220.40
N4	N6	19.22	2698.59
N5	N1	-47.66	0.00
N5	N2	-41.22	0.00
N5	N3	-25.89	3520.92
N5	N4	-14.66	1220.40
N5	N5	0.00	0.00
N5	N6	8.47	283.12
N6	N1	-61.12	0.00
N6	N2	-54.11	0.00
N6	N3	-37.41	0.00
N6	N4	-25.18	2698.59
N6	N5	-9.22	283.12
N6	N6	0.00	0.00
		15727.33	

Values of Conductance, Thermal Capacity, & Total

N1	N1	0.00	0.00	0.00	0.00
N1	N2	0.00	0.00	0.00	0.00
N1	N3	205.56	0.00	2055.55	0.00
N1	N4	0.00	0.00	0.00	0.00
N1	N5	0.00	0.00	0.00	0.00
N1	N6	0.00	0.00	0.00	0.00
N2	N1	0.00	0.00	0.00	0.00
N2	N2	0.00	0.00	0.00	0.00
N2	N3	144.76	0.00	1447.55	0.00
N2	N4	390.88	0.00	3908.82	0.00
N2	N5	0.00	0.00	0.00	0.00
N2	N6	0.00	0.00	0.00	0.00
N3	N1	205.56	0.00	2055.55	0.00
N3	N2	144.76	0.00	1447.55	0.00
N3	N3	0.00	0.00	0.00	0.00
N3	N4	59.24	0.00	592.39	0.00
N3	N5	352.09	0.00	3520.92	0.00
N3	N6	0.00	0.00	0.00	0.00
N4	N1	0.00	0.00	0.00	0.00
N4	N2	390.88	0.00	3908.82	0.00
N4	N3	59.24	0.00	592.39	0.00
N4	N4	0.00	0.00	0.00	0.00
N4	N5	122.04	0.00	1220.40	0.00
N4	N6	269.86	0.00	2698.59	0.00
N5	N1	0.00	0.00	0.00	0.00
N5	N2	0.00	0.00	0.00	0.00
N5	N3	352.09	0.00	3520.92	0.00
N5	N4	122.04	0.00	1220.40	0.00
N5	N5	0.00	0.00	0.00	0.00
N5	N6	28.31	0.00	283.12	0.00
N6	N1	0.00	0.00	0.00	0.00
N6	N2	0.00	0.00	0.00	0.00
N6	N3	0.00	0.00	0.00	0.00
N6	N4	269.86	0.00	2698.59	0.00
N6	N5	28.31	0.00	283.12	0.00
N6	N6	0.00	0.00	0.00	0.00
Surplus		22189.548			

Better Exercise: Find a consistent set (other than the obvious one!) of O–D transactions involving traders at passive Nodes 3 and 4.

	10	
(1)→	1000	→**(2)**
↓		↙↓
10	10	10
1000	1000	1000
↓ ↙		↓
(3)→	10	→**(4)**
	1000	
↓		↙↓
10	10	10
1000	1000	1000
↓ ↙		↓
(5)→	10	→**(6)**
	1000	

Figure 2. Link Admittances and Thermal Capacities in Example 610

	147.5		377.4	
	↓		↓	
$v_1 = 10$	**(1)**		**(2)**	$v_2 = 10$
	↓		↙↓	
	147.5	147.5	229.9	
	125.7	125.7	177.1	
	↓ ↙		↓	
$v_3 = 8.525$	**(3)**→	70.3	→**(4)**	$v_4 = 7.701$
		63.5		
	↓		↙↓	
	181.2	100.2	140.4	
	136.0	83.2	107.2	
	↓ ↙		↓	
$v_5 = 6.400$	**(5)**→	33.4	→**(6)**	$v_6 = 5.878$
		30.7		
	↓		↓	
	185.8		137.9	

Figure 3. Link Power Flows in Example 610

147.5		377.4
$\downarrow$		$\downarrow$
(1)		**(2)**
$\downarrow$		$\swarrow\downarrow$
147.5	142.8	234.6
94.4	91.4	137.9
$\downarrow \ \swarrow$		$\downarrow$
(5)		**(6)**
$\downarrow$		$\downarrow$
185.8		137.9

Figure 4. A Set of Optimal Origin–Destination Transactions for Example 610

$v_1 = 10$	**(1)**		**(2)**	$v_2 = 10$
$p_1 = 14.748$	$\downarrow$		$\swarrow\downarrow$	$p_2 = 18.871$
	30.50	26.38	31.80	
	$\downarrow \ \swarrow$		$\downarrow$	
$v_5 = 6.400$	**(5)**		**(6)**	$v_6 = 5.878$
$p_5 = 70.709$				$p_6 = 86.214$

Figure 5. Node Prices, Voltages, and O–D Tolls for the Transactions of Figure 4

8. **How are transmission tolls computed and what do they represent?** Transmission tolls can be computed in either a direct but rather difficult fashion or easier indirect one. The REX program employed for these examples takes the easy course. At the optimum marginal costs/values must satisfy (5) and (6). Loss ratios are known so the residuals in those equations are the proper transmission tolls. The direct procedure attempts to quantify externalities in detail. A marginal increase in power transmission from origin i to destination j causes power losses not captured in the i to j loss ratio. Loss ratios throughout the network are affected— even the i to j loss ratio— because the added loop current flows change voltage drops across the system. These voltage drop changes can be computed and their effect on given tranmissions assessed and valued. Explicit use of this procedure was tried with partial success in McGuire (1996) to compute tolls in an adjustment process where the indirect method (requiring optimality) would not work. Accurate computation of tolls was found to be difficult, however, and the difficulties would only increase with the more complicated cases dealt with here, so we stick with the easy indirect method. Notice that, as long as we confine attention to optimal outcomes, nothing (except better understanding!) is lost— the toll estimates are correct. Perhaps the important thing is that we understand in a rough qualitative way the source of the congestion externalities that the tolls represent. As higher current flows press upon the limited admittances of a network, losses increase of course but more importantly loss *ratios* change.

In this discussion— and in Example 310— there are no binding thermal constraints. We postpone discussion of this other form of congestion to Section 12.

9. **In a congested network the grid operator collects toll revenues. What incentive remains to reduce congestion by increasing network admittances?** Again refer to Example 310 in Table 2. Toll revenues for each link are given (twice) at Line 85. Marginal values of link conductances are listed at Line 74. These are computed from shadow prices arising in the REX optimization. The details of the computation are described in Section 4, but are not necessary to an understanding of the following argument.

One can as well assume that we have performed the tedious computation of adjusting, one at a time, each conductance by one unit and carrying out the OPF optimization to find the change in payoff.

At Line 98 in Table 3 are listed "total values" of conductance. The term is a bit misleading. The "total value" for a link is simply link conductance marginal value times conductance. If we suppose that the grid operator pays a competitive rent to the supplier of a certain link conductance, then the total value for that link is his total payment to that supplier.

Link-by-link comparison of toll revenue and conductance value reveals that the grid operator who pays competitive market rents for his components makes *zero profit on each link*. At the optimum the grid operator, were he in charge of choosing conductance levels, has no incentive to either to increase or decrease link conductances and consequent levels of congestion. If he were to change a conductance, the new optimum would display the very same equality of grid income and expense. The grid operator who receives and pays competitive prices is completely neutral.

If the link suppliers are in charge of choosing levels of conductance then competitive behavior on their part would lead to expansion (contraction) if their own marginal costs of suppling conductance were less (greater) than the marginal value of conductance which is the price they receive. Will such adjustments lead to an optimum network? We examine this question in Section 10.

10. **Optimum networks.** Even putting aside the fact that our whole analysis deals only with steady state systems— not a very useful context in which to study long-run investment— the above analysis does not deal adequately with network topology. Where no link exists no revenue accrues to the owner (if such he can be called) of this non-link. To be sure, there may well be an O–D toll across this link, but the revenue is distributed according to the impact of this O–D traffic on existing links. Cost impacts on a non-existing link are zero. Toll revenues therefore provide no incentive to this "owner" to expand his link from zero admittance.

Signalling failures on links yet to be built are perhaps not surprising. Do these tolls work properly to motivate owners of *existing* links to expand or contract? Intuition suggests that capacity decisions based on such signals ought to work properly *in the small*. They will lead to *locally* optimal networks, but not necessarily to globally optimal networks unless— as is unlikely— the grand social payoff function of link capacities is convex.

11. **How do buyers and sellers of real and reactive power behave in a bilateral system?** In Table 4 Example 390 we have a grid in which admittance is a complex link characteristic consisting of conductance (the G matrix) and susceptance (the H matrix). Power is also complex and both of its components, real and reactive power, are valuable and costly to produce. How do profitable bilateral deals relate marginal costs and values? The loss ratio for i to j transmission is the complex voltage ratio $\mathbf{r}_{ij} = \mathbf{v}_j/\mathbf{v}_i$. The real and imaginary parts of this ratio we denote r_{ij} and $i\hat{r}_{ij}$. Let c and $\hat{c}$ denote marginal costs of real and reactive power, and t and $\hat{t}$ denote tolls on real and reactive power. Rewriting marginal cost rules (5) and (6) we have

$$c_i = r_{ij}c_j + \hat{r}_{ij}\hat{c}_j - t_{ij} \tag{5'}$$

and

$$\hat{c}_i = -\hat{r}_{ij}c_j + r_{ij}\hat{c}_j - \hat{t}_{ij}. \tag{6'}$$

Thus in Example 390 we have for a 1-3 contract

$$\mathbf{r}_{1,3} = 7.756e^{i1.417}/10e^{i1.481} = .7740 + i(-.0196)$$

and

$$\begin{bmatrix} .7740 & -.0196 \\ .0496 & .7740 \end{bmatrix} \cdot \begin{bmatrix} 54.403 \\ 72.185 \end{bmatrix} - \begin{bmatrix} 19.511 \\ 11.118 \end{bmatrix} = \begin{bmatrix} 19.045 \\ 47.435 \end{bmatrix}.$$

TABLE 4. Example 390

```
TABLE 4.  EXAMPLE   390
 Type GH-PQ
=================================================
INPUTS  INPUTS  INPUTS  INPUTS
=================================================  6.0 < Voltage Magnitude <   10.0

- 1.5 < Phase Angles <  1.5

Thermal limit on total flow or loss? 0 or 1?
  PSI =        1.00

Real Value Parameters: Intercept, Slope, and Role
N1            0.00        0.05        1.00
N2            0.00        0.15        1.00
N3          100.00        0.10       -1.00

Reactive Value Parameters: Intercept & Slope (Role=0)
N1           20.00        0.10        0.00
N2           40.00        0.10        0.00
N3          100.00        0.10        0.00

Conductance, Susceptance, & Thermal Capacity--  PSI= 1
N1    N1             27.000         -20.00          0.000
N1    N2            -15.000           8.00       1000.000
N1    N3            -12.000          12.00       1000.000
N2    N1            -15.000           8.00       1000.000
N2    N2             23.000         -21.00          0.000
N2    N3             -8.000          13.00       1000.000
N3    N1            -12.000          12.00       1000.000
N3    N2             -8.000          13.00       1000.000
N3    N3             20.000         -25.00          0.000

=================================================
OUTPUTS  OUTPUTS  OUTPUTS  OUTPUTS
=================================================
Real & Reactive Power Flows & Real & Reactive Power Losses
N1    N1              0.000          0.000          0.000          0.000
N1    N2             50.678         62.267          3.345          1.784
N1    N3            330.232        212.079         64.179         64.179
N2    N1            -47.333        -60.482          3.345          1.784
N2    N2              0.000          0.000          0.000          0.000
N2    N3            220.189        179.434         30.270         49.188
N3    N1           -266.052       -147.899         64.179         64.179
N3    N2           -189.919       -130.246         30.270         49.188
N3    N3              0.000          0.000          0.000          0.000

Voltages & Phase Angles
N1             10.000          1.481
N2              9.566          1.500
N3              7.756          1.417

Real & Reactive Node Injections
N1            380.910        274.345
N2            172.856        118.952
N3           -455.972       -278.145

Real & Reactive Node Prices and Real & Reactive Mgl Costs/Values
N1             19.045         47.435         19.045         47.435
N2             25.928         51.895         25.928         51.895
N3             54.403         72.185         54.403         72.185

Real & Reactive Tolls
N1    N1           0.000       0.000
N1    N2           6.703       1.727
N1    N3          19.511      11.118
```

```
N2    N1           -6.971      -1.939
N2    N2            0.000       0.000
N2    N3           13.198      10.071
N3    N1          -26.015     -12.709
N3    N2          -17.248     -11.036
N3    N3            0.000       0.000

MV of Conductance, Susceptance, & Thermal Capacity
N1    N1             0.00       -0.04        0.00
N1    N2            32.44       -4.78        0.00
N1    N3           386.24      347.22        0.00
N2    N1            32.44       -4.78        0.00
N2    N2             0.00       -0.05        0.00
N2    N3           170.33      257.72        0.00
N3    N1           386.24      347.22        0.00
N3    N2           170.33      257.72        0.00
N3    N3             0.00       -0.08        0.00

Real & Reactive Toll Revenue
N1    N1             0.00        0.00        0.00
N1    N2           339.70      107.53      447.23
N1    N3          6443.02     2357.89     8800.92
N2    N1           329.97      117.27      447.23
N2    N2             0.00        0.00        0.00
N2    N3          2906.04     1807.07     4713.12
N3    N1          6921.25     1879.66     8800.92
N3    N2          3275.78     1437.34     4713.12
N3    N3             0.00        0.00        0.00

                 10107.88     3853.38    13961.27

Values of Conductance, Susceptance, Thermal Capacity, & Total
N1    N1             0.00        0.00        0.00        0.00
N1    N2           486.61      -38.28        0.00      448.34
N1    N3          4634.83     4166.59        0.00     8801.42
N2    N1           486.61      -38.28        0.00      448.34
N2    N2             0.00        0.00        0.00        0.00
N2    N3          1362.68     3350.41        0.00     4713.09
N3    N1          4634.83     4166.59        0.00     8801.42
N3    N2          1362.68     3350.41        0.00     4713.09
N3    N3             0.00        0.00        0.00        0.00

Surplus                 38563.994
```

12. **How can power be made to run "uphill"?** "Uphill" in the OPF context means "against the economic pressure". An uphill power flow goes from a high-price node to a lower-price node. Why should this ever be desirable? If it is desirable, how can it be implemented in a decentralized system that depends on the actions of profit-seeking buyers and sellers? Suppose the link from the low-price node to the high-price node has a thermal constraint that makes otherwise optimal power flows infeasible. Loop flow laws prevent the low-price node from freely redirecting its power to reduce the congestion painlessly. Fortunately, unlike traffic flows on congested freeways, forward power flows are cancelled by backflows. The thermal limit constrains the algebraic sum of the two flows. Thus

sometimes one wishes to increase the backflow (in this case from high-price to low-price) in order to reduce the congestion and relieve the low-price node from the costly effects of loop flow.

Example 316 in Table 5 exactly portrays such a situation.[19] Producers at Nodes 1 and 2 transmit power to the consumer at Node 3. Node 1 is a less expensive supplier than Node 2, and at the optimum we see that the price at Node 1 remains lower than at Node 2, yet since the injection at 2 is positive, loop flow says that some of it flows through Node 1 on its way to Node 3— an uphill flow.

Given, then, that optimal power flow sometimes calls for uphill flows, how are such flows to be managed? Oren *et al* (1995) find situations like this nearly fatal for management of bilateral systems. "In a pure bilateral framework such transactions will not be commercially viable, unless an ex-ante subsidy system is instituted." In Example 316 we see that the toll on 2-3 transmissions is, indeed, negative (-.046). They continue, "The problems in designing such a system are obvious." Are positive tolls feasible yet negative ones not? In the system described in this paper for bilateral management buyers and sellers are regularly informed of tolls and they regularly account for them in making their deals.

Example 316 is interesting in some other respects. The marginal value of thermal capacity on Link 1-2 is 0.48, indicating that expansion of this capacity— its cost aside— would be useful. But also notice that— *other things equal*— the marginal value of conductance is negative (-1.12). What guidance do these signals provide for grid investment decisions? The total toll revenue on Link 1-2 is 2.69, exactly the same as the "lq total value" of the link. But if— as in the earlier examples— the grid manager is imagined to pay competitive prices to suppliers of grid components, he must pay a total of 3.81 to the 1-2 thermal capacity supplier and he must *receive* a total of 1.12 from the supplier of 1-2 conductance. The latter supplier is being penalized for providing conductance that is not wanted. If,

[19] This example is a close approximation the one in Oren *et al*(1995). They phrase their example as an H-P model in order to keep it lossless. I have found it more convenient to phrase it as a G-P model with voltage limits so high that real power losses are negligible.

as is surely the case, a single supplier is responsible for both conductance and thermal capacity, then it would be important that he be given *both* marginal signals, since he is providing a two-dimensional commodity.

TABLE 5. Example 316

```
TABLE 5.  EXAMPLE   316
  Type G-P.  This mimics the lossless H-P model of Oren, et al.
================================================
INPUTS  INPUTS  INPUTS  INPUTS
================================================
 6.0 < Voltage Magnitude < 1000.0

- 1.5 < Phase Angles <  1.5

Thermal limit on total flow or loss? 0 or 1?
  PSI =        0.00

Real Value Parameters: Intercept, Slope, and Role
N1            0.00        0.20        1.00
N2            0.00        0.33        1.00
N3            3.30        0.00       -1.00

Reactive Value Parameters: Intercept & Slope (Role=0)
N1            0.00        0.00        0.00
N2            0.00        0.00        0.00
N3            0.00        0.00        0.00

Conductance, Susceptance, & Thermal Capacity--  PSI= 0
N1    N1             1.500          0.00         0.000
N1    N2            -1.000          0.00         8.000
N1    N3            -0.500          0.00        30.000
N2    N1            -1.000          0.00         8.000
N2    N2             4.000          0.00         0.000
N2    N3            -3.000          0.00        20.000
N3    N1            -0.500          0.00        30.000
N3    N2            -3.000          0.00        20.000
N3    N3             3.500          0.00         0.000

================================================
OUTPUTS  OUTPUTS  OUTPUTS  OUTPUTS
==================================================
Real & Reactive Power Flows & Real & Reactive Power Losses
N1    N1             0.000         0.000         0.000         0.000
N1    N2             8.000         0.000         0.006         0.000
N1    N3             7.025         0.000         0.010         0.000
N2    N1            -7.994         0.000         0.006         0.000
N2    N2             0.000         0.000         0.000         0.000
N2    N3            18.138         0.000         0.011         0.000
N3    N1            -7.016         0.000         0.010         0.000
N3    N2           -18.127         0.000         0.011         0.000
N3    N3             0.000         0.000         0.000         0.000

Voltages & Phase Angles
N1            100.710         0.000
N2            100.630         0.000
N3            100.570         0.000

Real & Reactive Node Injections
N1             15.025         0.000
N2             10.144         0.000
N3            -25.142         0.000
```

```
Real & Reactive Node Prices and Real & Reactive Mgl Costs/Values
N1              3.005          0.000          3.005          0.000
N2              3.344          0.000          3.348          0.000
N3              3.300          0.000          3.300          0.000

Real & Reactive Tolls
N1    N1           0.000       0.000
N1    N2           0.336       0.000
N1    N3           0.290       0.000
N2    N1          -0.336       0.000
N2    N2           0.000       0.000
N2    N3          -0.046       0.000
N3    N1          -0.291       0.000
N3    N2           0.046       0.000
N3    N3           0.000       0.000

MV of Conductance, Susceptance, & Thermal Capacity
N1    N1            0.00        0.00        0.00
N1    N2           -1.12        0.00        0.48
N1    N3            4.08        0.00        0.00
N2    N1            2.69        0.00        0.00
N2    N2            0.00        0.00        0.00
N2    N3           -0.28        0.00        0.00
N3    N1            4.08        0.00        0.00
N3    N2           -0.28        0.00        0.00
N3    N3            0.00        0.00        0.00

Real & Reactive Toll Revenue
N1    N1            0.00        0.00        0.00
N1    N2            2.69        0.00        2.69
N1    N3            2.04        0.00        2.04
N2    N1            2.69        0.00        2.69
N2    N2            0.00        0.00        0.00
N2    N3           -0.83        0.00       -0.83
N3    N1            2.04        0.00        2.04
N3    N2           -0.83        0.00       -0.83
N3    N3            0.00        0.00        0.00

                    3.90        0.00        3.86

Values of Conductance, Susceptance, Thermal Capacity, & Total
N1    N1            0.00        0.00        0.00        0.00
N1    N2           -1.12        0.00        3.81        2.69
N1    N3            2.04        0.00        0.00        2.04
N2    N1            2.69        0.00        0.00        2.69
N2    N2            0.00        0.00        0.00        0.00
N2    N3           -0.83        0.00        0.00       -0.83
N3    N1            2.04        0.00        0.00        2.04
N3    N2           -0.83        0.00        0.00       -0.83
N3    N3            0.00        0.00        0.00        0.00

Surplus                 43.415
```

13. **What is the role of reactive power in a "DC" model?** Aiming for simplicity, modelers commonly portray a system in terms of a "lossless network". The simplest of these nicknamed "DC" models assumes that conductance is zero on all links. As a result, the total of net real power losses is zero. In such models power is necessarily driven from node to node by AC voltage phase angle differences. In real-world systems with small

losses, such models are apparently not bad approximations.

Typically the analyst of a DC model pays no attention to injections and flows of reactive power. Yet for the student of economic behavior this neglect can lead to error. In a "lossless" system one can easily slip into the mistake of supposing that all power transactions are lossless. A lossless system is really a zero-sum loss system; some two-sided transactions lose power in transmission, others gain. Determination of proper transmission tolls depends on recognition of these gains and losses. Example 440 in Table 6 applies a DC model (Type H-P) to a two-producer two-consumer network. Optimal complex voltages and injections are listed there. Figure 6 shows optimal real and reactive power flows on links and Figure 7 displays real and reactive origin-destination power flows consistent with those OPF results. These O-D flows are of course not unique; they merely represent one set of bilateral deals that might have led to this optimal dispatch.

Now some arithmetic. In any bilateral transaction the loss ratio (i.e., the voltage ratio) times the dispatched power equals the received power. In the 1-4 transaction we have

$$(\mathbf{v}_j/\mathbf{v}_i)\mathbf{x}_{ij} = -\mathbf{x}_{ji}$$

or

$$(9.410e^{.002i}/9.405e^{.712i})(422+122i) = 400-182i.$$

Real power loss in this transaction is 22. Similar calculations for the 1-3 and the 2-3 flows show *negative* losses of -4 and -18, respectively, exactly countering the loss on 1-2. These local power gains and losses could be avoided by simply *defining* the loss ratio to be unity everywhere. For some purposes this might be acceptable, but certainly not for approximating a real near-lossless system. The tolls on real power are negative, keeping all node prices equal. Tolls on reactive power are positive and toll revenues on each link sum to zero. What economic sense does this make? Probably not much, since in this H-P model reactive power is (unrealistically!) a free good.[20]

[20] Sometimes a lossy model of type GH-P is also called a DC model. Chao-Peck (1996) and Schweppe *et al* (1985) and (1988) employ such models.

TABLE 6. Example 440

```
TABLE 6.  EXAMPLE   440
  Type H-P
================================================
INPUTS  INPUTS  INPUTS  INPUTS
================================================
 5.0 < Voltage Magnitude < 10.0

- 1.5 < Phase Angles < 1.5

Real Value Parameters: Intercept, Slope, and Role
N1            0.00        0.10        1.00
N2            0.00        0.30        1.00
N3           80.00        0.05       -1.00
N4          100.00        0.10       -1.00

Reactive Value Parameters: Intercept & Slope (Role=0)
N1            0.00        0.00        0.00
N2            0.00        0.00        0.00
N3            0.00        0.00        0.00
N4            0.00        0.00        0.00

Conductance, Susceptance 1
N1    N1              0.000         -12.00
N1    N2              0.000           4.00
N1    N3              0.000           0.00
N1    N4              0.000           8.00
N2    N1              0.000           4.00
N2    N2              0.000         -14.00
N2    N3              0.000          10.00
N2    N4              0.000           0.00
N3    N1              0.000           0.00
N3    N2              0.000          10.00
N3    N3              0.000         -16.00
N3    N4              0.000           6.00
N4    N1              0.000           8.00
N4    N2              0.000           0.00
N4    N3              0.000           6.00
N4    N4              0.000         -14.00

================================================
OUTPUTS  OUTPUTS  OUTPUTS  OUTPUTS
==================================================
Real & Reactive Power Flows & Real & Reactive Power Losses
N1    N1              0.000          0.000          0.000          0.000
N1    N2            138.832         39.155          0.000         58.810
N1    N3              0.000          0.000          0.000          0.000
N1    N4            461.168        170.452          0.000        341.612
N2    N1           -138.832         19.655          0.000         58.810
N2    N2              0.000          0.000          0.000          0.000
N2    N3            338.832         58.158          0.000        141.414
N2    N4              0.000          0.000          0.000          0.000
N3    N1              0.000          0.000          0.000          0.000
N3    N2           -338.832         83.257          0.000        141.414
N3    N3              0.000          0.000          0.000          0.000
N3    N4            -61.168         -3.725          0.000          7.271
N4    N1           -461.168        171.160          0.000        341.612
N4    N2              0.000          0.000          0.000          0.000
N4    N3             61.168         10.996          0.000          7.271
N4    N4              0.000          0.000          0.000          0.000

Voltages & Phase Angles
N1              9.405         0.712
N2              9.142         0.296
N3              9.278        -0.115
N4              9.410         0.002
```

```
Real & Reactive Node Injections
N1            600.000        209.608
N2            200.000         77.813
N3           -400.000         79.531
N4           -400.000        182.156

Real & Reactive Node Prices and Real & Reactive Mgl Costs/Values
N1             60.000          0.000          60.000          0.000
N2             60.000          0.000          60.000          0.000
N3             60.000          0.000          60.000          0.000
N4             60.000          0.000          60.000          0.000

Real & Reactive Tolls
N1    N1          0.000        0.000
N1    N2         -6.640       23.544
N1    N3        -19.898       43.538
N1    N4        -14.453       39.103
N2    N1         -3.528      -24.917
N2    N2          0.000        0.000
N2    N3         -4.175       24.325
N2    N4         -0.892       17.889
N3    N1        -18.796      -44.734
N3    N2         -5.803      -23.616
N3    N3          0.000        0.000
N3    N4          0.433       -7.105
N4    N1        -14.498      -39.064
N4    N2         -4.205      -16.887
N4    N3         -1.242        6.908
N4    N4          0.000        0.000

Real & Reactive Toll Revenue
N1    N1           0.00         0.00         0.00
N1    N2        -921.86       921.86         0.00
N1    N3           0.00         0.00         0.00
N1    N4       -6665.25      6665.25         0.00
N2    N1         489.74      -489.74         0.00
N2    N2           0.00         0.00         0.00
N2    N3       -1414.68      1414.68         0.00
N2    N4           0.00         0.00         0.00
N3    N1           0.00         0.00         0.00
N3    N2        1966.16     -1966.16         0.00
N3    N3           0.00         0.00         0.00
N3    N4         -26.47        26.47         0.00
N4    N1        6686.22     -6686.22         0.00
N4    N2           0.00         0.00         0.00
N4    N3         -75.97        75.97         0.00
N4    N4           0.00         0.00         0.00

                  18.95       -18.95         0.00

Surplus              36000.000
```

$600.0 + 209.6i$				$200.0 + 77.8i$
$\downarrow$				$\downarrow$
(1)	$\rightarrow$	$138.8 + 39.2i$		$\rightarrow$**(2)**
$\downarrow$		$138.8 - 19.7i$		$\downarrow$
$461.2 + 170.5i$				$338.8 + 58.2i$
$461.2 - 171.2i$				$338.8 - 83.3i$
$\downarrow$				$\downarrow$
(4)	$\rightarrow$	$61.2 - 11.0i$		$\rightarrow$**(3)**
$\downarrow$		$61.2 + 3.7i$		$\downarrow$
$400.0 - 182.2i$				$400.0 - 79.5i$

Figure 6. Link Power Flows in Example 440

$600.0 + 209.6i$			$200.0 + 77.8i$
$\downarrow$			$\downarrow$
(1)			**(2)**
$\downarrow$	$\searrow$		$\downarrow$
$421.9 + 122.5i$			$200.0 + 77.8i$
$400.0 - 182.2i$			$217.6 - 8.7i$
	$178.1 + 87.1i$		
	$182.2 - 71.1i$		
$\downarrow$		$\searrow$	$\downarrow$
(4)			**(3)**
$\downarrow$			$\downarrow$
$400.0 - 182.2i$			$400.0 - 79.5i$

Figure 7. A Set of Optimal O–D Transactions for Example 440

Suppose reactive power *is* a valued commodity as in the H-PQ Example 460 in Table 7 which differs only in that respect from Example 440. Now we find that tolls on real and reactive transmissions are generally non-zero and link revenues usually positive. There are still no real losses of course, but now the reactive losses count, and admittance investments have value. Even in "lossless" models there is a case for transmission tolls.

TABLE 7. Example 460

```
TABLE 7.  EXAMPLE  460
  Type H-PQ. Compare to EX440.
=================================================
INPUTS  INPUTS  INPUTS  INPUTS
=================================================
 5.0 < Voltage Magnitude <   10.0

- 1.5 < Phase Angles <  1.5

Real Value Parameters: Intercept, Slope, and Role
N1            0.00        0.10        1.00
N2            0.00        0.30        1.00
N3           80.00        0.05       -1.00
N4          100.00        0.10       -1.00

Reactive Value Parameters: Intercept & Slope (Role=0)
N1           20.00        0.10        0.00
N2           60.00        0.10        0.00
N3           50.00        0.10        0.00
N4           30.00        0.10        0.00

Conductance, Susceptance 1
N1    N1              0.000          -12.00
N1    N2              0.000            4.00
N1    N3              0.000            0.00
N1    N4              0.000            8.00
N2    N1              0.000            4.00
N2    N2              0.000          -14.00
N2    N3              0.000           10.00
N2    N4              0.000            0.00
N3    N1              0.000            0.00
N3    N2              0.000           10.00
N3    N3              0.000          -16.00
N3    N4              0.000            6.00
N4    N1              0.000            8.00
N4    N2              0.000            0.00
N4    N3              0.000            6.00
N4    N4              0.000          -14.00

=================================================
OUTPUTS  OUTPUTS  OUTPUTS  OUTPUTS
===================================================
Real & Reactive Power Flows & Real & Reactive Power Losses
N1    N1              0.000           0.000          0.000          0.000
N1    N2             50.147          51.684          0.000         12.965
N1    N3              0.000           0.000          0.000          0.000
N1    N4            310.804          63.612          0.000        125.807
N2    N1            -50.147         -38.719          0.000         12.965
N2    N2              0.000           0.000          0.000          0.000
N2    N3            199.333          -9.937          0.000         51.464
N2    N4              0.000           0.000          0.000          0.000
N3    N1              0.000           0.000          0.000          0.000
```

```
N3    N2           -199.333          61.401           0.000           51.464
N3    N3              0.000           0.000           0.000            0.000
N3    N4              4.078         -43.949           0.000            3.841
N4    N1           -310.804          62.195           0.000          125.807
N4    N2              0.000           0.000           0.000            0.000
N4    N3             -4.078          47.790           0.000            3.841
N4    N4              0.000           0.000           0.000            0.000

Voltages & Phase Angles
N1              10.000           0.400
N2               8.798           0.257
N3               9.194           0.008
N4               9.991           0.000

Real & Reactive Node Injections
N1             360.951         115.296
N2             149.187         -48.656
N3            -195.255          17.452
N4            -314.882         109.985

Real & Reactive Node Prices and Real & Reactive Mgl Costs/Values
N1              36.095          31.530          36.095          31.530
N2              44.756          55.134          44.756          55.134
N3              70.237          51.745          70.237          51.745
N4              68.512          40.998          68.512          40.998

Real & Reactive Tolls
N1    N1          0.000        0.000
N1    N2         -4.034       22.092
N1    N3          5.409       37.107
N1    N4         11.041       32.826
N2    N1          0.960      -25.508
N2    N2          0.000        0.000
N2    N3         13.057       15.364
N2    N4         18.699        9.635
N3    N1        -20.856      -35.051
N3    N2        -15.731      -11.169
N3    N3          0.000        0.000
N3    N4          3.881       -6.643
N4    N1        -22.957      -25.973
N4    N2        -18.078       -4.031
N4    N3         -3.527        6.140
N4    N4          0.000        0.000

Real & Reactive Toll Revenue
N1    N1           0.00         0.00         0.00
N1    N2        -202.29      1141.79       939.50
N1    N3           0.00         0.00         0.00
N1    N4        3431.53      2088.14      5519.67
N2    N1         -48.15       987.65       939.50
N2    N2           0.00         0.00         0.00
N2    N3        2602.61      -152.67      2449.94
N2    N4           0.00         0.00         0.00
N3    N1           0.00         0.00         0.00
N3    N2        3135.73      -685.80      2449.94
N3    N3           0.00         0.00         0.00
N3    N4          15.83       291.97       307.79
N4    N1        7135.03     -1615.36      5519.67
N4    N2           0.00         0.00         0.00
N4    N3          14.38       293.41       307.79
N4    N4           0.00         0.00         0.00

                8042.33      1174.56      9216.89

Surplus               26383.431
```

14. **Duality.** The real and imaginary parts of a complex variable $\mathbf{x} = x + i\hat{x}$ are reversed by the operation $ix^* = \hat{x} + ix$. Applying this operation to all the parts of the power injection equation $\mathbf{x} = \mathbf{v} \times (\mathbf{Y}\mathbf{v})^*$ we have

$$
\begin{aligned}
i\mathbf{v}^* \times (i\mathbf{Y}^* i\mathbf{v}^*)^* &= i^3\mathbf{v}^* \times (\mathbf{Y}^*\mathbf{v}^*)^* \\
&= -i\mathbf{v}^* \times (\mathbf{Y}\mathbf{v}) \\
&= -i\mathbf{v} \times (\mathbf{Y}\mathbf{v})^* \\
&= -(\hat{x} + ix)
\end{aligned}
$$

In words: exchange admittance & (negative) conductance, real & imaginary voltage, and economic parameters for real & reactive power and the result is that real and reactive power and all of their corresponding economic outputs from the OPF optimization will be found to have exchanged roles. The theory is completely symmetric with respect to real and reactive power. In a slight abuse of our model classification shorthand, a GH-PQ model is the dual of an HG-QP model. The popular lossless DC model of type H-P is the dual of the (ever used?) lossy model of type G-Q. Only in our asymmetric treatment of thermal losses (and in the REX model, the specification of "role") does the duality fail. Is this duality anything more than a curiosity? Might its recognition in some instances provide some insight?

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APPENDIX

The GAMS listing of the OPF program REX

```
*REX.GMS   AN OPTIMAL POWER FLOW ROUTINE
*          Output is listed in file REX.DAT.
*=========================================================
*  PASTE INPUT HERE in the fashion of template  EX300.TXT.
*Example 390 --- for instance
*  Type GH-PQ

SETS
  I /N1*N3/;
ALIAS(I,J);
SCALAR EXNUMBER / 390 /;
SCALAR EPSI / .0001 /;
SCALAR  VOLTTOP / 10  /;
SCALAR  VOLTBOT / 6 / ;
SCALAR ANGLELIM / 1.5  /;
SCALAR PSI  /1/;
PARAMETERS
  ID(I)
        /N1     1
         N2     1
         N3    -1 /
  A(I)  cost or value linear part above zero
        /N1     0
         N2     0
         N3   100 /
  B(I)  cost or value quadratic part
        /N1    .05
         N2    .15
         N3    .10 /
  AR(I)  /N1 20,N2 40,N3 100 /
  BR(I)  /N1 .10,N2 .10,N3 .10 /
  BBR(I) /N1 0,N2 0,N3 0 /;

TABLE C(I,J) thermal capacities of links
                  N1          N2         N3
         N1        0        1000       1000
         N2     1000           0       1000
         N3     1000        1000          0      ;
TABLE Y(I,J) admittance matrix
                  N1          N2         N3
         N1       27         -15        -12
         N2      -15          23         -8
         N3      -12          -8         20       ;
TABLE YR(I,J)
                  N1          N2         N3
         N1      -20           8         12
         N2        8         -21         13
         N3       12          13        -25;
*=========================================================
VARIABLES
  V(I)       voltage magnitude at node i
  T(I)       voltage phase angle in radians at node i
  U(I,J) real link ij current component from the conductance matrix G
  UR(I,J) real link ij current component from the susceptance matrix B
  W(I,J) reactive link ij current component from the G matrix
  WR(I,J) reactive link ij current component from the B matrix
  X(I)       real power injection at node i
  XR(I)      reactive power injection at node i
  Z          surplus (net social welfare);
```

```
* Upper and lower bounds imposed on voltage magnitudes and phase angles:
V.LO(I)=VOLTBOT;
*  If VOLTBOT is set too low the gams optimization sometimes hangs.
V.UP(I)=VOLTTOP;
*  Without this top limit surplus is unbounded.
T.LO(I)=-ANGLELIM;
T.UP(I)=ANGLELIM;

EQUATIONS
  ROLE(I)           Identifies node i as a producer consumer or neither
  LU(I,J)           These four equations constrain link current and
  LUR(I,J)          link voltage drops (both real and reactive) to
  LW(I,J)           conform to available link admittance (both conductance
  LWR(I,J)          and susceptance) Useful shadow prices result.
  INJECTION(I)      Real power injection at node i
  RINJECTION(I)     Reactive power injection at node i
  SURPLUS           Objective function (net social welfare)
  THERMAL(I,J)      Thermal constraint on link ij (see definition of PSI) ;

ROLE(I)    ..     X(I)*ID(I) =G= 0;
LU(I,J) ..        U(I,J)/(EPSI+COS(T(I)-T(J))) =E= Y(I,J);
LUR(I,J) ..       UR(I,J)/(EPSI+SIN(T(I)-T(J))) =E= YR(I,J);
LW(I,J) ..        W(I,J)/(EPSI+SIN(T(I)-T(J))) =E= Y(I,J);
LWR(I,J) ..       WR(I,J)/(EPSI+COS(T(I)-T(J))) =E= YR(I,J);

INJECTION(I) .. X(I) =E=V(I)*SUM(J,V(J)*(U(I,J)+UR(I,J)));
RINJECTION(I).. XR(I)=E=V(I)*SUM(J,V(J)*(W(I,J)-WR(I,J)));

SURPLUS ..  Z =E=  SUM(I,-A(I)*X(I) -.5*B(I)*SQR(X(I)))
                  +SUM(I,-AR(I)*XR(I)-.5*BR(I)*SQR(XR(I)))
                  +SUM(I,                          -.5*BBR(I)*X(I)*XR(I) );

* The next constraint is just REALFLOW(I,J)-REALFLOW(J,I). See below.
*  Written this way to save no. of equations in the optimization.
*  PSI=1 or 0 as you want thermal constraint on true link loss or on
*   total link flow, resp.
THERMAL(I,J) ..  C(I,J) =G= V(I)*
                 (
                   Y(I,J)*( V(J)*COS(T(I)-T(J))-V(I)
                          )+
                  YR(I,J)*( V(J)*SIN(T(I)-T(J))
                          )
                 ) + PSI*V(J)*

                 (
                   Y(J,I)*( V(I)*COS(T(J)-T(I))-V(J)
                          )+
                  YR(J,I)*( V(I)*SIN(T(J)-T(I))
                          )
                 ) ;

MODEL OPFLOW /ALL/;

SOLVE OPFLOW USING DNLP MAXIMIZING Z;
* Optimization is now complete. (DNLP is the name of the discrete
*   non-linear programming solver.)  Everything that follows below
*   is merely manipulation of the outputs for purposes of presentation.
* The GAMS term "PARAMETER", which we are compelled to use, is a misnomer;
*   these are just outputs.

PARAMETER MC(I)  marginal costs and values of real power;
MC(I) =A(I)+B(I)*X.L(I);
PARAMETER MCR(I)  marginal costs and values of reactive power;
MCR(I) =AR(I)+BR(I)*XR.L(I);
```

```
*  The four prices below are shadow prices from the injection equations.
*  They must be equal to marginal costs/values only if injection is non zero.
PARAMETER PRICE(I) real node price;
PRICE(I)=-injection.m(I);
PARAMETER RPRICE(I) reactive node price;
RPRICE(I)=-rinjection.m(i);

PARAMETER REALFLOW(I,J)  real power injected into link ij ;
REALFLOW(I,J) = V.L(I)*
                 (
                  Y(I,J)*( V.L(J)*COS(T.L(I)-T.L(J))-V.L(I)
                         )+
                 YR(I,J)*( V.L(J)*SIN(T.L(I)-T.L(J))
                         )
                 );

PARAMETER REACFLOW(I,J)   reactive power injected into link ij;
REACFLOW(I,J) = V.L(I)*
                 ( Y(I,J)*( V.L(J)*SIN(T.L(I)-T.L(J))
                         )-
                  YR(I,J)*( V.L(J)*COS(T.L(I)-T.L(J))-V.L(I)
                         )
                 );
*  The following two statements give per unit tolls on real and reactive
*    power flows from origin i to destination j.  Nodes i and j need not
*    be connected.  These tolls encompass opportunity costs and loss
*    spillovers from all sources of congestion.
PARAMETER REALTOLL(I,J)  transmission toll per unit of real poweri to j;
  REALTOLL(I,J) =(V.L(J)/V.L(I))*
                 ( COS(T.L(J)-T.L(I))*PRICE(J)
                  +SIN(T.L(J)-T.L(I))*RPRICE(J) )
                 -PRICE(I);
PARAMETER REACTOLL(I,J)  transmission toll per unit of reactive power i to j;
  REACTOLL(I,J) =(V.L(J)/V.L(I))*
                 (-SIN(T.L(J)-T.L(I))*PRICE(J)
                  +COS(T.L(J)-T.L(I))*RPRICE(J) )
                 -RPRICE(I);


*  Link revenues from tolls:
PARAMETER LINKREVA(I,J);
 LINKREVA(I,J)=REALFLOW(I,J)*REALTOLL(I,J);
PARAMETER LINKREVR(I,J);
 LINKREVR(I,J)=REACFLOW(I,J)*REACTOLL(I,J);
PARAMETER REVBOTH(I,J);
 REVBOTH(I,J)=LINKREVA(I,J)+LINKREVR(I,J);
PARAMETER TOLLREVA;
 TOLLREVA=.5*SUM((I,J),LINKREVA(I,J));
PARAMETER TOLLREVR;
 TOLLREVR=.5*SUM((I,J),LINKREVR(I,J));
PARAMETER GRIDREV;
 GRIDREV=-SUM(I,X.L(I)*MC(I) + XR.L(I)*MCR(I));

*  Link power losses, both real and reactive
PARAMETER LOSS(I,J)  real power losses on link ij;
  LOSS(I,J) = REALFLOW(I,J) + REALFLOW(J,I);
PARAMETER LOSSR(I,J) reactive power losses on link ij;
  LOSSR(I,J) = REACFLOW(I,J) + REACFLOW(J,I);

* The following four "parameters" are shadow prices that represent the
*   marginal values of link attributes:  thermal capacity, conductance
*   and thermal capacity together, and susceptance.  Succeeding statements
*   give value totals.
PARAMETER MVCAP(I,J);
 MVCAP(I,J)=-THERMAL.M(I,J);
PARAMETER MVCAPT(I,J);
```

```
 MVCAPT(I,J)=-C(I,J)*THERMAL.M(I,J);
PARAMETER MVCDUCCAP(I,J) marginal value of conductance and thermal;
 MVCDUCCAP(I,J)=-LU.M(I,J)-LU.M(J,I)+LU.M(I,I)+LU.M(J,J)
              -LW.M(I,J)-LW.M(J,I);
PARAMETER MVSUSCEP(I,J) marginal value of susceptance;
 MVSUSCEP(I,J)=LUR.M(I,J)+LUR.M(J,I)
              +LWR.M(I,J)+LWR.M(J,I)-LWR.M(I,I)-LWR.M(J,J);

* The following five lines separate mgl value of conductance from the
*  mgl value of thermal capacity.  Why this works is a bit mysterious,
*  but explicit tedious computation of marginal values verifies
*  that the procedure is correct.
PARAMETER TCONDUC(I,J);
 TCONDUC(I,J)=(-MVCDUCCAP(I,J)*Y(I,J))-MVCAPT(I,J);
 TCONDUC(I,I)=0;
PARAMETER MVCONDUC(I,J);
 MVCONDUC(I,J)=-TCONDUC(I,J)/Y(I,J);

PARAMETER TSUSCEP(I,J);
 TSUSCEP(I,J)=MVSUSCEP(I,J)*YR(I,J);
 TSUSCEP(I,I)=0;
PARAMETER TOTAL(I,J);
 TOTAL(I,J)=TCONDUC(I,J)+TSUSCEP(I,J)+MVCAPT(I,J);

*PARAMETER MVCONDUC(I,J) marginal value of conductance;
*  MVCONDUC(I,J)=-TCONDUC(I,J)/(EPSI+Y(I,J));
PARAMETER AMPS(I,J);
   AMPS(I,J)=V.L(I)*(COS(T.L(I))*Y(I,J)- SIN(T.L(I))*YR(I,J) );

*  From here on everything is instruction to the printer.  One reads the
*    results of a run by calling up the text file REX.DAT.

```