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Power-Grid Decentralization

Bart McGuire

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University of California Energy Institute
2539 Channing Way
Berkeley, California 94720-5180
www.ucei.berkeley.edu/ucei

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Bart McGuire
University of California Energy Institute
and
Goldman School of Public Policy UCB

ABSTRACT

Most of the literature concerned with decentralized operations in power grids has focussed on partitioning systems initially into two main parts, the transmission system and the market of users (the "ISO" and the "PX"), and asking how the actions of these two subgroups can be properly coordinated. While a number of studies have gone on to examine decentralized operations in the user market of producers and consumers, little attention has been given to how the operation of the transmission system itself might be decentralized. In this paper we propose an organizational procedure for coordinating the actions of the managers of given subgrids of a larger system. Non-invasive price-quantity dialogs are defined that, under fairly general circumstances, bring about perfect short-run coordination of the aggregate grid.

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Bart McGuire
University of California Energy Institute
and
Graduate School of Public Policy UCB

1. INTRODUCTION

Most of the literature concerned with decentralized operations in power grids has focussed on partitioning systems initially into two main parts, the transmission system and the market of users (the ISO and the PX), and asking how the actions of these two subgroups can be properly coordinated.[1] While a number of studies have gone on to examine decentralized operations in the user market of producers and consumers,[2] little attention has been given to how the operation of the transmission system itself might be decentralized.[3]

In this paper we propose an organizational procedure for coordinating the actions of the managers of given subgrids of a larger system. Although the context we assume is a very special one, we believe that the procedure can easily be extended to other contexts. In Section 2 two styles of price-quantity coordination are described, one of which is built upon in the body of the paper. In Section 3 we define an analytical tool, the ”link-impact matrix”, and we show its uses in determining appropriate congestion tolls in a power grid. Section 4 describes certain decompositions of this matrix which are relevant to the variety of grid decentralization patterns discussed in Sections 5, 6, and 7. Section 8 discusses some generalizations, and Section 9 some open questions.

Methodology. Effort has been exerted to be explicit in describing the working of the

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[1] Hogan (1992), Oren, *et al* (1995), Wu, *et al* (1996), Chao and Peck (1996), McGuire (1997a)

[2] Studies of market power, such as Borenstein and Bushnell (1998), fall into this class.

[3] A recent exception is Cadwalader, *et al* (1998). Other candidates are the studies of “transmission contract rights”. Coordination, however, is hardly the emphasis in these discussions. They do not contemplate anything other than centralized running of the grid (unless it be anarchy, which is little concerned with coordination.)

proposed procedures. Although a number of assertions are made about behavior of procedures, no proofs are attempted. The conclusions are plausible enough but lacking proofs we have relied on numerical examples. So that the serious reader can satisfy himself about our assertions by chasing through these not-always-simple numerical examples without too much tedium we have provided help. With each example will be found (usually in the Appendix) (1) a Matlab "notebook" page which defines the example and all the associated computations, and (2) the outputs from (1). Where the text description of procedure is unclear the reader can find a perfectly explicit description in the notebook. He can also easily vary some of the parameters to investigate his own similar example. None of the computations are long but few are quite pencil-and-paper calculations.

2. THE RAILROAD MODEL AND THE BLM MODEL

Policy debates in the last few years suggest that there are probably many equally good ways to organize the management of a power grid. For the analysis here it is useful to consider two models at rather opposite extremes which, for convenience, we nickname the "railroad model" and the "BLM model". In the railroad model the offerer of transmission services posts *prices*[4] on transport (*i.e.,* transmission) from a to b, c to d, *etc.* The market responds with transport damands. If demands cause unexpected congestion or unanticipated costs, railroad management boosts or lowers prices accordingly.[5] Demand responds, *etc.* , *etc.* [6] In the BLM model the Bureau of Land Management posts *quantities* of its services, the use of federally owned off-shore oil tracts, to be made available for lease. The market (*i.e.,* oil companies) responds by offering prices (in auctions) for the posted lands. Depending on the prices offered BLM counters by speeding up or slowing down the rate at which lands are put up for lease.

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[4] To avoid confusion between node price and transmission price we use the term "toll" for the latter.

[5] ICC here abolished by theoretical license.

[6] A fuller description of the possible working of such systems in the electric power context is given in McGuire(1997a) and (1997b).

All such systems— railroad or BLM— are *iterative:* they proceed step-by-step toward some (good or less-than-good) imagined equilibrium outcome. If the world changes slowly enough and and organization adjustment is fast, this equilibrium may in fact be reached. If change is fast and adjustment slow, equilibrium is a perpetually moving target, never reached, but perhaps worth chasing.

The current California power market clearly falls nearer to the BLM end of the organization spectrum in its style of operation: the market speaks prices, the grid speaks quantities. For the analysis in this paper we have found it a bit easier, nevertheless, to tell our story in terms of the railroad model. While we believe the results are relevant across the organizational spectrum, more work remains to be done to make a compelling case. Section 9 discusses this matter further.

Grid managers in this paper are benevolent: their actions are chosen to maximize social welfare rather than profits accruing to grids. This, again, is a convenient beginning point. Interestingly, however, the following analysis lays out a framework that may be quite appropriate for starting a study of competition among overlapping grids.

3. TOLL IMPUTATION AND THE LINK-IMPACT MATRIX

Treatments of power-grid management aimed at wide audiences often begin with what almost amounts to a proclamation of the difference between power flows in a transmission grid and vehicular traffic flows in a road network or, say, gas flows in a pipeline. Power flows must obey (both!) Kirchhoff's Laws. And therefore, unlike flows in other networks, the precise route that a given power transmission is to travel cannot be specified. This emphasis on difference is understandable for readers from the world of operations research.[7] Students of road congestion should, however, find this familiar territory. Electric current

[7] In the large body of OR literature on network flows that developed in the sixties and seventies mention of electricity networks and loop flows is surprisingly rare *e.g.*, Ford and Fulkerson(1962), Lawler(1976), Busacker and Saaty(1965). Rockafellar's otherwise comprehensive text (1984) at least recognizes electricity networks as falling outside the scope of his book. A recent elementary text by Dolan and Aldous (1993) is almost unique in its integrated treatment of the OR network literature *and* electric network theory.

flows along parallel paths to equate voltage drops; vehicular traffic divides along parallel roads to equate travel time. Both flows are governed by Kirchhoff's current law *and* his voltage law.[8]

A useful tool for the analysis of the distribution of power flows in a grid is the *link-impact matrix* (our term) defined below and more precisely in Appendix I. For a given transmission grid this matrix specifies exactly how the power transmitted from source A to destination B is divided among all parallel routes from A to B. This link-impact matrix is the central concept for all the analysis in this paper.

For the limited purposes of this analysis, a power grid is completely characterized by its *admittance matrix* Y and its *thermal capacity matrix* C. In an n-node grid Y is $n \times n$ and y_{ij} is the admittance of the link connecting node i with node j ($y_{ij} = 0$ if i and j are not connected). Admittance ordinarily is a complex quantity $x_{ij} + iy_{ij}$, the first component— the real part— representing *conductance* and the second component— the imaginary part— representing *susceptance*. In this paper we confine attention to lossless grids, so conductance must always be zero. Suppressing the imaginary i we will represent admittance then by the real scalar y_{ij}. By convenient convention y_{ij} is assigned a positive value and $y_{ii} = -\sum_{j \neq i} y_{ij}$. For simplicity we will also assume that $y_{ij} = y_{ji}$. The unit of admittance is the *mho* and where useful (mostly in diagrams) we indicate a mho by the symbol Y (appropriate anyway, but also the nearest thing in our font to an upside-down capital Omega). In the $n \times n$ capacity matrix C the element c_{ij} indicates the maximum power flow permissible through link ij; the unit of power (again, mostly for diagrams) denoted W. (Convention: $c_{ii} = 0$. Assumption: $c_{ij} = c_{ji}$.) In this paper the admittance matrix information will most often be presented in a network diagram.

As a function of the admittance matrix Y we may now define what we have chosen to call the *link-impact matrix* $M(Y)$ or simply M. For this purpose we must distinguish between *link* power flows across a link connecting one node to another and *origin-destination* or,

[8] Beckmann, *et al* (1955), Potts and Oliver(1972), Ferris and Pang(1997)

simply, O–D power flows from one node to another. The interpretations of these two terms are obvious; we may only note that, of course, O–D flows can occur between two nodes not connected by a single link. An element $m_{hi,jk}$ of link-impact matrix M indicates the power-flow impact on link hi of a one-unit increase in O–D flow jk.[9]

As described, M is a highly redundant matrix of size $n^2 \times n^2$, but of rank at most $n-1$, and often even less since (most importantly for the dicussion in Section 4!) not all node pairs are usually connected. We will sometimes find it convenient nevertheless to represent M in its redundant form and sometimes in its most compact form, confining both hi link pairs and jk O–D pairs to that set of directly connected pairs fg with $f < g$. In this latter use, the impacts of omitted O–D pairs can easily be constructed (by column operations) from those that are not omitted.

Even in simple examples M is difficult enought to compute by hand.[10] The Matlab function $M = LIMP(Y)$ in Appendix I should be useful to the reader who wishes to explore his own examples. The function can be adjusted to output either the redundant or the compact form of M. In most of what follows the context will indicate which form is appropriate.

Suppose we have a vector X of O–D flows as might be demanded in response to the announcement of a set of transmission prices. The implied vector x of link flows is then $x = MX'$. Suppose that some of the implied link flows exceed the thermal capacities defined in C. Tolls are imposed on these "congested" links to reduce flows appropriately. In a traffic network such tolls could be collected on the spot, but in a power network this would be awkward if not impossible. Let t be the chosen link-toll vector. Then an *imputed* vector T of O–D tolls is given by the product $T = tM$. These imputed tolls exacted from each unit O–D flow exactly replicate the desired link tolls; each O–D flow is charged for

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[9] The computation of M is described in McGuire(1997a) and in Appendix I. Strictly speaking, M relates *current* flows rather than power flows, but in a lossless system this distinction may be ignored.

[10] The rules of the relevant arithmetic are these: (1) Power moves along parallel paths in proportion to path admittances, (2) paths of admittances a and b connected in parallel have a combined admittance $a + b$, and (3) the same paths connected in series have a combined admittance $(a^{-1} + b^{-1})^{-1}$.

the sum of its dollar-valued link-flow impacts on the congested links.[11]

Example 3A. Define a simple four-node grid by its admittance matrix

$$Y = \begin{pmatrix} -6 & 2 & 4 & 0 \\ 2 & -6 & 0 & 4 \\ 4 & 0 & -6 & 2 \\ 0 & 4 & 2 & -6 \end{pmatrix}.$$

The corresponding compact link-impact matrix $M = LIMP(Y)$ is

$$M = \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} 4/6 & 1/6 & 3/6 & -3/6 & -1/6 & 2/6 \\ 2/6 & 5/6 & 3/6 & 3/6 & 1/6 & -2/6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2/6 & 1/6 & 3/6 & 3/6 & 5/6 & 2/6 \\ 2/6 & -1/6 & 3/6 & -3/6 & 1/6 & 4/6 \end{pmatrix} \end{array}.$$

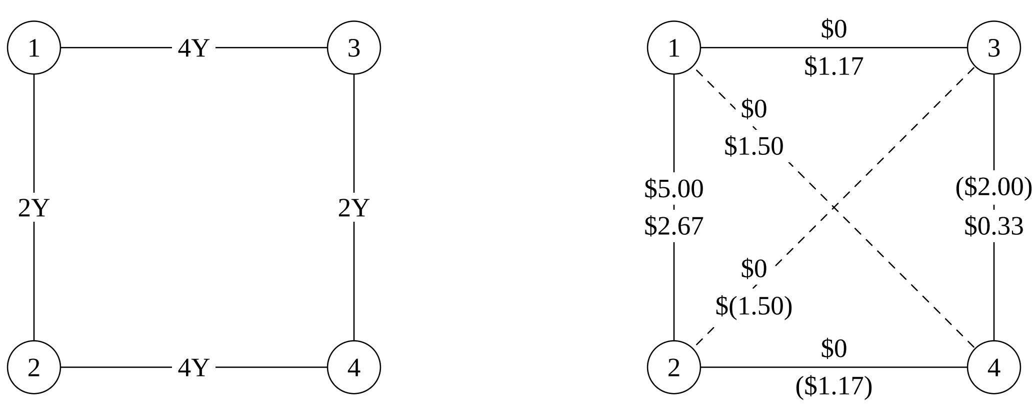

Figure 3.1. A Four-Node Grid: Admittances and Link & O–D Tolls

[11] Much discussion has centered on the difficulty/impossibility of defining a notion of capacity for a specific O–D (What should I call it? "Path" has other connotations.) "track". And then, consequently, of imposing appropriate O–D tolls on tracks that are in some sense "congested". Our discussion would suggest that all of these concerns are empty or beside the point. Under proper congestion pricing, a track toll goes up when the links which it impacts become more congested. A track toll can be negative if its impact on a congested link is opposite to the congesting flow.

The first display in Figure 3.1 shows this grid with link admittances; the second display shows the imputed O–D tolls corresponding to a *given* vector of link tolls. For each node-pair the given link toll is the top number and the O–D toll is the bottom number. A positive number indicates a toll *in the direction of lower-to-higher node index*, a negative number a toll in the opposite direction. Thus, for example, the top number ($2.00)— parentheses indicating a negative number— on the 3-4 line indicates that the link toll on link 3-4 is negative (*i.e.,* the 4-3 link toll is positive. The dashed lines show tolls for node-pairs not directly linked; these O–D tolls are found by simply adding the O–D tolls on parallel paths.

In an adjustment system following the "railroad" style of operation, the grid manager would (1) observe power flows, (2) raise/lower link tolls where flows were above/below link capacities, (3) employ M to find and then announce imputed O–D tolls, (4) return to step (1) with the newly demanded O–D flows *unless* flow changes are less than some predetermined small threshhold, in which case adjustment to the current environment is complete.[12]

Students of the current California system may object that our model is too strictly bilateral. "Scheduling Coordinators"— the marketing organizations that assemble deals between buyers and sellers of power— do not ordinarily submit their transmission demands in terms of specific O–D flows. Rather, they submit balanced or unbalanced vectors of node injections. (An injection vector $x = (x_1, .., x_n)$ is *balanced* if $\sum x_i = 0$.) Balanced submissions are easily incorporated in the model without any extension; the impact of unbalanced submissions on congestion calls for additional information, namely, the locations of compensating demands and supplies. We restrict attention here to balanced submissions. The treatment is as follows. If $x = (x_{11}, .., x_{ij}, .., x_{nn})$ and $z = (z_{11}, .., z_{ij}, .., z_{nn})$ are vectors of O–D flows and $\sum_i x_{ij} = \sum_i z_{ij}$ for all j then $Mx' = Mz'$. In words, vectors of O–D flows impact the grid equally if they give rise to the same vector of nodal

[12] An adjustment process operating under this toll-imputation procedure was described in McGuire(1997a).

injections: electric power is fungible. It follows that a given nodal injection vector is equally well represented by *any* O–D vector consistent with it, in particular by the O–D vector that sends all generation to Node 1 and supplies all consumption from Node 1. Returning to Example 3A, suppose a scheduling coordinator submits a node-injection vector $x = (11, 1, -4, -8)$ for transmission. A consistent O–D vector (in compact form) is $z = (z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}) = (-1, 4, 8, 0, 0, 0)$. Notice the simple relation between z and x. the The resulting link-impact vector is Mz'; the toll on the node-injection package is $tMz' = Tz'$. The scheduling coordinator could be informed merely of the lump-sum charge $T'z$ or, more helpfully for tailoring his package, the whole vector T of O–D tolls.

4. DECOMPOSITION OF THE LINK-IMPACT MATRIX

Although the railroad model and the BLM model envision different adjustment processes, they are identical in their observation of information privacy constraints: The grid manager knows the grid (Y and C) and nothing more; generators/consumers know supply/demand functions and nothing more, and subsequent messages between the two parties convey data that change this initial information endowment minimally.

Of course certain systems (not, with luck, the ones we study!) might consist of elements so intricately intertwined that the "minimal" exchange referred to above becomes a total exchange. In such a case, coordination cannot be achieved without, say, the grid manager knowing all demand and supply functions and operating in a completely centralized fashion. To the degree that achievement of coordination is informationally non-invasive (*i.e.,* consistent with the maintenance of information privacy) we say we have a decentralized system. And the more the privacy maintained the more the decentralization.

The grid/power-market (or ISO/PX) factoring of the grand power system has received a good deal of attention but until recently[13] no one has attempted extending the factoring to the grids themselves. In this section we lay the groundwork for a discussion in the next

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[13] Cadwalader, *et al* (1998)

section of decentralized grid operation. And remembering that grid operation calls for knowledge of the link-impact matrix we start our investigation there.

The link-impact matrix of a grid consisting of disconnected sub-grids, or of sub-grids sharing only one node, would of course be block diagonal. If two sub-grids share exactly two nodes we should expect the link-impact matrix to be *nearly* capable of block diagonalization in some sense.

Figure 4.1. Example 4A: Subgrids A and B with Link Admittances

We start with a numerical example, Example 4A. Figure 4.1 shows a 6-node grid with admittances indicated on each link. Sub-grid A, connecting nodes $A1, .., A3$ is loosely coupled to Sub-grid B, connecting nodes $B1, .., B3$. The two links, $A2 - B3$ and $A3 - B2$ connecting a A and B are supposed to have infinite admittances[14]—call these artificial links *connectors*. The link-impact matrix of the whole system is denoted M. We next construct two new grids A' and B' (Figure 4.2) by augmenting each of the sub-grids with virtual representations of their neighbors. A' is now seen to contain two new nodes $A4$ and $A5$ along with a new link $A4 - A5$ meant to represent Grid B. For proper representation of B, link $A4 - A5$ must have the same admittance as the whole of Grid B *between nodes* $B2$ *and* $B3$. Thus $y_{A4,A5} = 4 + 1/(1/2 + 1/2) = 5$. B' is defined analogously, with

[14] This convenient assumption is simply a way of modeling sub-grids with common nodes.

$y_{B4,B5} = 1 + 1/(1/4 + 1/4) = 3$. The new admittance matrices are denoted $Y_{A'}$ and $Y_{B'}$.

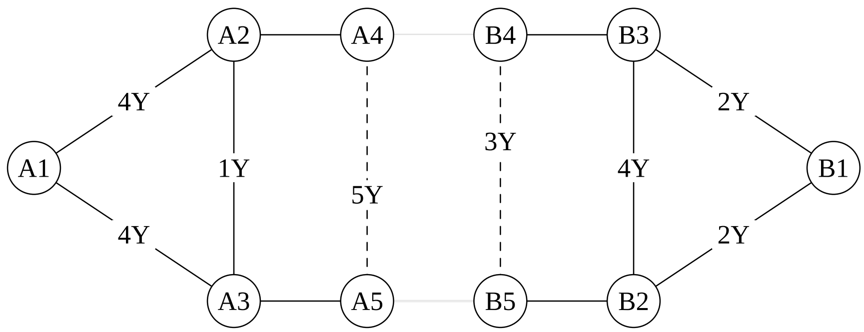

Figure 4.2. Example 4A: Subgrids A' and B' with Virtual Links

Some new notation: Let $Z(Y, i, j)$ be the total admittance between nodes i and j in a grid with admittance matrix Y *when all ij-paths in the grid are included in the calculation.*[15] Let $\bar{A}$ be the subgrid complementary to A (In the present case $\bar{A} = B$ of course.) Now we can write $Z(Y_{\bar{A}}, 2, 3)$ for the admittance of the virtual link in the augmented sub-grid A'. The appropriate admittance values for these virtual links are indicated in the diagrams of Figure 4.2. Let $M_{A'}$ and $M_{B'}$ be the link-impact matrices (compact form[16]) for each of these augmented sub-grids. Define a block-diagonal matrix Q by

$$Q = \begin{bmatrix} M_{A'} & 0 \\ 0 & M_{B'} \end{bmatrix}.$$

Now connect the two submatrices in the following fashion: Let the virtual link in Grid A' be at position a in the list of A' links and the same for B' and b. (In this example and most of the following ones link listing is chosen to make a come last. In this case $a = 4$ and $b = 8$.) Set $Q_{ab} = Z(Y_A, 4, 5)/Z(Y_{\bar{A}}, 4, 5)$ and $Q_{ba} = Z(Y_B, 4, 5)/Z(Y_{\bar{B}}, 4, 5)$. Equivalently, $Q_{ab} = Q_{bb}^{-1}$ and $Q_{ba} = Q_{aa}^{-1}$.

It can now be demonstrated that (after elimination of the rows and columns related to the

[15] In engineering terminology Z is the "one-port driving-point admittance at node-pair ij." Matlab function $ODY(Y, i, j)$ in Appendix I computes these virtual admittances.

[16] The connector links, of no interest, are also eliminated.

uninteresting connector links) the product $Q * Q * Q$ is identical to M. Table 4.1 lists the Matlab notebook instructions to compute all the values in this example. Table 4.2 displays the outputs.[17] The organizational interpretation of this result is the subject of the next section.

Table 4.1. Matlab Notebook Instructions to Compute Example 4A

```
%EXAMPLE OF DECOMPOSITION OF A 6-NODE GRID
%  Calls functions ADMAP (to input admittance data) and LIMP.

%Define Grid A:
va=[4 4 1];
ya=admat(3,va);

%Define Grid B:
vb=[2 2 4];
yb=admat(3,vb);

%Find one-link equivalents:
ea=ody(ya,2,3);
eb=ody(yb,2,3);

%Define Grid A':
Va=[4 4  0 0 1 10^8 0 0 10^8 eb];
Ya=admat(5,Va);

%Define Grid B':
Vb=[2 2 0 0 4 10^8 0 0 10^8 ea];
Yb=admat(5,Vb);

% Compute link-impact matrices:
Ma=limp(Ya);
Mb=limp(Yb);

  % Eliminate the connecting links:
    Ma=Ma([1 2 3 6],:); Ma=Ma(:,[1 2 3 6]);
    Mb=Mb([1 2 3 6],:); Mb=Mb(:,[1 2 3 6]);

%Array Q:
Q=[Ma 0*Ma ;  0*Mb Mb];
Q(4,8)=1/Q(8,8);
Q(8,4)=1/Q(4,4);

Q3=Q*Q*Q;

% Eliminate virtual links to produce M:
v=[1 2 3 5 6 7];
Qm=Q3(v,:);Qm=Qm(:,v);

% NOW THE CENTRALIZED CALCULATION:


v=[4 4 0 0 0 1 0 0 10^8 0 10^8 0 2 2 4];
Y=admat(6,v);
M=limp(Y);
N=M;
```

[17] A less symmetric, more interesting example, Example 4B, is presented in Appendix II.

```
  % Elim connectors:
    M=M([1 2 3 6 7 8],:);
    M=M(:,[1 2 3 6 7 8]);

```

Table 4.2. Computation Output for Example 4A

```
Y =
          8          -4          -4           0           0           0
         -4   100000005          -1           0           0  -100000000
         -4          -1   100000005           0  -100000000           0
          0           0           0           4          -2          -2
          0           0  -100000000          -2   100000006          -4
          0  -100000000           0          -2          -4   100000006
ya = 8    -4    -4         yb = 4    -2    -2
    -4     5    -1              -2     6    -4
    -4    -1     5              -2    -4     6
Ya =
          8          -4          -4           0           0
         -4   100000005          -1  -100000000           0
         -4          -1   100000005           0  -100000000
          0  -100000000           0   100000005          -5
          0           0  -100000000          -5   100000005
Yb =
          4          -2          -2           0           0
         -2   100000006          -4  -100000000           0
         -2          -4   100000006           0  -100000000
          0  -100000000           0   100000003          -3
          0           0  -100000000          -3   100000003
ea = 3    eb = 5
Ma =
    0.6250    0.3750   -0.2500   -0.2500
    0.3750    0.6250    0.2500    0.2500
   -0.0625    0.0625    0.1250    0.1250
   -0.3125    0.3125    0.6250    0.6250
Mb =
    0.5625    0.4375   -0.1250   -0.1250
    0.4375    0.5625    0.1250    0.1250
   -0.2500    0.2500    0.5000    0.5000
   -0.1875    0.1875    0.3750    0.3750
Q =
    0.6250    0.3750   -0.2500   -0.2500         0         0         0         0
    0.3750    0.6250    0.2500    0.2500         0         0         0         0
   -0.0625    0.0625    0.1250    0.1250         0         0         0         0
   -0.3125    0.3125    0.6250    0.6250         0         0         0    2.6667
         0         0         0         0    0.5625    0.4375   -0.1250   -0.1250
         0         0         0         0    0.4375    0.5625    0.1250    0.1250
         0         0         0         0   -0.2500    0.2500    0.5000    0.5000
         0         0         0    1.6000   -0.1875    0.1875    0.3750    0.3750
Qm =
    0.6250    0.3750   -0.2500    0.1250   -0.1250   -0.2500
    0.3750    0.6250    0.2500   -0.1250    0.1250    0.2500
   -0.0625    0.0625    0.1250   -0.0625    0.0625    0.1250
    0.0625   -0.0625   -0.1250    0.5625    0.4375   -0.1250
   -0.0625    0.0625    0.1250    0.4375    0.5625    0.1250
   -0.2500    0.2500    0.5000   -0.2500    0.2500    0.5000
M =
    0.6250    0.3750   -0.2500   -0.1250    0.1250    0.2500
    0.3750    0.6250    0.2500    0.1250   -0.1250   -0.2500
   -0.0625    0.0625    0.1250    0.0625   -0.0625   -0.1250
   -0.0625    0.0625    0.1250    0.5625    0.4375   -0.1250
    0.0625   -0.0625   -0.1250    0.4375    0.5625    0.1250
    0.2500   -0.2500   -0.5000   -0.2500    0.2500    0.5000
```

5. DECENTRALIZED COORDINATION OF INTERCONNECTED GRIDS

In Example 4A the manager of Grid A knows Y_A. He computes $Z(Y_A, 4, 5)$ and communicates this number, the admittance of a one-link virtual representation of his grid, to his counterpart in Grid B. The manager of B does the same. With this once-and-for-all[18] exchange, A can compute $M_{A'}$ and B can compute $M_{B'}$. Since $Z(Y, 4, 5) = Z(Y_A, 4, 5) + Z(Y_{\bar{A}}, 4, 5) = Z(Y_B, 4, 5) + Z(Y_{\bar{B}}, 4, 5)$ each can determine the off-diagonal numbers in Q. So much for the grid information trade: To Manager A Grid B is a single link with a given admittance; Manager B sees Grid A similarly. Now start the adjustment process for a new day (or hour?). A and B post beginning tentative transmission tolls $T^0_{A'}$ and $T^0_{B'}$. The market responds with O–D flow demands. The manager of A determines where congestion does/does not occur in his system and he raises/lowers *link* tolls by arbitrary small amounts to $t^1_{A'}$. With $M_{A'}$ in hand he computes imputed O–D tolls $T^1_{A'} = t^1_{A'} M_{A'}$. Manager B does the same. These parallel independent calculations are replicated by the operation $(T^1_{A'}, T^1_{B'}) = (t^1_{A'}, t^1_{B'})Q$. Before any communication takes place each manager must perform a scaling operation. The previous paragraph described a translation from link-tolls to O–D tolls; and, in particular, a translation from the vector of link-tolls to the O–D toll on the sub-grid's virtual link. We must now reverse the operation. Manager A must now ask, "What hypothetical link-toll on 4-5 would give rise to the given O–D toll T^1_{4-5}?" The answer is $t^2_{4-5} = T^1_{4-5}(Y_A + Y_{\bar{A}})/Y_{\bar{A}}$. This number— a virtual *link*-toll— is now communicated to Manager B who incorporates it in his link-toll vector to form t^2B'. With the parallel communication from B to A we have a new joint link-toll vector which can be written $(t^2_{A'}, t^2_{B'}) = (T^1_{A'}, T^1_{B'})Q$. Finally, the receiver of the new link-toll joins it with his other link-tolls to calculate new O–D tolls: $T^2_{B'} = t^2_{B'}Q$. Summing up (dropping the grid subscripts) we have

$$T^2 = t^2 Q = T^1 Q * Q = t^1 Q * Q * Q = t^1 M.$$

The new imputed O–D tolls are now announced to the market and the adjustment process continues through another iteration.

[18] So long as the grids do not change.

The decentralization process described here fits the simplest non-trivial instance of loosely-coupled sub-grids: two grids, connected at two points. Example 5A in Appendix II is another complete and somewhat more interesting example of this same type. We next consider three grids connected at two points each, and then two grids connected at three points. These two more complicated cases pretty much tell us how the general case can be handled.

6. THREE GRIDS, SIMPLY CONNECTED IN A TRIANGLE

New problems arise in Example 6A, a system of grids connected as in Figure 6.1. Each of the three grids is simply coupled (*i.e.,* at only two nodes) to the rest of the system but the grids are daisy-chained around a loop. To carry out the program of Section 4 Grid A must know the admittance $Y_{\bar{A}}$. But no single manager is in possession of such information. One approach— and the one we shall follow— is to introduce still another grid (a "supergrid") and manager, in this case Grid D. Manager D's initial information consists merely of the grid interconnection pattern with admittances (using the node indexing of Figure 4.2) $(Z(Y_A, 1, 2), Z(Y_B, 1, 3), Z(Y_C, 2, 3))$. Manager D can compute $Z(Y_D, 1, 2) - Z(Y_A, 1, 2)$ (which equals $Y_{\bar{A}}$) and communicate this quantity to A— just what he needs for the Section 4 program.

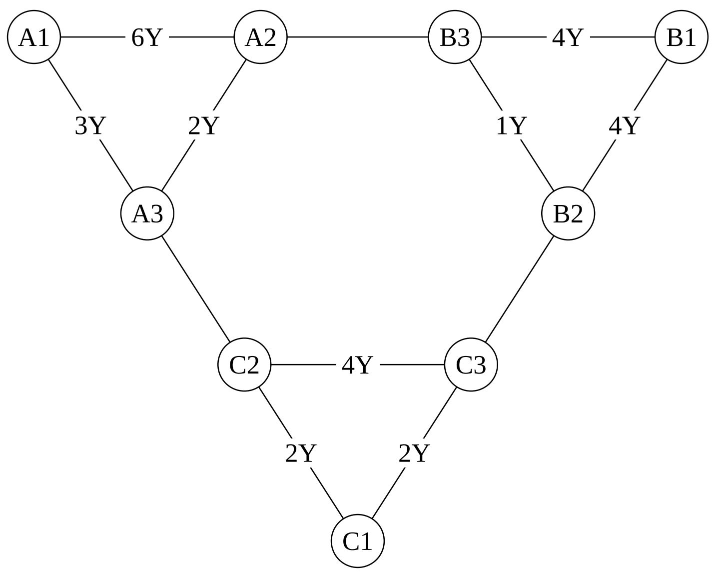

Figure 6.1. Example 6A: Subgrids A, B, and C with Admittances

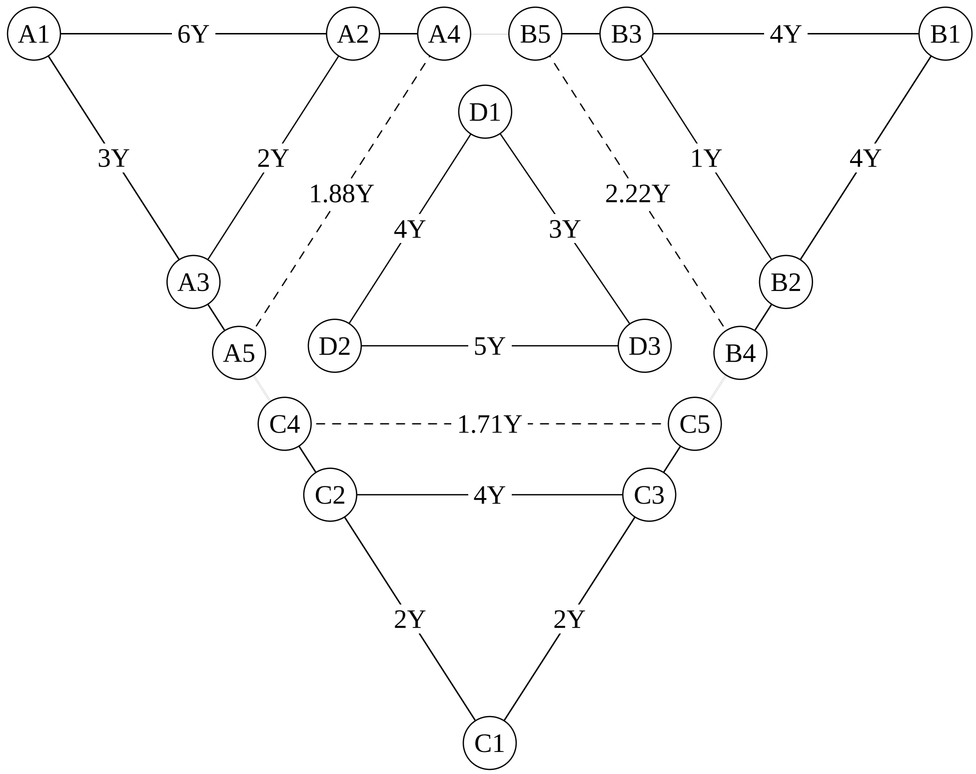

Figure 6.2. Example 6A: Supergrid D and Subgrids A', B',and C with Virtual links

Manager D plays a similar role in the iteration process, acting as an intermediate in the communication of toll information from one grid to another. None of the individual grid managers need know anything about the grand pattern of grid connection; effectively D is the only other grid in the system. The complete details of Example 6A are presented in Appendix III.

7. TWO GRIDS, CONNECTED AT THREE POINTS

Figure 7.1 portrays Example 7A: two five-node grids, A and B, connected at three points. For proper coordination A must obviously know the B-admittance across each of the three pairs of these contact nodes. To put it slightly differently, A must be able (at least) to visualize B as a known three-node grid (and, of course, vice versa). The appropriate three-node virtual grids for A and B are displayed in Figure 7.2.

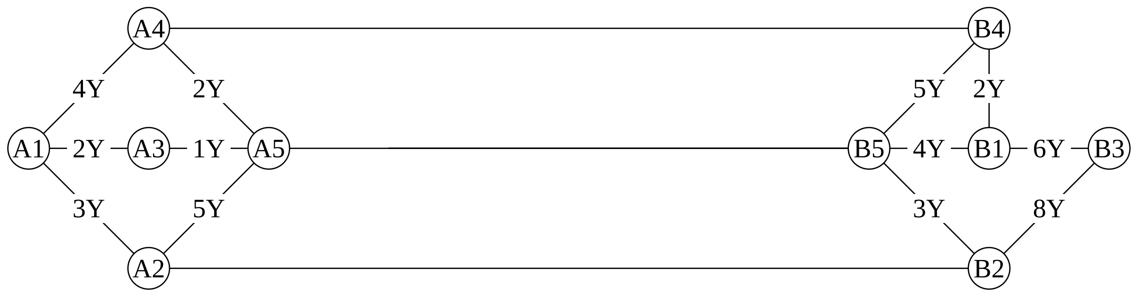

Figure 7.1. Example 7A: Subgrids A and B with Admittances

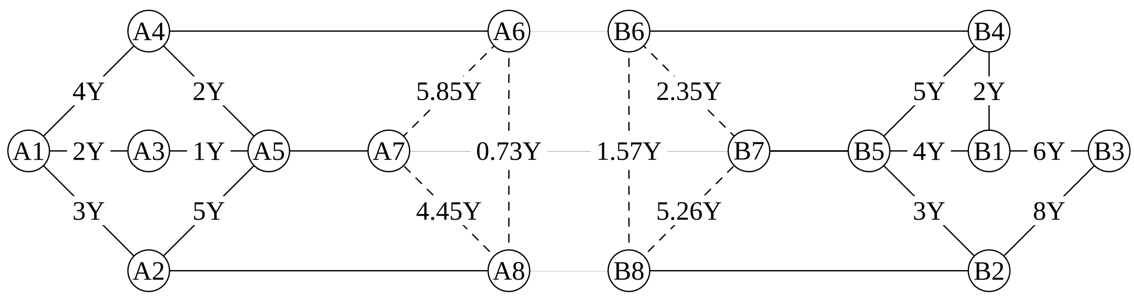

Figure 7.2. Example 7A: Subgrids A' and B' with Virtual links

The construction of such virtual grids is most easily illustrated in an example simpler than that of Figures 7.1 and 7.2.[19]

Figure 7.3(i) shows a four-node network with five links of equal admittance. One can easily verify that the internode, or O–D, admittances (the Z of the previous section) are as shown in Figure 7.3(ii). Figure 7.3(iii) shows the *link* admittances of a three-node system with

[19] Examples sufficiently simple to compute by hand without exhausting the reader's patience are often too symmetric to serve as arithmetic demonstrations of the validity of a computational procedure. Hence the complexity of Example 7A. Proofs would be better of course but....

the same O–D admittances.

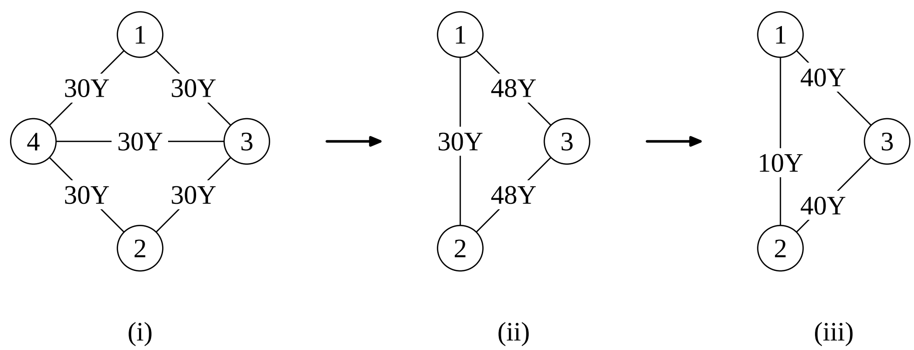

Figure 7.3. A 3-Node Equivalent of a 4-Node

The computation is as follows: Let a, b, and c stand for the three unknown link admittances and A, B, and C for the known O–D admittances. Obviously, then,

$$a + (1/(1/b + 1/c) = A$$

$$b + (1/(1/a + 1/c) = B$$

and

$$c + (1/(1/a + 1/b) = C.$$

This simple quadratic system is easily solved with a few iterations.[20] The correctness of the solution is affirmed by verifying that the O–D admittances of the (iii) triangle are the same as those shown as link admittances in the (ii) triangle.

The ease with which three-node representations can be found is a bit deceptive. Even a four-node representation is much harder to compute. Following the procedure for the three-node system, one can write the problem as a simultaneous system of six non-linear (quartic!) equations. Does a solution to such higher-order systems always exist? Can a simpler solution procedure be found? Both are open questions.

[20] The Matlab function VIRT in the Appendix solves these equations for $n = 3$. $n > 3$ is another matter!

Returning to Example 7A, the 2×5-node system of Figures 7.1 and 7.2, we now set up a near block-diagonal matrix Q in the fashion of Section 5. Since there are now three "extra" rows and columns in M_A (and M_B) instead of just one, the two off-diagonal parts of Q that serve to connect M_A and M_B are now 3×3 matrices. Denote these two by $Q_{A,B}$ and $Q_{B,A}$ (The first is in the NE quadrant of Q and the second in the SW quadrant.) Let $Q_{A,A}$ and $Q_{B,B}$ be those submatrices of M_A and M_B corresponding to virtual nodes. Figure 7.4 illustrates the arrangement.

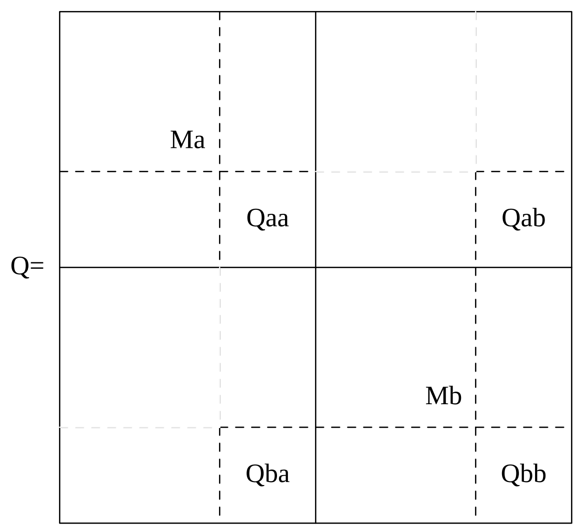

Figure 7.4. Layout of the Q Matrix

The job now is to find values of $Q_{A,B}$ and $Q_{B,A}$ that will achieve perfect coordination. Following the Section 5 example it is tempting to set $Q_{A,B} = Q^{-1}_{B,B}$, but this latter matrix is singular— at most rank 2. It turns out that a so-called "generalized inverse"[21] of $Q_{B,B}$ does the trick. Roughly: let G denote the inverse of a non-singular submatrix of $Q_{B,B}$. Set $Q_{A,B}$ equal to G bordered by zeros. Construct $Q_{B,A}$ analogously.

Now, as in Section 5, we find that $Q * Q * Q = M$ after elimination of rows and columns corresponding to virtual links. The organizational interpretation is that perfect coordi-

[21] A matrix G is a *generalized inverse* of a matrix A if $AGA = A$. Searle (1971)

nation is achieved by Managers A and B exchanging two[22] price signals reflecting their respective congestion costs.

All the computations— both procedures and results— for Example 7A are presented in the Appendix IV. The reader who wants to run his own examples can easily do so by following the instructions given there. Another similar example, Example 7B, is presented in Appendix V.

8. THE GENERAL CASE

The earlier discussion suggests two very different directions in which generalization might be pursued. The first would extend the example in Section 5 (where three grids were connected in a triangle) to many grids connected in more complicated patterns. The second direction would develop communication procedures appropriate to grids managed in the "BLM" style described in Section 2.

In the triangle example of Section 6, in order to keep local grid information local, we introduced the notion of a supergrid with one node for each connecting node of each of the interconnected subgrids. The supergrid itself has no real transmission links, only virtual links. The duty of the supergrid manager is to act toward each given subgrid as if it were the only other subgrid in the system. This interceding role can be performed properly by following the same operational rules as all of the other subgrids. (It means, however, that the supergrid will usually have many connecting nodes to the rest of the whole grid.) Question: In a complicated grid system, what information economies are to be gained by decentralizing the supergrid itself into sub-supergrids and super-supergrids? Such questions may have the ring of mathematical fancy, but surely management of very large systems must involve considerations not too distant from these.

In our decentralized version of the "railroad" model sub-grid managers communicate with

---

[22] Not three, since $Q_{B,B}$ is always singular. For proper construction of M_A it is, of course, still necessary for A to have a triangular representation of Subgrid B.

the market and with other sub-grid managers alike: in terms of transmission tolls. The intent of all of these "messages" at the beginning of an adjustment step is to— in a sense— inform other players about congestion. In obvious analogy, managers in the "BLM" models should communicate with their counterparts just as they do with the market: in terms of O–D power transmissions. And just as local link-impact matrices are used to translate link-tolls into O–D tolls in the railroad model, so the same matrices can be used to translate O–D power flows into link power flows in the BLM model. In loosely coupled systems of sub-grids the same possibilities exist for non-invasive *quantity* communication exchanges as for the non-invasive *price* exchanges that have been demonstrated in the earlier sections of this paper. The generalization suggested— and similar ones for mixed price-quantity systems— should not be difficult.

9. FURTHER QUESTIONS

Lossy Grids. In common with most of the recent economic studies of power grids, the discussion in this paper has treated thermal congestion as the sole source of transmission prices— as indeed it is in a lossless network. Transmission is a free good when thermal congestion is absent. In a lossy grid, in contrast, congestion can take other forms. A given O–D transmission suffers losses that are immediately apparent to the transmitter unless measures are taken by the grid manager to disguise them. But this same transmission imposes losses on *other* transmissions. And, conversely, unless measures (such as imposition of transmission tolls) are taken to make these spillover losses felt by the original transmitter everybody hurts everybody needlessly.

Transmission tolls to correct such inefficiencies are easily designed.[23]. The question here is whether (and how) the decentralized procedures described in this paper can be extended to this *lossy* context. Going one step further: How can the procedures be extended to encompass both real and reactive power management? We suspect that both of these extensions are routine but nonetheless worthy of attention.

---

[23] McGuire (1997a),(1997b)

Aggregation. Effective aggregation suppresses details in a non-costly manner. In this paper we have studied aggregation of *given* sub-grids into virtual smaller representations (at zero cost in terms of coordination). A grander question would ask how best to *choose* the subgrids into which a given larger grid is to be chopped. We have already seen that coordination in some such partitions calls for higher dimensional communications than others, as well as more complicated virtual representations of neighbors. Is dimensionality a good proxy for communication cost in this context? What are the costs of less-than-perfect coordination? If signals more-or-less inadequately indicate sources of congestion (as they surely often must in practice) is physical grid-damage the consequence or simply arbitrary rationing and economic inefficiency?

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Appendix I. The Link-Impact Matrix

I.1. The Link-Impact Calculation

Let an origin-destination (O-D) current flow from node i to node j be denoted x_{ij}. Current flow on link ij will be denoted z_{ij}. The superposition theorem of circuit theory[24] enables us to calculate precisely the unit impact $m_{(ij),(kl)}$ of O-D current flow x_{kl} on the link current flow z_{ij}. The $n^2 \times n^2$ matrix M of these impact coefficients we call the ***link-impact matrix.*** M is a function solely of the network admittance matrix Y and is constructed in the following fashion. Define I^h to be column h of the $n \times n$ identity matrix. Let matrix Y^h be Y with I^h replacing row h and column h. It then can be shown that $m_{(ij),(kl)} = g_k - g_l$ where $g \equiv (Y^i)^{-1} I^i \times y_{ij}$. The sequence of solutions for each ij-pair finally yields the link-impact matrix M. Since M is highly redundant, it will usually suffices for our purposes to deal only with a submatrix (for short, also called M) formed by keeping only those columns $m_{(..),(kl)}$ for which $k < l$ and only those rows $m_{(ij),(..)}$ for which $i < j$.

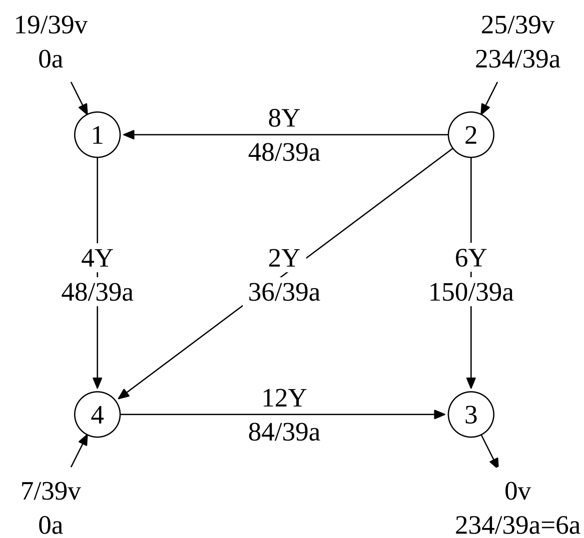

Figure I.1. O–D Voltage Drops = Impacts on Link 2–3

[24] Desoer and Kuh(1969), p. 658*ff*

Illustration. Consider the four-node grid in Figure I.1. Let us use the procedure above to determine O–D impacts on link 2-3. The equations above effectively state that current injections must be zero except at nodes 2 and 3, and that the total current flow from node 2 to node 3 (over all paths) must be equal in value (althought the units are different) to the 2-3 admittance: $6a = 6y$. We solve for the node voltages and link current flows consistent with these constraints. The results are displayed in the Figure. One verifies immediately that the O–D voltage drops in the Figure are exactly equal to the O–D impacts on link 2-3 as displayed in the 2-3 row of $limp(Y)$ below:

$$M = limp(Y) = \begin{array}{c} \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array} \begin{array}{c} \begin{array}{cccccc} 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \begin{pmatrix} 30 & 22 & 19 & -8 & -12 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 17 & 21 & 8 & 12 & 4 \\ -6 & 19 & 12 & 25 & 18 & -7 \\ -3 & 3 & 6 & 6 & 9 & 3 \\ -6 & -20 & 12 & -14 & 18 & 32 \end{pmatrix} \end{array} \times (1/39)$$

Further explanation of this interesting procedure is beyond the scope of this paper.[25]

I.2 The Matlab function LIMP, yielding the link-impact matrix

```
function [M]=limp(y)

%   Gives the link-impact matrix M as a function of
%   the admittance matrix y.
%   This is the compact version; uses the fn "uptri".
%   Related voltages are in the matrix Z, if wanted.

ny=size(y,1);
yI=eye(ny);
M=[];
Z=[];
 for i=1:ny,
  for j=1:ny,
   bb=yI(:,j);
   yy=y;
   yy(:,j)=bb;
   yy(j,:)=bb.';
   z=inv(yy)*yI(:,i)*(-y(i,j));
    % z is a vector of node voltages.
   m=z*ones(size(z))';
```

[25] But see Chen (1976), pp.227-228 and p.229*ff*.

```
    % The kl voltage drop in the matrix mm below
    % exactly represents the
    % impact of unit kl O-D flow on ij link flow.
   mm=m-m.';
   mv=mm(:)*abs((sign(i-j)));
   M=[M mv];
  end
 end
M=-M.';
ML=M;                   % Big version
short=uptri(ny);
ML=M(short,short);      % Compact version

% These remaining steps remove rows and columns corresponding
%  to ij pairs not directly linked.
y0=y~=0;
v=(1:ny)'*ones(1,ny);
v=v>v';
vv=v(:);
r=reshape((1:ny^2),ny,ny);
rr=r.*y0';
rrr=rr(:);
rrrr=select(vv',rrr');
sss=rrrr>0;
s=select(sss,rrrr);
M=M(s,:);
M=M(:,s);               % Compact-compact version
```

I.2.1 The Matlab function UPTRI used in LIMP

```
function [v]=uptri(n);

%  Fill an n by n matrix with 1,..,n^2.  Which numbers
%  are in the upper triangle?

a=reshape([1:n^2],n,n);
a=a';
b=diag(a)*ones(1,n);
c=(a>b);
c=c';
a=a';
v=select(c(:)',a(:)');

```

I.2.2 The Matlab function ODY to compute O–D admittances

```
function z=ody(y,i,j)

% What is the total O--D admittance between nodes i and j
%  in a grid with admittance matrix y?
% The function BIGLIMP is called.  This is the big
%   version link-impact matrix of size n^2 by n^2.

n=size(y,1);

yy=y;
yy(i,j)=yy(i,j)-1;
yy(j,i)=yy(j,i)-1;
yy(i,i)=yy(i,i)+1;
yy(j,j)=yy(j,j)+1;

```

```
m=biglimp(yy);
nij=(i-1)*n+j;

z=( (1-y(i,j))/m(nij,nij) ) - 1;
```

I.2.3 The Matlab function VIRT to compute "sufficient" grid representations

```

function y=virt(e)

% This computes the 3-node virtual grid with O--D
%  admittances given by 3-vector e.

H=[0 1 1;1 0 1;1 1 0];

% The following three items might need tuning sometimes.
N=10;
alpha=1;
c=(1/3)*e*H*e';

for i=1:N
  cc=c*[1 1 1]';
  m=diag(e)*H;
  y=inv(m)*cc;
  y=y';
  oldc=c;
  c=.5*y*H*y';
  c=oldc-alpha*(c-oldc);
end;
y;
```

Appendix II. Example 4B: A 2×5-Grid, Simply Connected

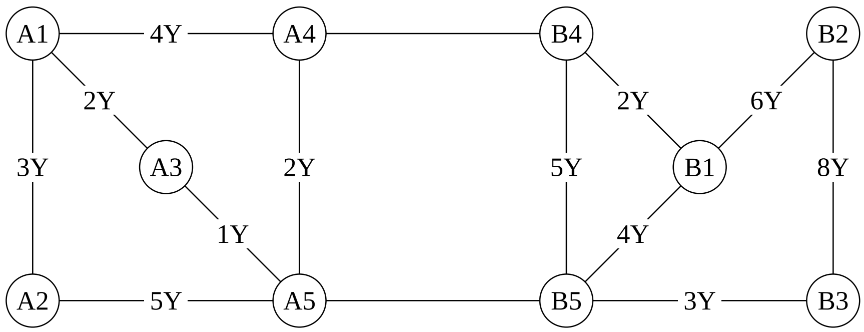

Figure II.1. Subgrids A and B with Admittances

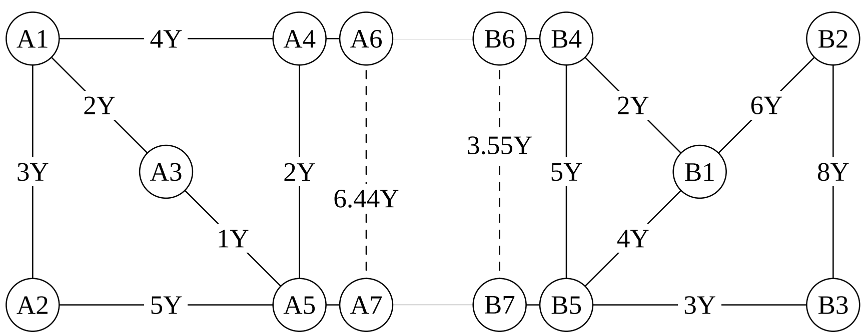

Figure II.2. Subgrids A' and B' with Virtual links

TABLE II.1. Example 4B: Matlab Instructions

```
%EXAMPLE OF DECOMPOSITION OF A 10-NODE GRID
%  Calls functions ADMAT (to input admittance data) and LIMP.

%Define Grid A:
va=[3 2 4 0 0 0 5 0 1 2];
ya=admat(5,va);

%Define Grid B:
vb=[6 0 2 4 8 0 0 0 3 5];
yb=admat(5,vb);

%Find one-link equivalents:
ea=ody(ya,4,5);
eb=ody(yb,4,5);

%Define Grid A':
Va=[3 2 4 0 0 0 0 0 5 0 0 0 1 0 0 2 10^8 0 0 10^8 eb];
Ya=admat(7,Va);

%Define Grid B':
Vb=[6 0 2 4 0 0 8 0 0 0 0 0 3 0 0 5 10^8 0 0 10^8 ea];
Yb=admat(7,Vb);

% Compute link-impact matrices:
Ma=limp(Ya);
Mb=limp(Yb);

  % Eliminate the connecting links:
    Ma=Ma([1 2 3 4 5 6 9],:); Ma=Ma(:,[1 2 3 4 5 6 9]);
    Mb=Mb([1 2 3 4 5 6 9],:); Mb=Mb(:,[1 2 3 4 5 6 9]);

%Array Q:
Q=[Ma 0*Ma ;  0*Mb Mb];
Q(7,14)=1/Q(14,14);
Q(14,7)=1/Q(7,7);

Q3=Q*Q*Q;

% Eliminate virtual links to produce M:
v=[1 2 3 4 5 6 8 9 10 11 12 13];
Qm=Q3(v,:);Qm=Qm(:,v);

% NOW THE CENTRALIZED CALCULATION:


v=[3 2 4 0 0 0 0 0 0 0 0 5 0 0 0 0 0 0 1 0 0 0 0 0 2 0 0 0 10^9 0 0 0 0 0 10^9];
vv=[6 0 2 4 8 0 0 0 3 5 ];
v=[v vv];
Y=admat(10,v);
M=limp(Y);
  % Elim connectors:
    M=M([1 2 3 4 5 6 9 10 11 12 13 14],:);
    M=M(:,[1 2 3 4 5 6 9 10 11 12 13 14]);
```

TABLE II.2. Example 4B: Output

```
ya =
     9    -3    -2    -4     0
    -3     8     0     0    -5
    -2     0     3     0    -1
    -4     0     0     6    -2
     0    -5    -1    -2     8
yb =
    12    -6     0    -2    -4
    -6    14    -8     0     0
     0    -8    11     0    -3
    -2     0     0     7    -5
    -4     0    -3    -5    12
ea =
    3.5541
eb =
    6.4737
Q =
  Columns 1 through 7
    0.5978    0.1188    0.2422   -0.2413    0.2377    0.1143    0.1143
    0.0792    0.7089    0.0861    0.0475   -0.5822    0.0407    0.0407
    0.3229    0.1722    0.6717    0.1938    0.3445   -0.1550   -0.1550
   -0.4022    0.1188    0.2422    0.7587    0.2377    0.1143    0.1143
    0.0792   -0.2911    0.0861    0.0475    0.4178    0.0407    0.0407
    0.0762    0.0407   -0.0775    0.0457    0.0813    0.1994    0.1994
    0.2467    0.1316   -0.2508    0.1480    0.2632    0.6456    0.6456
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0    1.5490
  Columns 8 through 14
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0    2.8214
    0.7924    0.1796    0.2216   -0.1557   -0.4152    0.0420    0.0420
    0.0599    0.3714    0.2245    0.0449    0.1197   -0.1470   -0.1470
    0.1477    0.4490    0.5539    0.1108    0.2954    0.1050    0.1050
   -0.2076    0.1796    0.2216    0.8443   -0.4152    0.0420    0.0420
   -0.2076    0.1796    0.2216   -0.1557    0.5848    0.0420    0.0420
    0.0350   -0.3674    0.1312    0.0262    0.0700    0.4986    0.4986
    0.0249   -0.2612    0.0933    0.0187    0.0497    0.3544    0.3544
Qm =
  Columns 1 through 7
    0.5978    0.1188    0.2422   -0.2413    0.2377    0.1143    0.0080
    0.0792    0.7089    0.0861    0.0475   -0.5822    0.0407    0.0029
    0.3229    0.1722    0.6717    0.1938    0.3445   -0.1550   -0.0109
   -0.4022    0.1188    0.2422    0.7587    0.2377    0.1143    0.0080
    0.0792   -0.2911    0.0861    0.0475    0.4178    0.0407    0.0029
    0.0762    0.0407   -0.0775    0.0457    0.0813    0.1994    0.0140
    0.0160    0.0086   -0.0163    0.0096    0.0171    0.0420    0.7924
   -0.0562   -0.0300    0.0571   -0.0337   -0.0599   -0.1470    0.0599
    0.0401    0.0214   -0.0408    0.0241    0.0428    0.1050    0.1477
    0.0160    0.0086   -0.0163    0.0096    0.0171    0.0420   -0.2076
    0.0160    0.0086   -0.0163    0.0096    0.0171    0.0420   -0.2076
    0.1906    0.1016   -0.1937    0.1143    0.2033    0.4986    0.0350
  Columns 8 through 12
   -0.0842    0.0301    0.0060    0.0160    0.1143
   -0.0300    0.0107    0.0021    0.0057    0.0407
    0.1142   -0.0408   -0.0082   -0.0218   -0.1550
```

```
   -0.0842    0.0301    0.0060    0.0160    0.1143
   -0.0300    0.0107    0.0021    0.0057    0.0407
   -0.1470    0.0525    0.0105    0.0280    0.1994
    0.1796    0.2216   -0.1557   -0.4152    0.0420
    0.3714    0.2245    0.0449    0.1197   -0.1470
    0.4490    0.5539    0.1108    0.2954    0.1050
    0.1796    0.2216    0.8443   -0.4152    0.0420
    0.1796    0.2216   -0.1557    0.5848    0.0420
   -0.3674    0.1312    0.0262    0.0700    0.4986
M =
  Columns 1 through 7
    0.5978    0.1188    0.2422   -0.2413    0.2377    0.1143    0.0080
    0.0792    0.7089    0.0861    0.0475   -0.5822    0.0407    0.0029
    0.3229    0.1722    0.6717    0.1938    0.3445   -0.1550   -0.0109
   -0.4022    0.1188    0.2422    0.7587    0.2377    0.1143    0.0080
    0.0792   -0.2911    0.0861    0.0475    0.4178    0.0407    0.0029
    0.0762    0.0407   -0.0775    0.0457    0.0813    0.1994    0.0140
    0.0160    0.0086   -0.0163    0.0096    0.0171    0.0420    0.7924
   -0.0562   -0.0300    0.0571   -0.0337   -0.0599   -0.1470    0.0599
    0.0401    0.0214   -0.0408    0.0241    0.0428    0.1050    0.1477
    0.0160    0.0086   -0.0163    0.0096    0.0171    0.0420   -0.2076
    0.0160    0.0086   -0.0163    0.0096    0.0171    0.0420   -0.2076
    0.1906    0.1016   -0.1937    0.1143    0.2033    0.4986    0.0350
  Columns 8 through 12
   -0.0842    0.0301    0.0060    0.0160    0.1143
   -0.0300    0.0107    0.0021    0.0057    0.0407
    0.1142   -0.0408   -0.0082   -0.0218   -0.1550
   -0.0842    0.0301    0.0060    0.0160    0.1143
   -0.0300    0.0107    0.0021    0.0057    0.0407
   -0.1470    0.0525    0.0105    0.0280    0.1994
    0.1796    0.2216   -0.1557   -0.4152    0.0420
    0.3714    0.2245    0.0449    0.1197   -0.1470
    0.4490    0.5539    0.1108    0.2954    0.1050
    0.1796    0.2216    0.8443   -0.4152    0.0420
    0.1796    0.2216   -0.1557    0.5848    0.0420
   -0.3674    0.1312    0.0262    0.0700    0.4986
```

Appendix III.) Example 6A: A Daisy-Chained 3×3-Grid

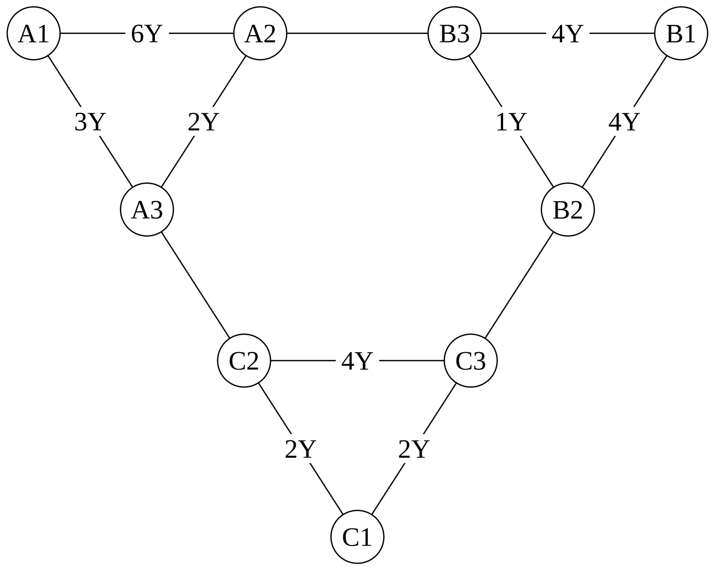

Figure III.1. Subgrids A, B, and C with Admittances

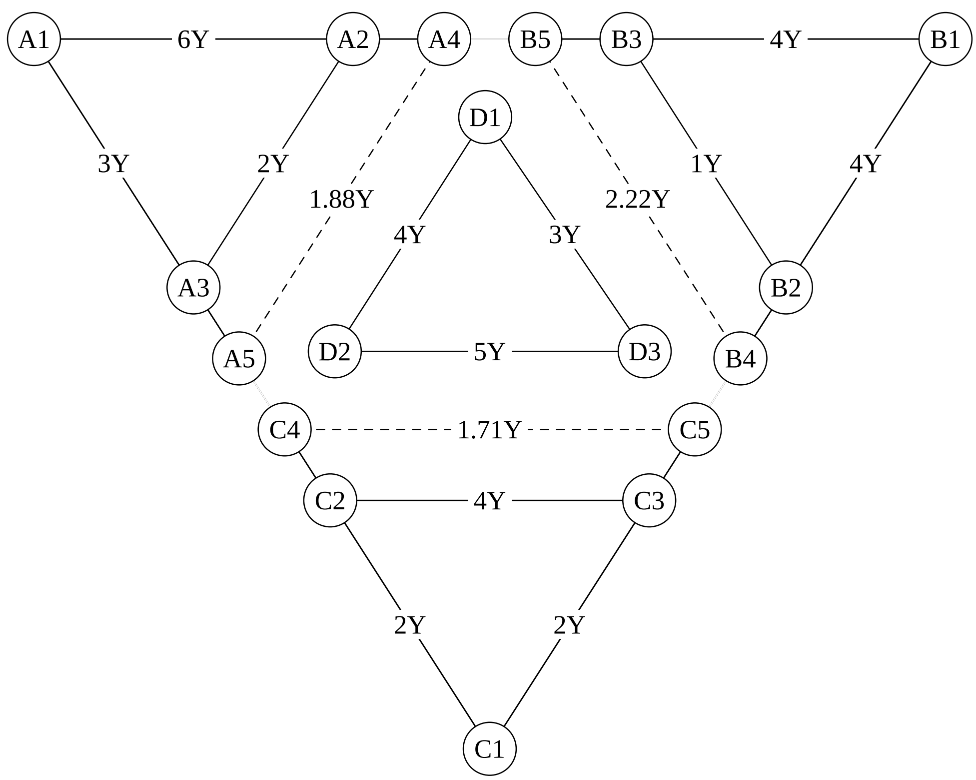

Figure III.2. Subgrids A', B', C' and Supergrid D

TABLE III.1. Example 6A: Matlab Instructions

```
% Example of 3-subgrid decentralization.

Y=[9 -6 -3 0 0 0 0 0 0
  -6 10000008  -2 0 0 -10000000 0 0 0
  -3 -2 10000005 0  0 0 0 -10000000 0
  0 0 0  8 -4 -4 0 0 0
  0 0 0 -4 10000005 -1 0 0 -10000000
  0 -10000000 0 -4 -1 10000005 0 0 0
  0 0 0 0 0 0 4 -2 -2
  0 0 -10000000 0 0 0 -2 10000006 -4
  0 0 0 0 -10000000 0 -2 -4  10000006];
 M=limp(Y);
 h=[1 2 3 6 7 8 10 11 12];
 M=M(h,:);  M=M(:,h);
va=[6 3 2];
vb=[4 4 1];
vc=[2 2 4];
ya=admat(3,va);
yb=admat(3,vb);
yc=admat(3,vc);

ea=ody(ya,2,3);
eb=ody(yb,2,3);
ec=ody(yc,2,3);

Yd=[7 -4 -3;-4 9 -5; -3 -5 8];
Md=limp(Yd);

va=[6 3 0 0 2 10^8 0 0 10^8 1.8750];
Ya=admat(5,va);
vb=[4 4 0 0  1 10^8 0  0 10^8 2.2222];
Yb=admat(5,vb);
vc=[2 2 0 0  4 10^8 0  0 10^8 1.71];
Yc=admat(5,vc);

Ma=limp(Ya);
Mb=limp(Yb);
Mc=limp(Yc);

v=[1 2 3 6];
Ma=Ma(v,v);
Mb=Mb(v,v);
Mc=Mc(v,v);

Q=[Ma  0*Ma  0*Ma; 0*Mb  Mb 0*Mb; 0*Mc 0*Mc Mc];
Q(4,8) =ody(Yd,1,2)/Yd(1,2);
Q(4,12)=Q(4,8);
Q(8,4) =ody(Yd,1,3)/Yd(1,3);
Q(8,12)=Q(8,4);
Q(12,4)=ody(Yd,2,3)/Yd(2,3);
Q(12,8)=Q(12,4);

Q3=Q*Q*Q;
hq=[1 2 3  5 6 7 9 10 11];
Qm=Q3(hq,hq);
```

TABLE III.2. Example 6A: Output

```
ya =
     9    -6    -3
    -6     8    -2
    -3    -2     5
yb =
     8    -4    -4
    -4     5    -1
    -4    -1     5
yc =
     4    -2    -2
    -2     6    -4
    -2    -4     6
ea =
    4.0000
eb =
     3
ec =
    5.0000
Q =
  Columns 1 through 7
    0.7801    0.4397   -0.3404   -0.3404         0         0         0
    0.2199    0.5603    0.3404    0.3404         0         0         0
   -0.1135    0.2270    0.3404    0.3404         0         0         0
   -0.1064    0.2128    0.3191    0.3191         0         0         0
         0         0         0         0    0.6915    0.3085   -0.3830
         0         0         0         0    0.3085    0.6915    0.3830
         0         0         0         0   -0.0957    0.0957    0.1915
         0         0         0   -1.7407   -0.2128    0.2128    0.4255
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0   -1.3429         0         0         0
  Columns 8 through 12
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
   -1.4688         0         0         0   -1.4688
   -0.3830         0         0         0         0
    0.3830         0         0         0         0
    0.1915         0         0         0         0
    0.4255         0         0         0   -1.7407
         0    0.5745    0.4255   -0.1490   -0.1490
         0    0.4255    0.5745    0.1490    0.1490
         0   -0.2981    0.2981    0.5961    0.5961
   -1.3429   -0.1274    0.1274    0.2548    0.2548
Qm =
  Columns 1 through 7
    0.7801    0.4397   -0.3404   -0.1064    0.1064    0.2128   -0.0637
    0.2199    0.5603    0.3404    0.1064   -0.1064   -0.2128    0.0637
   -0.1135    0.2270    0.3404    0.1064   -0.1064   -0.2128    0.0637
   -0.0709    0.1418    0.2128    0.6915    0.3085   -0.3830   -0.0849
    0.0709   -0.1418   -0.2128    0.3085    0.6915    0.3830    0.0849
    0.0355   -0.0709   -0.1064   -0.0957    0.0957    0.1915    0.0425
   -0.0213    0.0426    0.0639   -0.0426    0.0426    0.0852    0.5745
    0.0213   -0.0426   -0.0639    0.0426   -0.0426   -0.0852    0.4255
    0.0852   -0.1703   -0.2555    0.1703   -0.1703   -0.3406   -0.2981
  Columns 8 through 9
    0.0637    0.1274
   -0.0637   -0.1274
   -0.0637   -0.1274
    0.0849    0.1699
   -0.0849   -0.1699
   -0.0425   -0.0849
    0.4255   -0.1490
    0.5745    0.1490
```

```
     0.2981    0.5961
M =
  Columns 1 through 7
    0.7801    0.4397   -0.3404   -0.1064    0.1064    0.2128   -0.0638
    0.2199    0.5603    0.3404    0.1064   -0.1064   -0.2128    0.0638
   -0.1135    0.2270    0.3404    0.1064   -0.1064   -0.2128    0.0638
   -0.0709    0.1418    0.2128    0.6915    0.3085   -0.3830   -0.0851
    0.0709   -0.1418   -0.2128    0.3085    0.6915    0.3830    0.0851
    0.0355   -0.0709   -0.1064   -0.0957    0.0957    0.1915    0.0426
   -0.0213    0.0426    0.0638   -0.0426    0.0426    0.0851    0.5745
    0.0213   -0.0426   -0.0638    0.0426   -0.0426   -0.0851    0.4255
    0.0851   -0.1702   -0.2553    0.1702   -0.1702   -0.3404   -0.2979
  Columns 8 through 9
    0.0638    0.1277
   -0.0638   -0.1277
   -0.0638   -0.1277
    0.0851    0.1702
   -0.0851   -0.1702
   -0.0426   -0.0851
    0.4255   -0.1489
    0.5745    0.1489
    0.2979    0.5957
```

Appendix IV. Example 7A: A 2×5-Grid Triply Connected

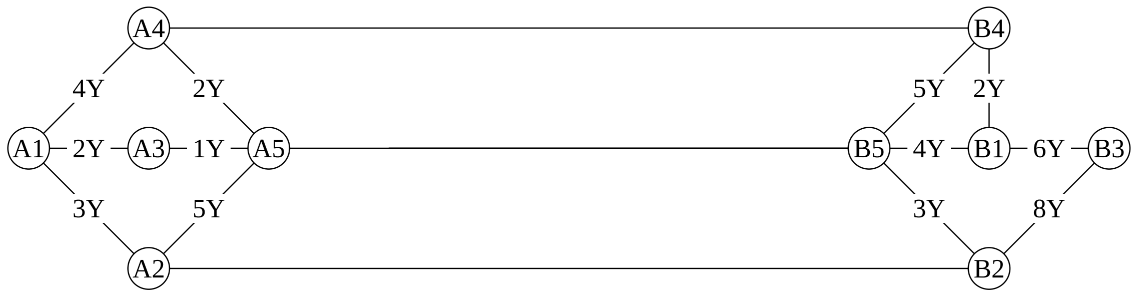

Figure IV.1. Subgrids A and B with Admittances

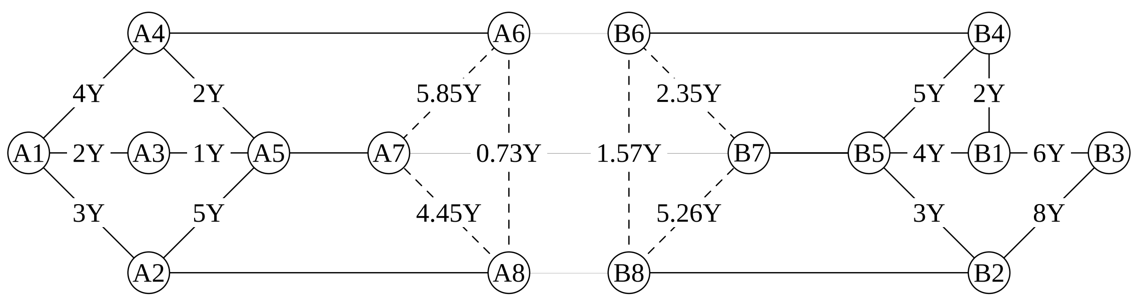

Figure IV.2. Subgrids A' and B' with Virtual links

TABLE IV.1. Example 7A: Matlab Instructions

```

%EXAMPLE OF DECOMPOSITION OF A 10-NODE GRID
%  Calls functions ADMAT (to input admittance data), VIRT, and LIMP.

%Define Grid A:
va=[3 2 4 0 0 0 5 0 1 2];
ya=admat(5,va);

%Define Grid B:
vb=[0 6 2 4 8 0 3 0 0 5];
yb=admat(5,vb);

%Find one-link equivalents:
ea45=ody(ya,4,5);
ea42=ody(ya,4,2);
ea52=ody(ya,5,2);
ea=[ea45 ea42 ea52];

eb45=ody(yb,4,5);
eb42=ody(yb,4,2);
eb52=ody(yb,5,2);
eb=[eb45 eb42 eb52];

ae=virt(ea);
be=virt(eb);

%Define Grid A':
Va=[3 2 4 0 0 0 0 0 0 5 0 0 10^9 0 1 0 0 0 2 10^9 0 0 0 10^9 0];
Va=[Va be];
Ya=admat(8,Va);

%Define Grid B':
Vb=[0 6 2 4 0 0 0 8 0 3 0 0 10^8 0 0 0 0 0 5 10^8 0 0 0 10^8 0];
Vb=[Vb ae];
Yb=admat(8,Vb);

% Compute link-impact matrices:
Ma=limp(Ya);
Mb=limp(Yb);

% Eliminate the connecting links:
    Ma=Ma([1 2 3 4 6 7 10 11 12],:); Ma=Ma(:,[1 2 3 4 6 7 10 11 12]);
    Mb=Mb([1 2 3 4 5 7 10 11 12],:); Mb=Mb(:,[1 2 3 4 5 7 10 11 12]);

%Array Q:

Q=[Ma 0*Ma ;  0*Mb Mb];
Sa=Ma(7:9,7:9);
Sb=Mb(7:9,7:9);

Ta=inv(Sa(1:2,1:2));
Tb=inv(Sb(1:2,1:2));
Ta=[Ta [0;0];[0 0 0]];
Tb=[Tb [0;0];[0 0 0]];

Q(7:9,16:18)=Tb;
Q(16:18,7:9)=Ta;

Q3=Q*Q*Q;

% Eliminate virtual links to produce M:
v=[1 2 3 4 5 6   10 11 12 13 14 15];
Qm=Q3(v,:);Qm=Qm(:,v);

% NOW THE CENTRALIZED CALCULATION:
```

```
v=[3 2 4  0 0 0 0  0 0 0 0  5 0 10^9 0 0 0 0 1 0 0 0 0 0];
vv=[2 0 0 0 10^9 0 0 0 0 0 10^9];
vvv=[0 6  2 4 8 0 3 0 0 5 ];
v=[v vv vvv];
Y=admat(10,v);
M=limp(Y);
  % Elim connectors:
    M=M(  [1 2 3 4 6 7 10 11 12 13 14 15],:);
    M=M(:,[1 2 3 4 6 7 10 11 12 13 14 15]);
```

TABLE IV.2. Example 7A: Output

```
ya =
     9    -3    -2    -4     0
    -3     8     0     0    -5
    -2     0     3     0    -1
    -4     0     0     6    -2
     0    -5    -1    -2     8
yb =
    12     0    -6    -2    -4
     0    11    -8     0    -3
    -6    -8    14     0     0
    -2     0     0     7    -5
    -4    -3     0    -5    12
ea= 3.5541    3.1886    6.2000       eb= 6.4737    3.2559    5.1014
ae= 2.3478    1.5652    5.2609       be= 5.8485    0.7273    4.4545
Q =
  Columns 1 through 7
    0.5330    0.1347    0.2830   -0.1290    0.2693    0.1210    0.1210
    0.0898    0.7064    0.0795    0.0293   -0.5873    0.0396    0.0396
    0.3773    0.1590    0.6376    0.0997    0.3180   -0.1606   -0.1606
   -0.2149    0.0732    0.1246    0.4345    0.1464    0.0950    0.0950
    0.0898   -0.2936    0.0795    0.0293    0.4127    0.0396    0.0396
    0.0807    0.0396   -0.0803    0.0380    0.0791    0.1990    0.1990
    0.2360    0.1157   -0.2348    0.1111    0.2314    0.5819    0.5819
    0.0606    0.0037   -0.0473   -0.0494    0.0075    0.0585    0.0585
    0.1915   -0.0652   -0.1110   -0.3871   -0.1304   -0.0846   -0.0846
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0    3.0626
         0         0         0         0         0         0   -1.6612
         0         0         0         0         0         0         0
  Columns 8 through 14
    0.2500    0.1290         0         0         0         0         0
    0.0103   -0.0293         0         0         0         0         0
   -0.2603   -0.0997         0         0         0         0         0
   -0.3396   -0.4345         0         0         0         0         0
    0.0103   -0.0293         0         0         0         0         0
    0.1610   -0.0380         0         0         0         0         0
    0.4708   -0.1111         0         0         0         0         0
    0.1079    0.0494         0         0         0         0         0
    0.3025    0.3871         0         0         0         0         0
         0         0    0.7040    0.2754    0.3063    0.2220   -0.1758
         0         0    0.0918    0.3369    0.1939   -0.0688    0.0333
         0         0    0.2042    0.3878    0.4998   -0.1531    0.1425
         0         0    0.2960   -0.2754   -0.3063    0.7780    0.1758
         0         0   -0.0879    0.0499    0.1069    0.0659    0.2607
         0         0    0.0258   -0.3574    0.1401   -0.0193    0.0950
  -13.3587         0    0.0121   -0.1678    0.0658   -0.0091    0.0446
```

```
   16.5109         0    0.0539   -0.1379   -0.0119   -0.0404   -0.1063
         0         0    0.1541   -0.0875   -0.1874   -0.1156   -0.4572
  Columns 15 through 18
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0    7.6291   -6.2071         0
         0   -4.1380    7.6717         0
         0         0         0         0
    0.0309    0.0309    0.2067    0.1758
   -0.1430   -0.1430   -0.1762   -0.0333
    0.1120    0.1120   -0.0305   -0.1425
   -0.0309   -0.0309   -0.2067   -0.1758
    0.0570    0.0570   -0.2037   -0.2607
    0.4975    0.4975    0.4025   -0.0950
    0.2336    0.2336    0.1890   -0.0446
    0.1260    0.1260    0.2323    0.1063
   -0.0999   -0.0999    0.3573    0.4572
Qm =
  Columns 1 through 7
    0.5330    0.1347    0.2830   -0.1290    0.2693    0.1210    0.0616
    0.0898    0.7064    0.0795    0.0293   -0.5873    0.0396   -0.0059
    0.3773    0.1590    0.6376    0.0997    0.3180   -0.1606   -0.0557
   -0.2149    0.0732    0.1246    0.4345    0.1464    0.0950   -0.1465
    0.0898   -0.2936    0.0795    0.0293    0.4127    0.0396   -0.0059
    0.0807    0.0396   -0.0803    0.0380    0.0791    0.1990    0.0103
    0.1231   -0.0176   -0.0836   -0.1758   -0.0351    0.0309    0.7040
   -0.0948   -0.0205    0.0814    0.0333   -0.0410   -0.1430    0.0918
   -0.0283    0.0381    0.0022    0.1425    0.0762    0.1120    0.2042
   -0.1231    0.0176    0.0836    0.1758    0.0351   -0.0309    0.2960
   -0.1290    0.0439    0.0748    0.2607    0.0878    0.0570   -0.0879
    0.2017    0.0989   -0.2008    0.0950    0.1978    0.4975    0.0258
  Columns 8 through 12
   -0.1423   -0.0212   -0.0462   -0.1290    0.1210
   -0.0205    0.0190    0.0044    0.0293    0.0396
    0.1628    0.0022    0.0418    0.0997   -0.1606
    0.0832    0.1782    0.1099    0.4345    0.0950
   -0.0205    0.0190    0.0044    0.0293    0.0396
   -0.1430    0.0560   -0.0077    0.0380    0.1990
    0.2754    0.3063    0.2220   -0.1758    0.0309
    0.3369    0.1939   -0.0688    0.0333   -0.1430
    0.3878    0.4998   -0.1531    0.1425    0.1120
   -0.2754   -0.3063    0.7780    0.1758   -0.0309
    0.0499    0.1069    0.0659    0.2607    0.0570
   -0.3574    0.1401   -0.0193    0.0950    0.4975
M =
  Columns 1 through 7
    0.5330    0.1347    0.2830   -0.1290    0.2693    0.1210    0.0616
    0.0898    0.7064    0.0795    0.0293   -0.5873    0.0396   -0.0059
    0.3773    0.1590    0.6376    0.0997    0.3180   -0.1606   -0.0557
   -0.2149    0.0732    0.1246    0.4345    0.1464    0.0950   -0.1465
    0.0898   -0.2936    0.0795    0.0293    0.4127    0.0396   -0.0059
    0.0807    0.0396   -0.0803    0.0380    0.0791    0.1990    0.0103
    0.1231   -0.0176   -0.0836   -0.1758   -0.0351    0.0309    0.7040
   -0.0948   -0.0205    0.0814    0.0333   -0.0410   -0.1430    0.0918
   -0.0283    0.0381    0.0022    0.1425    0.0762    0.1120    0.2042
   -0.1231    0.0176    0.0836    0.1758    0.0351   -0.0309    0.2960
   -0.1290    0.0439    0.0748    0.2607    0.0878    0.0570   -0.0879
    0.2017    0.0989   -0.2008    0.0950    0.1978    0.4975    0.0258
  Columns 8 through 12
   -0.1423   -0.0212   -0.0462   -0.1290    0.1210
   -0.0205    0.0190    0.0044    0.0293    0.0396
    0.1628    0.0022    0.0418    0.0997   -0.1606
```

118	0.0832	0.1782	0.1099	0.4345	0.0950
119	-0.0205	0.0190	0.0044	0.0293	0.0396
120	-0.1430	0.0560	-0.0077	0.0380	0.1990
121	0.2754	0.3063	0.2220	-0.1758	0.0309
122	0.3369	0.1939	-0.0688	0.0333	-0.1430
123	0.3878	0.4998	-0.1531	0.1425	0.1120
124	-0.2754	-0.3063	0.7780	0.1758	-0.0309
125	0.0499	0.1069	0.0659	0.2607	0.0570
126	-0.3574	0.1401	-0.0193	0.0950	0.4975

Appendix V. Example 7B: A 2×6-Grid Triply Connected

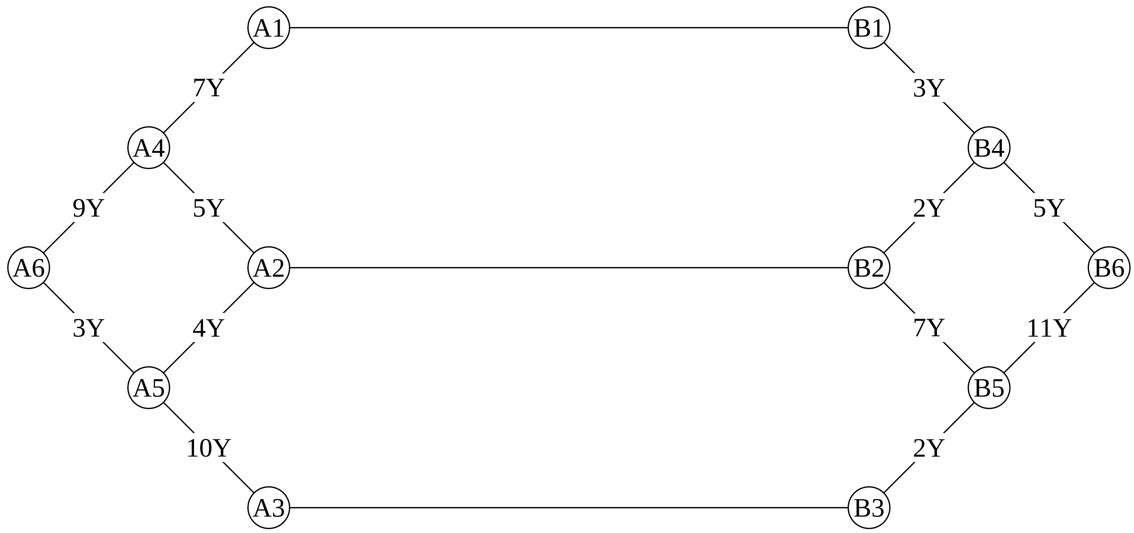

Figure V.1. Subgrids A and B with Admittances

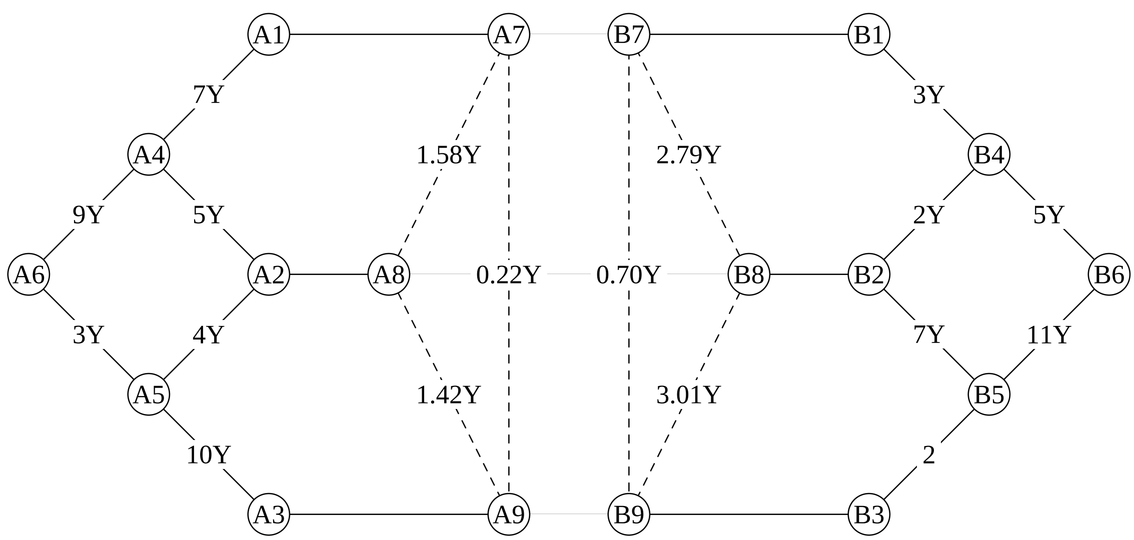

Figure V.2. Subgrids A' and B' with Virtual links

TABLE V.1. Example 7B: Matlab Instructions

```

%EXAMPLE OF DECOMPOSITION OF A 12-NODE GRID
%  Calls functions ADMAT (to input admittance data), VIRT, and LIMP.

%Define Grid A:
va=[0 0 7 0 0 0 5 4 0 0 10 0 0 9 3];
ya=admat(6,va);

%Define Grid B:
vb=[0 0 3 0 0 0 2 7 0 0 2 0 0 5 11];
yb=admat(6,vb);

%Find OD admittances:
ea12=ody(ya,1,2);
ea13=ody(ya,1,3);
ea23=ody(ya,2,3);
ea=[ea12 ea13 ea23];

eb12=ody(yb,1,2);
eb13=ody(yb,1,3);
eb23=ody(yb,2,3);
eb=[eb12 eb13 eb23];

% THE FOLLOWING NUMBERS COME FROM "VIRT":
ae=virt(ea);
be=virt(eb);

%Define Grid A':
Va=[0 0 7 0 0 10^9 0 0  0 5 4 0 0 10^9 0  0 10 0 0 0 10^9  0 9 0 0 0  3 0 0 0 0 0 0 ];
Va=[Va be];
Ya=admat(9,Va);

%Define Grid B':
Vb=[0 0 3 0 0 10^9 0 0   0 2 7 0 0 10^9 0   0 2 0 0 0 10^9    0 5 0 0 0   11 0 0 0 0 0 0];
Vb=[Vb ae];
Yb=admat(9,Vb);

% Compute link-impact matrices:
Ma=limp(Ya);
Mb=limp(Yb);

  % Eliminate the connecting links:
    Ma=Ma(  [1  3 4 6 8 9 10 11 12],:);
    Ma=Ma(:,[1  3 4 6 8 9 10 11 12]);
    Mb=Mb(  [1  3 4 6 8 9 10 11 12],:);
    Mb=Mb(:,[1  3 4 6 8 9 10 11 12]);

%Array Q:

Q=[Ma 0*Ma ;  0*Mb Mb];
Sa=Ma(7:9,7:9);
Sb=Mb(7:9,7:9);

Ta=inv(Sa(1:2,1:2));
Tb=inv(Sb(1:2,1:2));
Ta=[Ta [0;0];[0 0 0]];
Tb=[Tb [0;0];[0 0 0]];

Q(7:9,16:18)=Tb;
Q(16:18,7:9)=Ta;

Q3=Q*Q*Q;

% Eliminate virtual links to produce M:
v=[1 2 3 4 5 6 10 11 12 13 14 15 ];
```

```
Qm=Q3(v,:);
Qm=Qm(:,v);

% NOW THE CENTRALIZED CALCULATION:
U=zeros(6,6);
U(1:3,1:3)=-eye(3)*10^9;
Y=[ya U;U yb];
Y=Y+diag(10^9*[1 1 1 0 0 0 1 1 1 0 0 0 ]);
M=limp(Y);
  % Elim connectors:
    M=M(  [1  3 4 6 8 9 10 11 12 13 14 15],:);
    M=M(:,[1  3 4 6 8 9 10 11 12 13 14 15]);

```

TABLE V.2. Example 7B: Output

```
ya =
     7     0     0    -7     0     0
     0     9     0    -5    -4     0
     0     0    10     0   -10     0
    -7    -5     0    21     0    -9
     0    -4   -10     0    17    -3
     0     0     0    -9    -3    12
yb =
     3     0     0    -3     0     0
     0     9     0    -2    -7     0
     0     0     2     0    -2     0
    -3    -2     0    10     0    -5
     0    -7    -2     0    20   -11
     0     0     0    -5   -11    16
ea =
    3.3542    2.1438    3.5698
eb =
    1.7680    0.9675    1.6103
ae =
    2.7892    0.6954    3.0132
be =
    1.5765    0.2215    1.4161
Q =
  Columns 1 through 7
    0.8308    0.1742    0.0349    0.0174   -0.0348    0.1045    0.6566
    0.1244    0.6310    0.1980    0.0182   -0.1082    0.3247   -0.5065
    0.0199    0.1584    0.5724    0.0779    0.1035   -0.3105   -0.1385
    0.0249    0.0364    0.1948    0.8865    0.0396   -0.1187   -0.0116
   -0.0448   -0.1948    0.2329    0.0356    0.8569    0.4292    0.1501
    0.0448    0.1948   -0.2329   -0.0356    0.1431    0.5708   -0.1501
    0.1479   -0.1597   -0.0546   -0.0018    0.0263   -0.0789    0.3076
    0.0213   -0.0145    0.0197   -0.0156    0.0085   -0.0256    0.0358
    0.0035    0.0509    0.1751   -0.0980    0.0310   -0.0931   -0.0474
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0         0
         0         0         0         0         0         0    5.5788
         0         0         0         0         0         0   -2.8096
         0         0         0         0         0         0         0
  Columns 8 through 14
    0.6741    0.0175         0         0         0         0         0
   -0.3268    0.1798         0         0         0         0         0
    0.3560    0.4945         0         0         0         0         0
   -0.7033   -0.6917         0         0         0         0         0
    0.3473    0.1972         0         0         0         0         0
   -0.3473   -0.1972         0         0         0         0         0
```

```
    0.2548   -0.0528         0         0         0         0         0
    0.0711    0.0353         0         0         0         0         0
    0.2256    0.2730         0         0         0         0         0
         0         0    0.6052    0.2618    0.0695    0.0870   -0.1322
         0         0    0.1745    0.3358    0.1017    0.0464   -0.1610
         0         0    0.1623    0.3560    0.7047    0.4342    0.2398
         0         0    0.0580    0.0464    0.1241    0.4324    0.0534
         0         0   -0.2203   -0.4024    0.1713    0.1334    0.7069
         0         0    0.2203    0.4024   -0.1713   -0.1334    0.2931
  -19.9992         0    0.3193   -0.2249   -0.0772    0.0162    0.1016
   24.1389         0    0.0755   -0.0369    0.0076   -0.1032    0.0306
         0         0   -0.0175    0.0833    0.1164   -0.4645    0.0228
  Columns 15 through 18
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0    3.1533   -6.3698         0
         0   -1.5880    7.6883         0
         0         0         0         0
    0.0601    0.3434    0.3259   -0.0175
    0.0732   -0.1613   -0.1060    0.0553
   -0.1090   -0.1937    0.0768    0.2705
   -0.0243    0.0116   -0.2967   -0.3083
    0.1332    0.1821    0.2199    0.0378
    0.8668   -0.1821   -0.2199   -0.0378
   -0.0462    0.5442    0.4509   -0.0933
   -0.0139    0.1124    0.2232    0.1108
   -0.0104   -0.1008    0.4801    0.5809
Qm =
  Columns 1 through 7
    0.8308    0.1742    0.0349    0.0174   -0.0348    0.1045    0.3948
    0.1244    0.6310    0.1980    0.0182   -0.1082    0.3247   -0.2903
    0.0199    0.1584    0.5724    0.0779    0.1035   -0.3105   -0.0465
    0.0249    0.0364    0.1948    0.8865    0.0396   -0.1187   -0.0580
   -0.0448   -0.1948    0.2329    0.0356    0.8569    0.4292    0.1045
    0.0448    0.1948   -0.2329   -0.0356    0.1431    0.5708   -0.1045
    0.1692   -0.1742   -0.0349   -0.0174    0.0348   -0.1045    0.6052
   -0.0748    0.0865    0.0460   -0.0093   -0.0101    0.0304    0.1745
   -0.0695    0.1241    0.1836   -0.0868    0.0149   -0.0446    0.1623
   -0.0249   -0.0364   -0.1948    0.1135   -0.0396    0.1187    0.0580
    0.0944   -0.0877    0.0111   -0.0267    0.0247   -0.0741   -0.2203
   -0.0944    0.0877   -0.0111    0.0267   -0.0247    0.0741    0.2203
  Columns 8 through 12
   -0.2618   -0.0695   -0.0870    0.1322   -0.0601
    0.2162    0.0887   -0.0911   -0.0877    0.0399
    0.0920    0.1049   -0.3895    0.0089   -0.0040
   -0.0464   -0.1241    0.5676   -0.0534    0.0243
   -0.0456    0.0191   -0.1781    0.0445   -0.0202
    0.0456   -0.0191    0.1781   -0.0445    0.0202
    0.2618    0.0695    0.0870   -0.1322    0.0601
    0.3358    0.1017    0.0464   -0.1610    0.0732
    0.3560    0.7047    0.4342    0.2398   -0.1090
    0.0464    0.1241    0.4324    0.0534   -0.0243
   -0.4024    0.1713    0.1334    0.7069    0.1332
    0.4024   -0.1713   -0.1334    0.2931    0.8668
M =
  Columns 1 through 7
    0.8308    0.1742    0.0349    0.0174   -0.0348    0.1045    0.3948
    0.1244    0.6310    0.1980    0.0182   -0.1082    0.3247   -0.2903
    0.0199    0.1584    0.5724    0.0779    0.1035   -0.3105   -0.0465
    0.0249    0.0364    0.1948    0.8865    0.0396   -0.1187   -0.0580
   -0.0448   -0.1948    0.2329    0.0356    0.8569    0.4292    0.1045
    0.0448    0.1948   -0.2329   -0.0356    0.1431    0.5708   -0.1045
```

```
   0.1692   -0.1742   -0.0349   -0.0174    0.0348   -0.1045    0.6052
  -0.0748    0.0865    0.0460   -0.0093   -0.0101    0.0304    0.1745
  -0.0695    0.1241    0.1836   -0.0868    0.0149   -0.0446    0.1623
  -0.0249   -0.0364   -0.1948    0.1135   -0.0396    0.1187    0.0580
   0.0944   -0.0877    0.0111   -0.0267    0.0247   -0.0741   -0.2203
  -0.0944    0.0877   -0.0111    0.0267   -0.0247    0.0741    0.2203
 Columns 8 through 12
  -0.2618   -0.0695   -0.0870    0.1322   -0.0601
   0.2162    0.0887   -0.0911   -0.0877    0.0399
   0.0920    0.1049   -0.3895    0.0089   -0.0040
  -0.0464   -0.1241    0.5676   -0.0534    0.0243
  -0.0456    0.0191   -0.1781    0.0445   -0.0202
   0.0456   -0.0191    0.1781   -0.0445    0.0202
   0.2618    0.0695    0.0870   -0.1322    0.0601
   0.3358    0.1017    0.0464   -0.1610    0.0732
   0.3560    0.7047    0.4342    0.2398   -0.1090
   0.0464    0.1241    0.4324    0.0534   -0.0243
  -0.4024    0.1713    0.1334    0.7069    0.1332
   0.4024   -0.1713   -0.1334    0.2931    0.8668
```