POWER

PWP-089

Capacity Constrained Supply Function Equilibrium Models of Electricity Markets: Stability, Non-decreasing constraints, and Function Space Iterations

Ross Baldick and William Hogan

December, 2001
(Revised August 2002)

This paper is part of the working papers series of the Program on Workable Energy Regulation (POWER). POWER is a program of the University of California Energy Institute, a multicampus research unit of the University of California, located on the Berkeley campus.

University of California Energy Institute
2539 Channing Way
Berkeley, California 94720-5180
www.ucei.org

Capacity Constrained Supply Function Equilibrium Models of Electricity Markets: Stability, Non-decreasing Constraints, and Function Space Iterations

Ross Baldick
Department of Electrical and Computer Engineering
The University of Texas at Austin

William Hogan
Center for Business and Government
John F. Kennedy School of Government
Harvard University

August 23, 2002

Abstract

In this paper we consider a supply function model of an electricity market where strategic firms have capacity constraints. We show that if firms have heterogeneous cost functions and capacity constraints then the differential equation approach to finding the equilibrium supply function may not be effective by itself because it produces supply functions that fail to be non-decreasing. Even when the differential equation approach yields solutions that satisfy the non-decreasing constraints, many of the equilibria are unstable, restricting the range of the equilibria that are likely to be observed in practice. We analyze the non-decreasing constraints and characterize piece-wise continuously differentiable equilibria. To find stable equilibria, we numerically solve for the equilibrium by iterating in the function space of allowable supply functions. Using a numerical example based on supply in the England and Wales market in 1999, we investigate the potential for multiple equilibria and the interaction of capacity constraints, price caps, and the length of the time horizon over which bids must remain unchanged. We empirically confirm that the range of stable supply function equilibria can be very small when there are binding price caps. Even when price caps are not binding, the range of stable equilibria is relatively small. We find that requiring supply functions to remain fixed over an extended time horizon having a large variation in demand reduces the incentive to mark up prices compared to the Cournot outcome.

1 Introduction

Supply function equilibrium models were developed by Klemperer and Meyer in [1] to analyze markets where agents bid a schedule of price-quantities. The original motivation was to handle random shocks in demand that could be characterized by a continuous random variable having convex support. Their approach sets up a coupled differential equation that, under certain circumstances, characterizes the equilibrium. Klemperer and Meyer's work is in some respects similar in flavor to the auctions of shares literature and similar results can be found, for example, in [2, 3, 4].

In recent papers, supply function equilibrium models have been applied to analysis of electricity markets [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. This approach, pioneered by Green and Newbery [5], reinterprets the probability distribution of random shocks in [1] to be an electricity load-duration characteristic. The support of the probability distribution becomes the range of demands in the load-duration characteristic.

Unfortunately, without restrictive assumptions on the nature of the costs and capacity constraints, on the number of firms, or on the form of the allowed bid functions, it has proven difficult to find equilibria in supply functions. For example, in [6] the supply functions are restricted to being linear. (By linear in this context, we mean that the intercept of the supply function is zero.) In [7, 8, 10], the supply functions are affine but either the intercepts or the slopes are assumed constant. (By affine, we mean that there is a constant slope and a (possibly zero) intercept.) In [5, 6, 9], the marginal cost functions all have zero (or all have the same) intercept.

As a final example, in [11], to obtain a convenient characterization of the equilibrium, the authors assume that each bidder must submit either:

- an affine supply function or
- a piece-wise affine supply function where the number of pieces is relatively small.

In the case of minimum capacity constraints, [11] exhibits a piece-wise affine supply function equilibrium. Representing maximum capacity constraints prompts an *ad hoc* approach in [11] that attempts to approximate the equilibrium supply functions when there are maximum capacity constraints.

In this paper, we relax the assumption of [11] that the bidders submit a supply function consisting of a small number of pieces. We analyze the properties of the equilibrium and also numerically estimate candidate equilibrium supply functions by iterating in the function space of allowable bids; however, in practice this means that we still must approximate the supply functions with a piece-wise affine and continuous function, albeit having a large number of pieces.

We qualify the numerical estimates of the equilibria as "candidate" because the functions we calculate cannot be guaranteed to be equilibria without a further check of global optimality of each bidder's bid (given everyone else's bid.) We do not perform this even more computationally intensive calculation. However, as argued in [12], the "limited optimizing behavior" that we simulate may nevertheless be a reasonable model for gaining some insight about plausible bidder behavior.

We investigate one basic criticism of supply function equilibrium analysis: that there are multiple supply function equilibria so that the approach has little predictive value. As a first response to this criticism, Green and Newbery [5, § II.B] note that capacity constraints tend to limit the range of equilibria. They describe conditions for uniqueness in an extreme case where the capacity constraints are so tight that the price at peak demand in the supply function equilibrium is as high as the price under Cournot competition.

We find that although there is a continuum of equilibria in the uncapacitated case, the range of equilibria is less likely to be problematic when there are moderately tight capacity constraints and price caps. This echos the observations by Green and Newbery but goes further in that we find that the presence of price caps yields unique equilibria with prices well below the Cournot price.

Moreover, we show theoretically under somewhat restrictive assumptions that even when there is a wide range of equilibria, all but one of these equilibria are unstable and so are unlikely to be observed in practice. Our analysis confirms a suggestion made in [14] that "an equilibrium is less likely to be stable if it involves generators offering power at prices very much higher than their marginal costs" [14, page 20].

We then use the numerical calculations to explore the interaction of three issues:

1. the effect of price caps (set above the maximum marginal cost of production) in an institutional framework where firms are obliged to supply all their available generation capacity whenever the price reaches the price cap,

2. the effect of maximum capacity constraints on strategic behavior, and

3. the effect of requiring that supply function bids be fixed over an extended time horizon during which demand varies essentially continuously.

We discuss these issues in the following paragraphs.

The assumption that bidders must sell all their capacity at the price cap does not accurately represent those markets with price caps where either:

- the bidders have alternate sales opportunities that are not price-capped or

- the bidders can otherwise declare their capacity to be unavailable to the market.

However, the assumption should provide a lower bound on the amount of capacity withholding that might occur in a real market. Our assumption is intended to reflect the intent of regulatory authorities in setting price caps: presumably they expect that all capacity will be offered whenever the price reaches the price cap.

We also consider the alternative of a *bid cap,* where there is a limit on the bid prices but the market prices can rise above this level to limit the demand. Bid caps have been proposed as a means to limit market power when there are transmission constraints, while also allowing prices to rise to high levels to reflect the true cost of a constraint. We investigate their application in transmission unconstrained systems where the generation capacity is limited.

Generation maximum capacity constraints are pervasive in electricity markets. As discussed in [11], the presence of capacity constraints complicates the determination of conditions for profit maximization because the profit functions are typically non-concave. We will discuss this issue in the context of a profit function defined over a time horizon.

Some markets, such as the England and Wales market until 2001 and the Pennsylvania–New Jersey–Maryland (PJM) market, explicitly require bidders to submit a single supply function valid (essentially) for a whole day. Other markets, such as the New York market, have rules that limit the revision of supply functions. Requiring supply function bids to be fixed over an extended time horizon (or limiting the revisions to the supply functions) means that bidders must balance the desire to withhold capacity when prices are high against sales opportunities at lower prices. In contrast, other markets, such as the Australian National market and the (now defunct) California Power Exchange, allow different bid functions every hour.

Issues such as start-up costs, ramp rate limits, and environmental constraints couple generation costs from hour to hour. Moreover, capacity can change due to outages. However, the production cost function of an in-service generator may not change significantly on an hour by hour basis, so that the flexibility to bid different supply functions on an hour by hour basis is not obviously justified by technical issues, except to the extent that start-up costs, ramp rate limits, environmental constraints, and changing fuel costs are significant.

We consider the incentives due to requiring consistent bids over an extended time horizon; however, we do not consider how to handle start-up costs, ramp rates and environmental constraints nor the institutional oversight required to enforce bid consistency [5, § II.B]. There are admitted difficulties in trying to enforce consistency of bids. For example, in the England and Wales market, although bids were fixed over a day, declared capacities could be changed, effectively redefining the bid. Also, day-ahead markets typically have hourly or real-time markets. Implicit in our analysis is the assumption that most volume is traded in the day-ahead market.

We assume that the load-duration characteristic is continuous over the time horizon. This is analogous to the Klemperer and Meyer assumption that the random variable representing the demand shock has convex support [1].

To investigate the three issues of price caps, maximum capacity constraints, and the requirement to bid supply functions that are consistent over an extended time horizon, we perform numerical calculations using cost data that are based on that in [11] for the five strategic firm industry in England and Wales subsequent to the 1999 divestiture. Our demand and price cap assumptions are, however, for the most part fictitious and simply chosen to highlight the effects of capacity constraints, price caps, and an extended time horizon. Naturally, caution should be exercised in extrapolating the numerical results to other cases.

We assume that all energy is sold at the marginal clearing price. More recently, the England and Wales market has changed to a pay-as-bid structure; however, we have not modeled this new market structure.

The main findings of this work are:

- In markets with firms having heterogeneous cost functions and capacity constraints, the differential equation approach to finding the equilibrium supply function may not be effective by itself.

- The range of supply function equilibria can be very small when there are binding price caps. Even when price caps are not binding, the range of stable equilibria appears small compared to the difference between, say, the competitive and the Cournot outcomes.

- Requiring supply functions to remain fixed over an extended time horizon having a large and continuous variation in demand appears to reduce the incentive to mark up prices compared to the Cournot outcome.

- A single price cap imposed at all times may have significant effects both on- and off-peak.

The third observation is consistent with the results in [16], which used an "adaptive agent" approach to evaluate the incentives of daily and hourly bidding in the England and Wales market.

The outline of the paper is as follows. The formulation is described in section 2, with the assumptions and formulation essentially standard from the supply function equilibrium literature. Section 3 then explores the approach to solving the equilibrium conditions as a coupled differential equation. In section 4 we discuss some of the assumptions of the model in detail, highlighting three issues that are critical in the analysis of section 3:

1. consistency of bids across the time horizon,
2. continuity of the load-duration characteristic, and
3. the nature of the marginal cost functions.

We use a three firm example based on an example in [12] to illustrate the effect of requiring consistency of bids across the time horizon on the range of equilibria.

We next consider stability. There are various time scales in the operation of an electric power system, from sub-second to longer than a day. At the sub-second time scale, the electromechanical interactions must be analyzed for stability [17]. At a slightly slower time scale, short-term electric power markets have dynamics that can potentially interact with the electromechanical dynamics. Alvarado *et al.* analyze these interactions [18]. Our interest is in the stability of the economic equilibria. Alvarado considers electricity market stability in a quantity bidding context [19]. Anderson and Xu considers stability of supply function equilibria in [14].

In section 5, we analyze the stability of the supply function equilibria calculated using the differential equation approach and present a theorem that characterizes unstable supply function equilibria. This theorem sheds light on why the apparent multiplicity of supply function equilibria may not be as serious a problem as implied by the apparently wide range of possible solutions of the differential equation. We again use the three firm example to illustrate how the stability analysis restricts the range of equilibria that are likely to be observed in practice.

In section 6 we then present a theorem that suggests why the coupled differential equation approach by itself is not likely to be fruitful in the case of firms having capacity constraints and asymmetric cost functions. The reason is that the solutions of the differential equation will not usually satisfy the requirement that the supply functions be non-decreasing across the range of realized prices. We illustrate this theorem with a five firm example system based on the England and Wales system [11].

We complement the analysis in section 6 with a further analysis of the non-decreasing constraints in section 7. This analysis provides a characterization of the properties of piece-wise continuously differentiable SFEs. In particular, we show that while the range of the load-duration characteristic affects the set of possible supply function equilibria, the set of possible equilibria is not affected by the detailed functional form of the load-duration characteristic.

We use a two firm example system to illustrate an apparently paradoxical property of supply function equilibria. In particular, the non-decreasing constraints are not apparently binding on the equilibrium solutions in the sense that the equilibrium solutions are typically all strictly increasing. However, these constraints are actually binding in the sense that if the non-decreasing constraints were relaxed for a particular firm then its optimal response would be different. This apparent paradox is due to the fact that the profit function for a firm can be non-concave, so that apparently non-binding constraints actually cut off solutions that have higher profit than the feasible solutions.

Section 8 describes an approach to finding the SFEs that involves iterating in the function space of supply functions. Section 10 discusses the detailed assumptions in the numerical implementation, while case studies and results are presented in section 11 based on the five firm example system. The case studies first investigate numerically the issue of multiplicity of equilibrium solutions. Then the effect of varying price caps, capacities, the load factor, and demand are investigated. We conclude in section 12.

2 Formulation

In this section, we first discuss the demand, generation costs and capacities, and supply functions. Then we discuss price and price caps, assumptions on the form of the supply functions, the profit, and the equilibrium conditions. The development is standard.

2.1 Demand

Following Green [6], we assume that *demand* $D : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is a continuous function of the form:

$$\forall p \in \mathbb{R}_+, \forall t \in [0,1], D(p,t) = N(t) - \gamma p, \tag{1}$$

where:

- p is the price,
- t is the (normalized) time,
- $N : [0,1] \to \mathbb{R}_+$ is the *load-duration* characteristic, and
- $\gamma \in \mathbb{R}_+$ is minus the slope of the demand curve.

That is, the demand is assumed to be additively separable in its dependence on price and on time. The load-duration characteristic N represents the distribution of demand over a time horizon, with:

- the time argument t normalized so that it ranges from 0 to 1 and
- N non-increasing, so that $t = 0$ corresponds to peak conditions and $t = 1$ corresponds to minimum demand conditions.

For most of the theoretical analysis, we will additionally assume that N is continuous, with a discussion in section 4.3 of the implications if the assumption of continuity of N is relaxed. For the computational model that we develop, we will additionally assume that N is affine. Figure 1 illustrates an affine load-duration characteristic.

The assumption of a linear demand-price dependence and an affine load-duration characteristic is somewhat restrictive. More complicated continuous load-duration characteristics N can easily be accommodated in the computational model; however, as we will see, the functional form of the load-duration characteristic does not affect the set of equilibria. Other demand-price dependencies such as constant elasticity could also be represented, but this would require more substantial modifications.

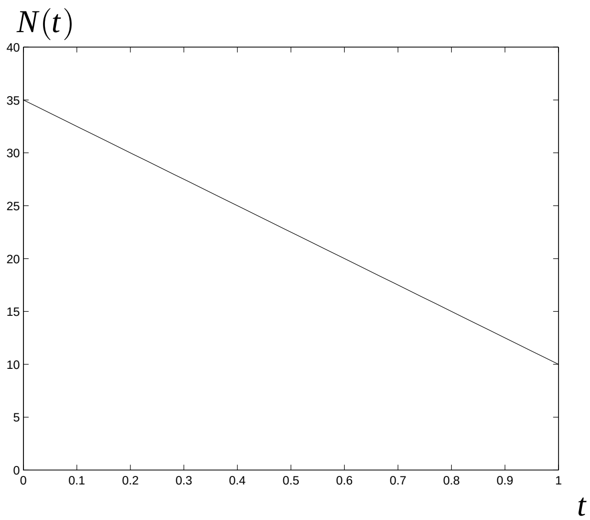

Figure 1: Example load-duration characteristic.

2.2 Generation costs and capacities

We assume that firms are labeled $i = 1, \ldots, n$, with $n \geq 2$. Following [11] and except as noted, we will assume that the *total variable generation cost function* $C_i : \mathbb{R}_+ \to \mathbb{R}$ of the i-th firm is quadratic and of the form:

$$\forall q_i \in \mathbb{R}_+, C_i(q_i) = \frac{1}{2} c_i q_i^2 + a_i q_i,$$

with $c_i \geq 0$ for each i so that the variable generation costs are convex. We therefore ignore issues such as start-up and minimum-load costs. We use superscript $'$ to represent differentiation and denote the marginal cost by C_i', so that:

$$\forall q_i \in \mathbb{R}_+, C_i'(q_i) = c_i q_i + a_i. \tag{2}$$

Each firm is assumed to be able to produce down to zero output, so that the minimum capacity constraints are all equal to zero. Each firm has a maximum capacity $\bar{q}_i$. That is, the *capacity constraints* for the firms require that:

$$\forall i, 0 \leq q_i \leq \bar{q}_i. \tag{3}$$

The cost function C_i represents the variable generation cost function of the whole firm i. Typical firms own several generation units, including several technologies such as coal, oil, and natural gas. Moreover, typical generation units have increasing marginal costs over their operating range of production. Therefore, C_i can be construed as resulting from optimal economic dispatch of the portfolio of generation owned by firm i, with issues such as start-up and no-load costs subsumed into the the function C_i.

The assumption of affine marginal costs does not capture jumps in marginal cost from, say, coal to gas technology and does not capture the rapid increase in marginal costs at high output close to the maximum capacity. However, it does represent the qualitative observation of increasing marginal cost with output. That is, we will usually have that $c_i > 0$.

More complicated marginal cost curves could easily be incorporated into the computational model. For example, a "barrier term" could be added to the cost function to represent a rapid rise in marginal costs as q_{it} approaches $\overline{q}_i$.

2.3 Supply functions

As discussed in the introduction, in the formulation of Klemperer and Meyer [1] a probability distribution characterizes a range of random demand outcomes. Bushnell and Wolak [20] use such a model to investigate optimal hourly responses in the California electricity market.

In contrast, Green and Newbery [5] and Green [6] model deterministic variation of demand over an extended time horizon. We follow this approach, assuming that each firm bids a *supply function* into the market; that is, a function $S_i : \mathbb{R}_+ \to \mathbb{R}$ that represents the amount of power it is willing to produce at each specified price per unit energy. (We will restrict the functional form of the S_i further in section 2.5 and definition 1.) The supply function applies throughout the time horizon specified by the load-duration characteristic. For example, in the England and Wales until 2001, a new supply function could be specified for each day so that the load-duration characteristic could be considered to be of one day duration.

Analysis of hybrid situations is also possible, where D represents the distribution of demand over a day but the demand is not completely deterministic. In such a hybrid case, S_i still applies throughout the time horizon and responds both to the variation of demand over the time horizon and also to the uncertainty of demand at each time.

We investigate how the load factor over the time horizon affects the equilibrium outcomes. We will observe that requiring bids to be consistent over extended time horizons that include the peak conditions and also lower demand conditions can have a significant effect on limiting price mark ups and equilibrium profits. However, we recognize that such non-cooperative equilibrium analyses also understate the level of market power because they neglect the possibility of collusion and the impact of repeated interactions.

2.4 Price cap and price minimum

Price caps are in place in many electricity markets. The detailed implementation of the price caps varies from market to market. To represent the effect of a generic price cap on the market, we follow von der Fehr and Harbord [21] and assume that the market rules specify a price cap $\overline{p}$ and that the firms are obliged to bid supply functions that satisfy:

$$\forall i, S_i(\overline{p}) = \overline{q}_i. \tag{4}$$

That is, each firm must be willing to operate at full output if the price reaches the price cap. Of course, firms might also bid so that they would be prepared to produce at full output for lower prices.

As discussed in [22, § V], enforcement of this requirement necessitates that the market operator be prepared to curtail demand and not breach the price cap. Furthermore, the market operator must be able to reliably estimate the maximum marginal cost of production by any firm in the market so that the price cap can be set above the maximum marginal cost of production.

We assume for convenience that there is a known minimum price $\underline{p}$ below which no firm would be prepared to bid any non-zero supply. For example, $\underline{p} = \min_i \{a_i\}$ is a suitable value since no firms will be willing to generate for a price that falls below the marginal operating costs at zero output of the cheapest generator.

2.5 Feasible and allowable supply functions

We require that each supply function be defined for every price in the interval $[\underline{p},\overline{p}]$. To be *feasible* the range of the supply function for firm i must be contained in the interval $[0,\overline{q}_i]$. That is, the supply function for firm i is a function $S_i : [\underline{p},\overline{p}] \to [0,\overline{q}_i]$.

Market rules require that supply functions be *non-decreasing* in order to be *allowable* as bids. That is, $p \leq p' \Rightarrow S_i(p) \leq S_i(p')$. Some authors appear to neglect this constraint. For example, Bolle [23] presents supply function equilibrium solutions that fail to satisfy the non-decreasing constraints. (See [23, Figure 2].) We will find that the non-decreasing constraints must be represented in the model. (However, we will also observe that the non-decreasing constraints are not apparently binding at the equilibrium.)

The requirement that each supply function be feasible and allowable is embodied in the following:

Definition 1 For each $i = 1,\ldots,n$, the set $\mathbb{S}_i$ is the function space of feasible and allowable supply functions for firm i having domain $[\underline{p},\overline{p}]$. That is, $\mathbb{S}_i$ is the set of functions with domain $[\underline{p},\overline{p}]$ that:

1. have range $[0,\overline{q}_i]$ (so that all bids are feasible for allowed prices) and
2. are non-decreasing over the domain $[\underline{p},\overline{p}]$, (so that the function is an allowable supply function).

□

In section 3 when we analyze differential equations with solutions that yield supply function equilibria, we will further restrict $\mathbb{S}_i$ to be the space of *differentiable* functions that are feasible and allowable. In this case, the non-decreasing constraints are equivalent to:

$$\forall i = 1,\ldots,n, \forall p \in [\underline{p},\overline{p}], S_i'(p) \geq 0,$$

where superscript $'$ denotes differentiation.

2.6 Price

At each time $t \in [0,1]$, the market is cleared based on the bid supply functions $S = (S_i)_{i=1,\ldots,n}$ and the demand. That is, at each time t, the price is determined by the solution of:

$$D(t,p) = N(t) - \gamma p = \sum_{i=1}^{n} S_i(p), \tag{5}$$

assuming a solution exists. All firms receive the marginal clearing price for their supply. We say that this price corresponds to the bid supply functions S.

If $\gamma > 0$ then for each t and each collection of choices of non-decreasing supply functions $S_i \in \mathbb{S}_i, i = 1,\ldots,n$ there is at most one solution to (5) having $\underline{p} \leq p \leq \overline{p}$. If there is a solution to (5) in this range, then this solution determines the price at time t.

For most of the supply functions we exhibit, the S_i are continuous and solutions to (5) exist except when rationing is necessary. However, if some of the S_i are discontinuous then, following Anderson and Xu [14, §2], we must modify the notion of "a solution to (5)" slightly to:

$$\inf_p \left\{ p \,\middle|\, D(t,p) \leq \sum_{i=1}^{n} S_i(p) \right\}. \tag{6}$$

See Anderson and Xu [14, §2] for a fuller discussion of this issue. We will only need to deal with this issue for some of the supply functions we exhibit.

If there is no solution to (5) in the range $\underline{p} \leq p \leq \overline{p}$, then rationing must occur and the realized price depends on whether the market is assumed to have price caps or bid caps. We discuss these two cases in the next sections.

2.6.1 Price caps

In the case of price caps, the market price is never allowed to rise above $\overline{p}$. If there is insufficient supply to meet the demand at some time $t \in [0,1]$ at a price $p = \overline{p}$ then demand must be rationed. In this case, we will assume that:

- demand at time t is rationed to the available supply and
- all energy is sold at time t at a price equal to the price cap.

For any particular choices $S_i, i = 1,\ldots,n$, we can therefore implicitly solve for price as a function of time. That is, assuming that the S_i are continuous, there is a function $P : [0,1] \to [\underline{p}, \overline{p}]$, which is parameterized by $S_j \in \mathbb{S}_j, j = 1,\ldots,n$, such that:

$$\forall t \in [0,1], D(t, P(t; S_j, j = 1,\ldots,n)) \geq \sum_i S_i(P(t; S_j, j = 1,\ldots,n)), \tag{7}$$

with equality between the left and right hand sides except at times when demand rationing occurs. For notational convenience, we will omit the explicit parameterization of the function P and just write it with one argument, namely, the normalized time t. Occasionally, we will need to consider price functions arising from alternative choices of supply functions. In this case, we will distinguish the price functions by superscripts. For example, in sections 5 and 7, we will consider supply functions $S_i^{\varepsilon}, i = 1,\ldots,n$. We will denote the resulting price function P^{ε}.

2.6.2 Bid caps

In this alternative market structure, prices can rise to higher than $p = \overline{p}$ in order to ration demand based on price. That is, there is a cap on bids but not on prices. To implement the bid caps, we implicitly extrapolate the supply functions to being functions $S_i : [\underline{p}, \infty) \to [0, \overline{q}_i]$ by defining:

$$\forall i, \forall p > \overline{p}, S_i(p) = \overline{q}_i.$$

Moreover, we relax the upper limit on price and only require that $p \geq \underline{p}$. In this case there is always a solution to (5) given that the S_i are continuous; however, the resulting price might exceed the bid cap $\overline{p}$.

Again, we can implicitly solve for the marginal clearing price as a function of time. In this case, the price is a function $P : [0,1] \to [\underline{p}, \infty)$. (In fact, with a linear demand-price relationship, if $\gamma > 0$ then the highest realized price is always below the "choke price" of $N(0)/\gamma$.)

2.7 Profit

By the discussion in 2.6, given a supply function S_i of firm i and also given the supply functions of the other firm, which we will denote by $S_{-i} = (S_j)_{j \neq i}$, we can determine the corresponding price function P. Moreover, at any time t the accrual of profit per unit (normalized) time to firm i is π_{it}:

$$\pi_{it} = S_i(P(t))P(t) - C_i(S_i(P(t))). \tag{8}$$

The profit π_i to firm i over the time horizon is then given by:

$$\begin{aligned} \forall S_j \in \mathbb{S}_j, j = 1, \ldots, n, \pi_i(S_i, S_{-i}) &= \int_{t=0}^{1} \pi_{it} dt, \\ &= \int_{t=0}^{1} S_i[(P(t))P(t) - C_i(S_i(P(t)))]dt. \end{aligned} \tag{9}$$

That is, the profit π_i is the integral of the profit per unit time over the time horizon.

2.8 Equilibrium definition

Following standard definitions, we make:

Definition 2 A collection of choices $S^\star = (S_i^\star)_{i=1,\ldots,n}$, with $S_i^\star \in \mathbb{S}_i, i = 1, \ldots, n$ is a *Nash supply function equilibrium* (SFE) if:

$$\forall i = 1, \ldots, n, S_i^\star \in \underset{S_i \in \mathbb{S}_i}{\operatorname{argmax}} \{\pi_i(S_i, S_{-i}^\star)\}, \tag{10}$$

where $S_{-i}^\star = (S_j^\star)_{j \neq i}$. □

3 Equilibrium conditions as a differential equation

In the following sections we paraphrase and interpret the supply function equilibrium derivations of Klemperer and Meyer [1], Green and Newbery [5], and Green [6], which lead to solutions of the SFE involving the solution of a differential equation. This approach to solving for the SFE as a vector differential equation has been used with considerable success by Green and Newbery in several cases [5, 6]:

1. all firms having the same marginal cost functions and having the same generation capacity constraints, which we refer to as the *symmetric capacitated* case,

2. firms having affine but different marginal cost functions but no capacity constraints, which we refer to as the *asymmetric affine marginal cost uncapacitated* case, and

3. two firms having asymmetric marginal cost functions and capacity constraints, which we refer to as the *asymmetric capacitated duopoly* case.

We develop this approach in order to highlight why an analogous approach is unsuccessful for calculating the SFE in the multi-firm, capacitated, asymmetric case.

3.1 Basic analysis

This section paraphrases the discussion in Klemperer and Meyer [1], Green and Newbery [5], and Green [6] into our notation. The approach in those papers to finding the SFE can be interpreted as:

1. assuming that for each firm i, the supply functions of all the other firms are infinitely differentiable,

2. solving the conditions on price and quantity, at each time t, for maximizing the contribution to profit per unit time for firm i as defined in (8) and

3. finding an infinitely differentiable supply function S_i that matches these conditions, if such a function exists.

We will initially consider a general functional form for the marginal cost function. Consider a firm i and suppose that each other firm $j \neq i$ has committed to an infinitely differentiable supply function S_j. At time t, the price for energy is determined by these supply functions and the production of firm i. Conversely, if firm i is committed to supplying the residual demand at any given price then the price p_t at time t determines the production q_{it} of firm i at time t according to:

$$\forall t \in [0,1], q_{it} = N(t) - \gamma p_t - \sum_{j \neq i} S_j(p_t),$$

where we ignore demand rationing for convenience. Since the supply functions $S_j, j \neq i$ are assumed differentiable, necessary conditions for maximizing the profit per unit time π_{it} at each time t over choices of price p_t are:

$$\forall t \in [0,1], q_{it} = (p_t - C_i'(q_{it})) \left(\gamma + \sum_{j \neq i} S_j'(p_t) \right), \tag{11}$$

which we can solve for each t to find a corresponding unique optimal p_t and q_{it} for firm i. If the implicit relationship between q_{it} and p_t is monotonically non-decreasing then we can define a non-decreasing function $S_i : \{p_t | t \in [0,1]\} \to [0, \overline{q}_i]$ that satisfies:

$$\forall t \in [0,1], S_i(p_t) = q_{it}. \tag{12}$$

Applying the implicit function theorem to (11) shows that for each p_t, the function S_i is infinitely differentiable. If, furthermore, each value of q_{it} in (11) satisfies the capacity constraints (3) then we have found a supply function $S_i \in \mathbb{S}_i$ for firm i that achieves the maximum profit per unit time for firm i and each time t, given the supply functions of the other firms. Consequently, this supply function also maximizes the integrated profit π_i for firm i over the time horizon and, moreover, the supply function can be calculated without reference to the load-duration characteristic N.

In summary, we seek a function $S_i \in \mathbb{S}_i$ that satisfies:

$$\forall p \in \mathbb{P}_i, S_i(p) = (p - C_i'(S_i(p))) \left(\gamma + \sum_{j \neq i} S_j'(p) \right), \tag{13}$$

where $\mathbb{P}_i = \{p_t | t \in [0,1]\}$; that is, $\mathbb{P}_i$ is the set of all prices for which S_i is defined by (12). An SFE obtains if we can find $S_i \in \mathbb{S}_i, i = 1, \dots, n$ that satisfy (13) for every firm i over a common interval of prices. That is, if there are differentiable non-decreasing functions $S_i^\star$ for $i = 1,\ldots,n$, a corresponding price function P, and a set of prices $\mathbb{P} = \{P(t) | t \in [0,1]\}$ satisfying:

$$\forall i = 1,\ldots,n, \forall p \in \mathbb{P}, S_i^\star(p) = (p - C_i'(S_i^\star(p))) \left(\gamma + \sum_{j \neq i} S_j^{\star\prime}(p)\right), \tag{14}$$

then $S_i^\star, i = 1,\ldots,n$, is an SFE. This is equation (4) of [6] transcribed into our notation. Somewhat surprisingly, the conditions for the SFE do not depend on the functional form of the load-duration characteristic N; however, we will see that the range of N affects the range of possible SFEs. This has important implications for the range of possible equilibria in electricity markets that will be touched on in section 3.3 and then discussed in detail in sections 5 and 6.

The set $\mathbb{P}$ is an interval because $P(t)$ is a non-decreasing function of t. If $\mathbb{P} = [P(1), P(0)]$ is strictly contained in $[\underline{p}, \overline{p}]$ then we can extend the $S_i^\star$ to being functions on the whole of $[\underline{p}, \overline{p}]$ by defining, for example:

$$\begin{aligned} \forall i = 1,\ldots,n, \forall p \in [\underline{p}, P(1)], S_i^\star(p) &= S_i^\star(P(1)), \\ \forall i = 1,\ldots,n, \forall p \in [P(0), \overline{p}], S_i^\star(p) &= S_i^\star(P(0)). \end{aligned}$$

Klemperer and Meyer [1] characterized the conditions for existence of an SFE in the case of symmetric cost functions with no capacity constraints and discuss the multiplicity of equilibria. In the next section we recall the affine solution in the case of affine marginal costs and no capacity constraints. In section 3.3 we recall Klemperer and Meyer's uniqueness conditions and discuss it in the context of electricity markets and then in section 3.4 we return to the more general asymmetric capacitated case. In section 3.5 we specialize to symmetric cost functions and then in sections 3.6 and 3.7 we discuss difficulties with numerically solving the differential equation.

3.2 Affine solutions for affine marginal cost functions

In [6, 9, 11], linear and affine SFE are exhibited for the case of affine marginal generation costs of the form (2). The affine SFE $S^{\star\text{affine}} = (S_i^{\star\text{affine}})_{i=1,\ldots,n}$ is of the form:

$$\forall i, \forall p \in \mathbb{P}, S_i^{\star\text{affine}}(p) = \beta_i (p - a_i), \tag{15}$$

where the slopes $\beta_i \in \mathbb{R}_{++}, i = 1,\ldots,n$ satisfy:

$$\forall i, \frac{\beta_i}{1 - c_i \beta_i} = \sum_{j \neq i} \beta_j + \gamma. \tag{16}$$

The affine SFE provides one SFE for the asymmetric affine marginal cost uncapacitated case.

In most of our numerical simulations we assume that $\gamma > 0$; however, even in the case $\gamma = 0$ there is typically a strictly positive solution to (16). For example, for symmetric cost functions, the solution for $\gamma = 0$ is:

$$\forall i, \beta_i = \frac{(n-2)}{(n-1)} \frac{1}{c_i}. \tag{17}$$

That is, even if $\gamma = 0$, demand clears at a finite price satisfying (5). Furthermore, at $\gamma = 0$, the sensitivity of β_i to γ is bounded and proportional to $\frac{1}{(n-1)^2}$. Moreover, as $n \to \infty$ the supply functions converge to competitive bidding with slopes given by:

$$\forall i, \beta_i^{\text{comp}} = \frac{1}{c_i}.$$

The limiting behavior as $\gamma \to 0$ for the affine SFE contrasts to the situation under Cournot competition, for example. Under Cournot competition, taking the limit as $\gamma \to \mathbf{0}$ yields unbounded prices unless there is a competitive fringe (or a mechanism to limit prices, such as a price cap.)

3.3 Conditions for uniqueness

Klemperer and Meyer also showed conditions under which the SFE is unique [1, Proposition 4]. Translated into the electricity market context, the conditions for uniqueness are that the load-duration characteristic is unbounded. With affine costs, the unique solution is the affine SFE.

Unfortunately, in the practical case that the load-duration characteristic is bounded, there are multiple SFEs. For affine marginal costs, for example, some of the multiplicity of SFEs are more competitive than the affine SFE and some are less competitive than the affine SFE. Multiplicity of equilibria weakens the usefulness of SFE analysis. Fortunately, we will see in section 5 that the wide range of equilibria is significantly reduced when we discard the unstable equilibria.

3.4 Manipulation into standard form

If the marginal costs are not affine or if non-affine SFEs are being sought then we must return to the conditions (14). As discussed in [11], these conditions are in the form of a coupled differential equation that is not in the standard form for a non-linear vector differential equation because of the summation of the derivatives in (13). In [11] it was shown that the conditions can be transformed into the following standard form of a non-linear vector differential equation:

$$S^{\star\prime}(p) = \left[\frac{1}{(n-1)}\mathbf{1}\mathbf{1}^{\dagger} - \mathbf{I}\right] \begin{bmatrix} \frac{S_1^{\star}(p)}{p - C_1'(S_1^{\star}(p))} \\ \vdots \\ \frac{S_n^{\star}(p)}{p - C_n'(S_n^{\star}(p))} \end{bmatrix} - \frac{\gamma}{(n-1)}\mathbf{1}, \tag{18}$$

where:

- $S^{\star} = (S_i^{\star})_{i=1,\ldots,n}$ is the vector of supply functions and $S^{\star\prime}$ is the derivative of this vector,
- $\mathbf{1}$ is a vector of all ones of length n,
- superscript † means transpose, and
- $\mathbf{I}$ is the identity matrix.

To find an SFE, a natural approach is to seek solutions $S_i^{\star}$ of the differential equation (18) that also satisfy $S_i^{\star} \in \mathbb{S}_i$. A natural "initial condition" for the differential equation to implement the price cap condition is (4), which specifies the values of the supply functions at $p = \overline{p}$. The differential equation can then in principle be solved "backwards" from $p = \overline{p}$ to $p = \underline{p}$.

The specification of an initial condition may partly resolve the issue of the multiplicity of equilibria that are typically possible with supply function equilibria. That is, the price cap provides a public signal to the firms that may allow them to coordinate on the equilibrium satisfying $\forall i, S_i^\star(\overline{p}) = \overline{q}_i$, which is presumably the equilibrium that yields the largest profit given the price cap. If the solution of the differential equation for this initial condition is non-decreasing and satisfies the capacity constraints, so that the solution of the differential equation specifies an SFE, and if there is only one such SFE then the SFE may be a plausible outcome for the market. We will see in section 6 that in the asymmetric case solutions of (18) typically violate the non-decreasing constraints. To avoid this issue, we will specialize to symmetric cost functions in the next section.

3.5 Symmetry

In the case of symmetric cost functions, if symmetric solutions are sought then the vector differential equation (18) can be reduced to a scalar differential equation:

$$S_i^{\star\prime}(p) = \frac{1}{(n-1)} \frac{S_i^\star(p)}{(p - C_i'(S_i^\star(p)))} - \frac{\gamma}{(n-1)}. \tag{19}$$

The solution of this differential equation generalizes the affine symmetric case specified in (15) and (17).

The form of the equilibrium conditions (19) has been presented in the context of:

- supply function equilibria by Klemperer and Meyer [1] and
- auctions of divisable goods by Wilson [2], Wang and Zender [3, Equation (2)], Nyborg [24, Equation (23)], and others, usually with $\gamma = 0$.

3.6 Singular equations

A difficulty with solving the differential equation (18) is related to the terms in its right hand side. For each firm i, we define the *marginal cost conditions* to be:

$$\forall p \in [\underline{p}, \overline{p}], C_i'(S_i(p)) \leq p.$$

The marginal cost conditions characterizes prices where a firm i is selling at an operating profit. In numerical experiments, we found that non-affine solutions to the differential equation typically approached the boundary of the marginal cost conditions. That is, the marginal costs approach the price for certain prices. At the boundary of these conditions, the differential equations (18) become singular because of the terms in the denominators of the entries on the right hand side of (18). Nearby to the boundary of the marginal cost conditions, the differential equation become difficult to solve because of numerical conditioning issues.

The singularity can be removed by augmenting the differential equation in a manner analogous to rearranging the equations into parametric form, as discussed for the symmetric, two firm case in [1, §4]. In particular, define a parametric variable u and consider the differential equation:

$$\begin{bmatrix} \frac{dS}{du} \\ \frac{dp}{du} \end{bmatrix} = \frac{1}{1+\sum_{i=1}^{n} f_i(S,p)} \begin{bmatrix} f(S,p) \\ 1 \end{bmatrix}, \tag{20}$$

where the function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ evaluates the right hand side of (18):

$$\forall S \in \mathbb{R}^n, \forall p \in \mathbb{R}, f(S,p) = \left[\frac{1}{(n-1)}\mathbf{1}\mathbf{1}^\dagger - \mathbf{I}\right] \begin{bmatrix} \frac{S_1}{p-C_1'(S_1)} \\ \vdots \\ \frac{S_n}{p-C_n'(S_n)} \end{bmatrix} - \frac{\gamma}{(n-1)}\mathbf{1},$$

and where it is understood that if any of the entries of f approach infinity then the ratio on the right hand side of (20) should be evaluated as a limit. The solution of this differential equation yields the relationship of S to p and avoids the singularities of (18). (With a more careful definition of the right hand side of (20), it is also possible to identify u with the normalized time variable.)

3.7 Marginal cost conditions and feasibility constraints

Even with the transformation described in section 3.6 to circumvent the problem of singular equations, the solutions to the differential equation will often reach and even violate the marginal cost conditions. We also found that solutions to the differential equation typically failed to satisfy the feasibility constraints. However, preventing the trajectory from violating the feasibility constraints or the marginal cost conditions poses serious conceptual problems, which we were not able to solve.

We considered a number of approaches to modifying the differential equation to avoid solutions that were not feasible or did not satisfy the marginal cost conditions. For example, we considered imposing the feasibility constraints explicitly in the maximization of profit per unit time to obtain a constrained version of the problem of maximizing profit per unit time. This would modify (11) to include a Lagrange multiplier. The basic difficulty in manipulating the resulting equations into the form of a differential equation is that the dependence of q_{it} on the $S_j', j \neq i$ is no longer invertible. That is, we can longer write an equation analogous to (18) with the derivatives of the supply functions given by a function of the supply functions.

We also tried to model the capacity limit by adding "barrier terms" to the cost function that rapidly increase as the capacity is reached. However, we were not able to reliably generate solutions to the differential equation that satisfied the non-decreasing and capacity constraints.

4 Discussion of assumptions

We discuss some of the assumptions of the model in detail, highlighting four issues that are critical to the analysis in section 3:

- consistency of bids across the time horizon,

Firm $i =$	1	2	3
c_i(pounds per MWh per MWh) $=$	0.5	0.5	0.5
a_i(pounds per MWh) $=$	9	9	9

Table 1: Cost and capacity data for three firm example system based on [12].

- continuity of the load-duration characteristic,
- strictly increasing marginal costs, and
- functional form of the supply functions.

Discussion of these issues will help to clarify where the SFE model is appropriate and where other models, such as Cournot, may be more useful. For example, despite the apparent match of SFE analysis to bid-based pool rules, it it not necessarily the case that SFE analysis is more appropriate than, say, Cournot analysis if the basic assumptions of the SFE model are not satisfied.

In section 3, we already indicated that the marginal cost conditions and the capacity constraints can provide some difficulty in solving the differential equation (18). To avoid the issues of marginal cost conditions, price caps, and capacity constraints for the discussion in this section, we will concentrate on a symmetric uncapacitated three firm system based on an example in [12]. We first present the example system in section 4.1 and then discuss the issues in sections 4.2–4.5.

4.1 Three firm example system

We consider a three firm electricity market, based on the example in [12], with each firm having the same cost function. The cost and capacity data is shown in table 1. In the symmetric case, c_i is the same for each firm and a_i is the same for each firm; however, we have kept the notation consistent with (2).

Following [12], we assume a demand slope of $\gamma = 0.125$ GW per (pound per MWh) and a base-case load duration characteristic of:

$$\forall t \in [0,1], N(t) = 7 + 20(1-t),$$

with quantities measured in GW. That is, N varies linearly from 27 to 7 GW.

Green and Newbery [5] exhibit the wide range of symmetric equilibria for this symmetric, uncapacitated, no price cap case. The range is defined by the peak demand function:

$$\forall p \in \mathbb{R}_+, D(p,0) = N(0) - \gamma p.$$

In particular, suppose that the competitive price p_0^{comp} at peak demand is calculated by solving:

$$N(0) - \gamma p_0^{\text{comp}} = \sum_{i=1}^{n} \frac{1}{c_i}(p_0^{\text{comp}} - a_i),$$

and the corresponding quantities are calculated according to:

$$\forall i = 1, \ldots, n, q_i^{\text{comp}} = \frac{1}{c_i}(p_0^{\text{comp}} - a_i).$$

The price p_0^{comp} and the quantities $q_i^{\text{comp}}, i = 1, \dots, n$ are used as a "competitive initial condition" to solve the differential equation (18) backwards from p_0^{comp} towards $p = \underline{p}$. The solution $S^{\star\text{comp}} = (S_i^{\star\text{comp}})_{i=1,\dots,n}$ provides one extreme of the range of SFE. We will call $S^{\star\text{comp}}$ the "most competitive symmetric SFE."

Similarly, a Cournot price p_0^{Cournot} for the peak demand can be calculated by solving:

$$N(0) - \gamma p_0^{\text{Cournot}} = \sum_{i=1}^{n} \frac{1}{(c_i + 1/\gamma)} (p_0^{\text{Cournot}} - a_i).$$

The corresponding quantities are calculated according to:

$$\forall i = 1, \dots, n, q_i^{\text{Cournot}} = \frac{1}{(c_i + 1/\gamma)} (p_0^{\text{Cournot}} - a_i).$$

The price p_0^{Cournot} and the quantities $q_i^{\text{Cournot}}, i = 1, \dots, n$ are used as a "Cournot initial condition" to solve the differential equation (18) backwards from p_0^{Cournot} towards $p = \underline{p}$. The solution $S^{\star\text{Cournot}} = (S_i^{\star\text{Cournot}})_{i=1,\dots,n}$ also satisfies the non-decreasing constraints. The SFE $S^{\star\text{Cournot}}$ provides the other extreme of the range of SFE. We will call $S^{\star\text{Cournot}}$ the "least competitive symmetric SFE."

At each price $p \in [a_i, p_0^{\text{comp}}]$, we have that $S^{\star\text{Cournot}}(p) \leq S^{\star\text{comp}}(p)$ with strict inequality except at $p = a_i$. The most and least competitive symmetric SFEs define a wide range, as illustrated in [5, Figure 3]. Figure 2 is based on [5, Figure 3] and shows the most and least competitive symmetric SFEs for the example system as solid lines. The price p_0^{Cournot} is more than five times larger than p_0^{comp} (and more than five times the marginal costs at peak) for this example system.

There is a continuum of equilibria intermediate between the most and least competitive symmetric SFEs. These intermediate symmetric SFEs are specified by intermediate choices of initial conditions for the differential equation (18) that are between the competitive and Cournot initial conditions. For example, the affine SFE $S^{\star\text{affine}}$ is intermediate between the most competitive and least competitive symmetric SFEs. For each $p \in [a_i, p_0^{\text{comp}}], S^{\star\text{Cournot}}(p) \leq S^{\star\text{affine}}(p) \leq S^{\star\text{comp}}$, with strict inequality except at $p = a_i$ (unless there is only one firm, in which case $S^{\star\text{Cournot}} = S^{\star\text{affine}}$.)

4.2 Consistency of bids across the time horizon

A fundamental assumption of the analysis in section 3 is that each firm must submit a single non-decreasing supply function that remains valid throughout the time horizon. The coupling effect throughout the time horizon limits the possible equilibria.

In the absence of a requirement to bid consistently over an extended time horizon, there is no such limitation on the range of equilibria. At one extreme, firms could behave as Cournot oligopolists at each time throughout the time horizon. Cournot prices at each time can lead to much higher prices on average than in the supply function equilibrium. At the other extreme, firms could bid competitively at each time throughout the time horizon. If there is no obligation to bid consistently over the time horizon, there is a wide range of possible equilibrium outcomes for each time. Because of this wide range of equilibria, the supply function equilibrium framework has very little predictive value if there is no obligation to bid consistently over an extended time horizon [25].

In addition to the equilibrium supply functions, figure 2 also shows two other supply functions:

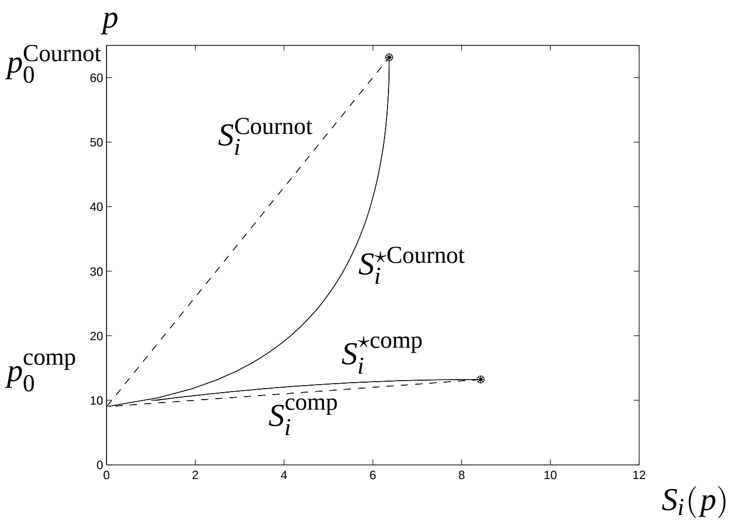

Figure 2: Least and most competitive symmetric SFEs $S_i^{\star\text{Cournot}}$ and $S_i^{\star\text{comp}}$, shown solid, together with Cournot and competitive supply functions S_i^{Cournot} and S_i^{comp}, shown dashed.
Source: This figure is based on [5, Figure 3], but uses the data for the symmetric three firm system.

- "competitive," S^{comp} where the supply functions are the inverses of the marginal cost functions:

$$\forall i, \forall p \geq a_i, S^{\text{comp}}(p) = \beta_i^{\text{comp}}(p - a_i); \beta_i^{\text{comp}} = \frac{1}{c_i}, \tag{21}$$

- "Cournot," S^{Cournot} where quantities and prices under Cournot competition are calculated for each $t \in [0,1]$ and a supply function drawn through the resulting price-quantity pairs:

$$\forall i, \forall p \geq a_i, S^{\text{Cournot}}(p) = \beta_i^{\text{Cournot}}(p - a_i); \beta_i^{\text{Cournot}} = \frac{1}{c_i + \frac{1}{\gamma}}. \tag{22}$$

For $a_i < p < p_0^{\text{comp}}, S_i^{\star\text{comp}}(p) < S_i^{\text{comp}}(p)$. For $a_i < p < p_0^{\text{Cournot}}, S_i^{\text{Cournot}}(p) < S_i^{\star\text{Cournot}}(p)$.

The functions S^{comp} and S^{Cournot} are shown dashed in figure 2. For $n > 1$, S^{Cournot} differs from the SFE $S^{\star\text{Cournot}}$. It is to be emphasized that S^{Cournot} (for $n > 1$) and S^{comp} are not SFEs. (We have ommitted the superscript $\star$ to denote this in the symbols S^{Cournot} and S^{comp}.) The Cournot supply function S^{Cournot} represents an extreme of behavior where each firm behaves as a Cournot oligopolist at each time. The competitive supply function S^{comp} represents the other extreme where each firm behaves competitively at each time. Green and Newbery's analysis shows that when firms must bid a single supply function that applies throughout the time horizon then the range of possible equilibrium outcomes is limited to being between $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$. As illustrated in figure 2, this range can be considerably smaller than the range between S^{Cournot} and S^{comp}.

Some analyses implicitly assume that the supply functions apply over time horizons that are much longer than the time between updates of bids allowed under pool rules. For example, [11] models the England and Wales market but the time horizon is considerably longer than a day. This analysis potentially understates the level of market power available to bidders that can update their supply functions arbitrarily day by day or even hour by hour. For example, in the (now defunct) California Power Exchange, bids could be updated hour by hour and consequently SFE analysis

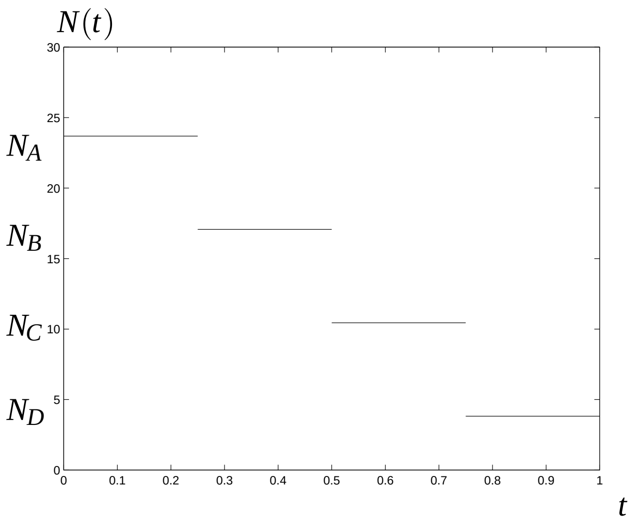

Figure 3: Piece-wise constant load-duration characteristic.

assuming consistency of the supply functions across a day is not applicable. In the extreme, if firms can update their bids very often then a Cournot model applied at each time may be more appropriate.

Nevertheless, even if there is no explicit requirement to bid consistently, implicit regulatory oversight or the bidders' limited ability to observe the other bidders' supply functions in a timely manner may limit the rapidity with which bids are updated. That is, even if there is no explicit market rule there may be some consistency between bids across time and so supply function equilibrium analysis may be applicable.

4.3 Continuity of the load-duration characteristic

Even if bid functions are required to be consistent over an extended time horizon, the supply function equilibrium model may not be suitable if the load-duration characteristic is not continuous. For example, suppose that there is no uncertainty in demand over the time horizon. Moreover, suppose that demand is represented by a small number of demand functions, each one applying throughout a period of time in the time horizon. That is, assume that the load-duration characteristic N is piece-wise constant. For example, suppose that there were just, say, four periods, say periods A, B, C, D. Such a piece-wise constant load-duration characteristic is illustrated in figure 3, taking on the four values $N_A > N_B > N_C > N_D$.

We can imagine such a load-duration characteristic being used in a day-ahead market with market rules specifying that a clearing price would be calculated for each of the four periods based on the demand function specified for each period. In this case, a supply function consisting of steps could be used to achieve the Cournot outcome in each of the four periods. For example, suppose that the Cournot prices in the four periods were, respectively, $p_A > p_B > p_C > p_D$ and that for firm i the corresponding Cournot quantities were $q_{iA} > q_{iB} > q_{iC} > q_{iD}$.

Figure 4 shows a bid supply function that will achieve the Cournot prices and quantities in each

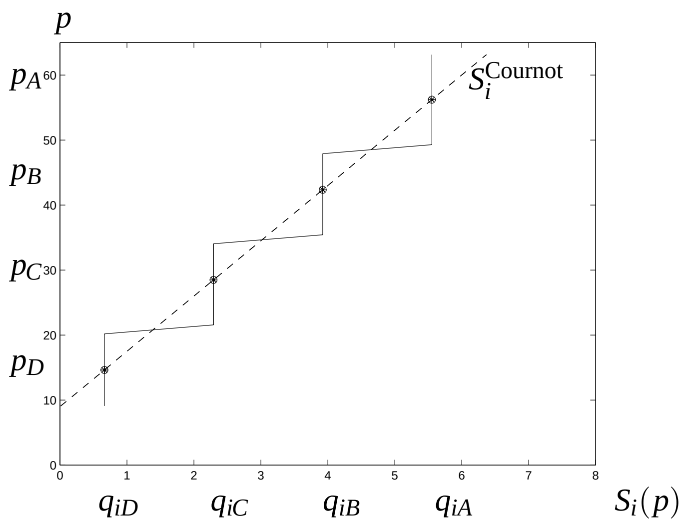

Figure 4: Bid supply function to achieve Cournot prices and quantities in a four period time horizon.

of the four periods. The dashed curve shows the Cournot supply function S_i^{Cournot}. The solid curve shows a bid function that is constant independent of price in each of four price bands around the prices $p_A > p_B > p_C > p_D$. In each band the bid supply is equal to the corresponding Cournot quantities at the prices $p_A > p_B > p_C > p_D$. If each player bids a similar step function then the Cournot outcomes can be achieved in each of the four periods.

In summary, if the demand is specified by a piece-wise constant load-duration characteristic and if there is no uncertainty in supply (that is, there are no "forced outages") then we must consider the possibility that there might be SFEs that result in Cournot prices in each period. That is, we must consider whether the assumption of a continuous load-duration characteristic is essential to proving that the SFEs lie between $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$.

In fact, for the example shown in figures 3 and 4, which is based on the symmetric three firm example of section 4.1, the exhibited supply functions are not an SFE. In particular, in period A for firm i, say, if the other firms each bid the supply function exhibited in figure 4 then the optimal response of firm i is to offer considerably more than q_{iA} because by doing so the price is depressed to the price band around p_B where the quantities offered by the other firms are significantly lower.

This is illustrated in figure 5, which shows the profit per unit time in period A versus quantity q_i supplied by firm i if the other firms each bid the supply function exhibited in figure 4. Abusing notation slightly, we write π_{iA} for the profit per unit time for firm i in period A. The bullet shows the Cournot quantity and profit for firm i in this period. It is clearly not the globally optimal profit for firm i given the supply functions of the other firms and shows that the supply function shown in figure 4 will not yield optimal profits for firm i over the time horizon.

More generally, we show in the following theorem that if the differences between the values of the load-duration characteristic in successive periods are sufficiently small then no SFE can result in Cournot outcomes in each period. We show this by constructing a response by firm i that depresses the price in each period τ to the Cournot price for the following period $\tau+1$. For convenience, the theorem is proven for the symmetric n-firm case, with the values of the load-

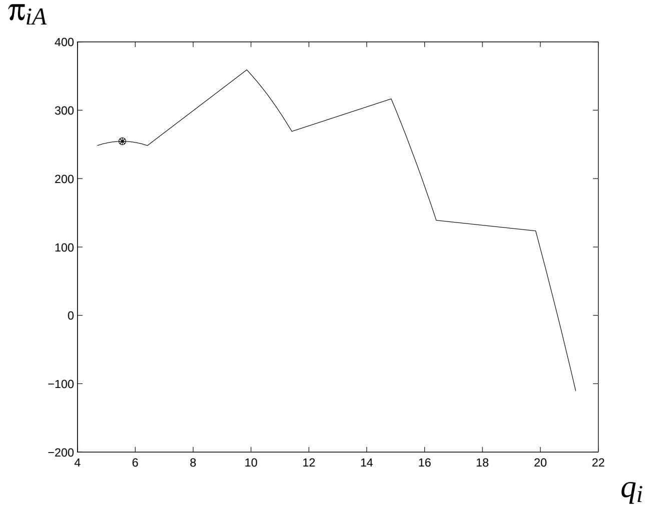

Figure 5: Profit per unit time π_{iA} versus quantity q_i for firm i in period A, assuming that the other firms each bid the supply function illustrated in figure 4.

duration characteristic evenly spaced, and assuming a linear demand-price relationship. However, the result appears to hold considerably more generally.

Theorem 1 *Suppose that there are n identical firms each with marginal cost function specified by (2) and suppose that demand is of the form (1) where the load-duration characteristic is piece-wise constant, taking on the values* $N_1 > \cdots > N_T$ *throughout each of the* $T \geq 2$ *periods in the time horizon. Moreover, suppose that the demand levels* $N_\tau, \tau = 1, \ldots, T$ *are approximately evenly spaced in the sense that:*

$$\forall \tau = 2, \ldots, T-1, (1+n+\gamma c_i)(N_{\tau-1} - N_\tau) > (n+\gamma c_i)(N_\tau - N_{\tau+1}). \tag{23}$$

Let the Cournot prices p_τ *and quantities* $q_{i\tau}, i = 1, \ldots, n$ *in each period* $\tau = 1, \ldots, T$ *be defined by the solution of:*

$$\begin{aligned} \forall \tau = 1, \ldots, T, N_\tau - \gamma p_\tau &= \sum_{i=1}^{n} \frac{1}{(c_i + 1/\gamma)}(p_\tau - a_i), \\ \forall \tau = 1, \ldots, T, \forall i = 1, \ldots, n, q_{i\tau} &= \frac{1}{(c_i + 1/\gamma)}(p_\tau - a_i). \end{aligned}$$

Suppose that the solution of these equations satisfies $q_{i\tau} > 0, \forall \tau = 1, \ldots, T, \forall i = 1, \ldots, n.$

Suppose that the firms offer supply functions $S_i, i = 1, \ldots, n$ *that are feasible and allowable and that satisfy:*

$$\forall \tau = 1, \ldots, T, \forall i = 1, \ldots, n, S_i(p_\tau) = q_{i\tau},$$

and which are continuous at $p = p_\tau, \tau = 1, \ldots, T$*. (These supply functions result in Cournot prices and quantities throughout each period, as in the four period example in figure 4.)*

If:

$$\exists \hat{T} \in \{2, \ldots, T\} \text{ such that } \forall \tau = 1, \ldots, \hat{T}-1, N_\tau - N_{\tau+1} < \frac{(N_{\tau+1} - a_i\gamma)(n + \frac{1}{2}\gamma c_i)}{\frac{1}{2}\gamma c_i(1+n+\gamma c_i)^2}, \tag{24}$$

then $S = (S_i)_{i=1,\dots,n}$ is not an SFE.

Proof On solving the conditions for the Cournot prices and quantities given the symmetric cost functions, we obtain:

$$\begin{aligned} \forall \tau = 1,\dots,T, p_\tau &= \frac{N_\tau(c_i + 1/\gamma) + na_i}{1+n+\gamma c_i}, \\ \forall \tau = 1,\dots,T, \forall j = 1,\dots,n, q_{j\tau} &= \frac{N_\tau - \gamma a_i}{1+n+\gamma c_i}. \end{aligned}$$

Note that by assumption on $N_\tau, \tau = 1,\dots,T$, we have that $p_1 > \cdots > p_T$ and $\forall i, q_{i1} > \cdots > q_{iT}$.

We consider a particular firm i and construct a response $\hat{S}_i$ by firm i to $S_{-i} = (S_j)_{j \neq i}$ that yields a profit $\pi_i(\hat{S}_i, S_{-i})$ that is higher than the profit $\pi_i(S_i, S_{-i})$. This is sufficient to show that S is not an SFE.

Define:

$$\forall \tau = 2,\dots,\hat{T}, \hat{q}_{i\tau} = N_{\tau-1} - N_\tau + q_{i\tau}.$$

We claim that:

$$\forall \tau = 2,\dots,\hat{T}-1, \hat{q}_{i\tau} > \hat{q}_{i,\tau+1}.$$

To see this, note that for $\tau = 2,\dots,\hat{T}-1$,

$$\begin{aligned} \hat{q}_{i\tau} &= N_{\tau-1} - N_\tau + q_{i\tau}, \text{ by definition,} \\ &= N_{\tau-1} - N_\tau + \frac{N_\tau - \gamma a_i}{1+n+\gamma c_i}, \text{ by direct calculation,} \\ &= N_{\tau-1} - N_\tau - \frac{N_\tau - N_{\tau+1} + N_{\tau+1} - \gamma a_i}{1+n+\gamma c_i}, \\ &> (N_\tau - N_{\tau+1})\frac{n+\gamma c_i}{1+n+\gamma c_i} + \frac{N_\tau - N_{\tau+1}}{1+n+\gamma c_i} + \frac{N_{\tau+1} - \gamma a_i}{1+n+\gamma c_i}, \\ &\quad \text{by the assumption of approximate even spacing (23),} \\ &= N_\tau - N_{\tau+1} + \frac{N_{\tau+1} - \gamma a_i}{1+n+\gamma c_i}, \\ &= \hat{q}_{i,\tau+1}, \text{ by definition.} \end{aligned}$$

Also, we have that:

$$\begin{aligned} \hat{q}_{i\hat{T}} &= N_{\hat{T}-1} - N_{\hat{T}} + q_{i\hat{T}}, \text{ by definition,} \\ &> q_{i\hat{T}}, \end{aligned}$$

since $N_{\hat{T}-1} > N_{\hat{T}}$.

Now let $\hat{S}_i$ be any feasible and allowable supply function for firm i that satisfies:

$$\begin{aligned} \forall \tau = 1,\dots,\hat{T}-1, \hat{S}_i(p_{\tau+1}) &= \hat{q}_{i,\tau+1}, \\ \forall p < p_{\hat{T}}, \hat{S}_i(p) &= S_i(p). \end{aligned}$$

where we note that, by construction:

- $p_2 > \ldots > p_{\hat{T}}$,
- $\hat{q}_{i2} > \ldots > \hat{q}_{i\hat{T}}$, and
- $\hat{S}_i(p_{\hat{T}}) = \hat{q}_{i\hat{T}} > q_{i\hat{T}} = S_i(p_{\hat{T}})$, so that there is a discontinuity in $\hat{S}_i$ at $p_{\hat{T}}$,

so that such feasible and allowable supply functions exist.

We consider the profits accruing in each period to firm i if it bids S_i and then consider the profits if it bids $\hat{S}_i$. With a slight abuse of notation, let $\pi_{i\tau}$ be the profit per unit time accruing to firm i in period $\tau = 1, \ldots, T$, if it bids S_i and the other firms bid S_{-i}. We note that $\pi_i(S_i, S_{-i})$ is a time-weighted average of the values $\pi_{i\tau}$, where the weights are given by the length of time of each period τ.

Similarly, let $\hat{\pi}_{i\tau}$ be the profit per unit time accruing to firm i in period $\tau = 1, \ldots, T$, if it bids $\hat{S}_i$ and the other firms bid S_{-i}. The profit $\pi_i(\hat{S}_i, S_{-i})$ is a time-wighted average of the values $\hat{\pi}_{i\tau}$, with the weights again equal to the length of each period τ.

We show that:

$$\begin{aligned} \hat{\pi}_{i\tau} &> \pi_{i\tau}, \tau = 1, \ldots, \hat{T} - 1, \\ \hat{\pi}_{i\tau} &= \pi_{i\tau}, \tau = \hat{T}, \ldots, T. \end{aligned}$$

Since $\pi_i(S_i, S_{-i})$ and $\pi_i(\hat{S}_i, S_{-i})$ are time weighted averages of the $\pi_{i\tau}$ and $\hat{\pi}_{i\tau}$, respectively, with the same weights, this will suffice to show that $\pi_i(\hat{S}_i, S_{-i}) > \pi_i(S_i, S_{-i})$ and that S cannot be an SFE.

We first note that since the supply functions are all non-decreasing there is at most one price that satisfies the market clearing conditions (6) in each period τ. In fact, by construction there is a solution of (6) in every period. We consider first the case where firm i bids S_i and then consider the case where firm i bids $\hat{S}_i$.

If firm i bids S_i, then note that:

$$\begin{aligned} \forall \tau = 1, \ldots, T, N_\tau - \gamma p_\tau &= \sum_{j=1}^{n} \frac{1}{(c_j + 1/\gamma)} (p_\tau - a_j), \text{ by definition of } p_\tau, \\ &= \sum_{j=1}^{n} q_{j\tau}, \text{ by definition of } q_{j\tau}, \\ &= \sum_{j=1}^{n} S_j(p_\tau), \text{ by definition of } S_j. \end{aligned}$$

That is, if firm i bids S_i then the clearing price in period τ is p_τ. With some calculation, we obtain that:

$$\forall \tau = 1, \ldots, T, \pi_{i\tau} = \frac{(N_\tau - a_i\gamma)^2 (\frac{1}{2} c_i \gamma + 1)}{\gamma (1 + n + \gamma c_i)^2}.$$

Now we consider the profit per unit time if firm i bids $\hat{S}_i$. There are three cases:

- In periods $\tau = \hat{T} + 1, \ldots, T$, because of the definition of $\hat{S}_i$, the clearing price is $p = p_\tau$, the quantity supplied by firm i is $q_{i\tau}$, and so $\hat{\pi}_{i\tau} = \pi_{i\tau}$.

- In period $\tau = \hat{T}$, we have that:

$$\begin{aligned}
\hat{S}_i(p_{\hat{T}}) &= \hat{q}_{i\hat{T}}, \\
N_{\hat{T}} - \gamma p_{\hat{T}} &= \sum_{j=1}^{n} q_{j\hat{T}}, \\
&< \hat{q}_{i\hat{T}} + \sum_{j\neq i} q_{i\hat{T}}, \\
&= \hat{S}_i(p_{\hat{T}}) + \sum_{j\neq i} S_j(p_{\hat{T}}), \\
N_{\hat{T}} - \gamma p_{\hat{T}} &= \lim_{p\uparrow p_{\hat{T}}} \left(\hat{S}_i(p) + \sum_{j\neq i} S_j(p) \right),
\end{aligned}$$

  where by $p \uparrow p_{\hat{T}}$ we mean the limit as price approaches $p_{\hat{T}}$ from below. Therefore, by (6), the clearing price for period $\hat{T}$ is $p = p_{\hat{T}}$ and, moreover, firm i will be called on to supply $q_{i\hat{T}} < \hat{S}_i(p_{\hat{T}})$ in this period. That is, $\hat{\pi}_{i\hat{T}} = \pi_{i\hat{T}}$.

- For periods $\tau = 1, \ldots, \hat{T} - 1$, we have that:

$$\begin{aligned}
\forall \tau = 1, \ldots, \hat{T}-1, N_\tau - \gamma p_{\tau+1} &= N_\tau - N_{\tau+1} + N_{\tau+1} - \gamma p_{\tau+1}, \\
&= N_\tau - N_{\tau+1} + \sum_{j=1}^{n} q_{j,\tau+1}, \\
&\quad \text{by definition of } p_{\tau+1} \text{ and } q_{j,\tau+1}, \\
&= N_\tau - N_{\tau+1} + q_{i,\tau+1} + \sum_{j\neq i} q_{j,\tau+1}, \\
&= \hat{q}_{i,\tau+1} + \sum_{j\neq i} q_{j,\tau+1}, \text{ by definition of } \hat{q}_{i,\tau+1}, \\
&= \hat{S}_i(p_{\tau+1}) + \sum_{j\neq i} S_j(p_{\tau+1}).
\end{aligned}$$

  That is, in period $\tau = 1, \ldots, \hat{T} - 1$, the market clearing price is $p = p_{\tau+1}$ and the quantity for firm i is $\hat{q}_{i,\tau+1}$. With some calculation, we obtain that:

$$\begin{aligned}
\forall \tau = 1, \ldots, \hat{T}-1, \hat{\pi}_{i\tau} &= \frac{(N_{\tau+1} - a_i\gamma)(\frac{1}{2}c_i\gamma + 1) - \frac{1}{2}c_i\gamma(1+n+\gamma c_i)(N_\tau - N_{\tau+1})}{\gamma(1+n+\gamma c_i)^2} \times \\
&\quad (N_\tau - \gamma a_i + (n+\gamma c_i)(N_\tau - N_{\tau+1})).
\end{aligned}$$

Now note that:

$$\begin{aligned}
&\forall \tau = 1, \ldots, \hat{T}-1, \\
&\frac{1}{(N_\tau - N_{\tau+1})\frac{1}{2}c_i}(\hat{\pi}_{i\tau} - \pi_{i\tau}) = -N_\tau + N_{\tau+1} + \frac{(N_{\tau+1} - a_i\gamma)(n + \frac{1}{2}\gamma c_i)}{\frac{1}{2}\gamma c_i(1+n+\gamma c_i)^2}, \\
&\qquad > 0,
\end{aligned}$$

by hypothesis. Therefore, $\hat{\pi}_i > \pi_i$. □

Corollary 2 *Consider the possible SFEs as T is varied. Suppose that, for each T, we define the periods so that $N_\tau - N_{\tau+1}$ is independent of τ (so that the even spacing condition (23) is satisfied) and so that the Cournot quantities are always strictly positive. Then, for sufficiently large number of periods T there can be no SFE that achieves Cournot prices and quantities in each period.*

Proof Note that in the hypothesis of theorem 1, the even spacing condition (23) is satisfied since:

$$\begin{aligned}\forall \tau = 2,\ldots,T-1, (1+n+\gamma c_i)(N_{\tau-1} - N_\tau) &= (1+n+\gamma c_i)(N_\tau - N_{\tau+1}),\\ &> (n+\gamma c_i)(N_\tau - N_{\tau+1}).\end{aligned}$$

Moreover, in (24) note that the right hand side is independent of T and strictly greater than zero since:

$$\begin{aligned}(\forall j = 1,\ldots,n, q_{j,\tau+1} > 0) &\Rightarrow N_{\tau+1} > \gamma p_{\tau+1},\\ &\Rightarrow N_{\tau+1} > \gamma a_i,\end{aligned}$$

since $q_{i,\tau+1} > 0$ by hypothesis. Consequently, for sufficiently large T we must have that (24) is satisfied. □

Corollary 2 shows that if there are sufficiently many periods then Cournot prices in each period will not be an equilibrium outcome if supply functions must be consistent across all periods, even if there is no uncertainty in demand in each period and no uncertainty in supply. For the example load-duration characteristic of figure 3, the profit function shown in figure 5 shows that as few as four periods can be enough to prevent Cournot prices from being an equilibrium in each period. In typical day-ahead markets there are usually many more than four demand periods, with 24 or 48 being typical. Corollary 2 suggests that Cournot prices in each period cannot be an equilibrium outcome of such markets.

We have not proved that the conditions for equilibria for a large number of periods converge, as the number of periods grows large, to the conditions for SFE with a continuous load-duration characteristic. However, corollary 2 is suggestive that such a result may hold.

Moreover, uncertainty in each period due to either:

- uncertainty in the demand functions or
- uncertainly in the supply of other firms due to “forced outages” of generation,

effectively acts to “smooth out” the load-duration characteristic, assuming that the supply functions must be bid in advance of the demand and forced outages becoming known. For example, in figure 6, uncertainty in demand in each of the four demand periods has been incorporated into the load-duration characteristic faced by the firms.

If the demand uncertainty in each period is large enough then the distribution of demand for successive periods can overlap. Similarly, if the uncertainty in the supply of other firms in each period is large enough then the residual demand faced by a firm for successive periods can overlap. With large enough uncertainty in each period, the residual demand faced by a firm would be distributed continuously, even if the market is cleared with a single price applying throughout each period. In this case, the supply function equilibrium is the appropriate equilibrium model.

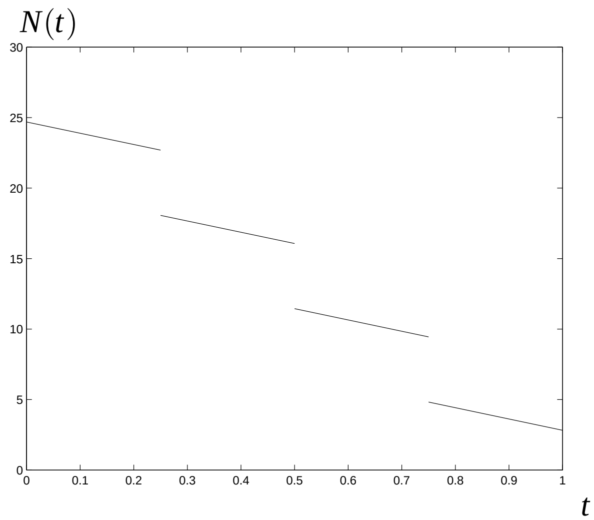

Figure 6: Piece-wise affine load-duration characteristic.

4.4 Strictly increasing marginal cost functions

The cost function (2) represents the variation of marginal cost with production. In contrast, some authors assume that the marginal costs are constant. For example, von der Fehr and Harbord assume that [21]:

- each firm has constant marginal cost across its full range of production and
- each firm's marginal cost is different from all other firms' marginal costs.

Transcribed into our notation, they assume:

$$a_i \neq a_j, 1 \leq i \neq j \leq n, \quad (25)$$
$$c_i = 0, \forall i. \quad (26)$$

As another example, some of the auctions of shares literature assumes that the marginal costs are constant and equal for all firms [2, 3, 24]. As discussed in the introduction, the assumption of constant marginal costs is not realistic for a firm that owns a portfolio of generation.

Nevertheless, suppose for the sake of argument that marginal costs are constant across production for each firm. That is, suppose that (26) holds. We claim that there is then no affine SFE solution in any market where at least two prices are realized and the demand slope is strictly positive. To see this, consider an affine function of the form:

$$\forall i, \forall p \in \mathbb{P}, S_i^{\star\text{affine}}(p) = \beta_i p - \alpha_i.$$

We substitute into (13) to obtain,

$$\forall i, \forall p \in \mathbb{P}, \beta_i p - \alpha_i = (p - a_i)\left(\gamma + \sum_{j \neq i} \beta_j\right).$$

Since this expression is identically true for all realized prices in the set $\mathbb{P}$ and since $\mathbb{P}$ has at least two elements by assumption, we can equate like coefficients of powers of p. Equating the coefficient of p^1 yields (16) for the particular case $c_i = 0$ for each firm. We obtain:

$$\forall i, \beta_i = \sum_{j \neq i} \beta_j + \gamma.$$

Summing this expression over all firms, yields:

$$\begin{aligned} \sum_{i=1}^{n} \beta_i &= \sum_{i=1}^{n} \sum_{j \neq i} \beta_j + n\gamma, \\ &= (n-1) \sum_{i=1}^{n} \beta_i + n\gamma. \end{aligned}$$

Rearranging, we obtain:

$$(n-2) \sum_{i=1}^{n} \beta_i + n\gamma = 0,$$

which has no solution for $n \geq 2$ in non-negative values of β_i given a strictly positive γ and has no solution for $n \geq 3$ in positive values of β_i given a non-negative γ.

In the special case of $n = 2$ firms and $\gamma = 0$, McAdams [4] (referring to Wilson [2]) illustrates that there are multiple equilibria. In particular, for any sufficiently large $\beta \in \mathbb{R}_{++}$, the functions:

$$\forall i, \forall p \in \mathbb{P}, S_i^{\star \text{affine}}(p) = \beta(p - a_i),$$

specify an affine SFE.

In summary, the assumption of constant marginal costs across the production range for each firm results in there being no affine equilibrium in the SFE framework, except in the special case of two firms and a price slope of $\gamma = 0$. In section 5, we will see that in some circumstances all equilibria besides the affine SFE are unstable. Combining these two observations, if marginal costs are constant then thre are no stable SFEs and so the SFE framework is inapplicable.

4.5 Functional form of the supply functions

In [21], von der Fehr and Harbord argue that "the equilibria found by Green and Newbery (1991) in their model do not generalise to the case in which individual generating sets are of positive size." That is, if the cost functions reflect economic dispatch of a portfolio of units having finite size, von der Fehr and Harbord claim that the Green and Newbery analysis is not applicable. The argument of von der Fehr and Harbord rests on their proposition 1 [21, pp 533–534], which assumes that each individual generating unit must offer all of its capacity at a single price. More generally, Nyborg shows that when certain discreteness requirements are placed on the bids then the equilibria that can arise differ qualitatively from the equilibria that we describe [24, Section 4].

While it is true that typical market rules limit the number of "blocks" that can be bid for a given generating unit, there is nevertheless considerable flexibility to offer generation capacity in several blocks having different prices. Moreover, the size of the blocks can usually be modified at will. The aggregate supply of a portfolio of such bids could approximate a smooth supply function to any required accuracy. It is ultimately an empirical question as to whether a smooth functional representation or a representation in terms of blocks best describes the way strategic players represent their decisions.

5 Stability of equilibria

In this section, we discuss the stability of equilibria and present conditions for an SFE to be unstable. In practice, an unstable equilibrium is unlikely to be observed. Consequently, we restrict attention to stable equilibria. In [14], Anderson and Xu present conditions for an equilibrium in a similar market structure to be stable. We have not adapted the Anderson and Xu analysis; however, its conclusions are consistent with ours.

To introduce the relevance of stability, recall the symmetric, uncapacitated, no price cap case discussed in section 4.1. As discussed in section 4.1, the range between the symmetric most competitive and symmetric least competitive SFEs can be very wide. We will show, however, that all of the SFEs between the most competitive SFE $S^{\star\text{comp}}$ and the least competitive symmetric SFE $S^{\star\text{Cournot}}$, except for the affine SFE $S^{\star\text{affine}}$, are unstable. Consequently, only the affine SFE $S^{\star\text{affine}}$ will be exhibited in practice.

In section 5.1, we develop the theorem characterizing stability in the context of an SFE where the cost functions are not necessarily symmetric. In section 5.2, we discuss the implications. The theorem as stated applies only to SFEs that are obtained as non-decreasing solutions to the differential equation (18). The reason for this restriction is due to the technical difficulty of characterizing optimal responses when the profit function for a player is non-concave. However, we hypothesize that the theorem holds in much more generality than we have stated it. In particular, the numerical results in sections 9 and 11 are essentially consistent with the conclusion of the theorem.

5.1 Analysis

In this section we first define some particular sets of functions, prove some technical lemmas and then use them in the main theorem. The basic approach involves considering a supply function equilibrium $S^\star = (S_i^\star)_{i=1,\dots,n}$ that is a non-decreasing solution of (18). We then define a perturbation $S_i^\varepsilon, i = 1,\dots,n$ of $S_i^\star, i = 1,\dots,n$. In the case that the SFE $S^\star$ is less competitive than the affine SFE, the perturbed functions S_i^ε involve "bending" the SFE functions $S_i^\star$ to be slightly more competitive. We then find that the optimal response by firm i to $S_j^\varepsilon, j \neq i$ involves an even larger bend. Similarly, in the case that the SFE is more competitive than the affine SFE, the perturbed functions are bent to be slightly less competitive. The optimal response is again an even larger bend. In summary, a small perturbation to the equilibrium results in a response with a larger perturbation so that equilibrium is not stable.

It is relatively easy to construct *an* optimal response by firm i to $S_j^\varepsilon, j \neq i$ that deviates more from $S_i^\star$ than does S_i^ε. However, there is a continuum of such optimal responses. Most of the technical effort in the the proofs involves showing that *every* optimal response by firm i to $S_j^\varepsilon, j \neq i$ deviates more from $S_i^\star$ than does S_i^ε.

We begin with:

Definition 3 Suppose that demand is of the form (1). Consider bid supply functions $S_i \in \mathbb{S}_i$ defined on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. Suppose that supply and demand intersect at the peak demand time $t = 0$ at a price $p_0 \in \mathbb{P}$. We call p_0 the "peak realized price for the bids $S_i, i = 1,\dots,n$." Suppose that supply and demand intersect at the minimum demand time $t = 1$ at a price $p_1 \in \mathbb{P}$. We call p_1 the "minimum realized price for the bids $S_i, i = 1,\dots,n$." $\square$

In the symmetric case, if the players bid the least competitive symmetric equilibrium $S^{\star\text{Cournot}}$ then the peak realized price is p_0^{Cournot}. If the players bid the most competitive symmetric equilibrium $S^{\star\text{comp}}$ then the peak realized price is p_0^{comp}.

Definition 4 Suppose that demand is of the form (1) and that firm i has marginal costs C_i' for $i = 1,\ldots,n$. Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1,\ldots,n$ of the differential equation (18) on an interval of prices $\mathbb{P} = [\underline{p},\overline{p}]$. Suppose that the $S_i^\star$ are non-decreasing and that the peak realized price for the bids $S_i^\star, i = 1,\ldots,n$ is $p_0^\star$. By definition of the differential equation, the $S_i^\star$ are continuously differentiable.

Let $\underline{p} < p^\varepsilon < p_0^\star$ and define $S^\varepsilon : [\underline{p},\overline{p}] \to \mathbb{R}^n$ by:

$$\forall i = 1,\ldots,n, \forall p \in [\underline{p},\overline{p}], S_i^\varepsilon(p) = \begin{cases} S_i^\star(p), & \text{if } \underline{p} \leq p < p^\varepsilon, \\ S_i^\star(p^\varepsilon) + \beta_i^\varepsilon(p - p^\varepsilon), & \text{if } p^\varepsilon \leq p \leq \overline{p}, \end{cases}$$

where $\beta_i^\varepsilon = S_i^{\star\prime}(p^\varepsilon), i = 1,\ldots,n$. For each firm i, $S_i^\varepsilon(p)$ equals $S_i^\star(p)$ for prices p between $\underline{p}$ and p^ε. For prices p greater than or equal to p^ε, the slope of $S_i^\varepsilon(p)$ is constant at $\beta_i^\varepsilon = S_i^{\star\prime}(p^\varepsilon)$. By definition, S_i^ε is continuously differentiable, since $S_i^\star$ is continuously differentiable.

We call S_i^ε the “linear continuation of $S_i^\star$ from price p^ε.” We call $S_i^\varepsilon(p_0^\star)$ the “maximum relevant supply of the linear continuation of $S_i^\star$.” □

Definition 4 is illustrated in figure 7 for a supply function that is concave. The two solid curves depict the functions:

- $S_i^\star$ and
- the residual demand faced by firm i at peak, $D(0,\bullet) - \sum_{j\neq i} S_j^\star(\bullet)$.

These functions intersect at the point $(p_0^\star, S_i^\star(p_0^\star))$, which is shown as the leftmost of the pair of bullets, $\bullet$, near the top of the figure. The point $(p^\varepsilon, S_i^\star(p^\varepsilon))$ is illustrated as the bullet that is towards the bottom of the figure. The dashed curve shows the function S_i^ε in the interval $[p^\varepsilon, p_0^\star]$, with the point $(p_0^\star, S_i^\varepsilon(p_0^\star))$ shown as the rightmost of the pair of bullets near the top of the figure.

The supply functions in figure 7 and in all subsequent figures are shown with price p on the vertical axis and the values of production S_i on the horizontal axis. In lemma 6 and subsequently, we will consider supply functions S_i that are strictly concave or strictly convex. Despite the pictorial representation of price versus quantity, when we specify that S_i is concave, for example, we mean that the function S_i is concave as a function of p.

Definition 5 Suppose that demand is of the form (1) and that firm i has marginal costs C_i' for $i = 1,\ldots,n$. Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1,\ldots,n$ of the differential equation (18) on an interval of prices $\mathbb{P} = [\underline{p},\overline{p}]$. Suppose that the $S_i^\star$ are non-decreasing and that the peak realized price for the bids $S_i^\star, i = 1,\ldots,n$ is $p_0^\star$.

Let $\underline{p} < p^\varepsilon < p_0^\star$ and let S_i^ε be the linear continuation of $S_i^\star$ from price p^ε. Suppose that firm i faces supply $S_j^\varepsilon, j \neq i$. In the following lemma, we will consider one particular profit maximizing feasible and allowable response by firm i to the functions $S_j^\varepsilon, j \neq i$. In general there can be a multiplicity of optimal responses by player i. We will construct one such function and write $\hat{S}_i \in \mathbb{S}_i$ for it. We call $\hat{S}_i(p_0^\star)$ the “maximum relevant supply of the firm i optimal response to $S_j^\varepsilon, j \neq i$.” □

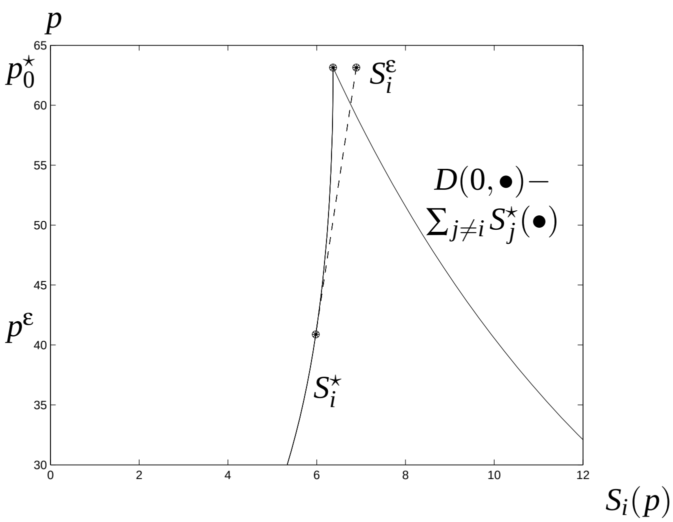

Figure 7: Illustration of definition 4.

Lemma 3 *Suppose that demand is of the form (1) and that each firm $i = 1,\dots,n$ has affine marginal costs C_i' of the form (2) and that the capacity of each firm is arbitrarily large. Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1,\dots,n$ of the differential equation (18) on an interval of prices $\mathbb{P} = [\underline{p},\overline{p}]$. Suppose that the $S_i^\star$ are non-decreasing so that $S_i^\star \in \mathbb{S}_i, i = 1,\dots,n$ and that the peak realized price for the bids $S_i^\star, i = 1,\dots,n$ is $p_0^\star$.*

Let $a_i < p^\varepsilon < p_0^\star$ and let S_i^ε be the linear continuation of $S_i^\star$ from price p^ε. We claim that the following function $\hat{S}_i$ is an optimal response to $S_j^\varepsilon, j \neq i$:

$$\forall p \in [\underline{p},\overline{p}], \hat{S}_i(p) = \begin{cases} S_i^\star(p), & \text{if } \underline{p} \leq p < p^\varepsilon, \\ S_i^\star(p^\varepsilon) + \hat{\beta}_i(p - p^\varepsilon), & \text{if } p^\varepsilon \leq p \leq \overline{p}, \end{cases} \tag{27}$$

where:

$$\forall i = 1,\dots,n, \hat{\beta}_i = \frac{\sum_{j\neq i} \beta_j^\varepsilon + \gamma}{1 + c_i(\sum_{j\neq i} \beta_j^\varepsilon + \gamma)}. \tag{28}$$

Proof As in the derivation of the equilibrium conditions in section 3.1, we first neglect the non-decreasing constraints and consider, for each p, the optimal response of firm i to the bids of the other firms. We then check that the function as defined satisfies the non-decreasing constraints.

We consider the two (just overlapping) intervals of prices $\underline{p} \leq p \leq p^\varepsilon$ and $p^\varepsilon \leq p \leq p_0^\star$ separately. For prices $\underline{p} \leq p \leq p^\varepsilon$, we claim that the quantity $\hat{S}_i(p) = S_i^\star(p)$ is the unique globally optimal response at price p to $\hat{S}_j(p), j \neq p$. This is true by definition of the differential equation (18) because in this range of prices we have that $S_j^{\varepsilon\prime} = S_j^{\star\prime}, j \neq i$. This verifies the first line of the right hand side of (27) and, in addition, shows that $\hat{S}_i(p^\varepsilon) = S_i^\star(p^\varepsilon)$. We will use this last fact to help evaluate terms in the optimal response for prices $p^\varepsilon \leq p \leq p_0^\star$. The function $\hat{S}_i$ is continuous at p^ε because of the continuity of the derivatives of S_j^ε at p^ε.

For prices $p^\varepsilon \leq p \leq p_0^\star$, the optimality condition (11) states that:

$$\hat{S}_i(p) = (p - a_i - c_i\hat{S}_i(p))\left(\gamma + \sum_{j \neq i} \beta_j^\varepsilon\right).$$

Rearranging this yields the unique globally optimal response at price p of:

$$\hat{S}_i(p) = \hat{\beta}_i(p - a_i),$$

where $\hat{\beta}_i$ is as defined in (28). Substituting in the price $p = p^\varepsilon$, we obtain:

$$\hat{S}_i(p^\varepsilon) = S_i^\star(p^\varepsilon) = \hat{\beta}_i(p^\varepsilon - a_i), \tag{29}$$

so that:

$$\begin{aligned} \forall p \in [p^\varepsilon, \overline{p}], \hat{S}_i(p) &= \hat{\beta}_i(p - a_i), \\ &= \hat{\beta}_i(p^\varepsilon - a_i) + \hat{\beta}_i(p - p^\varepsilon), \\ &= S_i^\star(p^\varepsilon) + \hat{\beta}_i(p - p^\varepsilon), \end{aligned}$$

by (29). This verifies the second line of the right hand side of (27).

Now we must check that $\hat{S}_i$, as defined, satisfies the non-decreasing constraints. By definition, $\hat{S}_i$ satisfies the non-decreasing constraints for $\underline{p} \leq p < p^\varepsilon$. Moreover, $\hat{S}_i$ satisfies the non-decreasing for $p^\varepsilon \leq p \leq \overline{p}$ because $\hat{\beta}_i \geq 0$ since it is the ratio of two positive numbers because $\gamma \geq 0$ and $\beta_i^\varepsilon \geq 0$. Since $\hat{S}_i$ is continuous it therefore satisfies the non-decreasing constraints for $\underline{p} \leq p \leq \overline{p}$.
□

Lemma 3 is illustrated in figure 8 for supply functions that are concave. (The price axis is scaled differently to figure 7.) As previously, the function $S_i^\star$ is shown solid. The function S_i^ε is shown dashed on the interval $[p^\varepsilon, p_0^\star]$ and the function $\hat{S}_i$ is shown dotted on the same interval. The points $(p_0^\star, S_i^\star(p_0^\star))$ and $(p^\varepsilon, S_i^\star(p^\varepsilon))$ are shown as bullets. Although $S^\star$ is an equilibrium, neither $S^\varepsilon = (S_i^\varepsilon)_{i=1,\ldots,n}$ nor $(\hat{S}_i)_{i=1,\ldots,n}$ are equilibria. However, for each i, $\hat{S}_i$ is an optimal response to $S_j^\varepsilon, j \neq i$.

Definition 6 Suppose that the assumptions of lemma 3 hold. Suppose that firm i bids the function $\hat{S}_i$ while the other firms bid the functions $S_j^\varepsilon, j \neq i$. We write $\hat{p}_{0i}$ for the peak realized price for these bids and we write $\hat{p}_{1i}$ for the minimum realized price for these bids. We call $\hat{S}_i(\hat{p}_{0i})$ the "peak realized supply given firm i optimal response to $S_j^\varepsilon, j \neq i$." □

Lemma 4 *Suppose that the assumptions of lemma 3 hold. Then the set of all optimal response functions for firm i to $S_j^\varepsilon, j \neq i$ is the set of all feasible non-decreasing functions on $[\underline{p}, \overline{p}]$ that match the function $\hat{S}_i$ on the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$, where $\hat{S}_i$ was defined in lemma 3.*

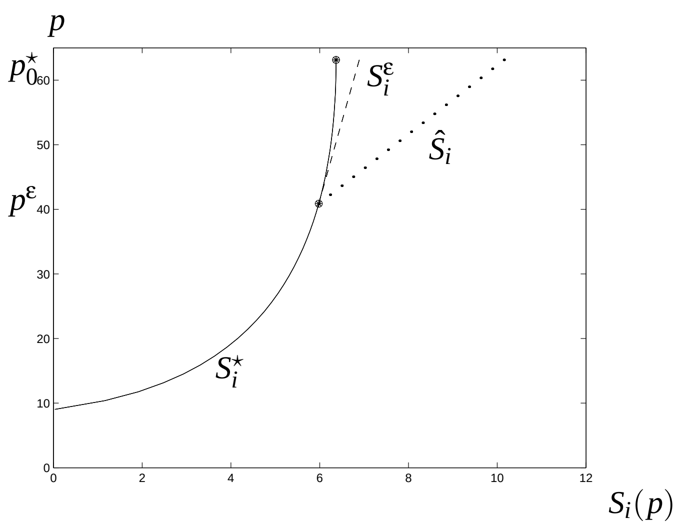

Figure 8: Illustration of lemma 3.

Proof Lemma 3 exhibits one possible optimal response by firm i to the bids $S_j^\varepsilon, j \neq i$, namely $\hat{S}_i$. For each price in the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$, the value of $\hat{S}_i(p)$ defined in lemma 3 is the unique globally optimal response at that price. That is, for prices in the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$, the values of the optimal response for firm i are uniquely determined. However, for prices lower than $\hat{p}_{1i}$ or higher than $\hat{p}_{0i}$, the value of $\hat{S}_i$ is irrelevant because prices outside the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$ are not realized. Any non-decreasing function that matches $\hat{S}_i$ on the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$ will also be an optimal response to $S_j^\varepsilon, j \neq i$ so long as the function does not violate the capacity constraints. □

We will be interested in considering an element $\underline{\hat{S}}_i$ of the set of optimal responses to $S_j^\varepsilon, j \neq i$ whose maximum value is minimized. This element $\underline{\hat{S}}_i$ will be the closest optimal response to $S_i^\star$ in the sense of a norm to be defined later. One such function $\underline{\hat{S}}_i$ is defined by:

$$\forall p \in [\underline{p}, \overline{p}], \underline{\hat{S}}_i(p) = \min\{\hat{S}_i(p), \hat{S}_i(\hat{p}_{0i})\}, \tag{30}$$

which matches $\hat{S}_i$ on the interval $[\underline{p}, \hat{p}_{0i}]$ but has constant value $\hat{S}_i(\hat{p}_{0i})$ for prices in the interval $[\hat{p}_{0i}, \overline{p}]$.

In the following lemmas and corollary, we consider the the variation of certain quantities with p^ε as it decreases from $p_0^\star$. We then turn to a formal definition of stability of equilibrium.

Lemma 5 *Suppose that the assumptions of lemma 3 hold. Consider the following expressions:*

- $S_i^\varepsilon(p_0^\star)$, *the maximum relevant supply of the linear continuation of* $S_i^\star$,
- $\hat{S}_i(p_0^\star)$, *the maximum relevant supply of the firm i optimal response to* $S_j^\varepsilon, j \neq i$.
- $\hat{p}_{0i}$, *the peak realized price given firm i optimal response to* $S_j^\varepsilon, j \neq i$, *and*
- $\hat{S}_i(\hat{p}_{0i})$ *the peak realized supply given firm i optimal response to* $S_j^\varepsilon, j \neq i$.

In each case, we view the expression as an implicit function of p^ε and consider the derivative of it with respect to p^ε, evaluated at $p_0^\star$. (Since some of the functions are not defined uniquely for prices greater than $p_0^\star$, strictly speaking we will evaluate the derivative only for movements in the direction of decreasing p^ε.) The derivatives of these expressions with respect to p^ε evaluated at $p^\varepsilon = p_0^\star$ are, respectively, equal to:

- 0,
- $\beta_i^\star - \hat{\beta}_i$,
- $-\frac{\beta_i^\star - \hat{\beta}_i}{\sum_{j \neq i} \beta_i^\star + \hat{\beta}_i + \gamma}$, *and*
- $\frac{(\beta_i^\star - \hat{\beta}_i)(\sum_{j \neq i} \beta_i^\star + \gamma)}{\sum_{j \neq i} \beta_i^\star + \hat{\beta}_i + \gamma}$,

where:

$$\forall i = 1, \ldots, n, \beta_i^\star = S_i^{\star\prime}(p_0^\star).$$

Proof For the first item, note that:

$$S_i^\varepsilon(p_0^\star) = S_i^\star(p^\varepsilon) + S_i^{\star\prime}(p^\varepsilon)(p_0^\star - p^\varepsilon).$$

Totally differentiating with respect to p^ε yields:

$$\begin{aligned} \frac{d[S_i^\varepsilon(p_0^\star)]}{dp^\varepsilon}(p^\varepsilon) &= S_i^{\star\prime}(p^\varepsilon) + S_i^{\star\prime\prime}(p^\varepsilon)(p_0^\star - p^\varepsilon) - S_i^{\star\prime}(p^\varepsilon), \\ &= S_i^{\star\prime\prime}(p^\varepsilon)(p_0^\star - p^\varepsilon), \end{aligned}$$

where the double superscript $\prime$ indicates the second derivative. Evaluating this expression at $p^\varepsilon = p_0^\star$ yields zero.

For the second item, note that:

$$\hat{S}_i(p_0^\star) = S_i^\star(p^\varepsilon) + \hat{\beta}_i(p_0^\star - p^\varepsilon).$$

Differentiating with respect to p^ε yields:

$$\frac{d[\hat{S}_i(p_0^\star)]}{dp^\varepsilon}(p^\varepsilon) = S_i^{\star\prime}(p^\varepsilon) + \frac{d\hat{\beta}_i}{dp^\varepsilon}(p^\varepsilon)(p_0^\star - p^\varepsilon) - \hat{\beta}_i.$$

Evaluating this expression at $p^\varepsilon = p_0^\star$ yields $\beta_i^\star - \hat{\beta}_i$

The third item involves the price that results at peak demand from bids. The price is implicitly determined by the solution of (5). We use the implicit function theorem to show that the price $\hat{p}_{0i}$ is a well-defined function of p^ε for p^ε in a neighborhood of $p_0^\star$ and to calculate the derivative.

At the peak demand and given that firm i bids $\hat{S}_i$ while the other firms bid the functions S_j^ε, $j \neq i$, equation (5) becomes, after rearranging:

$$\gamma\hat{p}_{0i} + \sum_{j \neq i} S_j^\varepsilon(\hat{p}_{0i}) + \hat{S}_i(\hat{p}_{0i}) - N(0) = 0.$$

For $p^{\varepsilon} = p_0^{\star}$, the solution to this equation is $\hat{p}_{0i} = p_0^{\star}$. Applying the implicit function theorem we obtain that $\hat{p}_{0i}$ is a well-defined and differentiable function of p^{ε} within a neighborhood of $p_0^{\star}$. In particular,

$$\frac{d\hat{p}_{0i}}{dp^{\varepsilon}}(p_0^{\star}) = -\frac{\beta_i^{\star} - \hat{\beta}_i}{\sum_{j \neq i} \beta_i^{\star} + \hat{\beta}_i + \gamma}.$$

For the last item, note that:

$$\hat{S}_i(\hat{p}_{0i}) = S_i^{\star}(p^{\varepsilon}) + \hat{\beta}_i(\hat{p}_{0i} - p^{\varepsilon}),$$

so that

$$\frac{d[\hat{S}_i(\hat{p}_{0i})]}{dp^{\varepsilon}}(p^{\varepsilon}) = S_i^{\star\prime}(p^{\varepsilon}) + \frac{d\hat{\beta}_i}{dp^{\varepsilon}}(p^{\varepsilon})(\hat{p}_{0i} - p^{\varepsilon}) + \hat{\beta}_i \left(\frac{d\hat{p}_{0i}}{dp^{\varepsilon}}(p^{\varepsilon}) - 1 \right).$$

Evaluating this at $p^{\varepsilon} = p_0^{\star}$ yields:

$$\begin{aligned} \frac{d[\hat{S}_i(\hat{p}_{0i})]}{dp^{\varepsilon}}(p_0^{\star}) &= \beta_i^{\star} + \hat{\beta}_i \left(-\frac{\beta_i^{\star} - \hat{\beta}_i}{\sum_{j \neq i} \beta_i^{\star} + \hat{\beta}_i + \gamma} - 1 \right), \\ &= \frac{(\beta_i^{\star} - \hat{\beta}_i)(\sum_{j \neq i} \beta_i^{\star} + \gamma)}{\sum_{j \neq i} \beta_i^{\star} + \hat{\beta}_i + \gamma}, \end{aligned}$$

on rearranging. □

Lemma 6 *Suppose that the assumptions of lemma 3 hold. If, for each firm i, $S_i^{\star}, i = 1, \ldots, n$ is strictly concave on the interval $[a_i, p_0^{\star}]$ then for each i, $\hat{\beta}_i > \beta_i^{\varepsilon}$. If, for each firm i, $S_i^{\star}, i = 1, \ldots, n$ is strictly convex on the interval $[a_i, p_0^{\star}]$ then for each i, $\hat{\beta}_i < \beta_i^{\varepsilon}$.*

Proof We first consider the case where each supply function is strictly concave. Consider the linear function defined for each p by:

$$S_i^{\star}(p^{\varepsilon}) + \hat{\beta}_i(p - p^{\varepsilon}). \tag{31}$$

This function matches the function $\hat{S}_i$ defined in lemma 3 for prices in the interval $[p^{\varepsilon}, \overline{p}]$. It intersects the function $S_i^{\star}$ at the point $(p^{\varepsilon}, S_i^{\star}(p^{\varepsilon}))$. In the proof of lemma 3, it was shown that the function defined in (31) is the same as the function defined for each p by:

$$\hat{\beta}_i(p - a_i).$$

We note that for $p = a_i$, we have that $\hat{\beta}_i(a_i - a_i) = 0$. Also, by definition of (18), $S_i^{\star}(a_i) = 0$. That is, the function (31) also intersects the function $S_i^{\star}$ at the point $(a_i, 0)$. In summary, the function (31) has slope $\hat{\beta}_i$ and intersects the increasing, strictly concave function $S_i^{\star}$ at two points, namely $p = a_i$ and $p = p^{\varepsilon}$, with $a_i < p^{\varepsilon}$. Therefore, $\hat{\beta}_i > S_i^{\star\prime}(p^{\varepsilon}) = \beta_i^{\varepsilon}$.

The argument in the case of each supply function being strictly convex is similar. □

Lemma 6 shows that the relative slopes of the functions S_i^{ε} and $\hat{S}_i$ are as depicted in figure 8 for concave $S_i^{\star}$.

Corollary 7 *Suppose that the assumptions of lemma 3 hold. First, suppose that for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly concave on the interval $[a_i, p_0^\star]$. Then the derivatives of the first and fourth quantities considered in lemma 5 are, respectively, zero and negative. As p^ε decreases from $p_0^\star$, the fourth quantity becomes strictly greater than the first quantity.*

On the other hand, suppose that for each firm the supply functions are strictly convex on the interval $[a_i, p_0^\star]$. Then the derivatives of the first and fourth quantities considered in lemma 5 are, respectively, zero and positive. As p^ε decreases from $p_0^\star$, the fourth quantity becomes strictly less than the first quantity.

Proof Note that for $p^\varepsilon = p_0^\star, S_i^\varepsilon(p_0^\star) = \hat{S}_i(p_0^\star)$.
□

Finally, we define the notion of unstable equilibrium and characterize conditions for an unstable equilibrium:

Definition 7 Let $\mathbb{S} = \bigtimes_{i=1}^n \mathbb{S}_i$ and suppose that $S^\star \in \mathbb{S}$ is an SFE. Let $\overline{\mathbb{S}}$ be the function space of integrable functions with domain $[\underline{p}, \overline{p}]$ and range $\mathbb{R}^n$ and let $\|\bullet\|$ be a norm on equivalence classes of elements of $\overline{\mathbb{S}}$ such that if $S \in \mathbb{S}$ and $\|S - S^\star\| = 0$ then the price function defined by (7) resulting from the supply functions S is the same as the price function resulting from supply functions $S^\star$. Then we say that $S^\star$ is an unstable equilibrium if for every $\varepsilon > 0$ there exists $S^\varepsilon = (S_i^\varepsilon)_{i=1,\ldots,n} \in \mathbb{S}$ such that:

- $\|S^\varepsilon - S^\star\| < \varepsilon$ and
- if, for each i, $\tilde{S}_i$ is any optimal response to $S_j^\varepsilon, j \neq i$ and we define $\tilde{S} = (\tilde{S}_i)_{i=1,\ldots,n}$ then $\|\tilde{S} - S^\star\| > \|S^\varepsilon - S^\star\|$.

□

That is, $S^\star$ is unstable if a small perturbation S^ε to $S^\star$ results in responses $\tilde{S}$ by the firms that deviate even more from $S^\star$. "Small perturbation" is defined by a norm on equivalence classes of elements of $\mathbb{S}$ that distinguishes the resulting price functions. The definition is "local" in the sense that it does not require that the best response to $\tilde{S}$ be even further from $S^\star$ than $\tilde{S}$.

Theorem 8 *Suppose that the assumptions of lemma 3 hold. Moreover, suppose that either:*

- *for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly concave on the interval $[a_i, p_0^\star]$ and that the capacity constraints are not binding at the price $p_0^\star$ or*
- *for each firm i, $S_i^\star, i = 1, \ldots, n$ is strictly convex on the interval $[a_i, p_0^\star]$.*

The the SFE $S^\star$ is unstable.

Proof We first consider the case where each supply function is strictly concave and capacity constraints are not binding. We define a norm on the equivalence classes of functions in $\overline{\mathbb{S}}$. In particular, define $\|\bullet\|$ by:

$$\forall S \in \overline{\mathbb{S}}, \|S\| = \max_{i=1,\ldots,n} \int_{\underline{p}}^{p_0^\star} |S_i(p)|\, dp.$$

Note that if $S \in \mathbb{S}$ and $\|S - S^\star\| = 0$ then S and $S^\star$ are identical up to price $p_0^\star$ (except possibly on a set of measure zero.) Consequently, the price function resulting from S is the same as the price function resulting from $S^\star$.

We show that an arbitrarily small perturbation (in the sense of the norm $\|\bullet\|$) to the SFE $S^\star$ will result in a response by the firms that deviates even more from $S^\star$. This will be sufficient to show that the equilibrium is unstable.

Let $\varepsilon > 0$ be given. By continuity of $S^\star$ and $S^{\star\prime}$ in the neighborhood of $p_0^\star$, let $a_i < p^\varepsilon < p_0^\star$ be large enough such that:

- $\|S^\varepsilon - S^\star\| < \varepsilon$, where $S^\varepsilon = (S_i^\varepsilon)_{i=1,\ldots,n}$ and S_i^ε is the linear continuation of $S_i^\star$ from price p^ε,
- By corollary 7, for each $i = 1,\ldots,n$, $\hat{S}_i(\hat{p}_{0i}) > S_i^\varepsilon(p_0^\star)$, where:
  - $\hat{S}_i$ is the firm i optimal response to $S_j^\varepsilon, j \neq i$,
  - the quantity $S_i^\varepsilon(p_0^\star)$ is the maximum relevant supply of the linear continuation of $S_i^\star$, and
  - the quantity $\hat{S}_i(\hat{p}_{0i})$ is the peak realized supply given firm i optimal response to $S_j^\varepsilon, j \neq i$,
- for each firm $\hat{S}_i(\hat{p}_{0i}) < \overline{q}_i$.

The function S^ε represents a perturbation from $S^\star$. By lemma 4, the optimal response of firm i to $S_j^\varepsilon, j \neq i$ is any non-decreasing function that matches the function $\hat{S}_i$ on the interval $[\hat{p}_{1i}, \hat{p}_{0i}]$. We show that the functions $\underline{\hat{S}} = (\underline{\hat{S}}_i)_{i=1,\ldots,n}$ defined in (30) are the optimal responses that are closest to $S^\star$ in the sense of the norm $\|\bullet\|$. Moreover, we show that: $\left\|\underline{\hat{S}} - S^\star\right\| > \|S^\varepsilon - S^\star\|$.

Because of the concavity of the $S_i^\star$ and by lemma 6, $\hat{S} \geq S^\varepsilon \geq S^\star$. Consequently, by the discussion after lemma 4, out of the set of optimal responses by firm i to the bids $S_j^\varepsilon, j \neq i$, the function that is closest to $S^\star$ in the sense of the norm $\|\bullet\|$ is the function $\underline{\hat{S}}_i$ defined in (30). We have that:

$$\begin{aligned}
\forall i = 1,\ldots,n, \forall p \in [p^\varepsilon, \hat{p}_{0i}], \underline{\hat{S}}_i(p) &= \hat{S}_i(p),\\
&\geq S_i^\varepsilon(p), \text{ by lemma 6},\\
&\geq S_i^\star(p), \text{ by concavity of } S^\star,\\
\forall i = 1,\ldots,n, \forall p \in [\hat{p}_{0i}, p_0^\star], \underline{\hat{S}}_i(p) &= \underline{\hat{S}}_i(\hat{p}_{0i}), \text{ by (30)},\\
&= \hat{S}_i(\hat{p}_{0i}), \text{ by (30)},\\
&> S_i^\varepsilon(p_0^\star), \text{ by construction},\\
&\geq S_i^\varepsilon(p), \text{ since } S_i^\varepsilon \text{ is non-decreasing},\\
&\geq S_i^\star(p).
\end{aligned}$$

Also:

$$\forall i = 1,\ldots,n, \forall p \in [\underline{p}, p^\varepsilon], S_i^\varepsilon(p) = S_i^\star(p).$$

Consequently, by definition of the norm, $\left\|\hat{\underline{S}} - S^\star\right\| > \|S^\varepsilon - S^\star\|$. Moreover, every vector $\tilde{S}$ of optimal responses to S^ε satisfies: $\left\|\tilde{S} - S^\star\right\| \geq \left\|\hat{\underline{S}} - S^\star\right\| > \|S^\varepsilon - S^\star\|$ and so the equilibrium $S^\star$ is unstable.

In the case of strictly convex S_i, $\hat{p}_{0i} > p_0^\star$ and the optimal response to S^ε is $\hat{S}$ throughout the interval $[\underline{p}, p_0^\star]$. □

Lemma 9 *Consider a solution $S^\star$ of (18) for demand of the form (1) and affine marginal costs of the form (2). Then:*

$$S^{\star\prime\prime}(p) = \left[\frac{1}{(n-1)}\mathbf{1}\mathbf{1}^\dagger - \mathbf{I}\right] \begin{bmatrix} \frac{S_1^{\star\prime}(p)(p-a_i)-S_1^\star(p)}{(p-C_1'(S_1^\star(p)))^2} \\ \vdots \\ \frac{S_n^{\star\prime}(p)(p-a_n)-S_n^\star(p)}{(p-C_n'(S_n^\star(p)))^2} \end{bmatrix}. \tag{32}$$

Proof Differentiate (18). □

Corollary 10 *Consider any symmetric non-decreasing solution $S^\star$ of (18) for demand of the form (1) and affine marginal costs of the form (2) where the marginal costs are the same for each firm. Suppose that either:*

- *the solution satisfies $S^\star < S^{\star\text{affine}}$ and the capacity constraints are strictly satisfied at all prices up to the peak realized price $p_0^\star$ or*
- *the solution satisfies $S^\star > S^{\star\text{affine}}$.*

Then $S^\star$ is unstable.

Proof If the solution satisfies $S^\star < S^{\star\text{affine}}$ then the terms in the numerator of the right hand side of (32) are all negative so that $\forall i = 1,\ldots,n, \forall p \in [a_i, p_0^\star], S_i^{\star\prime\prime}(p) < 0$ and so the supply functions are strictly concave. Furthermore, by assumption the capacity constraints are strictly satisfied.

If the solution satisfies $S^\star > S^{\star\text{affine}}$ then the terms in the numerator of the right hand side of (32) are all positive so that $\forall i = 1,\ldots,n, \forall p \in [a_i, p_0^\star], S_i^{\star\prime\prime}(p) > 0$ and so the supply functions are strictly convex. In either case, the SFE is not stable. □

5.2 Discussion

Corollary 10 shows that in the symmetric case every SFE between $S^{\star\text{Cournot}}$ and $S^{\star\text{affine}}$ (including $S^{\star\text{Cournot}}$ but not including $S^{\star\text{affine}}$) is unstable unless capacity constraints are just binding at the peak realized price. The corollary also shows that in the symmetric case every SFE between $S^{\star\text{affine}}$ and $S^{\star\text{comp}}$ (including $S^{\star\text{comp}}$ but not including $S^{\star\text{affine}}$) is unstable. Baldick and Kahn show that, under mild conditions, if the bid functions are required to be affine, then the affine SFE $S^{\star\text{affine}}$ is stable in the function space of affine SFEs [11]. We hypothesize the stronger result that, with respect to a suitable norm on $\overline{\mathbb{S}}$, the affine SFE is stable in $\mathbb{S}$. Although there is a wide range of equilibria in the symmetric unconstrained case, this wide range is unlikely to be observed in practice because the equilibria that are different to $S^{\star\text{affine}}$ are unstable.

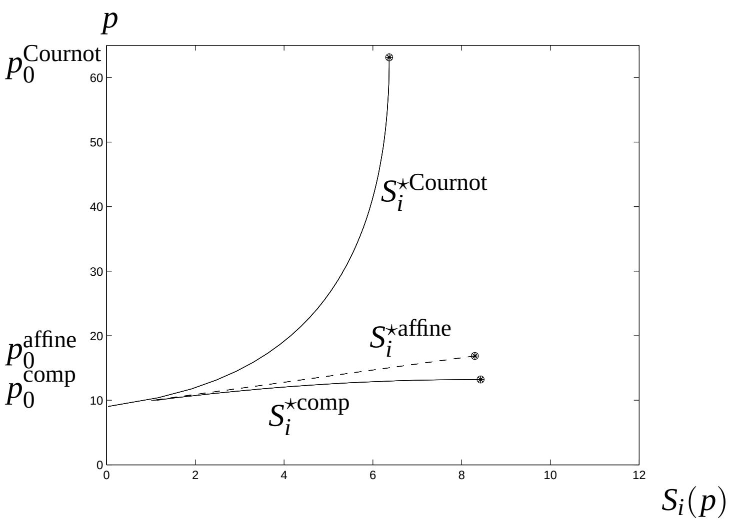

Figure 9: Illustration of corollary 10.

The situation is illustrated in figure 9 for the three firm example system discussed in section 4.1. Green and Newbery's analysis [5] suggests that any equilibrium between the least competitive symmetric SFE $S^{\star\text{Cournot}}$ and the most competitive symmetric SFE $S^{\star\text{comp}}$ can be observed. These supply functions are both shown solid in figure 9. However, corollary 10 shows that only the affine SFE $S^{\star\text{affine}}$ (shown dashed in figure 9) can be stable. Only stable equilibria are likely to be observed in practice.

Green and Newbery [5] use the least competitive SFE $S^{\star\text{Cournot}}$ for some of their analysis to estimate an upper bound on price mark-ups in the England and Wales system. Their calculations yield price mark-ups that are much higher than were observed. Corollary 10 suggests that $S^{\star\text{Cournot}}$ is not a tight bound on the equilibrium mark-ups.

We can also consider applying the previous analysis to SFEs that are not obtained as solutions to the differential equation and where the profit function is non-concave. In this case we can only guarantee that the response $\hat{S}_i$ that we construct to the bids $S_j^{\varepsilon}, j \neq i$ is a local but not necessarily globally optimal response. Nevertheless, even in the case that the profit function is non-concave, if the functions $S_i^{\star}$ are all concave or all convex then a similar construction can still be used to find a function $\hat{S}_i$ that is a better response than S_i^{ε} to the bids $S_j^{\varepsilon}, j \neq i$. However, we cannot in general show that $\hat{S}_i$ is the *best* response to the bids $S_j^{\varepsilon}, j \neq i$. This suggests, however, that if:

- supply functions are all strictly concave or all strictly convex in the vicinity of the maximum realized price and
- the capacity constraints are not binding (in the case of strictly convex supply functions),

then the equilibrium will be unstable. Moreover, if a local improvement algorithm is used by firms to respond to the supplies of other firms then such equilibria will not be observed in practice. Conversely, we expect that, in the vicinity of the peak price, stable equilibria will not involve all strictly concave supply functions unless capacity constraints are binding and will not involve all strictly convex supply functions.

The construction of bending the supply function also fails if the capacity constraints are binding. Supply functions satisfying the differential equation (18) that are less competitive than $S^{\star\text{affine}}$ can be stable only if the capacity constraints are just binding at the peak realized price. For example, Day and Bunn [12] use their numerical technique on the symmetric three firm case upon which our example is based and exhibit results that are consistent with the least competitive equilibrium $S^{\star\text{Cournot}}$. They apparently choose the capacity constraints to be exactly equal to the Cournot supply at the peak demand. That is, they are implicitly considering a capacitated case where the capacity constraints are just binding at the peak realized price. The least competitive symmetric equilibrium $S^{\star\text{Cournot}}$ is stable in that constrained case.

Finally, we observe that the stability results may not apply perfectly to equilibria calculated using the numerical framework that we will develop in section 8. This is because the proofs of equilibria being unstable rely on the construction of arbitrary differentiable functions. In the numerical framework we develop, we will use a finite dimensional parametrization of the supply functions. We have not investigated theoretically the conditions for an unstable equilibrium in this context, but speculate that the results would be less “clear cut” than the results we have developed here. Moreover, non-quadratic costs functions and binding capacity constraints may also alter the character of the stability results.

6 Allowable functions

Klemperer and Meyer [1] and Green and Newbery [5] show that if the cost functions are the same for each firm and if a non-affine symmetric solution is obtained for the differential equation (18) then for sufficiently high prices the solution will either violate the non-decreasing constraints (in the case of solutions that are less competitive than the affine SFE) or become vertical. However, so long as the realized prices do not exceed the price at which the solutions become decreasing or vertical then the solution of the differential equation provides an SFE.

In this section we will observe that it is generally very difficult to find solutions of (18) that are non-decreasing over all realized prices except in very special cases, namely:

- if the cost functions are the same for each firm, as explored by Klemperer and Meyer [1] and Green and Newbery [5],
- if the marginal costs are affine and there are no capacity constraints so that there are linear or affine solutions to (18), which was explored in [6, 9, 11], or
- if the load factor over the time horizon is very close to 100%.

In the general case, of firms having capacity constraints and asymmetric costs, solutions of (18) typically violate the non-decreasing requirements somewhere over the range of realized prices over the time horizon. The following theorem helps to explain why this is the case. It shows that the solutions of the differential equation must satisfy tight bounds in order for the solution to be non-decreasing over a range of prices. The theorem partially generalizes analysis in Klemperer and Meyer developed for the symmetric case [1, Proposition 1].

6.1 Analysis

Theorem 11 *Consider a solution $S_i^\star : \mathbb{P} \to \mathbb{R}, i = 1,\dots,n$ of the differential equation (18) on an interval of prices $\mathbb{P} = [\underline{p}, \overline{p}]$. If each function $S_i^\star, i = 1,\dots,n$ is non-decreasing on $\mathbb{P}$ then:*

$$\forall i = 1,\dots,n, \forall p \in \mathbb{P}, \gamma \leq \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} \leq \frac{1}{(n-1)} \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{(n-1)}. \qquad (33)$$

Proof We first prove the lower bound condition in (33). That is, we prove:

$$\forall i = 1,\dots,n, \forall p \in \mathbb{P}, \gamma \leq \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))}.$$

The differential equation (18) collects together and rearranges the conditions (13) applied to each firm. Rearranging (13), we obtain:

$$\begin{aligned} \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} &= \gamma + \sum_{j \neq i} S_j'(p), \\ &\geq \gamma, \end{aligned}$$

since $S_j'(p) \geq 0, \forall j$ by assumption.

We now prove the upper bound condition in (33). That is, we prove:

$$\forall i = 1,\dots,n, \forall p \in \mathbb{P}, \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} \leq \frac{1}{(n-1)} \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{(n-1)}.$$

Let $\mathbf{I}_i$ be the vector of all zeros, except in the i-th place where it is equal to 1. For any $p \in \mathbb{P}$,

$$\begin{aligned} 0 &\leq S_i^{\star\prime}(p), \\ &= [\mathbf{I}_i]^\dagger S^{\star\prime}(p), \\ &= \frac{1}{(n-1)} \mathbf{1}^\dagger \begin{bmatrix} \frac{S_1^\star(p)}{p - C_1'(S_1^\star(p))} \\ \vdots \\ \frac{S_n^\star(p)}{p - C_n'(S_n^\star(p))} \end{bmatrix} - \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} - \frac{\gamma}{(n-1)}, \text{ by (18)}, \\ &= \frac{1}{(n-1)} \sum_{j=1}^{n} \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} - \frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} - \frac{\gamma}{(n-1)}. \end{aligned}$$

Rearranging we obtain:

$$\frac{S_i^\star(p)}{p - C_i'(S_i^\star(p))} \leq \frac{1}{(n-1)} \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p - C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{(n-1)}.$$

□

6.2 Discussion

In theorem 11, the lower bound condition in (33) requires that γ be no larger than the smallest entry of the vector:

$$\begin{bmatrix} \frac{S_1^\star(p)}{p-C_1'(S_1^\star(p))} \\ \vdots \\ \frac{S_n^\star(p)}{p-C_n'(S_n^\star(p))} \end{bmatrix}. \tag{34}$$

Furthermore, the expression:

$$\frac{1}{(n-1)} \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))} \right\},$$

is equal to $\frac{n}{(n-1)}$ times the average of the entries in the vector (34). The upper bound condition in (33) in theorem 11 requires that each entry of the vector (34) is smaller than $\frac{n}{(n-1)}$ times the average of the entries. For n large, the ratio $\frac{n}{(n-1)}$ is only slightly greater than one. That is, the upper bound condition in theorem 11 dictates that the values of $\frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))}$ must fall in a narrow range in order for the solution to the differential equation be non-decreasing.

In the cases of:

1. symmetric cost functions and symmetric solutions to the differential equation or

2. affine solutions to the differential equation with affine marginal costs,

then the necessary conditions in theorem 11 are relatively mild as we will discuss in the following two sections. We will then discuss capacity constraints in section 6.2.3.

6.2.1 Symmetric cost functions

If the cost functions and the solutions to the differential equation are symmetric then the upper bound condition in (33) can be verified as follows:

$$\begin{aligned}
&\frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))} \\
&\quad = \left(\frac{n-1}{n-1}\right) \frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))} \\
&\quad = \frac{1}{(n-1)} \sum_{j=1}^{n} \frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))} - \frac{1}{(n-1)} \frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))}, \\
&\qquad \text{since the cost functions and solutions are symmetric,} \\
&\quad \leq \frac{1}{(n-1)} \sum_{j=1}^{n} \left\{ \frac{S_j^\star(p)}{p-C_j'(S_j^\star(p))} \right\} - \frac{\gamma}{(n-1)},
\end{aligned}$$

where the inequality is true if the lower bound condition in (33) in theorem 11 is satisfied. That is, the upper bound condition on $\frac{S_i^\star(p)}{p-C_i'(S_i^\star(p))}$ is automatically satisfied if the lower bound condition is

Firm $i =$	1	2	3	4	5
c_i(pounds per MWh per MWh) $=$	2.687	4.615	1.789	1.93	4.615
a_i(pounds per MWh) $=$	12	12	8	8	12

Table 2: Cost data based on five firm industry described in [11].

satisfied. This means that the non-decreasing constraints are easier to satisfy in the symmetric case than in the asymmetric case. In fact, as Klemperer and Meyer show [1, Proposition 1], a necessary and sufficient condition for a symmetric solution of the differential equation to be an SFE is that the lower bound condition in (33) be satisfied. In the symmetric case, the equilibrium supply functions $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$ satisfy the non-decreasing constraints over the range of realized prices. Moreover every symmetric equilibrium between these equilibria also satisfies the non-decreasing constraints.

6.2.2 Affine solutions for affine marginal cost functions

The affine SFE $S^{\star\text{affine}}$ was exhibited in (15). Each function $S_i^{\star\text{affine}}$ has slope $\beta_i \in \mathbb{R}_+$ satisfying (16). Since the $\beta_i \in \mathbb{R}_+$, the affine functions are guaranteed to be non-decreasing.

6.2.3 Capacity constraints

To interpret theorem 11 in the case of capacity constraints (3), we will assume that the marginal costs effectively increase very rapidly as capacity constraints are approached. This means that entries in the vector (34) change rapidly with p as capacity constraints are approached so that the upper bound condition will not be satisfied unless all firms reach their capacity at the same price. We conjecture that this in unlikely except in the case of symmetric cost functions and capacities. That is, in the asymmetric capacitated case, the solution to the differential equation will typically violate the non-decreasing constraints at some price.

6.3 Five firm example system

To illustrate theorem 11, we consider a five firm example system based on the cost data presented in [11] for the five strategic firm industry in England and Wales subsequent to the 1999 divestiture. Table 2 shows the cost parameters. Firms 2 and 5 have identical cost functions. The demand slope is 0.1 GW per (pound per MWh).

Solving (16) for the cost parameters in table 2 and a demand slope of $\gamma = 0.1$ GW per (pound per MWh), we find that the slopes of the affine solutions are:

$$\beta = \begin{bmatrix} 0.2840 \\ 0.1857 \\ 0.3718 \\ 0.3550 \\ 0.1857 \end{bmatrix},$$

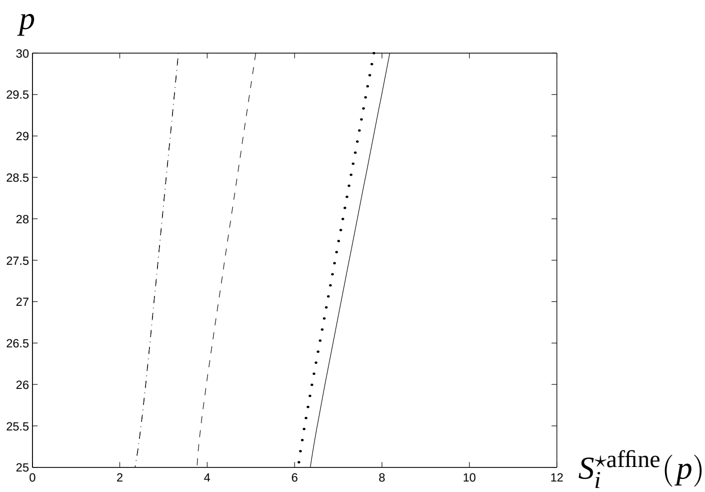

Figure 10: Solution of (18) that matches affine SFE. Firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.

and that the affine SFE is given by:

$$\forall p \in [12,\infty), S^{\star\text{affine}}(p) = \begin{bmatrix} 0.2840(p-12), \\ 0.1857(p-12), \\ 0.3718(p-8), \\ 0.3550(p-8), \\ 0.1857(p-12) \end{bmatrix}. \quad (35)$$

(For prices below $p = 12$ pounds per MWh, the minimum capacity constraint is binding on firms 1, 2, and 5, so we only define the affine solution for $p \geq 12$ pounds per MWh. A piece-wise affine SFE for this case is derived in [11] and described in detail in section 11.2.)

Using any initial condition for the differential equation (18) of the form $(\overline{p}, S^{\star\text{affine}}(\overline{p}))$, with $\overline{p} > 12$ pounds per MWh and $S^{\star\text{affine}}$ as defined in (35), will yield an affine solution that is identical to $S^{\star\text{affine}}$. For example, using $\overline{p} = 30$ pounds per MWh and integrating backwards yields figure 10. (In this figure and most subsequent figures illustrating the five firm example, firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.) The numerical solution of the differential equation differs very slightly from (35) because of numerical conditioning issues in the solution of the differential equation. However, the correspondence with the exact affine solution is very close.

To illustrate that the solution of (18) will violate the non-decreasing constraints when the solution is non-affine, we considered initial conditions that differed only very slightly from the initial condition of $\overline{p} = 30$ pounds per MWh and $S^{\star\text{affine}}(\overline{p})$. In particular, we considered the 32 vertices of the hypercube whose vertices are specified by:

$$S_i(\overline{p}) = 0.999 \times S_i^{\star\text{affine}}(\overline{p}), 1.001 \times S_i^{\star\text{affine}}(\overline{p}), i = 1, \ldots, 5.$$

That is, we successively decreased and increased each entry in $S_i^{\star\text{affine}}(\overline{p})$ by 0.1% and used the resulting vector as the initial condition to integrate backwards from $p = \overline{p}$.

The results of integrating from these 32 initial conditions are shown in figure 11. Each initial condition was integrated from $\overline{p}$ backwards until a price p' was reached where the non-decreasing constraints were violated significantly for one of the firms. In each case, the trajectory for all five

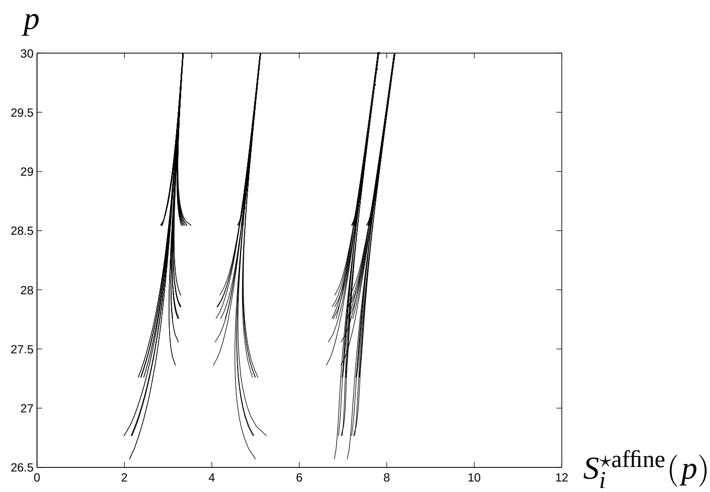

Figure 11: Solution of (18) from 32 initial conditions that are slight perturbations of a point satisfying the affine SFE.

firms was plotted for $[p', \overline{p}]$. Since the values of p' varied with the initial condition, the trajectories for most of the initial conditions can be individually distinguished in figure 11.

As previously, firms 2 and 5 have identical costs. Their trajectories are shown as the leftmost bundle of curves in figure 11. Whenever firms 2 and 5 are started with different initial conditions, the resulting trajectories for them will diverge. Firms 3 and 4 are the rightmost pair of bundles of curves in figure 11. Firm 1 appears as the middle bundle of curves in figure 11.

As shown in this figure, for every one of the 32 initial conditions, the supply of either firm 1 or firm 2 or firm 5 violates the non-decreasing constraints for some prices between 26.5 and 30. In summary, in this example the differential equation (18) yields solutions that violate the non-decreasing constraints when the initial conditions differ even slightly from satisfying the affine SFE conditions. Although this is not a proof in general, it suggests why solutions of (18) may violate the non-decreasing constraints.

6.4 Sensitivity to demand slope

The numerical results in the previous section used a demand slope of $\gamma = 0.1$ GW per (pound per MWh.) As discussed in section 3.2, the slopes of the affine SFE are not very sensitive to the value of γ. For example, solving (16) for the cost functions in table 2 and for a demand slope of $\gamma = 0$, we find that the slopes of the affine solutions are:

$$\beta = \begin{bmatrix} 0.2756 \\ 0.1825 \\ 0.3561 \\ 0.3409 \\ 0.1825 \end{bmatrix},$$

which differ from the slopes for $\gamma = 0.1$ GW per (pound per MWh) by only a few percent. Moreover, similar results to that in figure 11 can be obtained with different values of γ.

6.5 Summary

The most serious difficulty with the differential equation approach to solving for the SFE is that solutions of the differential equation do not "automatically" satisfy the capacity or non-decreasing conditions. Theorem 11 implies that unless the cost functions are all very similar or there are no capacity constraints then the non-decreasing constraints will typically be violated in a solution of the differential equations, unless the range of realized prices is small enough to only cover a segment of the solution that happens to be non-decreasing. The example in section 6.3 shows that even a very slight deviation from the affine solution results in solutions of (18) that are non-decreasing only over a narrow range of prices. If the load factor over the time horizon were very close to 100% then such a solution of (18) would be an equilibrium. However, if the load factor is significantly below 100% then most such solutions would violate the non-decresing constraints over the range of realized prices.

This analysis provides two observations. First, the usual approach to solving differential equations to obtain the SFE may not work in the case of heterogeneous portfolios of generation with capacity constraints when the load factor deviates significantly from 100%. In this case, we must explicitly impose the non-decreasing constraints.

Second, as discussed in the introduction, a basic criticism of the SFE approach is that there are multiple equilibria. Certainly, if *every* possible specification of the initial conditions for the differential equation (18) yielded an equilibrium then this extreme multiplicity of equilibria would limit the predictive value of the SFE approach. However, when the load factor deviates significantly from 100%, many of these putative equilibria are ruled out by the non-decreasing constraints. This strengthens the observations by Klemperer and Meyer in [1] that were made for the symmetric case concerning the multiplicity of equilibria. Moreover, the price cap condition (4), when it is binding on the behavior of firms, further limits the range of potential equilibria.

Solutions such as shown in figure 11 could form part of an equilibrium only if either:

1. the range of realized prices was very restricted, or,

2. there were a discontinuity in the derivative of the supply functions.

The first case could occur if the load factor were close to 100%. In this case, there would be a multiplicity of equilibria, with the range depending on the range of the function N, but not on the detailed dependence of $N(t)$ on t. Conversely, extended time horizons having load factors well below 100% rule out many of the solutions of (18) from being supply functions.

In the second case, we can imagine a discontinuous change in the behavior of the firms due to, for example, a binding capacity constraint being reached at a particular price. In this case, we can imagine equilibrium solutions consisting of the union of solutions of (18) that are "pasted" together at various break-points. We will confirm this observation theoretically in the next section and then see in section 11 that the numerical solutions have this appearance.

7 Strict satisfaction of non-decreasing constraints

In this section we show that although it is necessary to represent the non-decreasing constraints, they will be strictly satisfied at typical equilibria. The intuition behind this apparently paradoxical observation is that once the non-decreasing constraints are enforced, the profit maximizing response of a firm is strictly increasing. If the non-decreasing constraints were relaxed then the profit maximizing response would no longer be increasing because of the non-concavity in the profit function. This observation allows us to characterize SFEs in more detail. In section 7.2 we illustrate these observations with a two firm example.

7.1 Analysis

We first make some definitions to clarify the nature of "binding constraints."

Definition 8 Consider supply functions S and suppose that $\mathbb{P}$ is the interval of realized prices corresponding to S. Also suppose that for some firm i and for some interval $[\hat{p}, \check{p}] \subset \mathbb{P}$ we have that:

1. $S_j, j \neq i$ is differentiable on $(\hat{p}, \check{p})$,
2. S_i is constant on $[\hat{p}, \check{p}]$, with $0 < S_i(p) = q_i < \overline{q}_i, \forall p \in [\hat{p}, \check{p}]$, and
3. the profit function is increasing with price in $(\hat{p}, \check{p})$ in the following sense:

$$\text{for almost all } p \in (\hat{p}, \check{p}), q_i - \left(p - C'(q_i)\right)\left(\gamma + \sum_{j \neq i} S'_j(p)\right) > 0.$$

(That is, the set of points in $(\hat{p}, \check{p})$ for which $q_i - (p - C'(q_i))\left(\gamma + \sum_{j \neq i} S'_j(p)\right) \leq 0$ is of measure zero.)

Then we say that the non-decreasing constraints are *manifestly binding* for firm i on $[\hat{p}, \check{p}]$. □

Definition 9 Consider supply functions S and suppose that $\mathbb{P}$ is the interval of realized prices corresponding to S. Suppose that for some firm i and for some interval $[\hat{p}, \check{p}] \subset \mathbb{P}$ we have that:

$$\forall p \in (\hat{p}, \check{p}), S_i(p) - (p - C'(S_i(p)))\left(\gamma + \sum_{j \neq i} S'_j(p)\right) = 0.$$

Then we say that the non-decreasing constraints are *not apparently binding* for firm i on $[\hat{p}, \check{p}]$. □

Definition 10 Consider supply functions S and suppose that for firm i, S_i is the optimal non-decreasing response to $S_j, j \neq i$. Consider relaxing the non-decreasing constraints on the supply function of firm i. If the globally optimal response of firm i to $S_j, j \neq i$, given the relaxed constraints, is not equal to S_i then we say that non-decreasing constraints are *actually binding* for firm i. □

The adjective "manifestly" is used in definition 8 to emphasize that the choice of the supply function has been palpably restricted by the non-decreasing constraints. Definition 9 of "not apparently binding" covers the case where the the choices of supply function for firm i locally maximize the profit function for a given price.

Definition 10 of "actually binding" covers the case where relaxing the non-decreasing constraints would cause a different response. In principle, this could occur because either:

- the non-decreasing constraints were manifestly binding or

- the non-decreasing constraints were not apparently binding but yet the non-decreasing constraints ruled out other responses having higher profits.

The following theorem shows that under relatively mild conditions the non-decreasing constraints cannot be manifestly binding. We show that it is impossible for the non-decreasing constraints to be:

- not apparently binding up to some price $\hat{p}$ and

- manifestly binding for prices above $\hat{p}$.

That is, it is impossible for the supply function to become "flat" over an interval of prices. Moreover, this means that the non-decreasing constraints will always be not apparently binding.

As we will show in the example in section 7.2, the non-decreasing constraints can be actually binding. The conclusion is that while the non-decreasing constraints will be not apparently binding they will, however, be actually binding.

Theorem 12 *Let $\gamma > 0$. Consider piece-wise continuously differentiable supply functions S and suppose that $\mathbb{P}$ is the interval of realized prices corresponding to S. Also suppose that for firm i, S_i is the optimal non-decreasing response to $S_j, j \neq i$. Consider prices $\tilde{p}, \hat{p}, \check{p} \in \mathbb{P}$ such that either:*

- *$\tilde{p} < \hat{p} < \check{p}$ or*

- *$\hat{p}$ is equal to the minimum realized price and $\tilde{p} = \hat{p} < \check{p}$.*

Suppose that the non-decreasing constraints are not apparently binding for firm i on $[\tilde{p}, \hat{p}]$. (If $\tilde{p} = \hat{p}$ this condition is null.) Then the non-decreasing constraints cannot be manifestly binding for firm i on $[\hat{p}, \check{p}]$.

Proof Suppose that the non-decreasing constraints were manifestly binding for firm i on $[\hat{p}, \check{p}]$, with $S_i(p) = q_i, \forall p \in [\hat{p}, \check{p}]$. By adjusting $\tilde{p}$ and $\check{p}$ if necessary we can assume that $S_j, j = 1, \ldots, n$ are continuously differentiable on the intervals $(\tilde{p}, \hat{p})$ and $(\hat{p}, \check{p})$. (That is, the functions $S_j, j = 1, \ldots, n$ may fail to be continuously differentiable on $(\tilde{p}, \check{p})$ only at $p = \hat{p}$.) The situation is shown in figure 12. The function S_i is illustrated with the solid line. (Note that the function S_i is drawn on the horizontal axis while its argument is drawn on the vertical axis.)

Let $P : [0,1] \to \mathbb{P}$ be the realized prices at each time in the time horizon. Let $\tilde{t}, \hat{t}, \check{t}$ be the times corresponding to $\tilde{p}, \hat{p}, \check{p}$, respectively. By assumption, either:

- $\tilde{t} > \hat{t} > \check{t}$ or

- $\tilde{p} = \hat{p} < \check{p}$ and $\tilde{t} \geq \hat{t} > \check{t}$.

The residual demands $N(t) - \gamma p - \sum_{j \neq i} S_j(p)$ faced by firm i at times $t = \tilde{t}, \hat{t}, \check{t}$ are shown by the dotted lines in figure 12.

Consider a parameter $\varepsilon \geq 0$ and the following construction of functions $S_i^\varepsilon : [\underline{p}, \overline{p}] \to [0, \overline{p}_i]$ and $P^\varepsilon : [0,1] \to [\underline{p}, \overline{p}]$. The functions S_i^ε and P^ε are parametrized by ε.

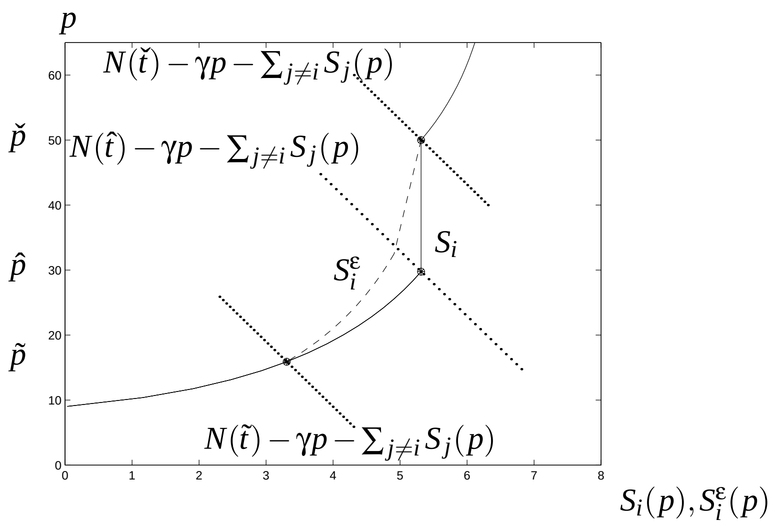

Figure 12: The functions S_i (shown solid) and S_i^ε (shown dashed) defined in proof of theorem 12.

First, for each $p \in [\underline{p}, \tilde{p}]$ and each $p \in [\check{p}, \overline{p}]$, let $S_i^\varepsilon(p) = S_i(p)$, so that S_i^ε matches S_i except on the interval $[\tilde{p}, \check{p}]$. Similarly, for each $t \in [0, \check{t}]$ and each $t \in [\tilde{t}, 1]$, let $P^\varepsilon(t) = P(t)$.

Second, for each $t \in [\check{t}, \hat{t}]$ find p such that:

$$N(t) - \gamma p - \sum_{j \neq i} S_j(p) = q_i - \varepsilon \left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right), \tag{36}$$

and define:

$$S_i^\varepsilon(p) = q_i - \varepsilon \left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right),$$

and $P^\varepsilon(t) = p$. The left hand side of (36) is illustrated in figure 12 for $t = \check{t}$ and $t = \hat{t}$ by the dotted lines. (By assumption, since $\gamma > 0$, the left hand side of (36) is strictly decreasing with p so that there is a solution.) By construction, note that $p \in [P^\varepsilon(\hat{t}), \check{p}] \subset [\hat{p}, \check{p}]$ and that S_i^ε is non-decreasing on $[P^\varepsilon(\hat{t}), \check{p}]$ and that S_i^ε is continuous at $p = \check{p}$. The function S^ε is shown dashed in figure 12. Also P^ε is non-increasing on $[0, \hat{t}]$.

Furthermore, by the implicit function theorem we have that the derivatives of these functions with respect to ε, evaluated at $\varepsilon = 0$ are, respectively:

$$\begin{aligned}
\forall t \in [\check{t}, \hat{t}], \frac{d[P^\varepsilon(t)]}{d\varepsilon}(0) &= \frac{1}{\gamma + \sum_{j \neq i} S_j'(P(t))} \left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right), \\
&\geq 0, \\
\forall t \in [\check{t}, \hat{t}], \frac{d[S_i^\varepsilon(P^\varepsilon(t))]}{d\varepsilon}(0) &= -\left(\frac{t - \check{t}}{\hat{t} - \check{t}} \right), \\
&= -\left(\gamma + \sum_{j \neq i} S_j'(P(t) \right) \frac{d[P^\varepsilon(t)]}{d\varepsilon}(0), \\
&\leq 0.
\end{aligned}$$

Third, for each $t \in [\hat{t}, \tilde{t}]$ find p such that:

$$N(t) - \gamma p - \sum_{j \neq i} S_j(p) = S_i(p) - \varepsilon \left(\frac{\tilde{t} - t}{\tilde{t} - \hat{t}} \right), \tag{37}$$

and define:

$$S_i^\varepsilon(p) = S_i(p) - \varepsilon \left(\frac{\tilde{t} - t}{\tilde{t} - \hat{t}} \right),$$

and $P^\varepsilon(t) = p$. The left hand side of (37) is illustrated in figure 12 for $t = \hat{t}$ and $t = \tilde{t}$ by the dotted lines. By construction, note that $p \in [\tilde{p}, P^\varepsilon(\hat{t})] \subset [\tilde{p}, \check{p}]$ and that S_i^ε is non-decreasing on $[\tilde{p}, P^\varepsilon(\hat{t})]$ and that S_i^ε is continuous at $p = \tilde{p}$ and at $p = P^\varepsilon(\hat{t})$. Also, The function S^ε is shown dashed in figure 12. The function P^ε is non-increasing on $[\hat{t}, 1]$.

Again, by the implicit function theorem we have that the derivatives of these functions with respect to ε, evaluated at $\varepsilon = 0$ are, respectively:

$$\begin{aligned}
\forall t \in [\hat{t}, \tilde{t}], \frac{d[P^\varepsilon(t)]}{d\varepsilon}(0) &= \frac{1}{\gamma + \sum_j S_j'(P(t))} \left(\frac{\tilde{t} - t}{\tilde{t} - \hat{t}} \right), \\
&\geq 0, \\
\forall t \in [\hat{t}, \tilde{t}], \frac{d[S_i^\varepsilon(P^\varepsilon(t))]}{d\varepsilon}(0) &= -\left(\gamma + \sum_{j \neq i} S_j'(P(t) \right) \frac{d[P^\varepsilon(t)]}{d\varepsilon}(0), \\
&\leq 0.
\end{aligned}$$

We now consider the change in profit accruing to firm i by changing its bid from S_i to S_i^ε. In particular, we calculate the derivative of the profit with respect to ε, evaluated at $\varepsilon = 0$. We have that:

$$
\begin{aligned}
& \frac{d[\pi_i]}{d\varepsilon}(0) \\
= \; & \frac{d\left[\int_{t=\check{t}}^{\tilde{t}} \pi_{it}\, dt\right]}{d\varepsilon}(0), \\
= \; & \frac{d\left[\int_{t=\check{t}}^{\tilde{t}} [S_i^\varepsilon(P^\varepsilon(t))P^\varepsilon(t) - C_i(S_i^\varepsilon(P^\varepsilon(t)))]dt\right]}{d\varepsilon}(0), \\
= \; & \int_{t=\check{t}}^{\tilde{t}} \left[\frac{d[S_i^\varepsilon(P^\varepsilon(t))P^\varepsilon(t) - C_i(S_i^\varepsilon(P^\varepsilon(t)))]}{d\varepsilon}(0)\right] dt, \\
& \text{ since the terms in the integral are differentiable,} \\
= \; & \int_{t=\check{t}}^{\tilde{t}} \left[S_i(P(t))\frac{d[P^\varepsilon(t)]}{d\varepsilon}(0) + (P(t) - C_i'(S_i(P(t)))\frac{d[S_i^\varepsilon(P^\varepsilon(t))]}{d\varepsilon}(0)\right] dt, \\
= \; & \int_{t=\check{t}}^{\hat{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))}\left[q_i - (P(t) - C_i'(q_i))\left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right]\left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt \\
& + \int_{t=\hat{t}}^{\tilde{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))} \\
& \times \left[S_i(P(t)) - (P(t) - C_i'(S_i(P(t))))\left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right]\left(\frac{\tilde{t}-t}{\tilde{t}-\hat{t}}\right) dt, \\
= \; & \int_{t=\check{t}}^{\hat{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))}\left[q_i - (P(t) - C_i'(q_i))\left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right]\left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt \\
& + \int_{t=\hat{t}}^{\tilde{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))}[0]\left(\frac{\tilde{t}-t}{\tilde{t}-\hat{t}}\right) dt, \\
& \text{since the non-decreasing constaints are not apparently binding for firm } i \text{ on } [\tilde{p}, \hat{p}], \\
= \; & \int_{t=\check{t}}^{\hat{t}} \frac{1}{\gamma + \sum_{j\neq i} S_j'(P(t))}\left[q_i - (P(t) - C_i'(q_i))\left(\gamma + \sum_{j\neq i} S_j'(P(t))\right)\right]\left(\frac{t-\check{t}}{\hat{t}-\check{t}}\right) dt, \\
> \; & 0,
\end{aligned}
$$

since the integrand is strictly positive over almost all of the interval $[\check{t}, \hat{t}]$ because the non-decreasing constraints are manifestly binding on $[\hat{p}, \check{p}]$. But this contradicts the hypothesis that S_i is an optimal response to $(S_j)_{j\neq i}$. Contradiction. □

Corollary 13 *Let $\gamma > 0$. Suppose that $S^\star$ is an SFE with each function $S_i^\star, i = 1, \ldots, n$ piece-wise continuously differentiable on the range of realized prices $\mathbb{P}$. Consider a firm i and prices $\tilde{p}, \hat{p}, \check{p} \in \mathbb{P}$ such that either:*

- *$\tilde{p} < \hat{p} < \check{p}$ or*
- *$\hat{p}$ is equal to the minimum realized price and $\tilde{p} = \hat{p} < \check{p}$.*

Suppose that the non-decreasing constraints are not apparently binding for firm i on $[\tilde{p}, \hat{p}]$. (If $\tilde{p} = \hat{p}$ this condition is null.) Then the non-decreasing constraints cannot be manifestly binding for firm i on $[\hat{p}, \check{p}]$. □

Note that corollary 13 does not preclude jumps in the supply functions; it only rules out regions where the supply functions are constant. It is therefore consistent with results in the non-linear pricing literature where tariff functions (that is, functions from quantity to price) can sometimes be constant over ranges of quantities [26].

The following corollary allows us to characterize SFEs:

Corollary 14 *Let $\gamma > 0$. Consider a piece-wise continuously differentiable SFE $S^\star = (S_i^\star)_{i=1,\dots,n}$. Consider any interval $[\hat{p}, \check{p}]$ of prices such that:*

- *the $S^\star$ are continuously differentiable,*
- *the capacity constraints of firms $i_1, i_2, \dots, i_m$ are not binding, and*
- *the capacity constraints of the other firms are binding.*

Then the supplies of firms $i_1, i_2, \dots, i_m$ on $[\hat{p}, \check{p}]$ match a solution of (18) where instead of having n firms with cost functions $C_1, \dots, C_n$, respectively, there are m firms with cost functions given by the cost functions $C_{i_1}, C_{i_2}, \dots, C_{i_m}$ of the m firms $i_1, i_2, \dots, i_m$.

Proof Note that by corollary 13, the non-decreasing constraints cannot be manifestly binding for firms $i_1, i_2, \dots, i_m$ on $[\hat{p}, \check{p}]$. Since the supply functions are continuously differentiable on this interval, they must satisfy the optimality conditions (11). But rearranging these optimality conditions, and noting that $S_j^{\star\prime}(p) = 0$ for $p \in [\hat{p}, \check{p}]$ and $j \neq i_1, i_2, \dots, i_m$, we find that $S_i^\star, i = i_1, i_2, \dots, i_m$ must satisfy an m firm version of (18). □

Corollary 14 allows us to characterize piece-wise continuously differentiable SFEs. In particular, as suggested in section 6.5, such SFEs involve the pasting together of solutions of (18). The points of non-differentiability in the SFE occur where the solutions of (18) for adjacent intervals are pasted together. Unfortunately, since we do not in general know where the break-points of the pieces of $S^\star$ will lie, we cannot usually use corollary 14 to directly construct an SFE. Because the solutions in each interval satisfy (18), it is only the range of the load-duration characteristic N, and not its exact functional form, that determines the possible equilibria as shown in:

Corollary 15 *Let $\gamma > 0$. The set of possible piece-wise continuously differentiable equilibria depends on the range of the load-duration characteristic but not on its exact form.*

Proof Consider a piece-wise continuously differentiable SFE $S^\star$ corresponding to a load-duration characteristic N_1. By assumption, we can partition the range of realized prices into intervals such that $S^\star$ is continuously differentiable on the interior of the interval and is a non-decreasing solution of (18).

Suppose that N_2 is another load-duration characteristic that has the same range as N_1. But since the range of N_2 is the same as the range of N_1, $S^\star$ is piece-wise continuously differentiable and non-decreasing over the (identical) range of realized prices for N_2. That is, $S^\star$ is an SFE corresponding to the load-duration characteristic N_2. □

Although the set of equilibria is independent of the exact functional form of the load-duration characteristic, in a numerical framework where we consider convergence to (one particular) equilibrium, it may be the case that the form of the load-duration characteristic affects which of the equilibria is exhibited by the numerical framework.

7.2 Two firm example system

To see the implications of theorem 12 and its corollaries, we will consider the following two firm market. To motivate the necessity of explicitly representing the non-decreasing constraints, we will postulate a supply function for firm 2 and then consider the optimal reaction of firm 1.

The demand is:

$$\forall p \in \mathbb{R}_+, \forall t \in [0,1], D(p,t) = 20 + 4.6(1-t) - 0.1p.$$

Firm 2 has a maximum capacity of $\overline{q}_2 = 17.1$ and is assumed to have bid a supply function of:

$$\forall p \in \mathbb{R}_+, S_2(p) = \begin{cases} 0.9p, & \text{if } p \leq 19, \\ 17.1, & \text{if } p > 19. \end{cases}$$

This function is non-decreasing. In the context of a multi-firm market, we can also think of S_2 as being the aggregate supply of all firms besides firm 1.

The cost function for firm 1 is:

$$\forall q_1 \in \mathbb{R}_+, C_1(q_1) = \frac{1}{7}(q_1)^2 + 4q_1,$$

with marginal cost $C_1'(q_1) = \frac{2}{7}(q_1) + 4$. We will assume that firm 1 has the same capacity as firm 2, so that $\overline{q}_1 = 17.1$. We will consider the optimal response of firm 1 to the given supply function of firm 2. (The resulting pair of supply functions S_1 and S_2 is not necessarily an equilibrium unless we make further assumptions but serves to illustrate the importance of the non-decreasing constraints.)

7.2.1 Ignoring the non-decreasing constraints

We first consider the optimal response by firm 1, ignoring the non-decreasing constraints. This simply amounts to maximizing the profit per unit time for firm 1 at each time. To maximize the profit per unit time to firm 1 for various times, we first observe that the profit function is *piece-wise* concave, with the pieces defined by whether or not the price is above $p = 19$. In fact, for some times t, the profit per unit time has two local maxima and so we must search over both pieces to find the value of q_{1t} that globally maximizes the profit per unit time of firm 1. We will consider the conditions for maximizing profit per unit time at two particular times: namely $t = 0$ and $t = 1$. This will suffice to demonstrate that a function S_1 that globally maximizes profit at each price would not be non-decreasing.

For $t = 1$, the maximum profit per unit time for firm 1 in the region $p \leq 19$ occurs for $p_1 = 13$ and $q_{11} = 7$. The corresponding profit is $\pi_{11} = 56$. For the region $p > 19$, it can be verified that the profit is always decreasing with p, and the profit is continuous across the regions as a function of p. Therefore, the globally optimal profit occurs at $p_1 = 13$ and $q_{11} = 7$.

At $t = 0$, the maximum profit per unit time for firm 1 in the region $p \le 19$ occurs for $p \approx 15.59$, with corresponding profit of 92.83. For prices $p > 19$, the profit is maximized for $p_0 = 40$, with corresponding quantity $q_{10} = 3.5$ and profit 124.25. Therefore, the globally optimal profit $\pi_{10} = 124.25$ occurs at $p_0 = 40$ and $q_{10} = 3.5$.

The significance of this example is that if we seek to use the pairs (p_0, q_{10}) and (p_1, q_{11}) to define points in the supply function S_1 for firm 1, we have just found that the resulting function will violate the non-decreasing constraint. The example relies on the particular choice of cost and demand, but many similar choices will yield similar results. For example, Anderson and Philpott [13] provide another example.

A further complicating issue is that we must perform a maximization over a non-concave profit function, so that the necessary conditions obtained from differentiating the profit are not sufficient. In general, at any price where the supply function of a firm $j \neq i$ changes slope, there will be a break-point (and potentially a non-concavity) in the profit function for firm i.

In this example, the break-point in the profit function of firm 1 is due to the change in the slope of the supply function of firm 2 as it reaches its full capacity $\overline{q}_2$. Such a break-point can also occur due to the capacity constraints of fringe firms. This issue prompted an *ad hoc* approach in [11].

7.2.2 Including non-decreasing constraints

We now consider the optimal response S_1 of firm 1 to S_2 considering the non-decreasing constraints. Assume a price cap of $\overline{p} = 50$. To approximate the optimal response of firm 1, we approximate the function space $\mathbb{S}_1$ by a subspace $\underline{\mathbb{S}}_1$ of $\mathbb{S}_1$ and choose S_1 from $\underline{\mathbb{S}}_1$. We specify $\underline{\mathbb{S}}_i$ as the set of piece-wise affine non-decreasing continuous functions with break-points at $p = 0, 4, 10, 13, 16, 19, 40, 50$. Since the marginal cost of firm 1 at zero production is 4, the optimal response of firm 1 must involve zero production up to price $p = \underline{p} = 4$. At a price of $p = \overline{p} = 50$, we specify that the firm must produce at full output $\overline{q}_1$, so this leaves the values $S_1(10), S_1(13), S_1(16), S_1(19)$ and $S_1(40)$ of the supply function at prices $p = 10, 13, 16, 19, 40$ to be specified. For the resulting supply function to be non-decreasing, we impose:

$$0 \le S_1(10) \le S_1(13) \le S_1(16) \le S_1(19) \le S_1(40) \le \overline{q}_1.$$

We calculated the profit π_1 of firm 1 according to (9), given the assumptions on demand and S_2. Exact integration was used. The profit is not concave as a function of $S_1(10), S_1(13), S_1(16), S_1(19)$, and $S_1(40)$. For example, figure 13 shows profits versus choices of $S_1(19)$ and $S_1(40)$ for $S_1(10) = S_1(13) = S_1(16) = 1$. The maximum profit point given $S_1(10) = S_1(13) = S_1(16) = 1$ is shown as a bullet. Maximum profit occurs for $S_1(19) = 1, S_1(40) = 5$. The profit curves up as $S_1(19)$ decreases.

Because of the non-concavity of the profit function, we used a grid search to find the (approximate) globally optimal choice for $S_1(10), S_1(13), S_1(16), S_1(19), S_1(40)$. We found that the maximum profit occurs for $S_1(13) \approx 7, S_1(16) \approx 9$, with the realized prices being contained in the interval $[13, 16]$. Moreover, given $S_1(13) = 7, S_1(16) = 9$, the profit π_1 is independent of $S_1(10), S_1(19)$, and $S_1(40)$ for values of $S_1(10), S_1(19)$, and $S_1(40)$ that satisfy:

$$0 \le S_1(10) \le S_1(13), S_1(16) \le S_1(19) \le S_1(40) \le \overline{q}_1.$$

Figure 14 shows the profit π_1 of firm 1 for $S_1(10) = 1, S_1(19) = S_1(40) = 17$ and versus choices

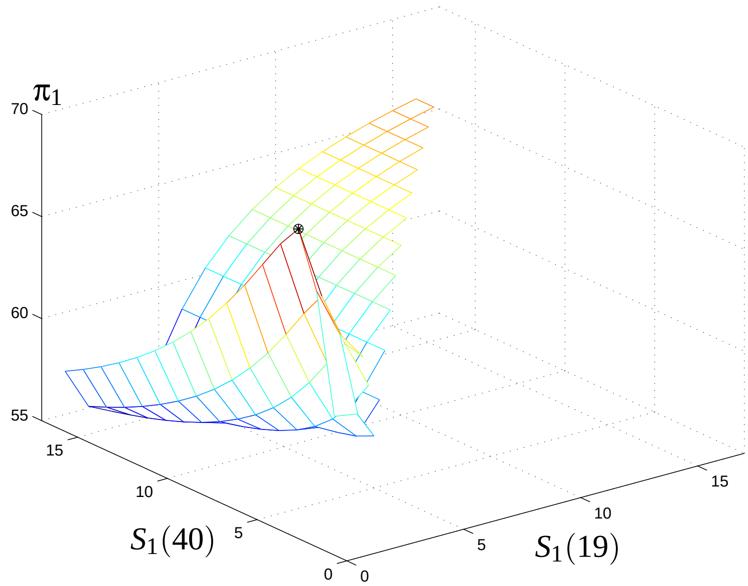

Figure 13: Profit for firm 1 for $S_1(10) = S_1(13) = S_1(16) = 1$ and versus choices of $S_1(19)$ and $S_1(40)$ in the range $1 \leq S_1(19) \leq S_1(40) \leq 17$.

of $S_1(13)$ and $S_1(16)$ in the range $1 \leq S_1(13) \leq S_1(16) \leq 17$. The maximum profit point is shown as a bullet.

Since the optimal response satisfies $S_1(13) \approx 7 < S_1(16) \approx 9$, the optimal supply function of firm 1 is strictly increasing. That is, the non-decreasing constraints are not manifestly binding over the range of realized prices. That is, S_1 satisfies theorem 13. However, the discussion in section 7.2.1 shows that the optimal response would change if the non-decreasing constraints were relaxed for firm 1. That is, the non-decreasing constraints are actually binding.

As demonstrated by figure 13, the profit function for firm 1 is not concave as a function of $S_1(10), S_1(13), S_1(16), S_1(19), S_1(40)$ when the supply function is piece-wise affine with break-points at $p = 4, 10, 13, 16, 19, 40$. The profits for small values of $S_1(19)$ and $S_1(40)$ bend up as $S_1(19)$ approaches zero. *A fortiori* the profit of firm 1 is not concave as a function of $S_1 \in \mathbb{S}_1$. However, the integration of the profit function over time in (9) has "smeared" out the non-concavities of the underlying profit per unit time functions. In particular, recalling the optimal behavior for firm 1 *just considering time* $t = 0$, we found previously that firm 1 should bid a quantity $q_{10} = 3.5$ at a price of $p_0 = 40$. That is, $S_1(40) = 3.5$, which would require that $S_1(19) \leq 3.5$ to satisfy the non-decreasing constraint. This strategy corresponds to values of $(S_1(19), S_1(40))$ that are near the origin in figure 13. However, the implications of this choice at other times is to significantly reduce the overall profit: for this reason, larger values of $S_1(19)$ and $S_1(40)$ actually yield the global optimum profit for firm 1.

In the next section we discuss an approach to numerically estimating equilibria when the cost functions are asymmetric, while taking explicit account of the non-decreasing and capacity constraints and the price cap. This will allow us to empirically investigate the issue of multiplicity of equilibria. We will see that the implications of the theorem proved in section 7.1 are corroborated by the numerical results:

- the solutions are piece-wise differentiable and appear to match solutions of (18) between points of non-differentiability;

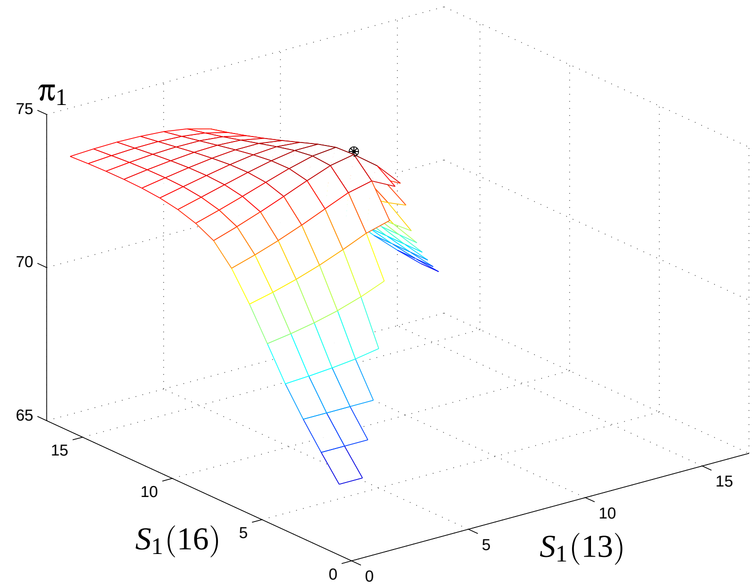

Figure 14: Profit for firm 1 versus choices of $S_1(13)$ and $S_1(16)$ in the range $1 \leq S_1(13) \leq S_1(16) \leq 17$.

- the non-decreasing constraints are never manifestly binding over the range of realized prices; however, the non-decreasing constraints are actually binding and their representation is essential in order to calculate the equilibria.

8 Iterations in function space

Because of the difficulties with the differential equation approach to seeking the SFE in general, we take an iterative numerical approach. An iterative numerical approach to finding the *Cournot* equilibrium in a transmission constrained electricity system is described in [27, Appendix]. Here, we describe an iterative numerical approach to finding the SFE in a transmission unconstrained system. Such numerical approaches can usually be expected to yield only stable equilibria, unless started at an equilibrium or unless the iterative process produces a particular iterate that happens to be an equilibrium. In the following sections, we describe the step direction, update, and step size and the computational issues involved.

8.1 Step direction

Given a current estimate of the equilibrium supply functions, denoted $S_i^{(\nu)}$ at iteration ν, we calculate the following step directions:

$$\forall i, \Delta S_i^{(\nu)} \in \underset{\Delta S_i}{\operatorname{argmax}} \{\tilde{\pi}_i(S_i^{(\nu)} + \Delta S_i, S_{-i}^{(\nu)}) | S_i^{(\nu)} + \Delta S_i \in \mathbb{S}_i\}, \tag{38}$$

where:

- $\tilde{\pi}_i$ is an approximation to π_i,
- $S_{-i}^{(\nu)} = (S_j^{(\nu)})_{j \neq i}$, and

- $\underline{\mathbb{S}}_i$ is a finite dimensional convex subset of $\mathbb{S}_i$.

8.2 Supply function subspace

The set $\underline{\mathbb{S}}_i$ consists of piece-wise affine non-decreasing functions with break-points evenly spaced between $(\underline{p}+0.1)$ pounds per MWh and $(\overline{p}-0.1)$ pounds per MWh, where $\underline{p}$ is the price minimum and $\overline{p}$ is the price cap. At $p=\underline{p}$, we define $S_i(p)=0$. At $p=\overline{p}$, we require $S_i(p)=\overline{q}_i$. That is, $\underline{\mathbb{S}}_i$ is convex.

For most cases, we used 40 break-points. We also tested some of the cases using functions with other numbers of break-points to investigate whether any of the results were an artifact of the number of break-points.

8.3 Update and step size

An initial guess $S_i^{(0)}, i=1,\ldots,n$ was used as a starting function to begin the iterations. We then update the iterates according to:

$$\forall \nu, \forall i, S_i^{(\nu+1)} = S_i^{(\nu)} + \alpha \Delta S_i^{(\nu)},$$

where $\alpha \in (0,1]$ is a step-size. Since $S_i^{(\nu)}$ and $S_i^{(\nu)} + \Delta S_i^{(\nu)}$ are both elements of the convex set $\underline{\mathbb{S}}_i$, then so is $S_i^{(\nu+1)}$.

We tested several step-size rules, including an elaborate "Armijo"-like rule [28] that sought to find directions at each iteration that guaranteed improvement in the profit of all firms. However, we found that a fixed step-size of $\alpha = 0.1$ performed satisfactorily.

Day and Bunn [12] take a similar approach, except that they only find an approximate local maximizer of (38) at each iteration and use a step size of $\alpha = 1$ at each iteration. Their approach requires less effort per iteration, but because of the inflexibility of the unity step size does not appear to converge [12, §4].

8.4 Profit function approximation

We estimated the integral in the profit function by dividing the time horizon into intervals having end-points at:

- $t=0$,
- the times corresponding to the realized prices at the break-points of the supply function, and
- $t=1$.

Linear interpolation was used to find the prices corresponding to $t=0$ and $t=1$, while (5) was used to evaluate the time corresponding to each price break-point. (If a price break-point corresponded to a "negative" time or to a time greater than one, it was simply discarded. Only realized prices, that is, prices for which $0 \leq t \leq 1$ in (5), are relevant in calculating the profit over the time horizon in (9).)

In some cases, we used the trapezoidal rule to approximate the integral on each interval. In other cases, we integrated the quadratic function on each interval exactly.

8.5 Computational issues

Iterating in the function space of supply functions requires considerable computational effort at each iteration and is subject to the drawback that the problem of finding the search direction may have multiple local optima. In practice, we use an iterative local search algorithm to seek the solution of (38) and do not guarantee to find the global optimum of (38). Consequently, even if the sequences of iterates $\{S_i^{(\nu)}\}_{\nu=0}^{\infty}$ converge this does not by itself prove that an equilibrium has been found. We do not perform the necessary global optimization checks to verify that an equilibrium has been found.

As we argued in section 7.2, because of the integrated profit function this issue may be less problematic in the supply function space than it appears at first. This is because the non-concavity shown in the two firm example system in section 7.2 involved a supply bid by firm 2 that was extreme in that it became nearly flat at high prices. If a good initial guess of the solution of (38) can be used, such as a known equilibrium of a similar problem, then the low profit regions such as $S_1(19), S_1(40) \approx 0$ in the example in section 7.2 can be avoided.

All software was implemented using Matlab and the Matlab Optimization Toolbox.

9 Three firm numerical results

We used the symmetric three firm example to illustrate the results on stability of equilibria from section 5. In the following section we discuss the demand, price cap and price minimum, the supply functions, the starting functions, and the results.

9.1 Demand

We assumed a base-case demand slope of $\gamma = 0.125$ GW per (pound per MWh) and a base-case load duration characteristic of:

$$\forall t \in [0,1], N(t) = 7 + 20(1-t),$$

with quantities measured in GW. That is, N varied linearly from 27 to 7 GW.

9.2 Price cap and price minimum

A price cap of $\overline{p} = 20$ pounds per MWh and a price minimum of $\underline{p} = 9$ pounds per MWh was used.

9.3 Supply functions

We used 40 break-points for most cases, with 20 break-points used to test the sensitivity of the results on the number of break-points.

9.4 Starting functions

In the case of symmetric cost functions and no capacity constraints nor price caps, we have already exhibited the range of equilibria between $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$. We used a range of such equilibria

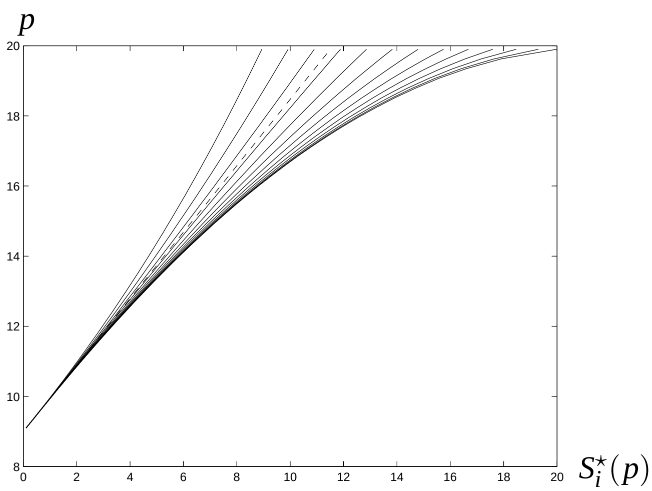

Figure 15: Starting functions for symmetric three firm example.

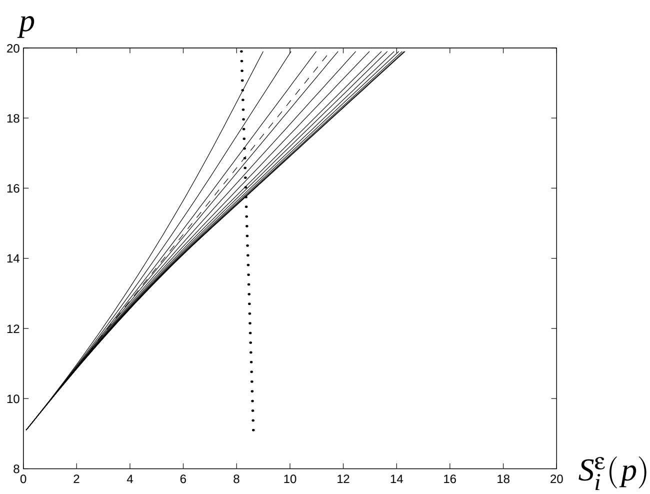

Figure 16: Perturbed starting functions constructed according to definition 4.

as starting functions. We calculated the equilibria using (18) and included SFEs that were more competitive and also SFEs that were less competitive than the affine SFE $S^{\star\text{affine}}$. Fourteen such starting functions are illustrated in figure 15. The affine SFE is shown dashed, while the others are shown solid. Since each SFE is symmetric, each supply function illustrated represents the supply functions of all three firms for that equilibrium.

We also used the construction in definition 4, with $p^\varepsilon \approx p_0^\star - 1$ pound per MWh, to perturb the SFEs slightly. These perturbed SFEs are shown in figure 16. The nearly vertical dotted line in figure 16 shows the vicinity of the peak realized prices and corresponding quantities for these supply functions. The perturbed SFEs are almost indistinguishable from the SFEs for prices up to the peak realized prices for the supply functions.

9.5 Results using SFEs as starting functions

The results of using the SFEs as starting functions are shown in figure 17. The figure shows profits versus iteration for one of the firms (the profits are identical for each firm) for each of the fourteen

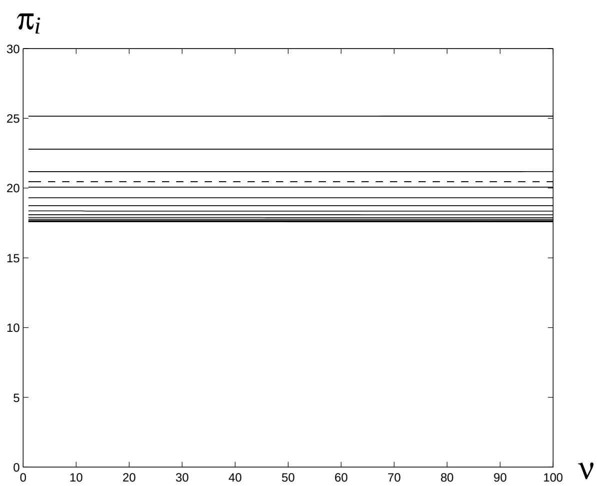

Figure 17: Profits versus iteration for SFE starting functions.

starting functions. The case of the affine SFE is shown dashed. In every case, the profits are identical at each iteration. This shows that the numerical framework evaluates the profits correctly for these starting functions.

9.6 Results using perturbed SFEs as starting functions

The results of using the perturbed SFEs as starting functions are shown in figure 18. The figure shows profits versus iteration for a firm for each of the starting functions. The results are very different to those shown in figure 17. In particular, except for the affine SFE and the two SFEs either side of it, (which are only "just" strictly convex and "just" strictly concave, respectively) the sequence of profits differs significantly from the starting profits. For all but these three starting functions, the sequence of profits appears to be drifting towards a band of profits that is lower than the profits for the affine SFE. This result is, however, dependent on the details of the numerical calculation. For example, figure 19 shows the results using similar starting functions but only 20 break-points in the functions. The sequence of profits is rather different.

By corollary 10, all SFEs produced according to (18) except the affine SFE are unstable. However, from a numerical perspective, it is not surprising that the SFEs that are "close" to the affine SFE appear to be stable on the basis of numerical calculations. Interestingly, the numerical results seem to also suggest that there is a band of stability involving SFEs that yield lower profits than the affine SFE. This may be a numerical artifact of the use of piece-wise affine approximations to the functions, since the band seems to be dependent, for example, on the number of break-points.

10 Simulation assumptions for five firm example

In the following sections we discuss the costs and capacities, the price cap and price minimum, the starting functions, and the criterion for assessing whether or not there are multiple equilibria.

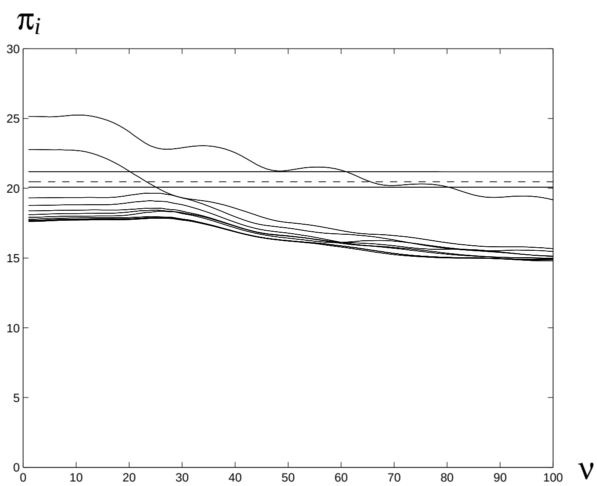

Figure 18: Profits versus iteration for perturbed SFE starting functions.

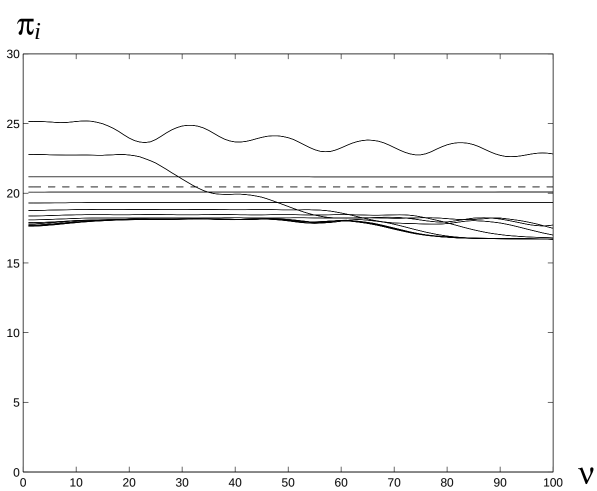

Figure 19: Profits versus iteration for perturbed SFE starting functions, with 20 breakpoints in supply functions.

Firm $i =$	1	2	3	4	5
$\overline{q}_i$(GW) $=$	5.70945	3.35325	10.4482	9.70785	3.3609

Table 3: Capacity data based on five firm industry described in [11].

10.1 Cost functions and capacities

We again consider the five firm example first introduced in section 6.3. The cost data is as in table 2. We initially investigate the uncapacitated case ("No capacity constraints," section 11.1) and then the case where non-negativity constraints are binding on production ("Minimum capacity constraints," section 11.2).

We then impose maximum capacity constraints, using the capacity data presented in [11] for the five strategic firm industry in England and Wales subsequent to the 1999 divestiture. Table 3 shows the maximum capacities. The total installed capacity is approximately 32.6 GW and the marginal cost at maximum production is roughly 27 pounds per MWh for all firms. Firms 2 and 5 are nearly identical and have the smallest capacity. Firms 3 and 4 have the largest capacity. These capacities were used for the base-case simulations ("Base-case," section 11.3). We also considered the effect of a capacitated fringe ("Fringe capacity constraints," section 11.4) based on the fringe capacity in the system investigated in [11] and considered the effect of increased capacities for the strategic players ("Increased capacities," section 11.5).

10.2 Demand

We assumed a base-case demand slope of $\gamma = 0.1$ GW per (pound per MWh) and a base-case load duration characteristic of:

$$\forall t \in [0,1], N(t) = 10 + 25(1-t),$$

with quantities measured in GW. That is, N varied linearly from 35 to 10 GW. This load-duration characteristic is illustrated in figure 1. The load factor is approximately 30%. (This is considerably smaller than a typical daily load factor. However, the five firms that we model from England and Wales do not include the baseload nuclear generation nor any fringe capacity, so that the N we use is actually a residual after baseload and fringe is subtracted. Alternatively, we can imagine that there has been some forward contracting of baseload capacity [22, 29].)

At a demand of $D(p,t) = 30$ GW and a price of 30 pounds per MWh, the price elasticity of demand is 0.1. The "choke price" at peak is $N(0)/\gamma = 350$ pounds per MWh, while the "choke price" at minimum demand is $N(1)/\gamma = 100$ pounds per MWh.

As sensitivity cases having different load factors, we also considered N varying linearly from:

1. 35 to 20 GW, ("Peak conditions," section 11.7.1,)

2. 20 to 10 GW, ("Off-peak conditions," section 11.7.2,)

3. 40 to 10 GW, ("Increased demand," section 11.9,)

4. 10 to 1 GW, ("Minimum capacity constraints," section 11.2,) and

5. 43.5 to 14.3 GW with capacitated competitive fringe capacity, (“Fringe capacity constraints,” section 11.4) and

6. sub-ranges of 43.5 to 14.3 GW with capacitated competitive fringe capacity, (“Increased load factor with capacitated fringe,” section 11.8.)

The first and second sensitivity cases divide the base-case time horizon into peak (35–20 GW) and off-peak (20–10 GW) times. Combining the results from both allows an evaluation of how the load factor affects the equilibrium profits and prices. The third sensitivity case requires demand rationing. The fourth sensitivity case investigates demand levels such that minimum capacity constraints are binding on some of the firms.

The fifth sensitivity case was used to investigate the effect of a capacitated fringe and was chosen to approximately reproduce the demand conditions investigated in [11]. The sixth sensitivity case divides the load-duration characteristic from [11] into peak, mid-load, and off-peak conditions to investigate how the load factor affects the equilibrium profits and prices when there is a capacitated fringe.

The assumption of an affine load-duration characteristic is not realistic, but simplifies the computational implementation because N can be inverted easily. By corollary 15, the set of SFEs depends on the range of N but not on its detailed functional form and so the candidate equilibria we obtain could also be used to estimate profits with a more realistic load-duration characteristic. Nevertheless, the assumption of an affine N may affect which equilibrium is exhibited by the numerical framework if there are multiple equilibria.

10.3 Price cap and price minimum

A price cap of $\overline{p} = 40$ pounds per MWh was used as the base-case price cap (“Base-case,” section 11.3.) Since the maximum marginal cost of production is approximately 27 pounds per MWh, the base-case price cap is nearly 50% higher than the maximum marginal production cost.

Sensitivity cases with price caps in the range of 30–80 pounds per MWh were also considered (“Varying the price cap,” section 11.6.) We also considered the case of bid caps at $\overline{p} = 40$ pounds per MWh (“Bid caps,” section 11.9.4.)

Since the maximum capacities of the five firms sums to approximately 32.6 GW and the price cap was 30 pounds per MWh or above, there is enough capacity to meet the peak demand at a price that is below the price cap. For the price cap of 30 pounds per MWh, the peak demand can only just be met. For price caps up to approximately 60 pounds per MWh, each firm is “pivotal” in that if any firm withdrew all its capacity from the market then the price would rise to the price cap at some times around peak demand and non-economic rationing would result.

For the cases with $N(1) \geq 10$ (and no fringe capacity), even competitive bids by all the players would result in prices above 12 pounds per MWh. A price minimum of $\underline{p} = 12$ pounds was used for most of these cases. A sensitivity case using $\underline{p} = 8$ pounds per MWh was used to verify that the choice of $\underline{p}$ did not tangibly affect results. The price minimum of $\underline{p} = 8$ pounds per MWh was also used for the cases with $N(1) = 1$ GW.

10.4 Starting functions

In the case of symmetric cost functions and no capacity constraints nor price caps, we have already exhibited a range of equilibria, including the three equilibria: $S^{\star\text{Cournot}}, S^{\star\text{affine}}$, and $S^{\star\text{comp}}$. Unfortunately, for the asymmetric cost functions we consider, supply functions $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$ constructed using (18) with Cournot and competitive initial conditions, respectively, both violate the non-decreasing constraints for prices below the peak realized price.

The functions $S_i^{\star\text{Cournot}}, i = 1, \ldots, n$ are illustrated in figure 20. They violate the non-decreasing constraints for prices less than about 64 pounds per MWh. (We continue to use a superscript $\star$ for these functions, although they are not even allowable supply functions if demand results in prices being realized for which the functions are not non-decreasing.)

At a price of 64 pounds per MWh, the total supply $\sum_i S_i^{\star\text{Cournot}}(64)$ is approximately 22 GW. This corresponds to a value on the load-duration characteristic of $N(t) = 28$ GW. That is, $S_i^{\star\text{Cournot}}$ could be an SFE for a system with load-duration characteristic that had range $[28, 35]$, which is a load factor of about 80%. (We note that even in this case, $S_i^{\star\text{Cournot}}$ is concave for firms 1, 3, 4, so that the equilibrium may be unstable.) For load factors below 80%, as in our example cases, $S_i^{\star\text{Cournot}}$ violates the non-decreasing constraints over the range of realized prices and therefore is not an equilibrium for such load-duration characteristics.

Were were unable to solve the differential equation starting from the competitive initial condition to obtain $S^{\star\text{comp}}$. However,

$$\lim_{p \uparrow p_0^{\text{comp}}} S^{\star\text{comp}\prime}(p) = \begin{bmatrix} -\infty \\ -\infty \\ +\infty \\ +\infty \\ -\infty \end{bmatrix},$$

where $p \uparrow p_0^{\text{comp}}$ means the limit from below. That is, $S^{\star\text{comp}}$ must also violate the non-decreasing constraints. We were able to solve the differential equation for initial conditions nearby to the competitive initial condition. One such solution is illustrated in figure 21. All such nearby solutions violate the non-decreasing constraints.

The function $S^{\star\text{affine}}$ is well-defined in both the symmetric and asymmetric cases and we use it as a starting function. However, since $S^{\star\text{Cournot}}$ and $S^{\star\text{comp}}$ are not allowable functions, we defined two other starting functions, one less and the other more competitive than the affine SFE $S^{\star\text{affine}}$. In particular, for the unconstrained and no price cap case we used three different starting functions:

- "uncapacitated competitive," S^{comp} where the supply functions are the inverses of the marginal cost functions, as specified in (21),
- "uncapacitated affine SFE," $S^{\star\text{affine}}$ where the supply functions are given by the solution of the affine SFE (15), with coefficients β_i satisfying (16), and
- "uncapacitated Cournot," S^{Cournot} where quantities and prices under Cournot competition are calculated for each $t \in [0, 1]$ as specified in (22) and a supply function drawn through the resulting price-quantity pairs.

For the maximum capacity constrained and price-capped cases, we used the following three starting functions:

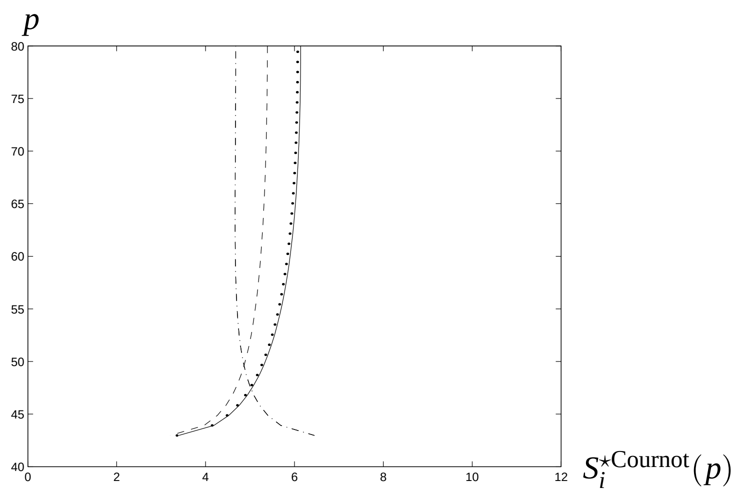

Figure 20: Supply function $S^{\star\text{Cournot}}$. Firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.

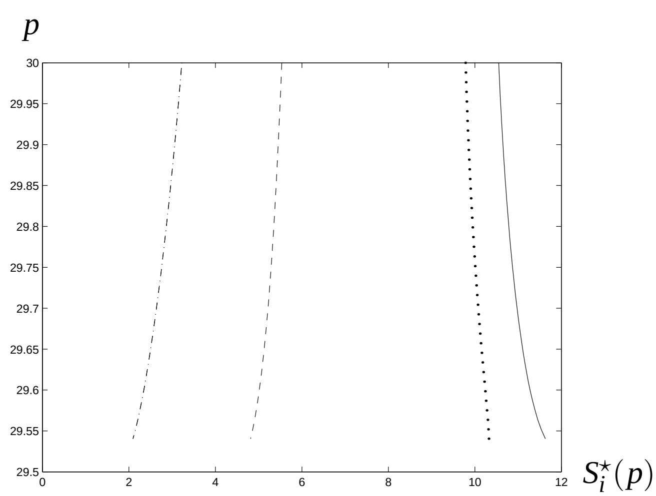

Figure 21: Supply function obtained using initial condition nearby to competitive initial condition. Firm 1 is shown as a dashed line, firms 2 and 5 are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.

- “capacitated competitive,” where the supply functions are the inverses of the marginal cost functions, but limited by the maximum capacity, as shown in figure 22,

- “capacitated affine SFE,” where the supply functions are given by the solution of the affine SFE, except that the values of S_i are limited by the maximum capacity, as shown in figure 23, and

- “price-capped Cournot,” where Cournot quantities and prices are calculated for each $t \in [0,1]$ and a supply function drawn through the resulting price-quantity pairs, but modified to satisfy the price cap condition (4), as shown in figure 24.

(In each case, we have graphed the supply function only for prices greater than 12 pounds per MWh, to avoid the issue of minimum capacity constraints under the assumption that realized prices are always at least 12 pounds per MWh. We will discuss this issue, and provide a generalization of $S^{\star\text{affine}}$ for the case of binding minimum capacity constraints in section 11.2.) In summary, the starting functions for the capacitated and price-capped cases are obtained by calculating a supply curve under the assumption of no capacity constraints and then truncating the supply curve to satisfy the capacity constraints and then (in the case of price-capped Cournot) redefining the supply function at the price $p = \overline{p}$ so that it satisfies (4).

Firms 2 and 5 are essentially identical and their supply functions appear superimposed as the leftmost dash-dot curve in figures 22–24 and in all subsequent figures. Firms 3 and 4 have the largest capacity and their supply functions appear as the solid and dotted curves, respectively, at the right of figures 22–24 and in all subsequent figures. (In figure 24, the supply functions of firms 3 and 4 are almost superimposed.) The supply function of firm 1 appears as the dashed curve in the middle of figures 22–24 and in all subsequent figures.

Although the starting functions are not equilibrium supply functions for the capacitated and price-capped cases, we can still consider the resulting prices if the firms were to bid these supply functions. The price-duration curves for the base-case demand conditions corresponding to the firms bidding the capacitated competitive, the capacitated affine SFE, and the price-capped Cournot supply functions, respectively, are shown in figures 25–27, respectively. Given bids equal to the capacitated competitive supply function, no firm ever reaches its capacity and so the price-duration curve in figure 25 has constant slope.

Given bids equal to the capacitated affine SFE supply functions, capacity constraints are reached for firms 2 and 5 at a price of about 30 pounds per MWh, so that the price-duration curve in figure 26 bends upward for peak demand times near to $t = 0$. A reasonable hypothesis is that the capacitated affine SFE starting function is in the vicinity of the equilibrium for the base-case since it is the equilibrium if the capacity constraints are not binding.

Given bids equal to the price-capped Cournot supply function, no firm ever reaches its capacity. However, the price cap is binding over most of the time horizon as shown in figure 27. As suggested in the introduction, the prices and profits corresponding to the price-capped Cournot supply function may be a reasonable prediction of the equilibrium behavior when firms face the price cap but are not required to bid consistently across the time horizon. We will use these “price-capped Cournot” prices and profits as a benchmark to evaluate the effect of requiring supply function bids to be consistent over the time horizon. (For comparison, the Cournot price corresponding to the peak time and with no price cap is around 80 pounds per MWh.)

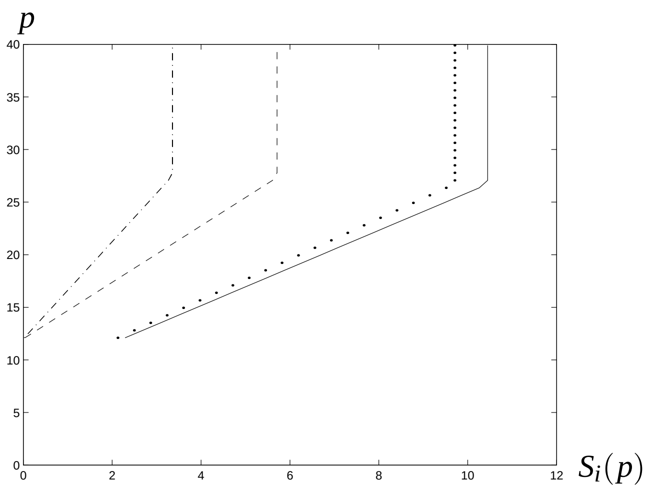

Figure 22: “Capacitated competitive” supply function.

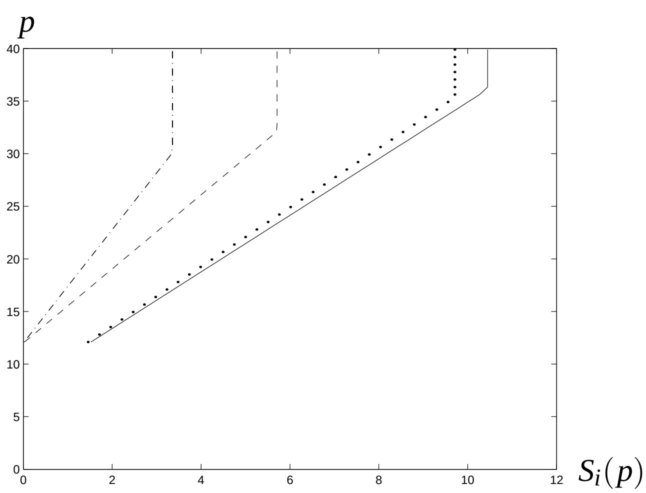

Figure 23: “Capacitated affine SFE” supply function.

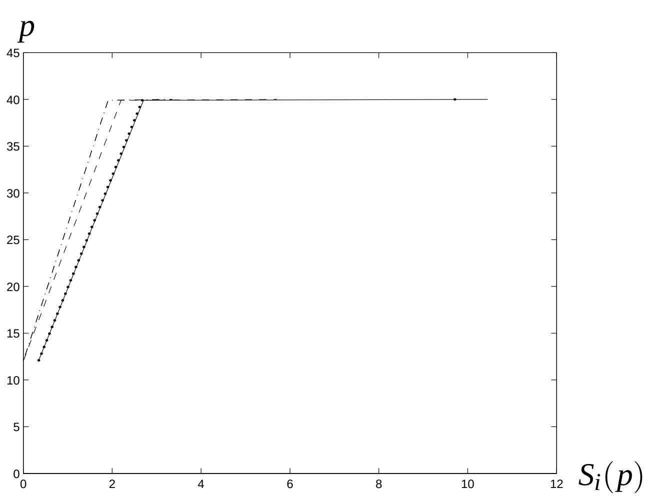

Figure 24: “Price-capped Cournot” supply function. (Note change in price axis compared to previous figures.)

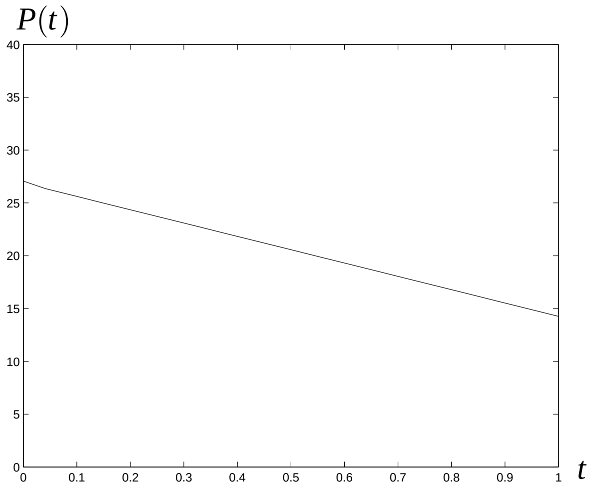

Figure 25: Price-duration curve for “capacitated competitive” supply function.

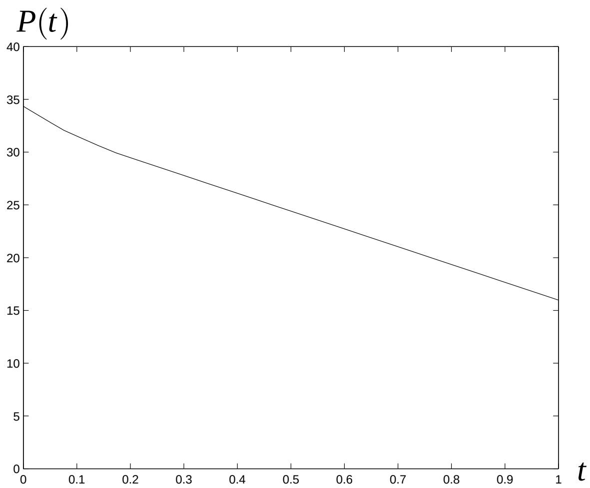

Figure 26: Price-duration curve for “capacitated affine SFE” supply function.

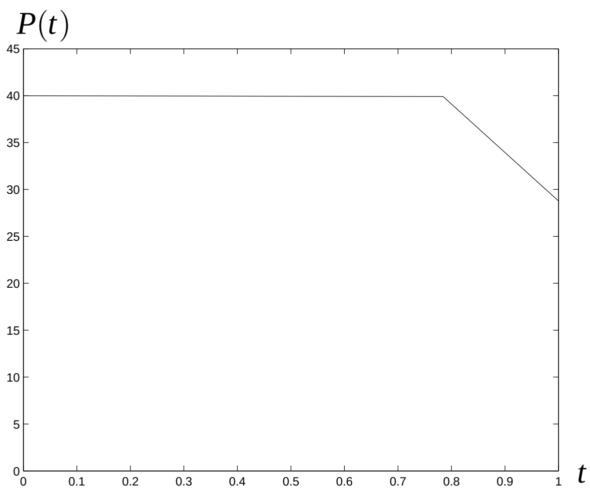

Figure 27: Price-duration curve for “price-capped Cournot” supply function. (Note change in price axis compared to previous figures.)

10.5 Criterion for assessing existence of multiple equilibria

In experiments, we found that even after a large number of iterations, the values of the supply functions were still changing by significant amounts from iteration to iteration. In particular, the L_1 norm of the difference between successive iterates was on the order of a few percent of the L_1 norm of the iterate itself. Moreover, supply functions change visibly from iteration to iteration, with the position of features such as points of non-differentiability in the supply functions slowly drifting over successive iterations.

In contrast, profit at each iteration showed much steadier progress. Defining profit at iteration ν according to (9) with the supply functions $S_i^{(\nu)}$ used to specify the price function through (7), we found that the profits typically changed by less than 0.1% from iteration to iteration after 100 iterations. Moreover, the profits typically reach a quasi-steady state level within about 20 iterations.

As suggested by [12], the changes in bid functions from iteration to iteration may be evidence of Edgeworth cycles. However, the steadiness of the profit functions suggests that the changes in the supply functions may simply be an artifact of the numerical calculations.

In assessing whether or not there are multiple equilibria, we must distinguish differences due to numerical artifacts of the calculations from truly different equilibria. For example in the symmetric three firm example in section 9, we know theoretically that all equilibria except the affine SFE are unstable. Nevertheless, we observed some apparently stable equilibria besides the affine SFE, as shown by the bands of stability in figures 18 and 19, and so these results are presumably an artifact of the numerical framework.

We apply the following *ad hoc* criterion to assess whether or not there are multiple equilibria. We deem two candidate equilibria to be the same if:

- for each firm, the profits are within 2% in each candidate equilibrium,
- for each firm, the supply functions have the same general shape in each candidate equilibrium over the range of realized prices, and
- the price-duration curves have the same general shape in each candidate equilibrium (and, in particular, have the same peak realized price.)

For each case, we iterated 100 times from the starting function and used the results from iteration 100 to assess whether or not candidate equilibria were the same or different.

When multiple equilibria are observed, we consider the range of equilibrium outcomes. In assessing whether the range of profits is relatively large or small, we compare the range of profits for the various equilibria to the range between:

- the profits that would accrue if all firms bid the capacitated competitive supply function, shown in figure 22 and
- the profits that would accrue if all firms bid the price-capped Cournot supply function, shown in figure 24.

That is, the difference between the competitive and Cournot profits provides a scale for assessing the relative spread of profits when there are multiple equilibria.

11 Five firm numerical results

In this section, we report results of several cases:

- no capacity constraints, section 11.1,
- minimum capacity constraints, section 11.2,
- base-case demand and supply conditions, section 11.3,
- capacitated fringe, section 11.4,
- increased capacities, section 11.5,
- changed price caps, section 11.6,
- increased load factor, sections 11.7 and 11.8, and
- increased demand, section 11.9.

We investigate empirically the conditions for the results to exhibit multiple equilibria and also the qualitative effects of the changes compared to the base-case. We briefly summarize the characteristics of the results in section 11.10 and compare them to the theoretical properties of the solutions.

11.1 No capacity constraints

If market rules require that an affine supply function be bid by each firm, then in the case of no capacity constraints the affine SFE $S^{\star\text{affine}}$ is the unique SFE. If market rules allow nonlinear supply functions, then in the case of no capacity constraints there is a continuum of supply function equilibria, with the affine solution $S^{\star\text{affine}}$ being one of them.

We used the software to solve the no capacity constraints, no price cap, and nonlinear bid supply function case for the base-case demand. We used starting functions equal to, respectively:

- the uncapacitated competitive supply function, S^{comp},
- the uncapacitated affine SFE supply function $S^{\star\text{affine}}$, and
- the uncapacitated Cournot supply function, S^{Cournot}.

The test run serves to verify the operation of the software on a problem for which we know one of the equilibria, namely the affine SFE. Using the affine SFE as a starting function serves to verify that the software evaluates the profit correctly.

11.1.1 Uncapacitated competitive starting function

Figure 28 shows the profits versus iteration ν for the no capacity limit case starting from the uncapacitated competitive supply function. (In this figure and all subsequent figures illustrating the five firm example, firm 1 is shown as a dashed line, firms 2 and 5 have identical costs and capacities and are shown superimposed as a dash-dot line, firm 3 is shown as a solid line and firm 4 is shown as a dotted line.) The leftmost points in figure 28 show the profits if each firm were to bid the uncapacitated competitive supply function. That is, these are the profits if each firm bid competitively.

Figure 29 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 30. The peak realized price is $p_0^\star = 29$ pounds per MWh.

The supply function for each firm i at iteration 100 is "just" strictly convex as a function of price on the interval $[a_i, p_0^\star]$. So, if the solutions satisfy (18) then by theorem 8 the SFE is unstable. We note, however, that similar to the three firm numerical results, this apparent stability may be an artifact of the numerical implementation.

11.1.2 Uncapacitated affine SFE starting function

Figure 31 shows the profits versus iteration ν for the no capacity limit case starting from the uncapacitated affine SFE supply function. Profits are identical in every iteration. The leftmost points in figure 31 show the profits if each firm were to bid the uncapacitated affine SFE supply function. That is, these are the equilibrium profits if the firms are required to bid affine supply functions.

Figure 32 shows the corresponding supply functions at iteration 100, which are identical to the uncapacitated affine SFE. The price-duration curve for iteration 100 is shown in figure 33. The peak realized price is between 32 and 33 pounds per MWh.

11.1.3 Uncapacitated Cournot starting function

Figure 34 shows the profits versus iteration ν for the no capacity limit case starting from the uncapacitated Cournot supply function. The leftmost points in figure 34 show the profits if each firm were to bid the uncapacitated Cournot supply function. That is, these are the profits if Cournot competition occurs at each time in the time horizon without any obligation to bid a supply function that is consistent across the whole time horizon.

Figure 35 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 36. The peak realized price is again between 32 and 33 pounds per MWh.

11.1.4 Summary

From the perspectives of:

- the profit;
- the shape of the supply functions over the range of realized prices; and,
- the price-duration curves,

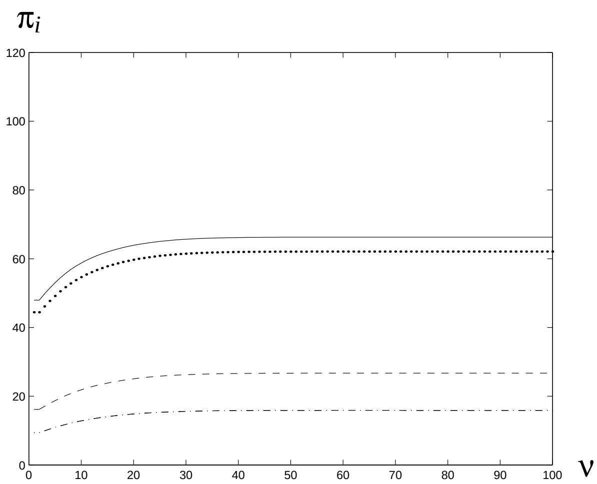

Figure 28: Profits versus iteration for case of no capacity constraints, starting from the uncapacitated competitive supply function.

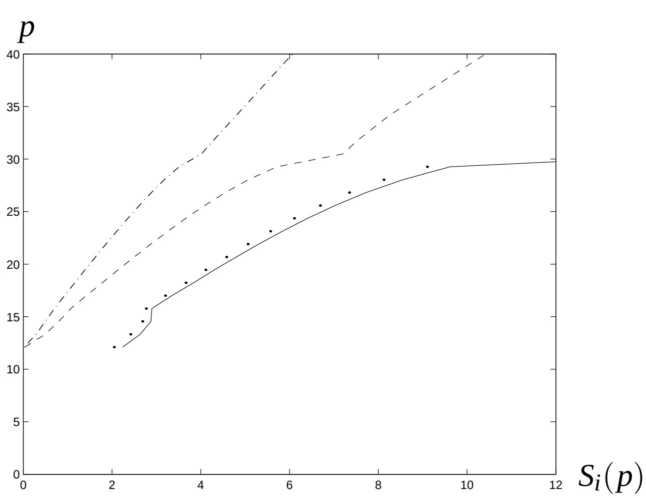

Figure 29: Supply functions at iteration 100 for case of no capacity constraints, starting from the uncapacitated competitive supply function.

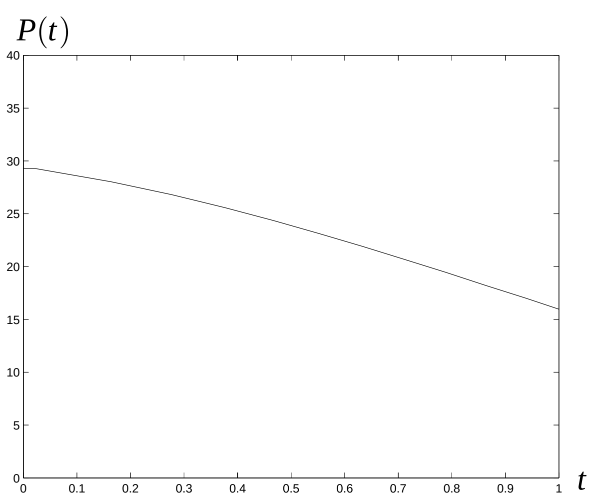

Figure 30: Price-duration curve at iteration 100 for case of no capacity constraints, starting from the uncapacitated competitive supply function.

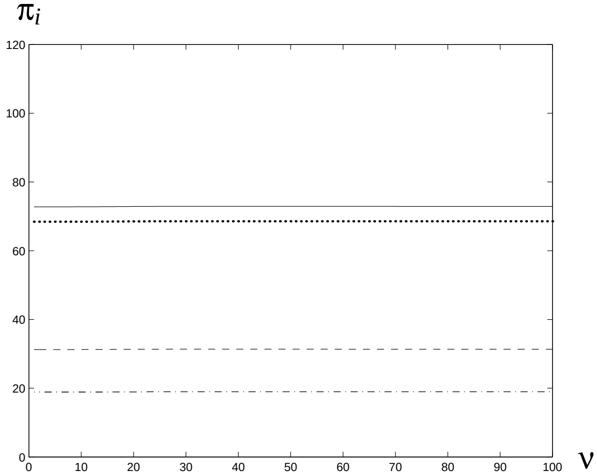

Figure 31: Profits versus iteration for case of no capacity constraints, starting from the uncapacitated affine SFE supply function.

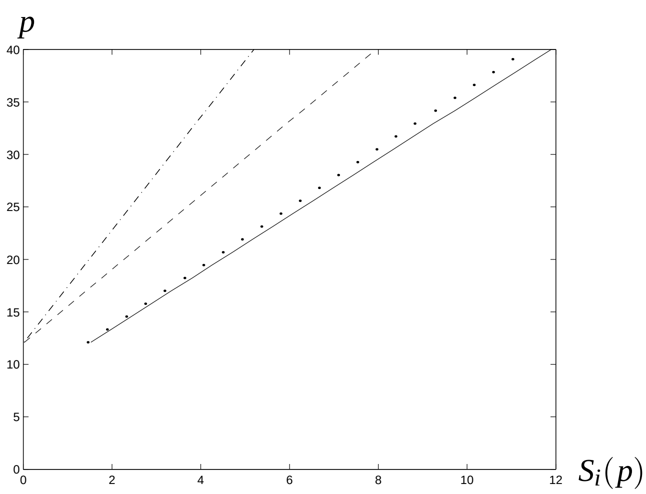

Figure 32: Supply functions at iteration 100 for case of no capacity constraints, starting from the uncapacitated affine SFE supply function.

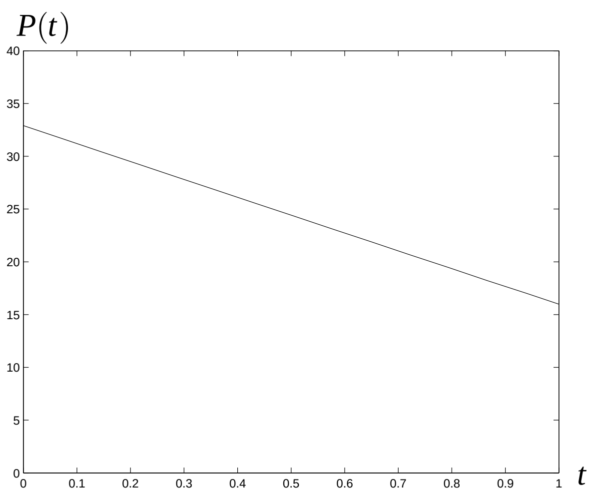

Figure 33: Price-duration curve at iteration 100 for case of no capacity constraints, starting from the uncapacitated affine SFE supply function.

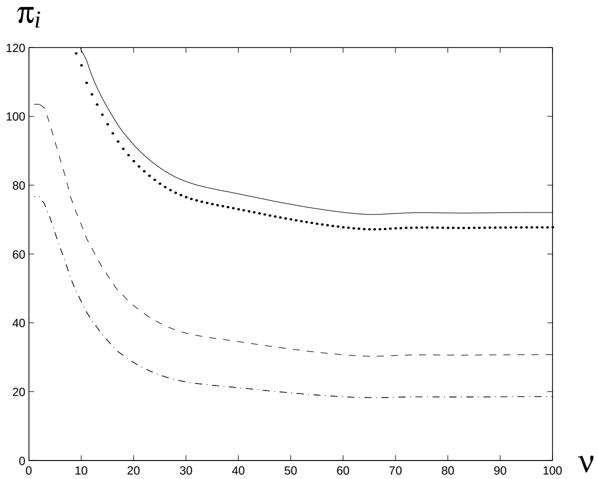

Figure 34: Profits versus iteration for case of no capacity constraints, starting from the uncapacitated Cournot supply function.

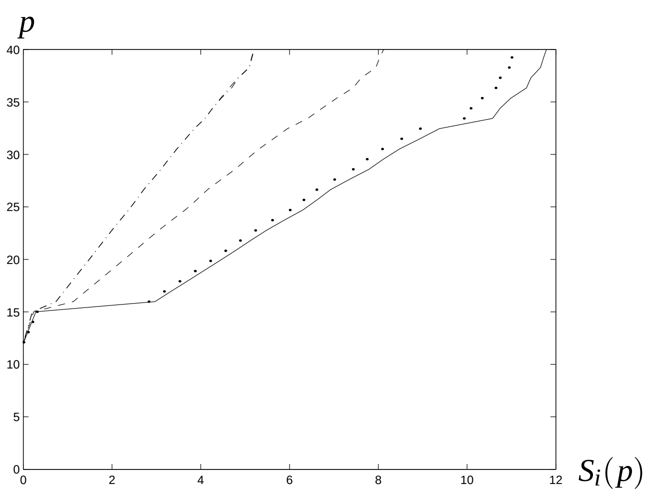

Figure 35: Supply functions at iteration 100 for case of no capacity constraints, starting from the uncapacitated Cournot supply function.

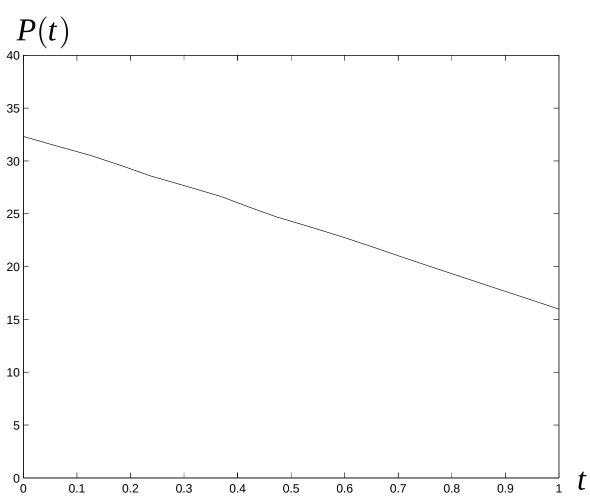

Figure 36: Price-duration curve at iteration 100 for case of no capacity constraints, starting from the uncapacitated Cournot supply function.

the results at iteration 100 starting from the uncapacitated affine SFE $S^{\star\text{affine}}$ and the uncapacitated Cournot functions are very similar. However, these two results differ from the results at iteration 100 starting from the uncapacitated competitive supply function. In particular, compared to the results at iteration 100 starting from the affine SFE or Cournot supply functions:

- the profits at iteration 100 starting from the uncapacitated competitive starting function are about 15% lower, as can be seen in figure 37, which shows the profit versus iteration for all starting functions combined,
- the values at iteration 100 of $S_i(p)$ starting from the uncapacitated competitive starting function are considerably higher for prices above about 20 pounds per MWh, as can be seen in figure 38, which shows the supply functions at iteration 100 for all starting functions combined, and
- the peak price at iteration 100 starting from the uncapacitated competitive starting function is considerably lower, as can be seen in figure 39, which shows the price-duration curve at iteration 100 for all starting functions combined.

The numerical results at iteration 100 show two candidate equilibria and there may be a continuum of equilibria between these two. We consider the relative range of the profits for the two candidate equilibria. For firm 1, for example, the range of profits at iteration 100 over the various start functions is from about 27 to 32, a range of 5.

The profits that would accrue to firm 1 if all firms bid the uncapacitated Cournot supply function are about 104. The profits that would accrue to firm 1 if all firms bid the uncapacitated competitive supply function are about 16. This is a range of about 88.

Combining these observations, the range of profits at iteration 100 for firm 1 over the various start functions is only about 6% of the range of profits for firm 1 between uncapacitated competitive and uncapacitated Cournot outcomes. That is, the range of SFE profits is relatively small. Similar observations apply for the other firms. Again, the range of apparently stable equilibria may be an artifact of the numerical framework.

11.2 Minimum capacity constraints

In this section, we use a reduced demand with $N(0) = 10$, $N(1) = 1$ in order to investigate the effects of *minimum* capacity constraints (that is requiring production to be non-negative) during off-peak times. In [11], piece-wise affine (but not continuous) SFEs are exhibited in the case of minimum capacity constraints. In this SFE, the equilibrium supply function of a firm i is discontinuous at any price p where a firm $j \neq i$ has cost function satisfying $a_j = p$. Using the results from [11] for the five firm example system results in the supply functions shown in figure 40.

We used the software to solve the minimum capacity constraints, no price cap, and nonlinear bid supply function case for the demand specified by $N(0) = 10$, $N(1) = 1$. Because the supply functions shown in figure 40 are an equilibrium in piece-wise affine functions, we used this as one of the starting functions (and will refer to it as $S^{\star\text{affine}}$.) Since we use a piece-wise affine and continuous representation of functions, we can only approximate the jump in this function at $p = 12$ pounds per MWh. We also used the competitive and Cournot starting functions and represented the minimum capacity limits in these functions by requiring the functions to be non-negative.

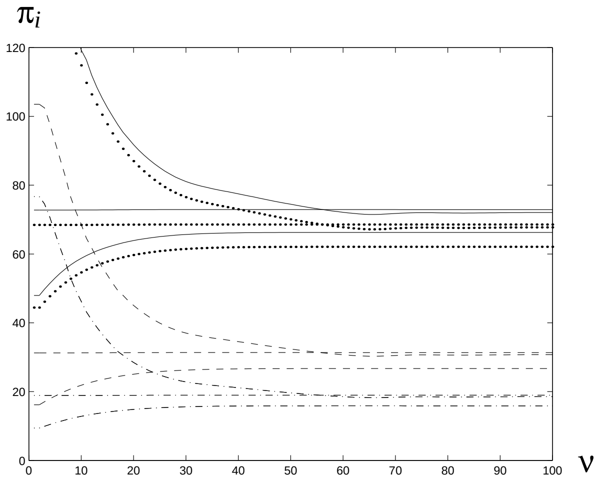

Figure 37: Profits versus iteration for case of no capacity constraints for all starting functions combined.

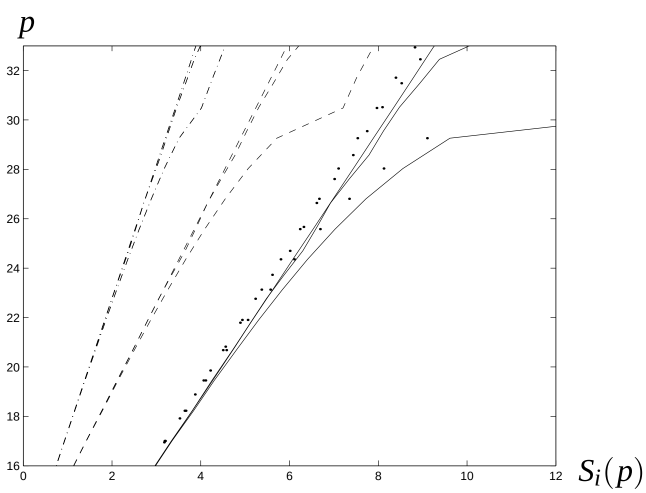

Figure 38: Supply functions at iteration 100 for case of no capacity constraints for all starting functions combined.

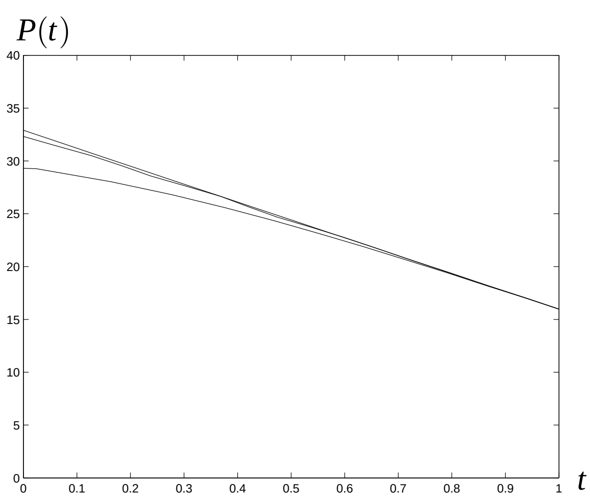

Figure 39: Price-duration curve at iteration 100 for case of no capacity constraints for all starting functions combined.

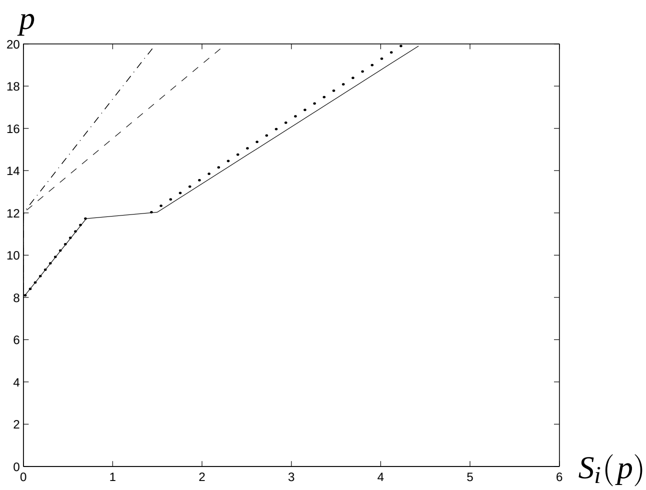

Figure 40: Piece-wise affine SFE constructed according to [11].

11.2.1 Competitive starting function

Figure 41 shows the profits versus iteration ν for the minimum capacity limit case starting from the uncapacitated competitive supply function. The leftmost points in figure 41 show the profits if each firm were to bid competitively. (The axes of the graphs in this section differ from that in section 11.1.)

Figure 42 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 43. The peak realized price is 15 pounds per MWh.

As in the no capacity constraints case, the supply functions at iteration 100 starting from the competitive starting function are all strictly convex and again are unstable based on application of the analysis in section 5. Again, the functions are only "just" strictly convex and their apparent stability may be an artifact of the numerical framework.

11.2.2 Piece-wise affine SFE starting function

Figure 44 shows the profits versus iteration ν for the minimum capacity limit case starting from the uncapacitated affine SFE supply function. Profits are almost identical in every iteration.

Figure 45 shows the corresponding supply functions at iteration 100, which are similar to the piece-wise affine SFE $S^{\star\text{affine}}$, except that the discontinuity at $p = 12$ pounds per MWh in $S^{\star\text{affine}}$ is smoothed off in the numerical results at iteration 100. Figure 44 shows that the smoothing off had essentially no effect on the profits of the firms. The price-duration curve for iteration 100 is shown in figure 46. The peak realized price is about 16 pounds per MWh.

11.2.3 Cournot starting function

Figure 47 shows the profits versus iteration ν for the minimum capacity limit case starting from the Cournot supply function. The leftmost points in figure 47 show the profits if each firm were to bid the Cournot supply function. That is, these are the profits if Cournot competition occurs at each time in the time horizon without any obligation to bid a supply function that is consistent across the whole time horizon.

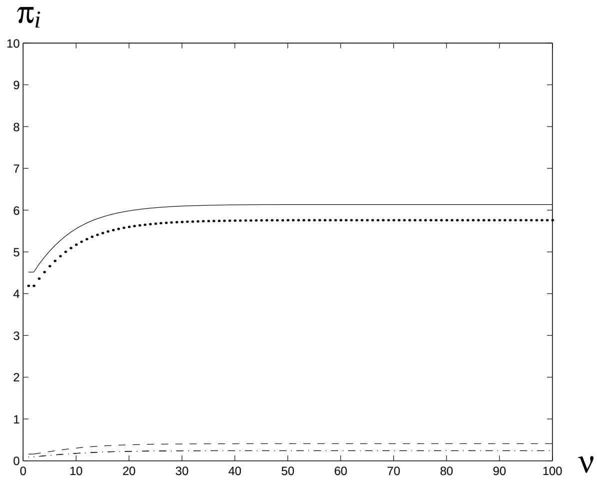

Figure 41: Profits versus iteration for case of minimum capacity constraints, starting from the competitive supply function.

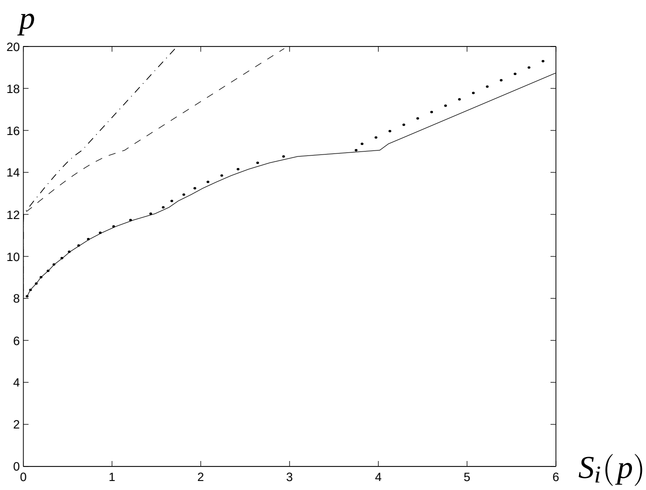

Figure 42: Supply functions at iteration 100 for case of minimum capacity constraints, starting from competitive supply function.

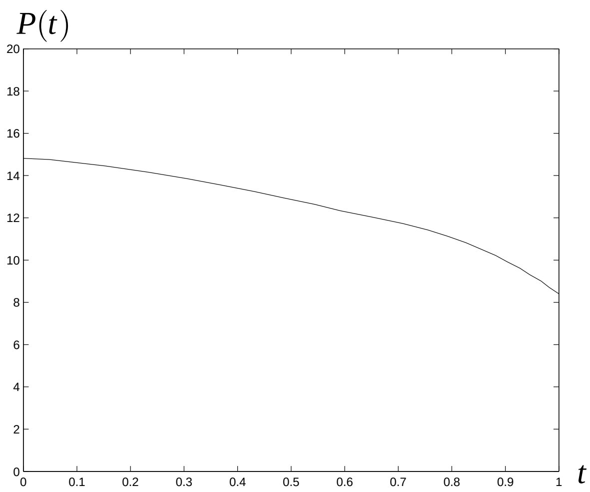

Figure 43: Price-duration curve at iteration 100 for case of minimum capacity constraints, starting from uncapacitated competitive supply function.

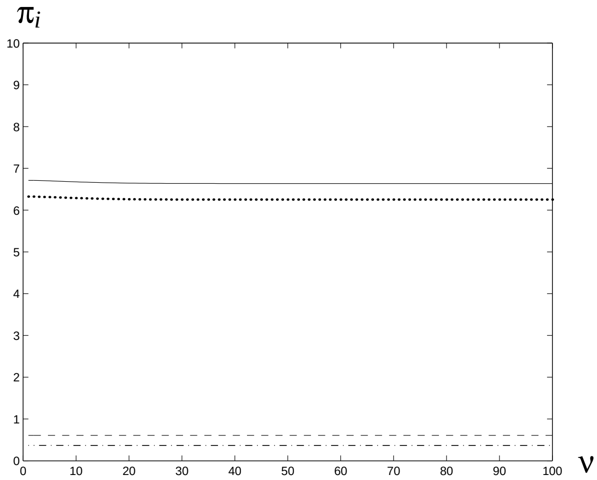

Figure 44: Profits versus iteration for case of minimum capacity constraints, starting from the piece-wise affine SFE supply function.

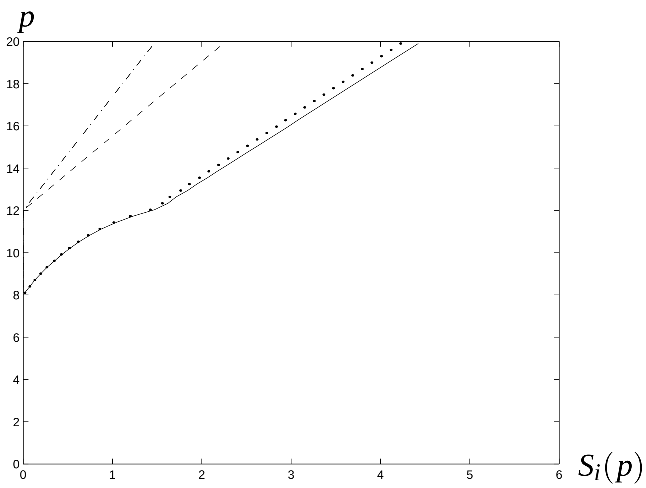

Figure 45: Supply functions at iteration 100 for case of minimum capacity constraints, starting from the piece-wise affine SFE supply function.

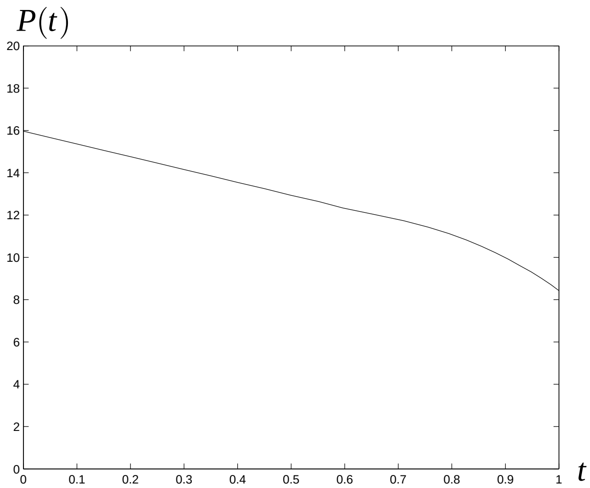

Figure 46: Price-duration curve at iteration 100 for case of minimum capacity constraints, starting from the piece-wise affine SFE supply function.

Figure 48 shows the corresponding supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 49. The peak realized price is about 16 pounds per MWh.

11.2.4 Summary

From the perspectives of:

- the profit;
- the shape of the supply functions over the range of realized prices; and,
- the price-duration curves,

the results at iteration 100 starting from the piece-wise affine SFE $S^{\star\text{affine}}$ and the Cournot functions are very similar. However, these two results differ from the results at iteration 100 starting from the competitive supply function. In particular, compared to the results at iteration 100 starting from the affine SFE or Cournot supply functions:

- the profits at iteration 100 starting from the uncapacitated competitive starting function are again somewhat lower,
- the values at iteration 100 of $S_i(p)$ starting from the uncapacitated competitive starting function are higher for prices above about 12 pounds per MWh, and
- the peak price at iteration 100 starting from the uncapacitated competitive starting function is lower.

These observations are apparent in the combined plots shown in figures 50–52, respectively.

The numerical results at iteration 100 again show two candidate equilibria. However, the range of SFE profits is again relatively small.

11.3 Base-case

We used the software to seek the equilibrium for the base-case assumptions, which involves capacity constraints and a price cap.

11.3.1 Starting from capacitated competitive

Figure 53 shows the profits versus iteration ν for the base-case assumptions starting from the capacitated competitive supply function. The leftmost points in figure 53 show the profits if each firm were to bid the capacitated competitive supply function. That is, these are the profits if the firms bid competitively at all times. The price-duration curve if each firm were to bid the capacitated competitive supply function is shown in figure 25. (The axes of the graphs in this section differ from that in section 11.2, but are similar to that in section 11.1.)

Profits at iteration 100 are considerably higher than in the uncapacitated case and more than double the profits that would accrue if the capacitated competitive supply functions were bid. As previously, firms 2 and 5 have identical costs and capacities, so they appear superimposed as the dash-dot curve.

Figure 54 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 55.

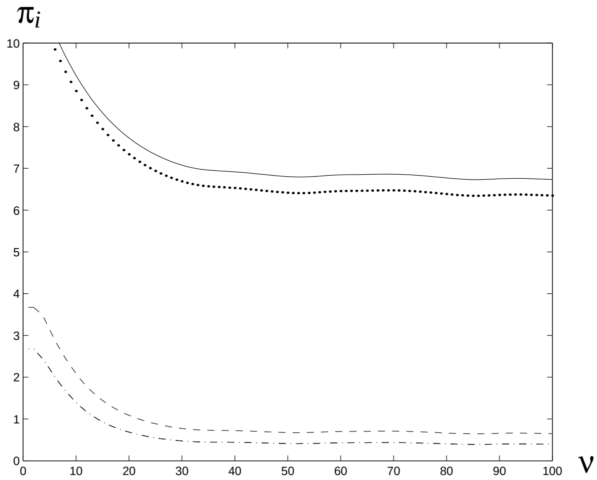

Figure 47: Profits versus iteration for case of minimum capacity constraints, starting from Cournot supply function.

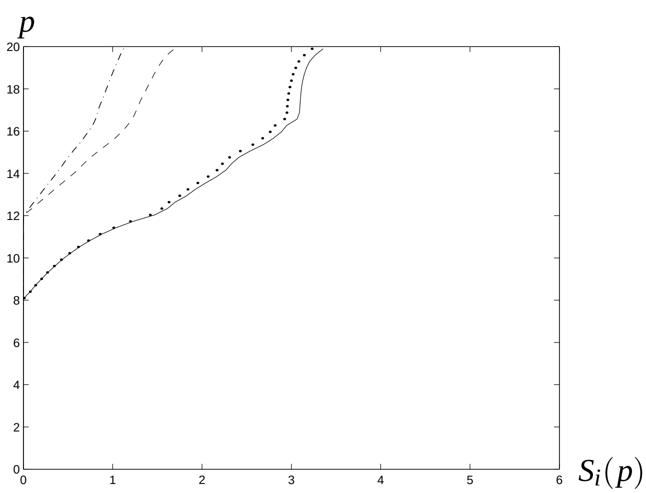

Figure 48: Supply functions at iteration 100 for case of minimum capacity constraints, starting from Cournot supply function.

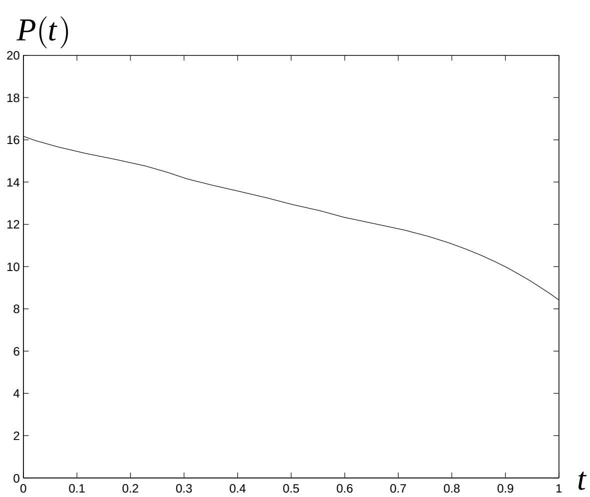

Figure 49: Price-duration curve at iteration 100 for case of minimum capacity constraints, starting from Cournot supply function.

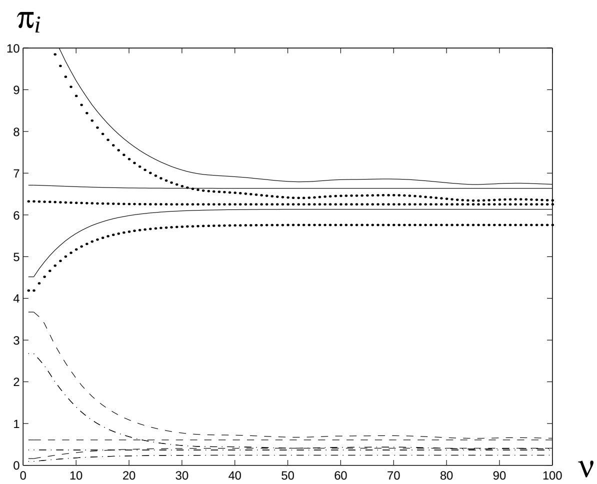

Figure 50: Profits versus iteration for case of minimum capacity constraints for all starting functions combined.

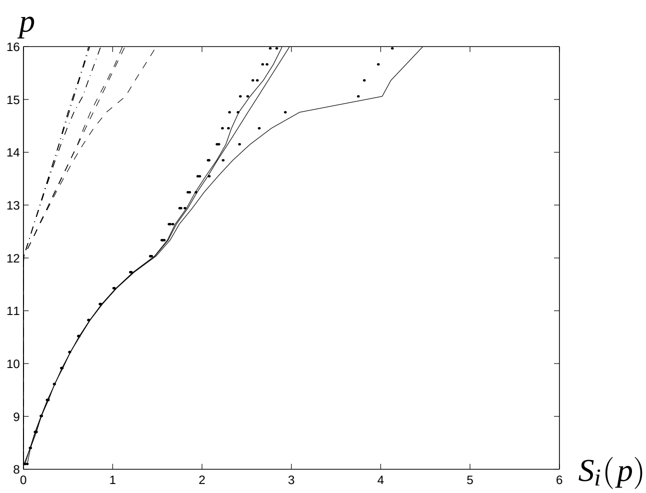

Figure 51: Supply functions at iteration 100 for case of minimum capacity constraints for all starting functions combined.

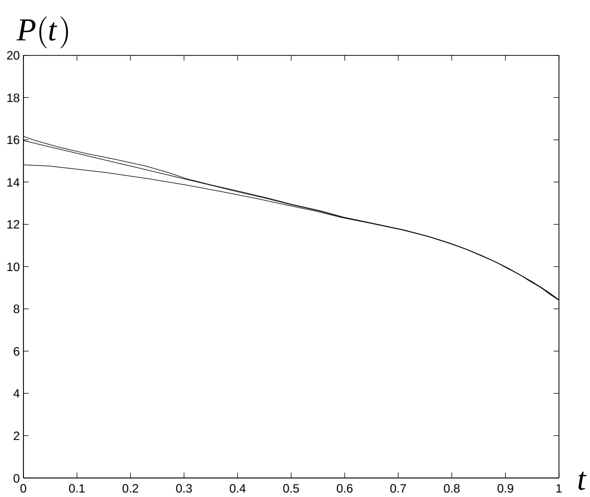

Figure 52: Price-duration curve at iteration 100 for case of minimum capacity constraints for all starting functions combined.

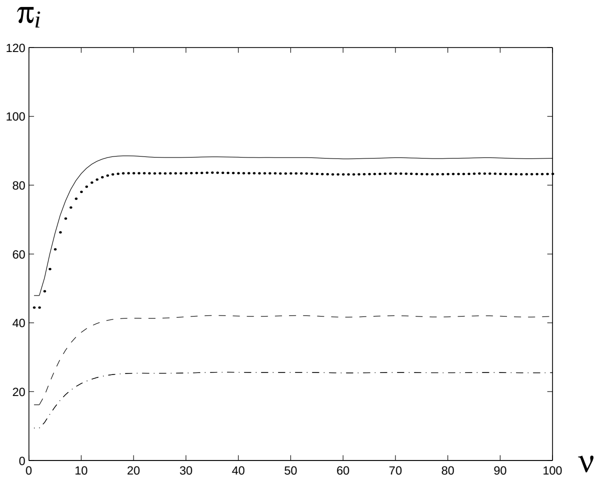

Figure 53: Profits versus iteration for base-case assumptions starting from capacitated competitive.

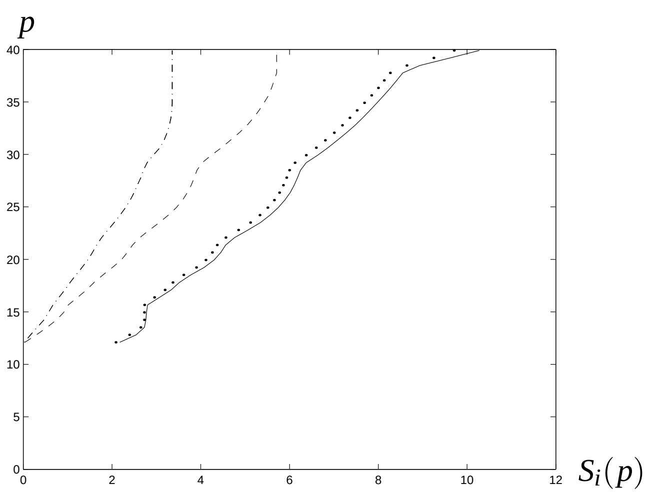

Figure 54: Supply functions at iteration 100 for base-case assumptions starting from capacitated competitive.

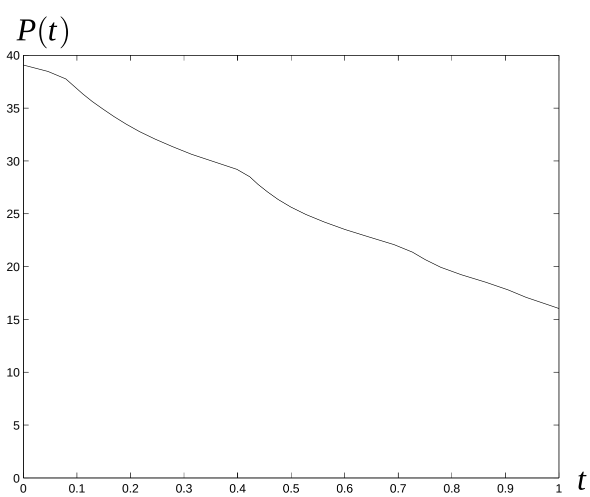

Figure 55: Price-duration curve at iteration 100 for base-case assumptions starting from capacitated competitive.

11.3.2 Starting from capacitated affine SFE

Figure 56 shows the profits versus iteration ν for the base-case assumptions starting from the capacitated affine SFE supply function. The leftmost points in figure 56 show the profits if each firm were to bid the capacitated affine SFE supply function. The price-duration curve if each firm were to bid the capacitated affine SFE supply function is shown in figure 26.

Profits at iteration 100 are again considerably higher than if all firms bid the capacitated affine starting function.

Figure 57 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 58. The results at iteration 100 starting from capacitated affine SFE are similar to the case of starting from the capacitated competitive supply function.

11.3.3 Starting from price-capped Cournot

Figure 59 shows the profits versus iteration ν for the base-case assumptions starting from the price-capped Cournot supply function. The leftmost points in figure 59 show the profits if each firm were to bid the price-capped Cournot supply function. That is, these are the equilibrium profits if price-capped Cournot competition occurs at each time in the time horizon without any obligation to bid a supply function that is consistent across the time horizon. The price-duration curve if each firm were to bid the price-capped Cournot supply function is shown in figure 27.

Profits at iteration 100 are considerably lower than if all firms bid the price-capped Cournot supply function.

Figure 60 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 61. The supply curves differ significantly from the previous cases for prices less than 16 pounds per MWh; however, these prices are below the minimum realized price and so are not relevant in the calculation of profits.

11.3.4 Starting from price-capped Cournot with reduced price minimum

Figure 62 shows the profits versus iteration ν for the base-case assumptions starting from the price-capped Cournot supply function, except that the price minimum was reduced to $\underline{p} = 8$ pounds per MWh. Figure 63 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 64. The results are similar to figures 59–61 except that the price-duration curve is slightly different for prices between 25 and 35 pounds per MWh.

11.3.5 Reduced number of break-points

Figure 65 shows the profits versus iteration ν for the base-case assumptions starting from the capacitated affine SFE supply function but with only 20 break-points in the supply function. Figure 66 shows the supply functions at iteration 100.

Figures 57 and 66 both show the results at iteration 100 starting from the capacitated affine SFE starting function. The difference is that figure 57 involves supply functions with 40 break-points while figure 66 involves supply functions with 20 break-points. The differences between the supply functions in these figures is an artifact of the numerical technique. The differences seem qualitatively to be of the same magnitude as the differences between these figures and the results at iteration 100 for the other starting functions. Consequently, we hypothesize that the differences

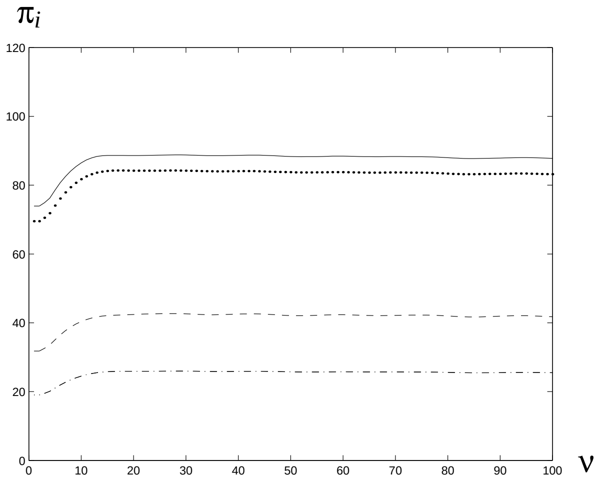

Figure 56: Profits versus iteration for base-case assumptions starting from capacitated affine SFE.

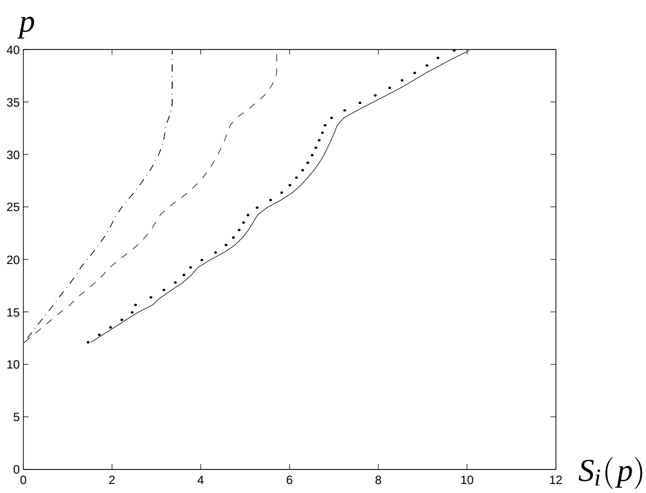

Figure 57: Supply functions at iteration 100 for base-case assumptions starting from capacitated affine SFE.

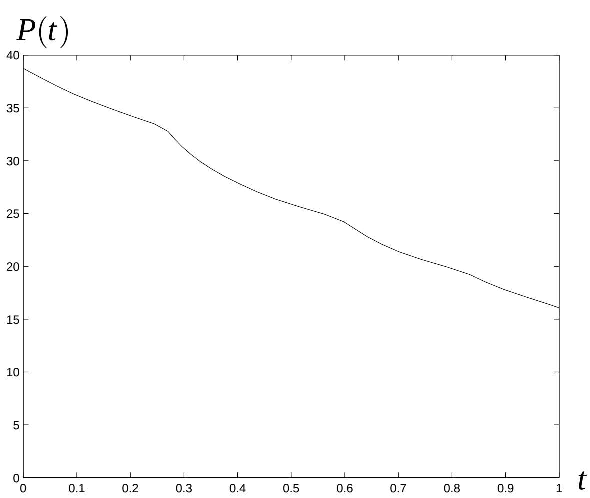

Figure 58: Price-duration curve at iteration 100 for base-case assumptions starting from capacitated affine SFE.

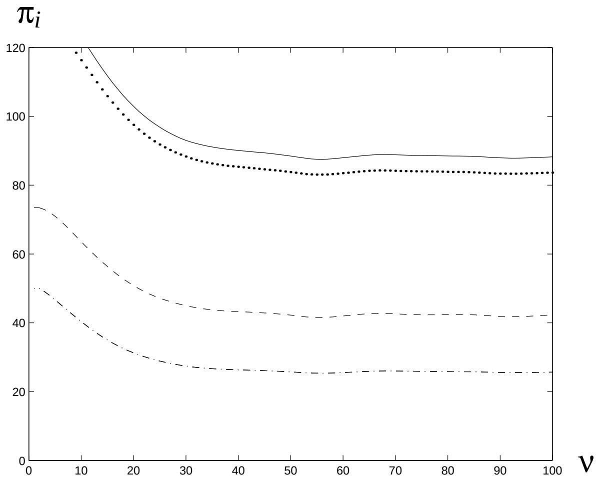

Figure 59: Profits versus iteration for base-case assumptions starting from price-capped Cournot.

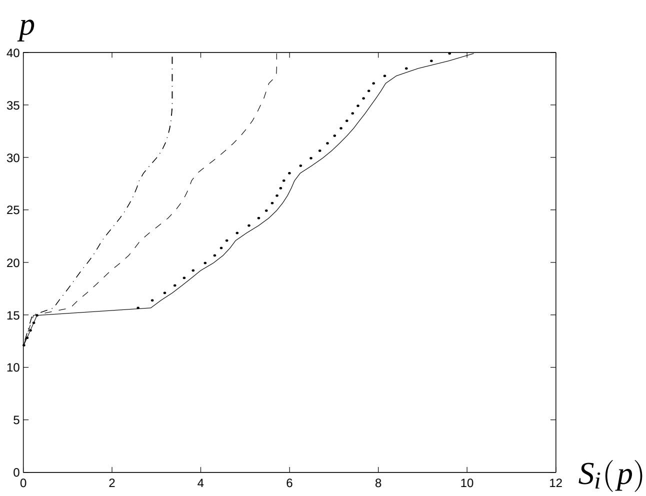

Figure 60: Supply functions at iteration 100 for base-case assumptions starting from price-capped Cournot.

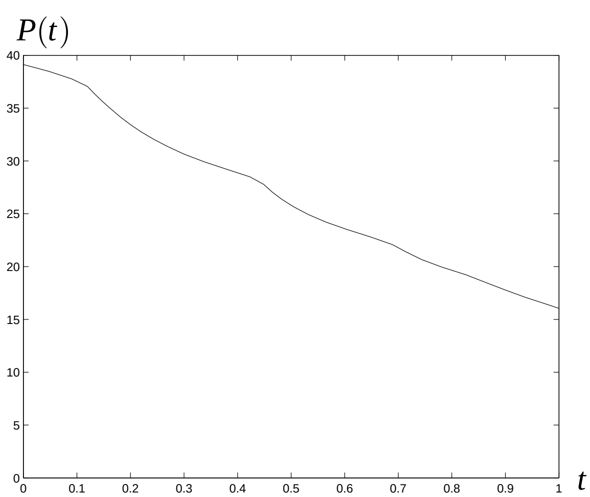

Figure 61: Price-duration curve at iteration 100 for base-case assumptions, starting from price-capped Cournot.

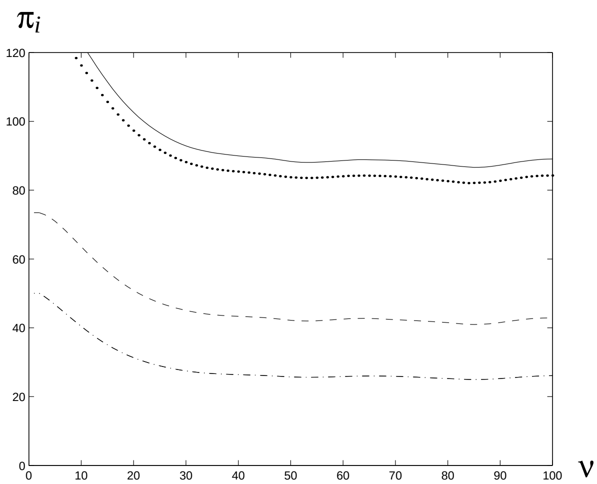

Figure 62: Profits versus iteration for base-case assumptions starting from price-capped Cournot but with reduced price minimum of $\underline{p} = 8$ pounds per MWh.

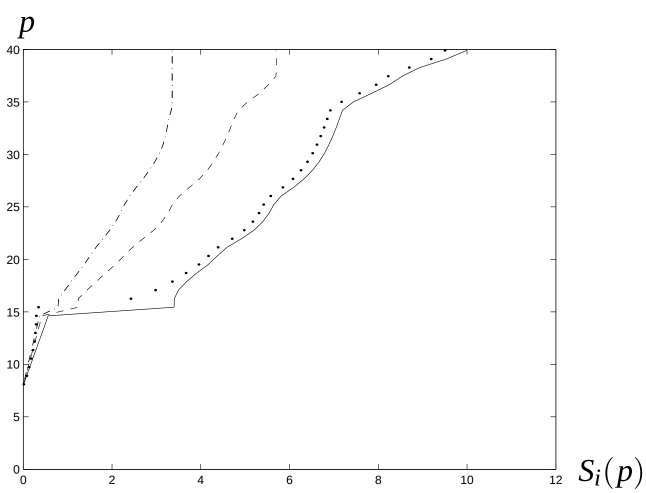

Figure 63: Supply functions at iteration 100 for base-case assumptions starting from price-capped Cournot but with reduced price minimum of $\underline{p} = 8$ pounds per MWh.

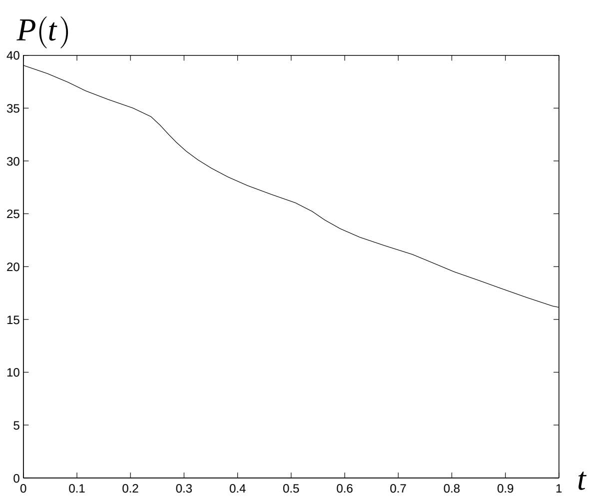

Figure 64: Price-duration curve at iteration 100 for base-case assumptions, starting from price-capped Cournot but with reduced price minimum of $\underline{p} = 8$ pounds per MWh.

in the supply functions at iteration 100 for the various starting functions are all artifacts of the numerical technique and not indicative of multiple equilibria.

The price-duration curve for iteration 100 starting from the capacitated affine SFE supply function with 20 break-points is shown in figure 67. The results at iteration 100 are slightly different from the previous results.

11.3.6 Summary

Given the base-case supply and demand configuration, the results at iteration 100 from all the starting functions, as shown in figures 53–67, are similar from the perspectives of:

- the profits, as can be seen in figure 68, which shows the profit versus iteration for all starting functions combined,
- the general shape of the supply functions over the range of realized prices, (between about 16 and 40 pounds per MWh), as can be seen in figure 69, which shows the supply functions at iteration 100 for all starting functions combined, and
- the form of the price-duration curve, as can be seen in figure 70, which shows the price-duration curve at iteration 100 for all starting functions combined.

The supply functions at iteration 100 differ in detail over the range of realized prices depending on the starting function. For example, there are points of apparent non-differentiability in the supply functions and the location of these points differs from starting function to starting function. However, all starting functions have evidently converged towards similar equilibria. That is, for the base-case there is only a very small range of multiple equilibria or it may even be the case that there is only one equilibrium and that the observed range is an artifact of the numerical method.

All firms have roughly the same marginal costs at peak capacity of approximately 27 pounds per MWh. However, in the supply functions at iteration 100, the largest two firms, 3 and 4, maximize their profits by withholding capacity so that prices are well in excess of 27 pounds per MWh for more than 45% of the time horizon.

The smallest two firms, 2 and 5, (represented by the leftmost of the supply function curves) bid in all their capacity when prices reach about 33 pounds per MWh. In contrast, the largest two firms do not provide all their capacity until the price reaches the price cap of 40 pounds per MWh.

The supply functions of firms 1, 2, and 5 are strictly concave over most of the range realized prices. These firms are at their capacity constraints at the peak realized capacity, so the concavity of their supply functions does not indicate an unstable equilibrium. On the other hand, firms 3 and 4 are not at their capacity constraints. Note that their supply functions at prices near to the maximum realized price are approximately linear and therefore neither strictly concave nor strictly convex. This is also consistent with the stability analysis in section 5.

The price at peak demand is just below the price cap. The prices at lower demands are significantly lower. SFE competition combined with a price cap has prevented prices from staying near to the price cap, except at peak demand.

For prices below about 20 pounds per MWh, corresponding to the right hand third of the price-duration curve, the supply functions and the price-duration curve are similar in appearance to the uncapacitated case. (Compare, for example, to figures 32 and 33, respectively.) However, it is

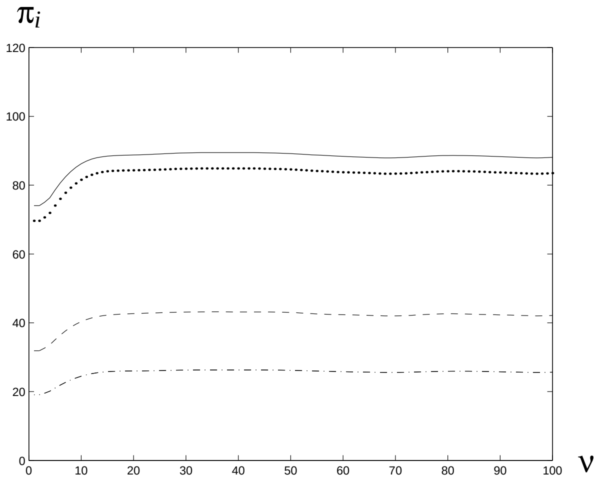

Figure 65: Profits versus iteration for base-case assumptions starting from capacitated affine SFE, except that supply functions have 20 break-points.

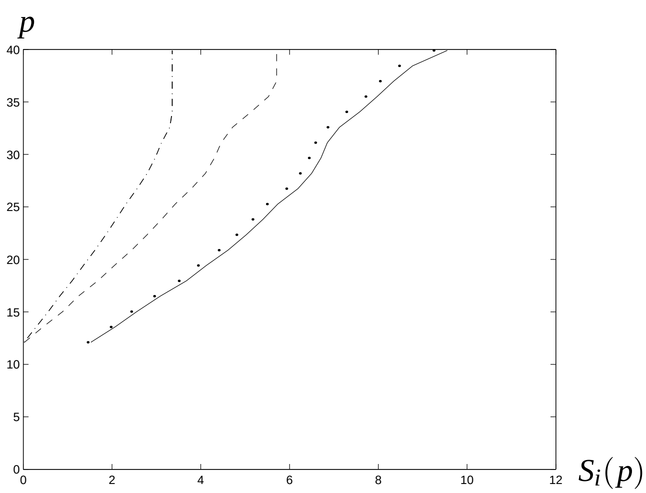

Figure 66: Supply functions at iteration 100 for base-case assumptions starting from capacitated affine SFE, except that supply functions have 20 break-points.

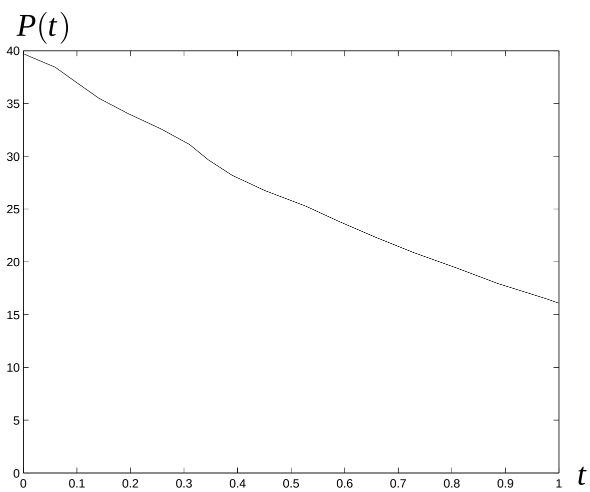

Figure 67: Price-duration curve at iteration 100 for base-case assumptions starting from capacitated affine SFE, except that supply functions have 20 break-points.

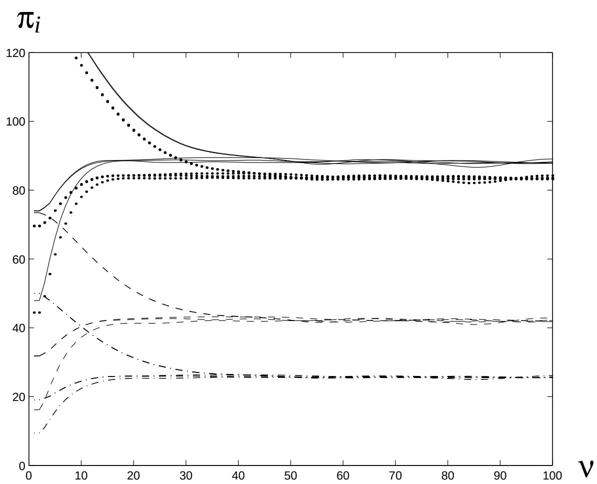

Figure 68: Profits versus iteration for base-case assumptions for all starting functions combined.

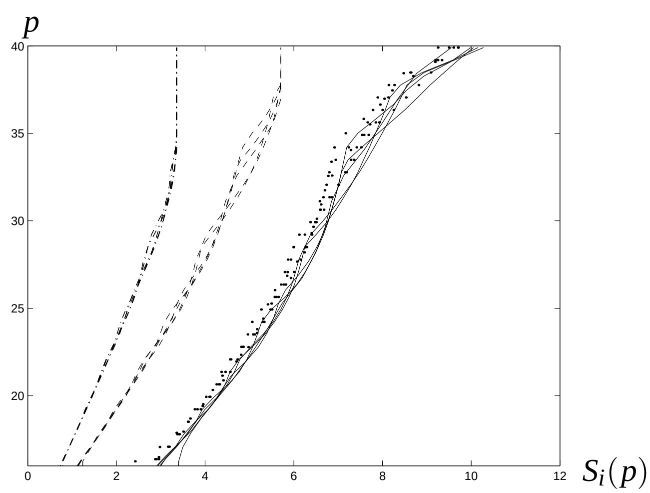

Figure 69: Supply functions at iteration 100 for base-case assumptions for all starting functions combined.

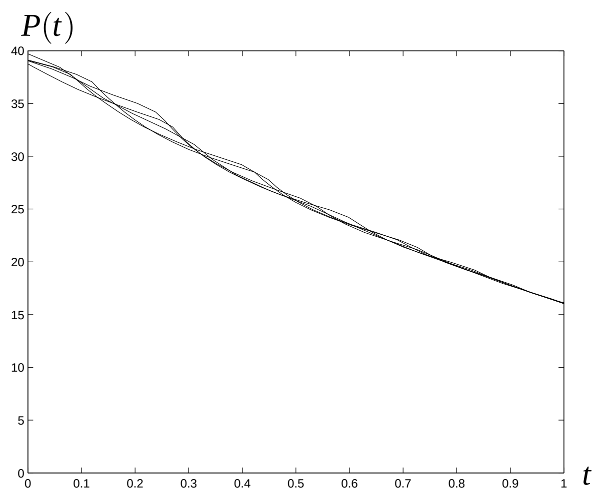

Figure 70: Price-duration curve at iteration 100 for base-case assumptions for all starting functions combined.

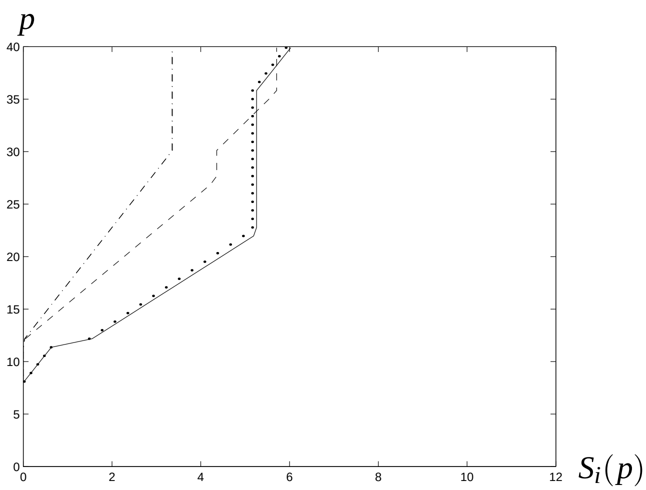

Figure 71: Supply function constructed according to recipe in [11] for capacitated strategic firms.

clear that the capacity constraints have caused a significant shift in the supply function for prices above 20 pounds per MWh even though production at this price is only less than half of capacity. The presence of capacity constraints causes significant price mark-ups even at demands far below the peak.

Despite the considerable mark-ups, the prices are considerably lower than if the firms were to bid the price-capped Cournot supply function. (Compare the prices to figure 27.) The requirement that the bids be consistent across the time horizon has significantly affected the outcome, reducing equilibrium profits to about half what they would be if the price-capped Cournot supply functions were bid.

Conversely, the prices are considerably higher for much of the time horizon than if each firm were to bid the capacitated affine SFE starting function. (Compare the prices to figure 26.) This confirms that it is important to explicitly consider the effect of the capacity constraints on the equilibrium and that the equilibrium supply functions are not well approximated by naively truncating an uncapacitated SFE solution.

In [11], an *ad hoc* approach is taken to incorporating capacity constraints. Applying the recipe in [11] for constructing supply functions results in figure 71. The recipe in [11] provides a reasonable estimate of the equilibrium supply bids in this case for firms 1,2, and 5 (the smallest three firms). However, the recipe predicts less supply than the calculated equilibria for firms 3 and 4 at high prices and, moreover, violates corollary 13.

The recipe in [11] does not explicitly consider the load-duration characteristic. The recipe sets supply at high prices based only on competition between firms 3 and 4 at high prices, but the effect of this is to limit the supply of these generators at lower prices. (See the flat part of the supply curves for firms 3 and 4 between about 22 and 37 pounds per MWh in figure 71.) The recipe fails to fully value the sales opportunities for firms 3 and 4 at prices between 22 and 37 pounds per MWh. In general, any recipe that seeks to define the supply function independently of the load-duration characteristic will fail to make the profit maximizing trade-off between withholding at high prices and sales opportunities at low prices.

Finally, we note that in [11] a capacitated fringe was considered and demand conditions were such that maximum capacity constraints of the strategic players were never binding. The demand

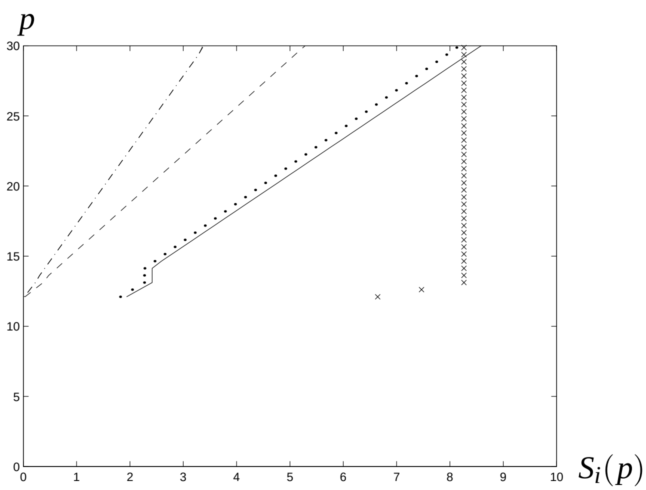

Figure 72: Supply function constructed according to recipe in [11] for capacitated fringe. The supply function of the fringe is shown with crosses. (Note that the axes are scaled differently compared to figure 71.)

conditions that we have considered in this section, however, were chosen so that maximum capacity constraints were binding. In the next section, we consider a capacitated fringe and demand conditions that are closer to those in [11].

11.4 Fringe capacity constraints

In [11], a capacitated fringe was modeled. We considered the same system as in [11], a similar range of load-duration characteristic, ranging from 43.5 to 14.3 GW, and a demand slope of $\gamma = 0.25$ GW per (pound per MWh). Again applying the recipe in [11], results in figure 72. We used this starting function and the capacitated competitive and price-capped Cournot starting functions.

11.4.1 Starting from capacitated competitive

Figure 73 shows the profits versus iteration ν for the capacitated fringe assumptions starting from the capacitated competitive supply function. The leftmost points in figure 73 show the profits if each firm were to bid the capacitated competitive supply function. That is, these are the profits if the firms bid competitively at all times.

Figure 74 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 75.

11.4.2 Starting from supply function calculated by recipe

Figure 76 shows the profits versus iteration ν for the base-case assumptions starting from the recipe described in [11] and illustrated in figure 72. The leftmost points in figure 76 show the profits if each firm were to bid the supply function calculated according to the recipe. The profits change very little over the course of the iterations, indicating that the recipe yields supply functions that are a useful predictor of the equilibrium profits for these supply and demand conditions.

Figure 77 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 78.

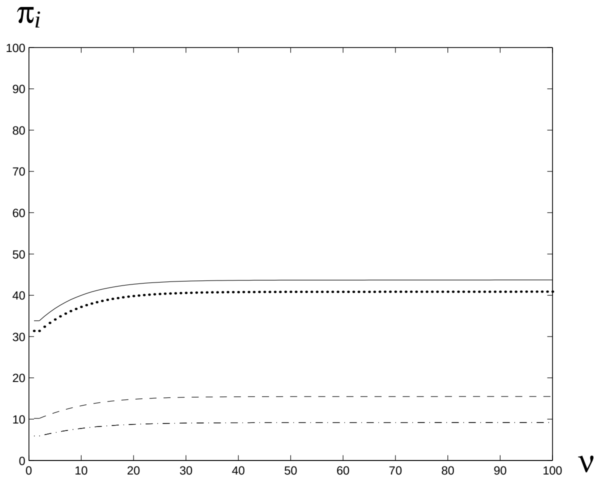

Figure 73: Profits versus iteration for capacitated fringe assumptions starting from capacitated competitive.

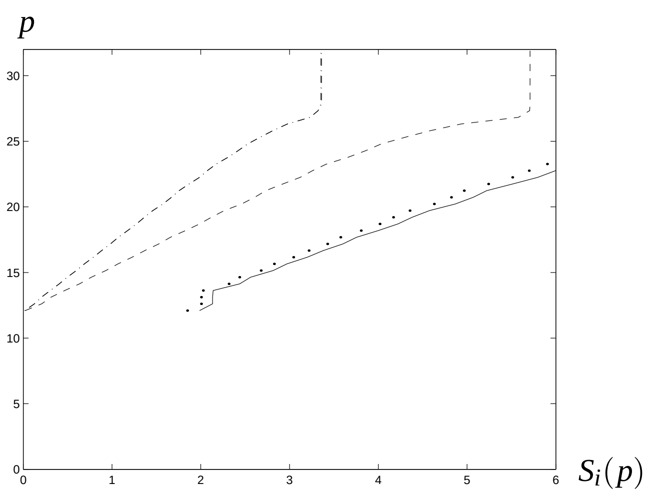

Figure 74: Supply functions at iteration 100 for capacitated fringe assumptions starting from capacitated competitive.

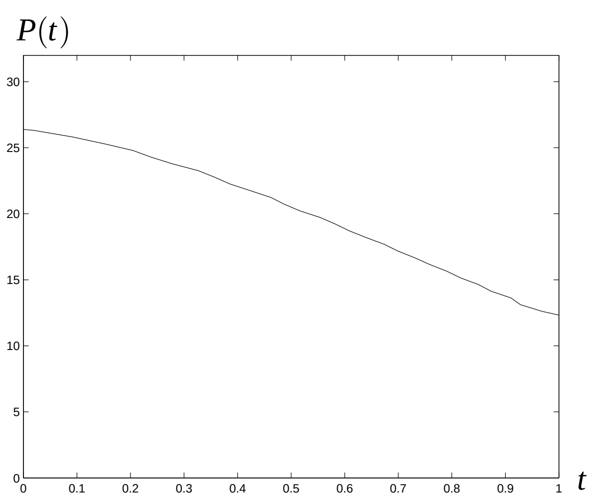

Figure 75: Price-duration curve at iteration 100 for capacitated fringe assumptions starting from capacitated competitive.

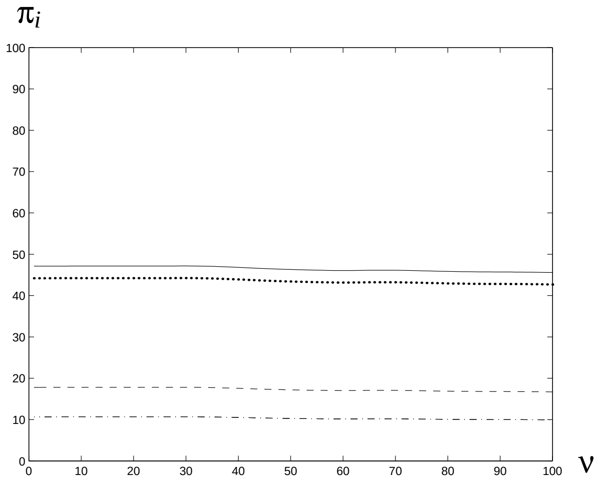

Figure 76: Profits versus iteration for capacitated fringe assumptions starting from supply function constructed according to recipe.

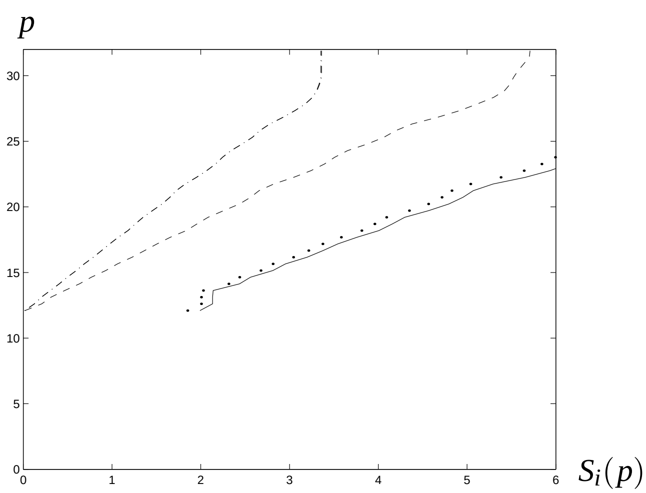

Figure 77: Supply functions at iteration 100 for capacitated fringe assumptions starting from supply function constructed according to recipe.

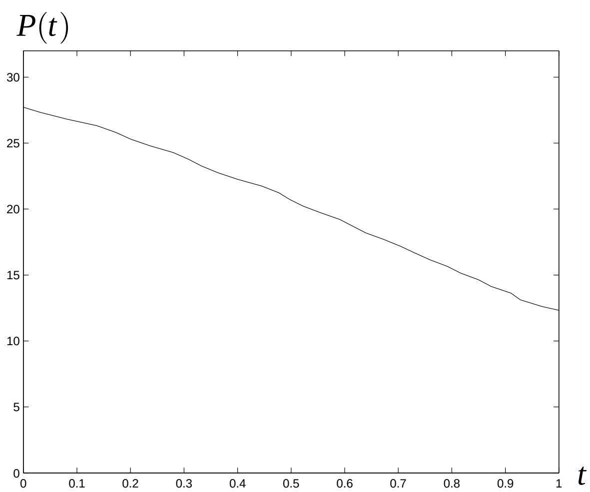

Figure 78: Price-duration curve at iteration 100 for capacitated fringe assumptions starting from supply function constructed according to recipe.

11.4.3 Starting from price-capped Cournot

Figure 79 shows the profits versus iteration ν for the base-case assumptions starting from the price-capped Cournot supply function. Figure 60 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 81.

11.4.4 Summary

The profits at iteration 100 starting from the competitive starting function are slightly lower than for the other starting functions. This is shown in the combined plot of profits versus iteration in figure 82. Similarly, the supply functions at iteration 100 starting from the competitive starting function are somewhat higher at prices above 25 pounds per MWh, as shown in figure 83. The peak price at iteration starting from the competitive starting function is somewhat lower than for the other starting functions as shown in figure 84.

11.5 Increased capacities

We increased the capacity of all firms by 5% compared to the base-case. The results for the competitive starting function are shown in figures 85–87. The results for the price-capped Cournot starting function are shown in figures 88–90. The profits at iteration 100 are approximately 20% lower for the capacitated competitive starting function compared to the price-capped Cournot starting function. The range of equilibrium profits is about 12% of the difference in profits between the price-capped Cournot and capacitated competitive supply functions.

In this case, firms 1, 2, and 5 reach their capacity below the peak realized price, but the price cap is not binding. There is apparently a range of equilibria in this case.

11.6 Varying the price cap

In this section we consider varying the price cap.

11.6.1 Starting from price-capped Cournot

Figure 91 shows the profits versus iteration ν for the base-case assumptions starting from the price-capped Cournot supply function, except that the price cap was increased to 50 pounds per MWh. Figure 92 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 93. (Note that the price axes on these graphs has a different scale to the previous graphs.)

Compared to the results for the base-case price cap of 40 pounds per MWh, the supply functions for the increased price cap case are significantly different for prices above 30 pounds per MWh. That is, the price cap affects supply at prices well below the price cap. In particular, raising the price cap yields further withholding of supply compared to the base-case even at prices well below the base-case price cap. Profits are up to 20% higher than in the base-case, due primarily to the withholding of supply until prices become close to the price cap. This suggests that there is considerable value in being able to estimate the maximum marginal cost of generation to set a fairly tight price cap.

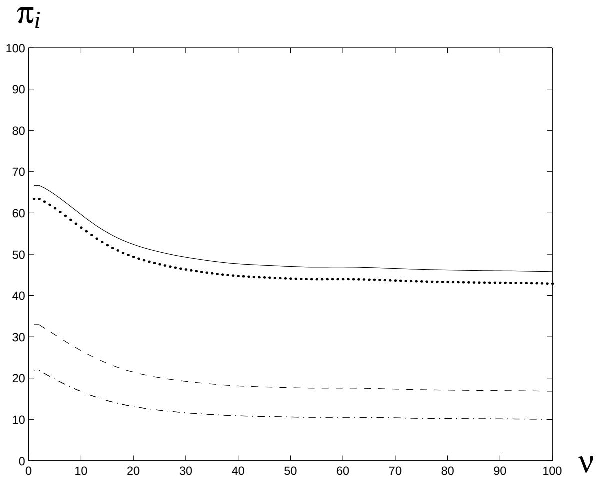

Figure 79: Profits versus iteration for capacitated fringe assumptions starting from price-capped Cournot.

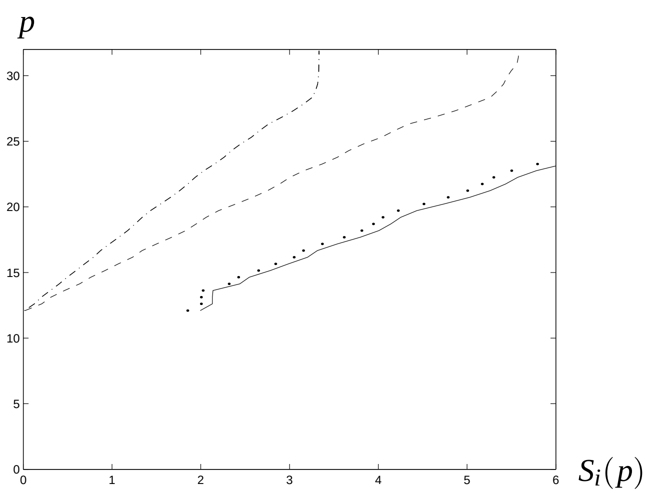

Figure 80: Supply functions at iteration 100 for capacitated fringe assumptions starting from price-capped Cournot.

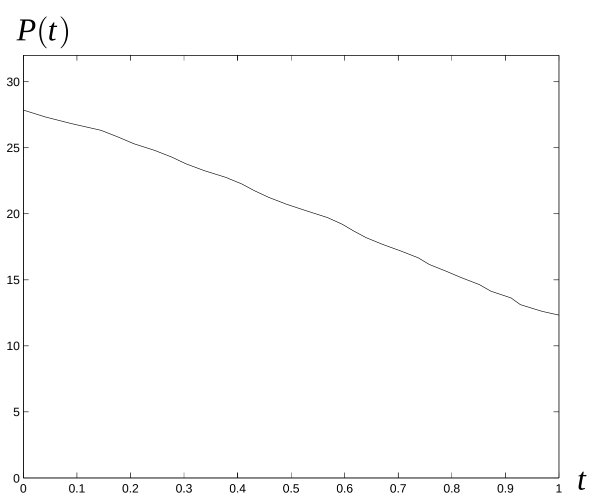

Figure 81: Price-duration curve at iteration 100 for capacitated fringe assumptions, starting from price-capped Cournot.

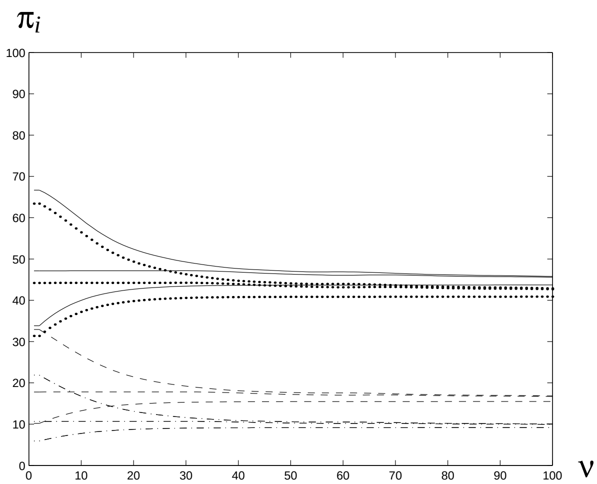

Figure 82: Profits versus iteration for capacitated fringe assumptions for all starting functions combined.

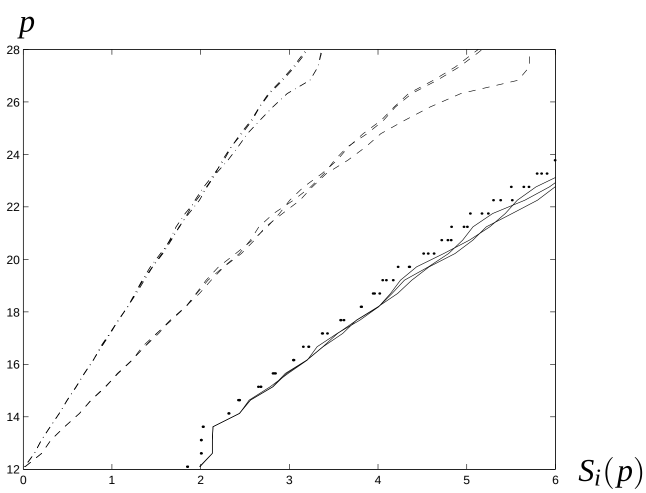

Figure 83: Supply functions at iteration 100 for capacitated fringe assumptions for all starting functions combined.

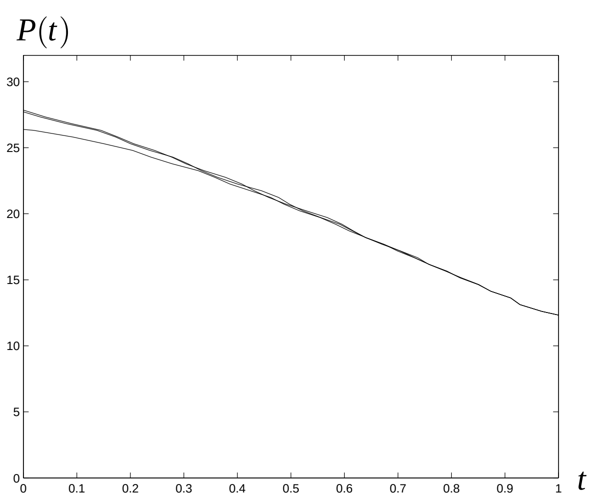

Figure 84: Price-duration curve at iteration 100 for capacitated fringe assumptions for all starting functions combined.

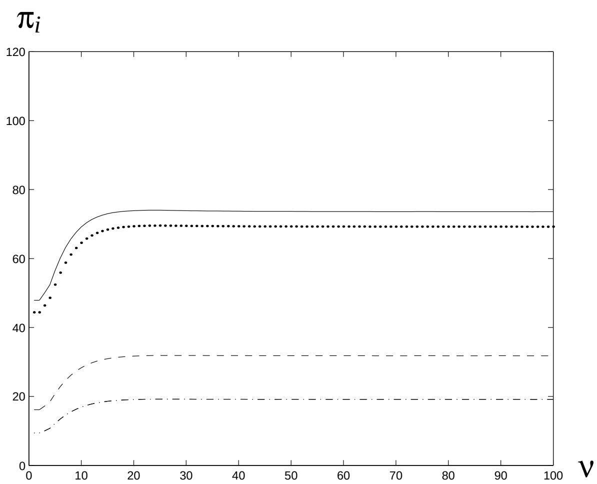

Figure 85: Profits versus iteration for base-case assumptions except for 5% increase in all capacities, starting from capacitated competitive.

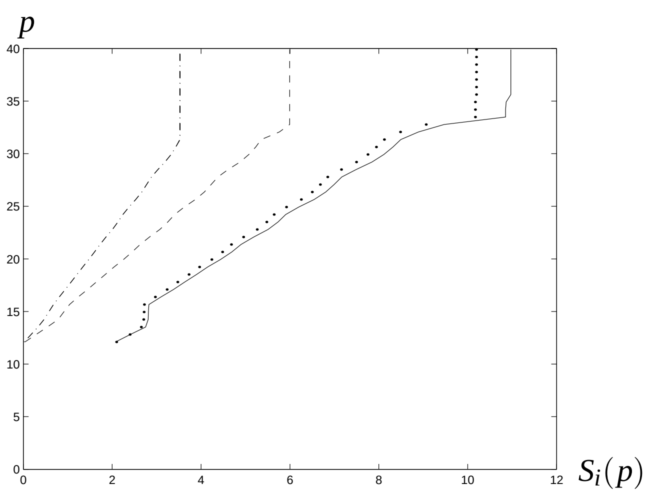

Figure 86: Supply functions at iteration 100 for base-case assumptions except for 5% increase in all capacities, starting from capacitated competitive.

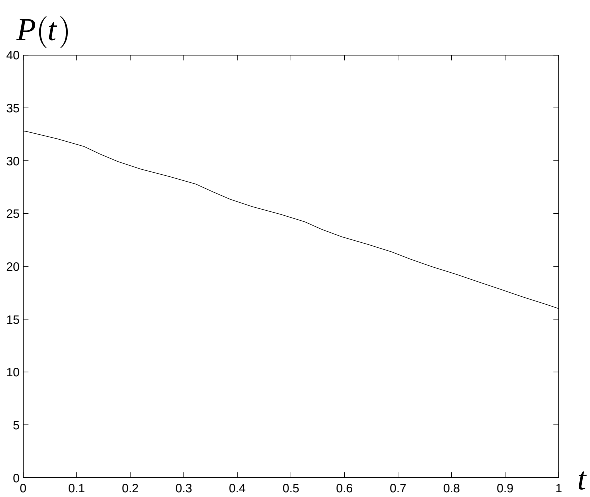

Figure 87: Price-duration curve at iteration 100 for base-case assumptions except for 5% increase in all capacities, starting from capacitated competitive.

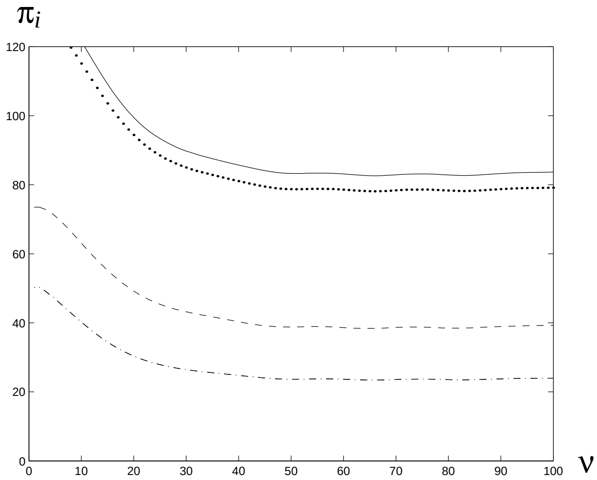

Figure 88: Profits versus iteration for base-case assumptions except for 5% increase in all capacities, starting from price-capped Cournot.

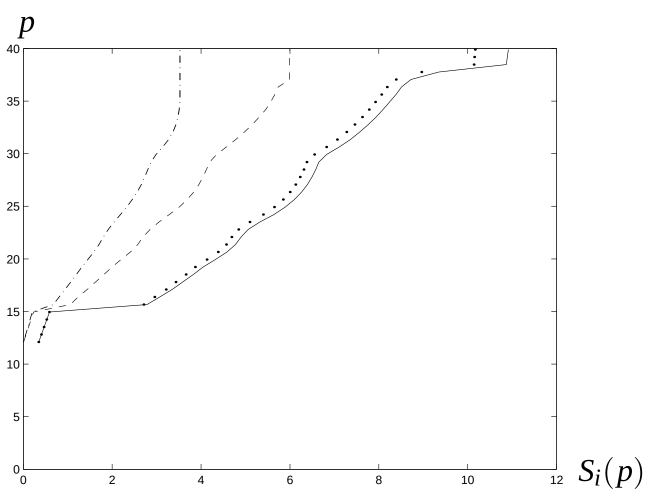

Figure 89: Supply functions at iteration 100 for base-case assumptions except for 5% increase in all capacities, starting from price-capped Cournot.

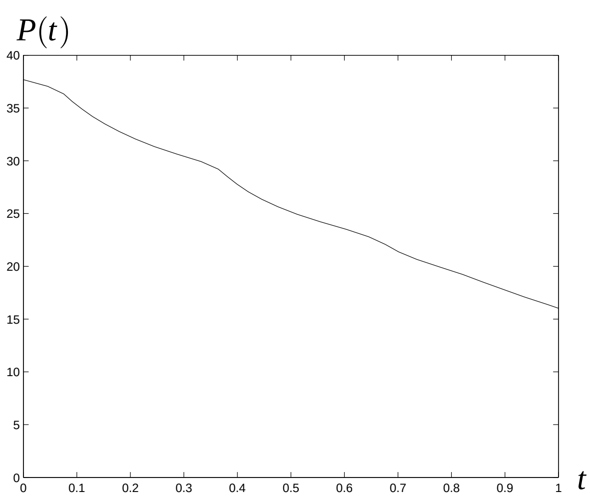

Figure 90: Price-duration curve at iteration 100 for base-case assumptions except for 5% increase in all capacities, starting from price-capped Cournot.

The observation that price caps deter the exercise of market power is well-known from single period models of interaction [22]. In the SFE case, a further issue is that the price stays well below the price cap at off-peak times. That is, the imposition of a single price cap applying at all times together with the requirement that supply functions remain fixed over an extended horizon has a similar effect to price caps that vary with demand conditions.

11.6.2 Price cap and multiple equilibria

Figure 94 shows the peak realized price at iteration 100 versus the price cap for price caps in the range of 30 pounds per MWh to 80 pounds per MWh. For each price cap, the result at iteration 100 for the price-capped Cournot starting function is shown as a cross while the result at iteration 100 for the capacitated competitive starting function is shown as a circle.

For price caps below about 40 pounds per MWh the peak realized price comes within about 1 pound per MWh of the price cap. The firms can coordinate on achieving close to the price cap. Moreover, for a given price cap the results at iteration 100 are very similar for both the price-capped Cournot and capacitated competitive starting functions. (The profits and supply functions at iteration 100 are also similar for the price-capped Cournot and the capacitated competitive starting functions for each value of the price cap below about 40 pounds per MWh.) That is, when the price cap is binding, there appears to be only a small range of equilibria exhibited.

In contrast, for values of the price cap above about 50 pounds per MWh the peak realized price at iteration 100 is in the range of around 45–50 pounds per MWh and there are non-trivial differences between the peak realized prices at iteration 100 for the price-capped Cournot and the capacitated competitive starting functions. (However, in some of the cases, the profit functions were still changing by more than 0.1% at each iteration, so some of the difference between the Cournot and competitive starting functions may be because the results at iteration 100 are not close enough to equilibrium.) This suggests that when the price cap is not binding there is a range of exhibited equilibria.

As discussed in section 6.2.1, we can calculate competitive and Cournot outcomes for the peak demand conditions. The price at peak demand for competitive bids is $p_0^{\text{comp}} \approx 27$ pounds per MWh. The price at peak demand under Cournot competition is $p_0^{\text{Cournot}} \approx 80$ pounds per MWh.

Even when the price cap is raised to 80 pounds per MWh, the peak realized price at iteration 100 for the supply function bids is far below 80 pounds per MWh for either starting function. The range of peak realized prices at iteration 100 for the price-capped Cournot and the capacitated competitive starting functions is relatively small compared to the peak Cournot price of 80 pounds per MWh.

In summary, when the price cap is binding on behavior, the range of exhibited equilibria seems to be very narrow. The price-capped Cournot and the capacitated competitive starting functions yield essentially the same results at iteration 100. When the price cap is not binding on behavior, there is a range of equilibrium outcomes; however, this range is relatively small compared to the difference between the price-capped Cournot and the capacitated competitive starting functions.

11.7 Increased load factor

The load duration characteristic in the base-case has a relatively small load factor of around 30% implying that the supply functions were required to be set for a very long period or that a significant

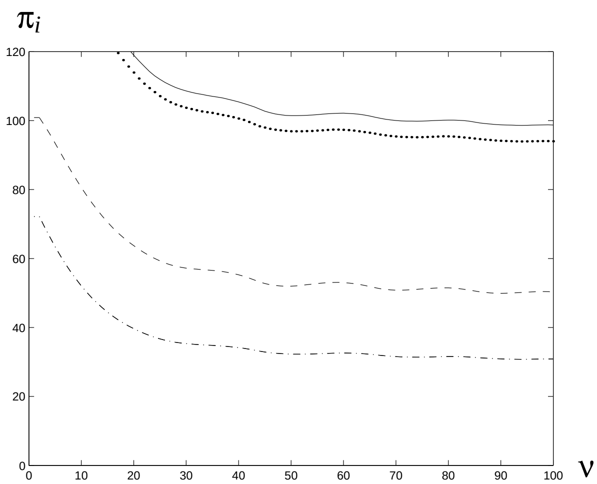

Figure 91: Profits versus iteration for base-case assumptions starting from price-capped Cournot, except for increased price cap.

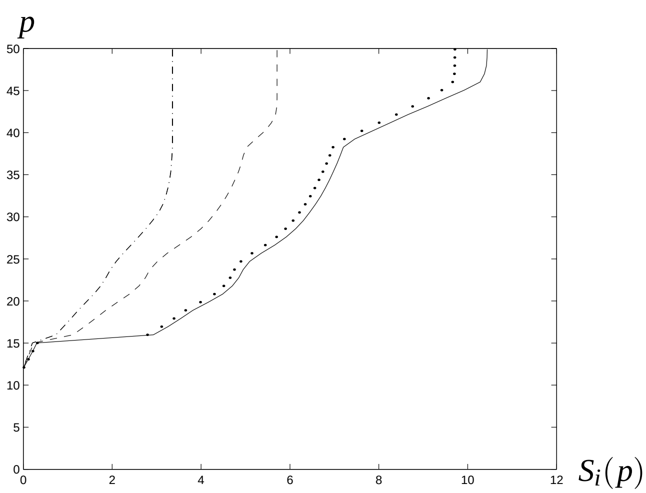

Figure 92: Supply functions at iteration 100 for base-case assumptions starting from price-capped Cournot, except for increased price cap. (Note that that price axis is scaled differently compared to previous figures.)

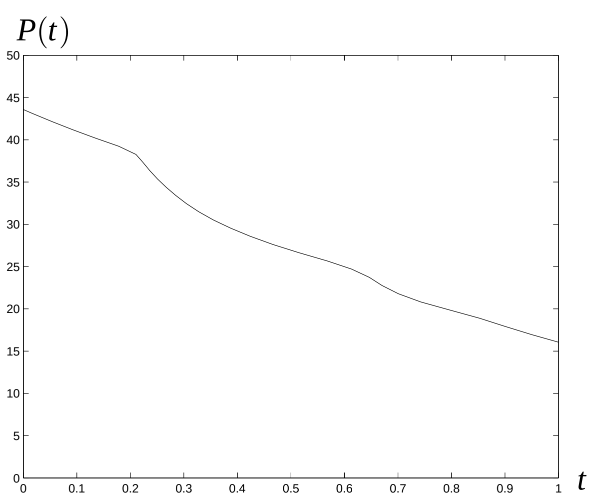

Figure 93: Price-duration curve at iteration 100 for base-case assumptions starting from price-capped Cournot, except for increased price cap. (Note that the price axis is scaled differently compared to previous figures.)

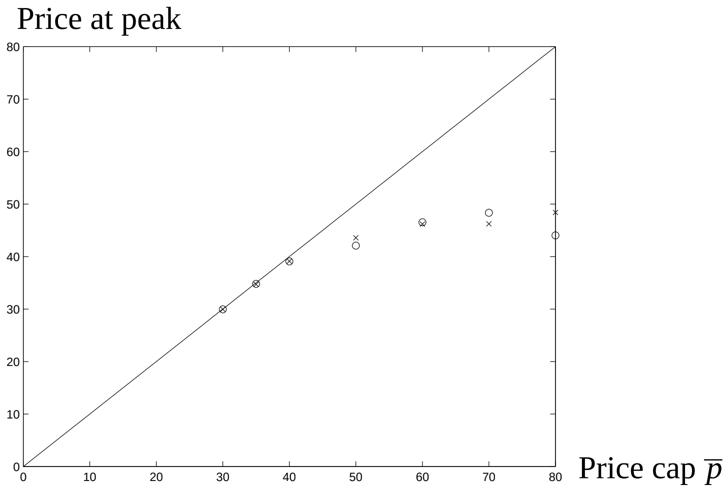

Figure 94: Price at peak versus price cap. Results starting from price-capped Cournot are shown with crosses, while results starting from capacitated competitive are shown with circles.

amount of demand was supplied by baseload capacity at prices at or below the price minimum $\underline{p}$ or that much of the demand was supplied under forward contracts. We divided the time horizon into peak and off-peak conditions and considered the case where bids were made separately for peak and off-peak conditions. That is, we increased the load factor over the time horizon.

11.7.1 Peak conditions

We investigated a case where the load duration characteristic ranged linearly from 20 to 35 GW. This implies a load factor of around 60%. That is, we shortened the time horizon compared to the base-case by omitting the off-peak times, but the time horizon still covered the peak conditions.

Figure 95 shows the profits versus iteration ν for the base-case assumptions starting from the price-capped Cournot supply function, except that the load-duration characteristic has been changed so that $N(1) = 20$. (The value $N(0)$ was kept at 35.) As previously, firms 2 and 5 have identical costs and capacities, so they appear superimposed. The profit functions are not directly comparable to previous cases since the demand conditions have changed.

Figure 96 shows the supply functions at iteration 100. The supply functions at iteration 100 are very similar to the base-case supply functions at iteration 100, over the range of realized prices. The price-duration curve for iteration 100 is shown in figure 97.

The results at iteration 100 for the capacitated competitive starting function are essentially the same as for the price-capped Cournot starting function. That is, it appears that the increase in the load factor has not significantly increased the range of equilibria for peak conditions.

11.7.2 Off-peak conditions

We also investigated a case where the load duration characteristic ranged linearly from 10 to 20 GW. That is, we shortened the time horizon compared to the base-case by omitting the peak times. In this case, the price cap is not binding and so, as in the uncapacitated case and the increased capacity case, there are multiple equilibria having a range of profits. The range of profits at iteration 100 is around 10% of the difference between the profits for the capacitated competitive and price-capped Cournot supply functions.

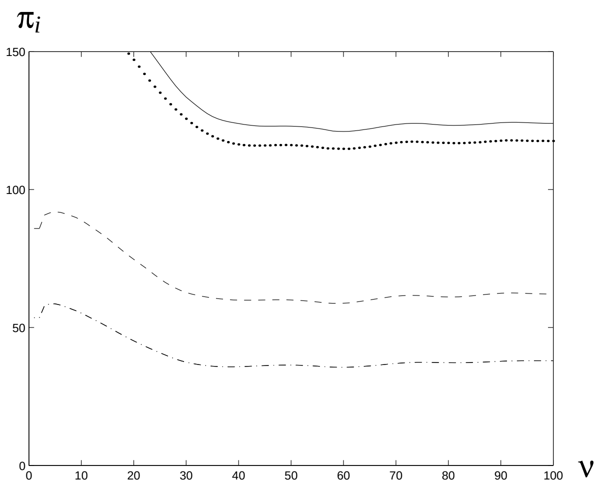

Figure 95: Profits versus iteration for base-case assumptions, except for increased value of $N(1)$. (Note that the profit axis is scaled differently compared to previous figures.)

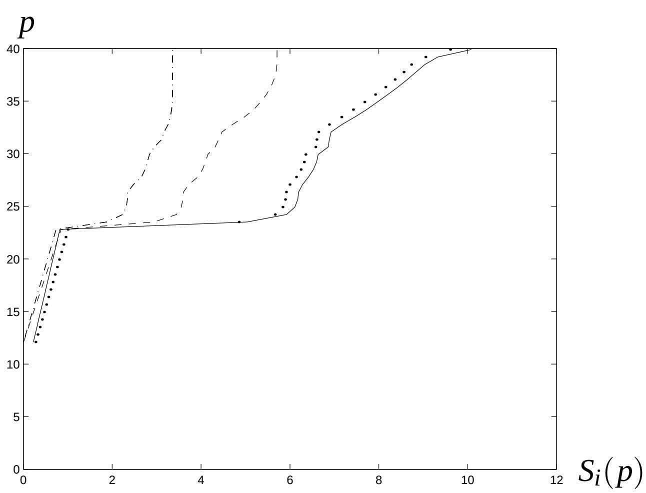

Figure 96: Supply functions at iteration 100 for base-case assumptions, except for increased value of $N(1)$.

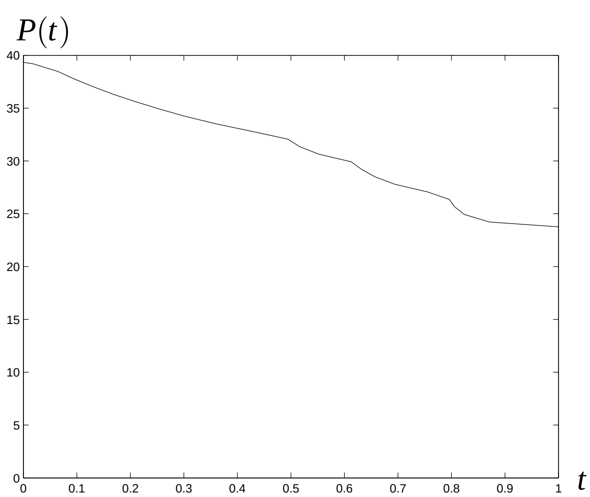

Figure 97: Price-duration curve at iteration 100 for base-case assumptions, except for increased value of $N(1)$.

11.8 Increased load factor with capacitated fringe

We consider again the system with a capacitated fringe from section 11.4. However, we considered load-duration characteristics varying in the ranges:

- peak conditions, with N varying from 28.9 to 43.5 GW,
- mid-load conditions, with N varying from 21.6 to 36.2 GW, and
- off-peak conditions, with N varying from 14.3 to 28.9 GW.

These ranges are sub-ranges of the "full" range of the load-duration characteristic used in section 11.4. For each range, we used the three starting functions described in section 11.4.

Figure 98 shows the supply functions at iteration 100 for all ranges (including the "full" range 14.3 to 43.5 GW considered in section 11.4) for all starting functions combined. That is, twelve supply functions have been combined in figure 98 corresponding to three starting functions for each of four demand ranges. For clarity, each supply function is plotted only over the range of realized prices. In each case, the supply functions at iteration 100 starting from the capacitated competitive starting function are somewhat higher at the upper end of realized prices.

Figure 99 shows the price-duration curves at iteration 100 for all ranges (again including the range 14.3 to 43.5 GW) for all starting functions combined. That is, there are twelve price-duration curves shown in figure 99. The price-duration curves for the sub-ranges of the full load-duration characteristic have been plotted so that they correspond to the appropriate sub-ranges of the normalized time. That is, the price-duration curve for N varying from 28.9 to 43.5 GW has been plotted for time varying from 0.5 to 1 and similarly for the other curves. Consequently, the price-duration curves can all be compared. The price-duration curves show only a relatively small variation depending on starting function and sub-range of load-duration characteristic.

Figures 98 and 99 show that the supply functions obtained at iteration 100 on the basis of the smaller ranges of demand are very similar to the supply functions obtained at iteration 100 using the full range of load-duration characteristic from 14.3 to 43.5 GW. That is, although corollary 15 suggests that the range of equilibria depends on the range of the load-duration characteristic, this dependence is fairly weak. Moreover, extended periods can be approximately analyzed by simply combining the load-duration characteristics. For example, as in [11], behavior in England and Wales over a year could be analyzed approximately with reference to a yearly load-duration characteristic, even though bids could be updated on a daily basis in England and Wales.

11.9 Increased demand

Finally, we consider an increase in demand with the same supply conditions as the base-case. The demand was increased so that rationing was required.

11.9.1 Starting from capacitated affine SFE

Figure 100 shows the profits versus iteration ν for the base-case assumptions starting from the capacitated affine SFE supply function, except that the load-duration characteristic has been changed so that $N(0) = 40$. (The value $N(1)$ was kept at 10.) In this case there is not enough capacity to

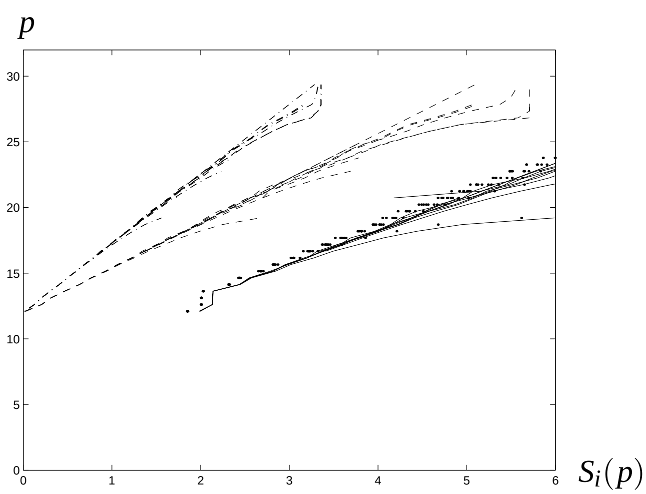

Figure 98: Supply functions at iteration 100 for capacitated fringe assumptions for all ranges of demand and for all starting functions combined.

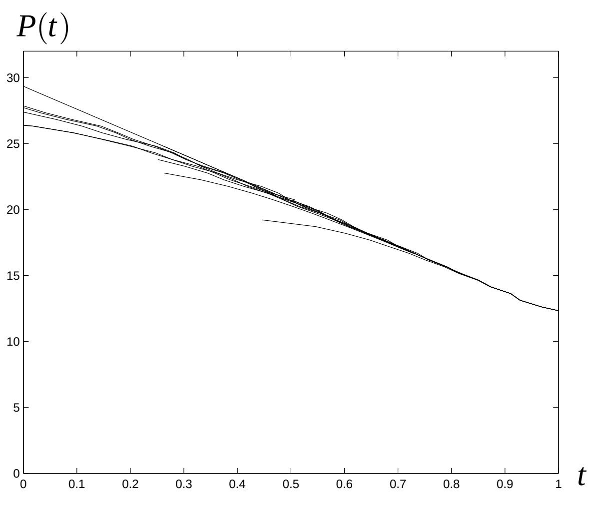

Figure 99: Price-duration curve at iteration 100 for capacitated fringe assumptions for all ranges of demand and for all starting functions combined.

meet demand at the peak. As previously, firms 2 and 5 have identical costs and capacities, so they appear superimposed.

Figure 101 shows the supply functions at iteration 100. The supply functions at iteration 100 are similar to the base-case. That is, the difference in profits compared to the base-case is primarily due to the higher demand in this case, rather than due to changed behavior because of tightened demand conditions. The price-duration curve for iteration 100 is shown in figure 102.

11.9.2 Starting from price-capped Cournot

Figure 103 shows the profits versus iteration ν for the base-case assumptions starting from the price-capped Cournot supply function, except that $N(0) = 40$. Figure 104 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 105. The results at iteration 100 are very similar to the case of starting from the capacitated affine SFE supply function.

11.9.3 Starting from capacitated affine SFE with high price cap

Figure 106 shows the profits versus iteration ν for the base-case assumptions starting from the capacitated affine SFE supply function, except that $N(0) = 40$ and the price cap is set to $\overline{p} = 50$ pounds per MWh. Note that the profit axis has changed compared to previous figures because the profits are considerably higher. Figure 107 shows the supply functions at iteration 100. The price-duration curve for iteration 100 is shown in figure 108. Note that the price axes have been changed compared to some of the previous figures.

11.9.4 Bid caps

The previous cases were tested with the alternate rule of market wide bid caps instead of price caps. The bid supply functions were not significantly different in this case; however, profits were higher than for price caps because prices exceeded the bid cap whenever supply is tight.

11.9.5 Summary

Profits are considerably higher than in the previous cases. However, for the price cap of 40 pounds per MWh, most of the difference in profits compared to the base-case is due to increased demand alone rather than changes in bid behavior. Despite the greater potential for exploitation of market power due to the need for rationing, the presence of the price cap and the requirement to bid consistently across the time horizon has limited the scope to increase profits.

In the case of the high price cap, however, the combination of the need for rationing and the increased price cap has led to even higher profits. The two firms with large capacity can withhold capacity until high prices are reached. This again demonstrates the value of a fairly tight price cap.

11.10 Characteristics of solutions

Corollary 10 of section 5 showed that in the absence of capacity constraints and with a symmetric system there could only be one stable SFE. In some of our numerical results, there is a range of

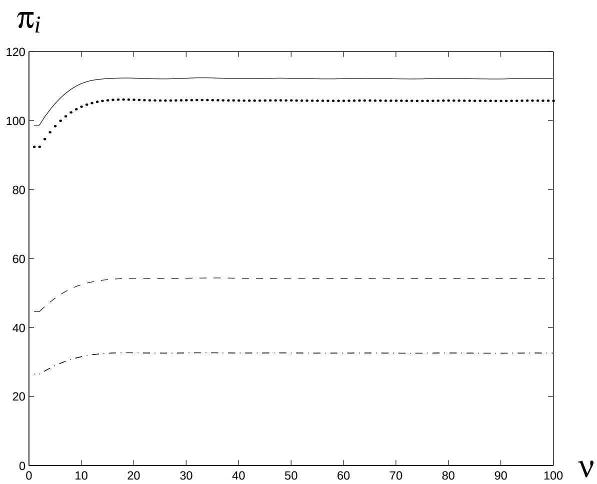

Figure 100: Profits versus iteration for case of rationing, starting from capacitated affine SFE.

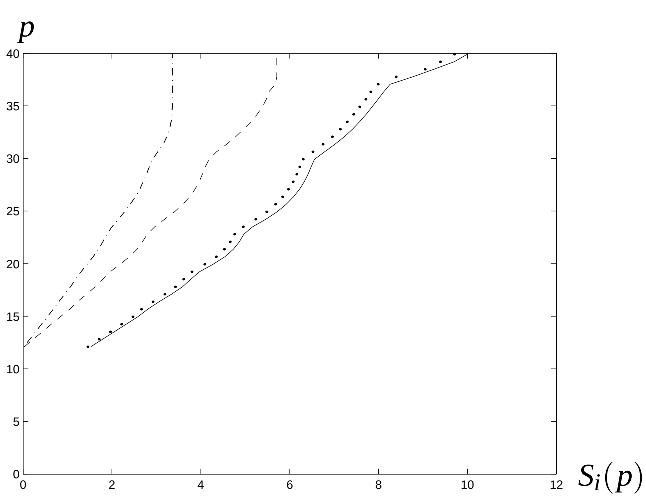

Figure 101: Supply functions at iteration 100 for case of rationing, starting from capacitated affine SFE.

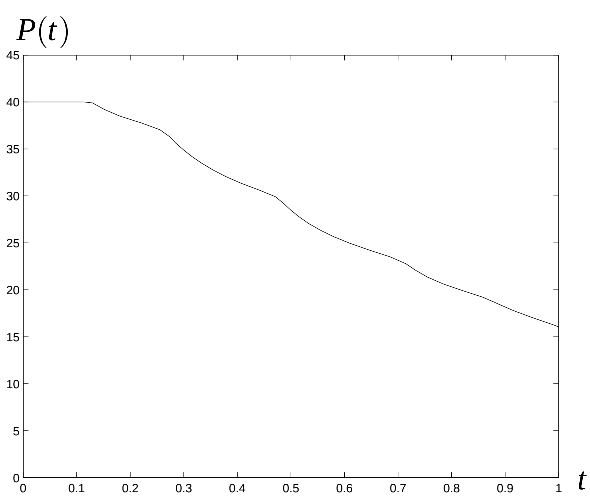

Figure 102: Price-duration curve at iteration 100 for case of rationing, starting from capacitated affine SFE.

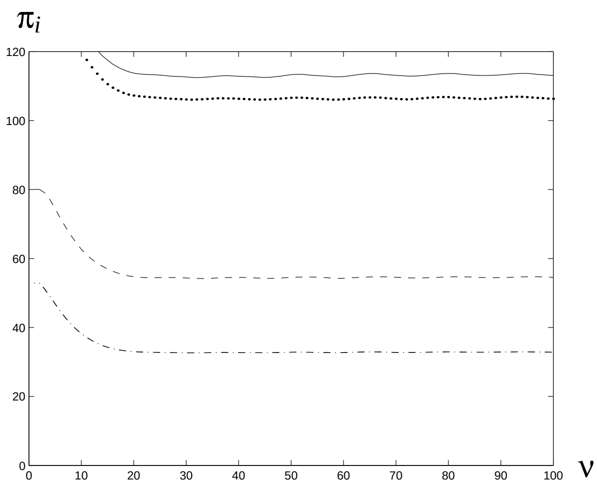

Figure 103: Profits versus iteration for case of rationing starting from price-capped Cournot.

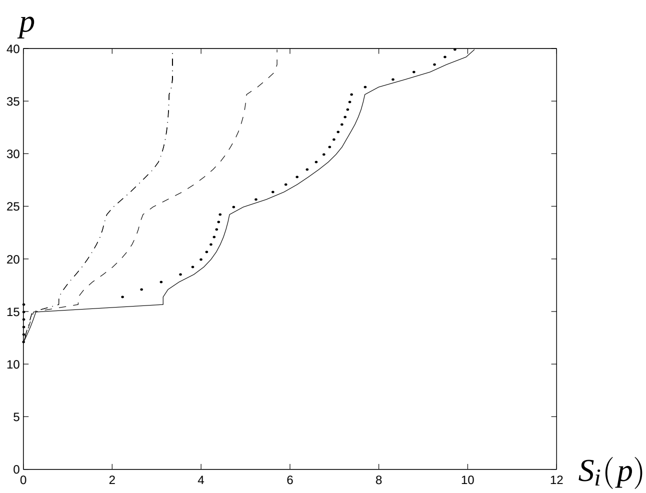

Figure 104: Supply functions at iteration 100 for case of rationing starting from price-capped Cournot.

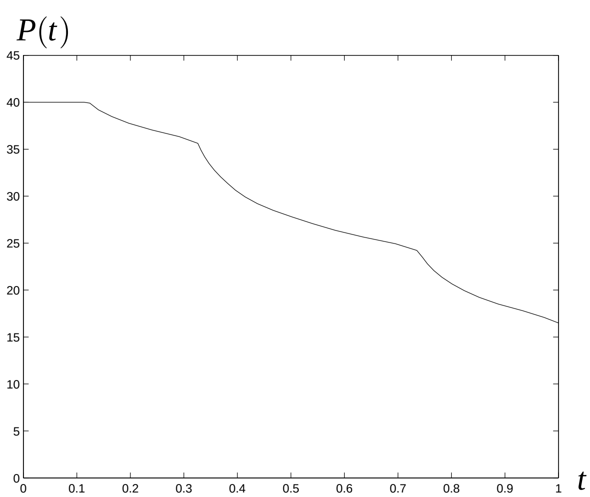

Figure 105: Price-duration curve at iteration 100 for case of rationing starting from price-capped Cournot. (Note that the price axis is scaled differently compared to previous figures.)

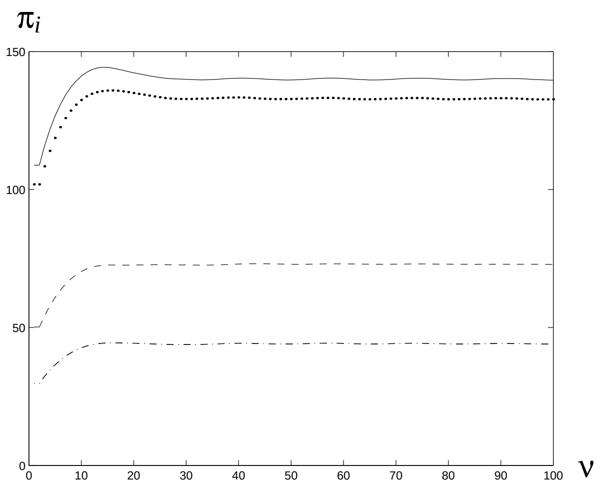

Figure 106: Profits versus iteration for high demand and high price cap. (Note that the profit axis is scaled differently compared to previous figures.)

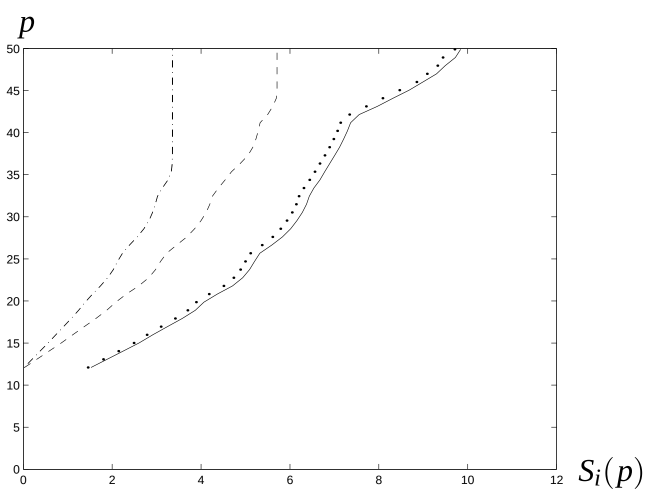

Figure 107: Supply functions at iteration 100 for high demand and high price cap. (Note that the price axis is scaled differently compared to previous figures.)

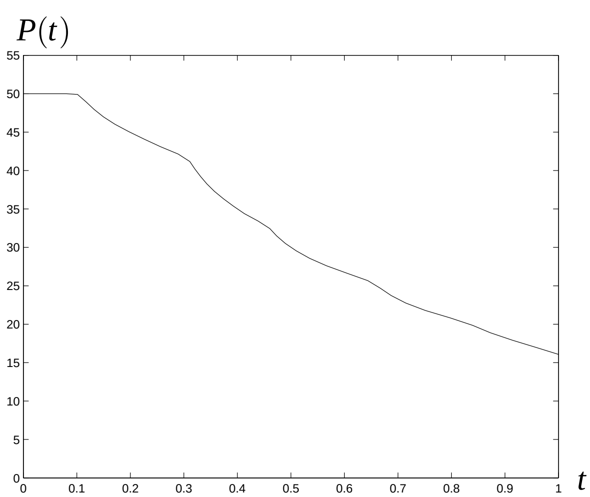

Figure 108: Price-duration curve at iteration 100 for high demand and high price cap. (Note that the price axis is scaled differently compared to previous figures.)

apparently stable SFEs. However, the range is relatively narrow and may be an artifact of the numerical framework.

Consistent with corollary 13 of section 7, the numerical results show solutions that are always strictly increasing in price, except when capacity constraints are binding. As suggested in section 6.5, the numerical supply functions exhibit discontinuities in their derivatives. Consistent with corollary 14, between the points of discontinuity of the derivatives, the solutions appear to be consistent with solutions of (18) corresponding to a subset of the firms.

12 Conclusion

The main results of this paper are:

- In markets with heterogeneous firms and capacity constraints, the differential equation approach to finding the equilibrium supply function may not be effective by itself because the non-decreasing constraints, which couple decisions across the time horizon, are likely to be binding. An alternate approach, of iterating in the space of supply functions, is computationally intensive and has theoretical drawbacks of its own. However, based on the case studied, it appears to produce consistent and useful results.

- The range of supply function equilibria may be very small when capacity is fairly tight and there are binding price caps. This market condition is the most critical from a market power perspective. Even when price caps are not binding, the range of stable equilibria appears relatively small compared to the difference between the competitive and the Cournot outcomes. This strengthens the case for SFE analysis when market rules require consistent bids across a time horizon, particularly when capacity constraints and price caps are binding.

- Requiring supply functions to remain fixed over an extended time horizon appears to reduce the incentive to mark up prices compared to the Cournot outcome. SFEs that achieve profits that are close to Cournot profits are unstable and consequently should not be observed in the market.

- A single price cap imposed at all times may have significant effects both on- and off-peak.

As discussed in Borenstein [22], there are various problems facing wholesale electricity markets. Borenstein discusses the value of long-term contracting, real-time pricing, and price caps to a smoothly functioning electricity market. As well as the advantages cited in [22], long-term contracting can also reduce the effective load factor in the day-ahead market, which can rule out some of the least competitive equilibria. In this paper, the analysis of stability and the numerical studies suggest that requiring bid functions to be consistent over an extended time horizon having a large variation of demand may also be valuable in mitigating extreme prices and market power.

Acknowledgment

Much of this work was performed during visits by the first author to the John F. Kennedy School of Government, Harvard University in Fall 2000 and to the University of California Energy Institute in Summer 2001. The first author gratefully acknowledges the hospitality of the John F.

Kennedy School of Government and the University of California Energy Institute. The first author was funded in part by the University of Texas Faculty Development Program, the National Science Foundation under grant ECS-0082506, and the University of California Energy Institute. The authors gratefully acknowledge the support of the Harvard Electricity Policy Group and the Harvard-Japan Project on Energy and Environment. The views expressed here are those of the authors and not necessarily those of any of these organizations. The authors would like to thank Matt Barmack, Severin Borenstein, Jim Bushnell, Ed Kahn, Shmuel Oren, Alex Rudkevich, and Frank Wolak for comments and suggestions.

References

[1] Paul D. Klemperer and Margaret A. Meyer. Supply function equilibria in oligopoly under uncertainty. *Econometrica*, 57(6):1243–1277, November 1989.

[2] Robert Wilson. Auctions of shares. *Quarterly Journal of Economics*, 93(4):675–689, November 1979.

[3] James J. D. Wang and Jaime F. Zender. Auctioning divisible goods. *Economic Theory*, 19:673–705, 2002.

[4] David McAdams. Modifying the uniform-price auction to eliminate 'collusive-seeming equilibria'. Working paper, February 2002.

[5] Richard Green and David M. Newbery. Competition in the British electricity spot market. *Journal of Political Economy*, 100(5):929–953, October 1992.

[6] Richard Green. Increasing competition in the British electricity spot market. *The Journal of Industrial Economics*, XLIV(2):205–216, June 1996.

[7] Carolyn A. Berry, Benjamin Hobbs, William A. Meroney, Richard P. O'Neill, and William R. Stewart, Jr. Analyzing strategic bidding behavior in transmission networks. In H. Singh, editor, *IEEE Tutorial on Game Theory Applications in Power Systems*, pages 7–32. IEEE, 1999.

[8] Benjamin Hobbs, Carolyn Metzler, and Jong-Shi Pang. Strategic gaming analysis for electric power networks: An MPEC approach. *IEEE Transactions on Power Systems*, 15(2):638–645, May 2000.

[9] Aleksandr Rudkevich. Supply function equilibrium in poolco type power markets: Learning all the way. Technical Report TCA Technical Paper Number 1299-1702, Tabors Caramanis and Associates, December 1999.

[10] J. D. Weber and T. J. Overbye. A two-level optimization problem for analysis of market bidding strategies. In *Proceedings of the IEEE Power Engineering Society Summer Meeting, Edmonton, Alberta*, July 1999.

[11] Ross Baldick, Ryan Grant, and Edward P. Kahn. Linear supply function equilibrium: Generalizations, application, and limitations. University of California Energy Institute POWER Paper PWP-078, www.ucei.berkeley.edu/ucei/PDF/pwp078.pdf, August 2000.

[12] Christopher J. Day and Derek W. Bunn. Divestiture of generation assets in the electricity pool of England and Wales: A computational approach to analyzing market power. *Journal of Regulatory Economics*, 19(2):123–141, 2001.

[13] E. J. Anderson and A. B. Philpott. Using supply functions for offering generation into an electricity market. To appear in *Operations Research*, 2001.

[14] E. J. Anderson and H. Xu. Nash equilibria in electricity markets. Technical Report Working paper 01-005, Australian Graduate School of Management, The University of New South Wales, Sydney, NSW 2052, Australia, July 2001.

[15] E. J. Anderson and H. Xu. Undercutting and overcutting for generator offers in an electricity market. Technical Report Working paper, Australian Graduate School of Management, The University of New South Wales, Sydney, NSW 2052, Australia, 2001.

[16] John Bower and Derek Bunn. Experimental analysis of the efficiency of uniform-price versus discriminatory auctions in the England and Wales electricity market. *Journal of Economic Dynamics and Control*, 25:561–592, 2001.

[17] Arthur R. Bergen and Vijay Vittal. *Power Systems Analysis*. Prentice-Hall, Upper Saddle River, NJ, second edition, 2000.

[18] Fernando L. Alvarado, Jianping Meng, Christopher L. DeMarco, and Wellington S. Mota. Stability analysis of interconnected power systems coupled with market dynamics. *IEEE Transactions on Power Systems*, 16(4):695–701, November 2001.

[19] Fernando Alvarado. The stability of power system markets. *IEEE Transactions on Power Systems*, 14(2):505–511, May 1999.

[20] James B. Bushnell and Frank A. Wolak. Regulation and the leverage of local market power in the California electricity market. University of California Energy Institute POWER Paper PWP-070, www.ucei.berkeley.edu/ucei/PDF/pwp070.pdf, September 1999.

[21] Nils-Henrik Morch von der Fehr and David Harbord. Spot market competition in the UK electricity industry. *The Economic Journal*, 103(418):531–546, May 1993.

[22] Severin Borenstein. The trouble with electricity markets (and some solutions). University of California Energy Institute POWER Paper PWP-081, www.ucei.berkeley.edu/ucei/PDF/pwp081.pdf, January 2001.

[23] Friedel Bolle. Supply function equilibria and the danger of tacit collusion: The case of spot markets for electricity. *Energy Economics*, pages 94–102, April 1992.

[24] Kjell G. Nyborg. Underpricing and market power in uniform price auctions. London Business School, March 2002.

[25] Ross Baldick. Electricity market equilibrium models: The effect of parametrization. To appear in *IEEE Transactions on Power Systems*, 2002.

[26] Shmuel S. Oren, Stephen A. Smith, and Robert B. Wilson. Competitive nonlinear tariffs. *Journal of Economic Theory*, 29:49–71, 1983.

[27] William W. Hogan. A market power model with strategic interaction in electricity markets. *The Energy Journal*, 18(4):107–141, 1997.

[28] C. T. Kelley. *Iterative Methods for Linear and Nonlinear Equations*. SIAM, Philadelphia, 1995.

[29] Richard Green. The electricity contract market in England and Wales. *The Journal of Industrial Economics*, XLVII(1):107–124, March 1999.