

# Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market

**Andrew E.B. Lim**

Department of Industrial Engineering and Operations Research

Columbia University

New York, N.Y. 10027

Email: lim@ieor.columbia.edu

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## **Abstract**

This paper concerns the problems of *quadratic hedging* and *pricing*, and *mean-variance portfolio selection* in an *incomplete market*. Asset prices are geometric Brownian motion, and parameters describing the market model may be random processes. We approach these problems from the perspective of *linear-quadratic (LQ) optimal control* and *backward stochastic differential equations* (BSDEs); that is, we focus on the so-called *stochastic Riccati equation* (SRE) associated with the problem. Excepting certain special cases, solvability of the SRE remains an open question. Our primary theoretical contribution is a proof of existence and uniqueness of solutions of the SRE associated with the quadratic hedging and mean-variance problems. In addition, we derive closed form expressions for the optimal portfolios and efficient frontier in terms of the solution of the SRE. A generalization of the *Mutual Fund Theorem* is also obtained.

**Key words**– quadratic hedging; mean-variance portfolio selection; incomplete markets; linear-quadratic optimal control; stochastic Riccati equation; backward stochastic differential equations; Mutual Fund Theorem; efficient frontier.

## **1 Introduction**

This paper concerns the problems of *quadratic hedging* and *pricing*, and *mean-variance portfolio selection* in an *incomplete market*. Asset prices are geometric Brownian motion, and parameters

describing the market model may be random processes. We approach these problems from the perspective of *linear-quadratic (LQ) optimal control* and *backward stochastic differential equations* (BSDEs); that is, our study will focus on the so-called *stochastic Riccati equation* (SRE) associated with the problem. Excepting certain special cases, solvability of the SRE remains an open question. Our primary theoretical contribution is a proof of existence and uniqueness of solutions of the SRE associated with the quadratic hedging and mean-variance problems. In addition, we derive closed form expressions for the optimal portfolios and efficient frontier in terms of the solution of the SRE. A generalization of the *Mutual Fund Theorem* is also obtained.

An alternative approach to the quadratic hedging problem is based on the projection theorem and duality; see, for example, Follmer and Sonderman (1986), Duffie and Jackson (1990), Duffie and Richardson (1991), Delbaen and Schachermayer (1996), Schweizer (1996), Gouriéroux, Laurent and Pham (1998) and Laurent and Pham (1999). One advantage of this approach is that asset prices need not be restricted to geometric Brownian motion, but may be semimartingales. On the other hand, the analysis becomes quite involved. The work of Schweizer (1996) and Delbaen and Schachermayer (1996) proved the existence of the so-called *Variance Optimal Martingale Measure* (VMM). The VMM is the equivalent martingale measure associated with the quadratic hedging problem and plays a central role in the duality based approach to this problem. Not surprisingly, there is a close relationship between the VMM and the SRE. Finally, for a duality based approach to optimal portfolio selection for a general class of utility functions, the reader may consult Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987), Pliska (1986) or Cvitanic and Karatzas (1992).

On the other hand, when asset prices are geometric Brownian motions, the associated wealth equation is a linear stochastic differential equation (SDE) and the problem may be regarded as a LQ optimal control problem.

LQ control is one of the cornerstones of stochastic control. Traditionally, when studying LQ problems, the parameters describing the dynamics and cost are assumed to be deterministic; see for example Anderson and Moore (1989). In this situation, the LQ problem boils down to solving the so-called *Riccati ordinary differential equation* (ODE). The solution of the Riccati equation plays the role, in the LQ problem, analogous to that of the Hessian matrix when minimizing a quadratic. That is, the LQ problem can be solved and the optimal cost and control can be determined if the associated Riccati equation has a positive definite solution. For this reason, much effort has gone into studying the properties of the Riccati ODE, and our understanding of this equation, in the case of deterministic equations, is fairly complete; see Rami, Chen, Moore and Zhou (2001) for some recent developments in this topic.

We approach the quadratic hedging and mean-variance problems as LQ problems and obtain the solution by studying the associated Riccati equation. In Zhou and Li (2000) (which

builds on the discrete time results of Li and Ng (2000)) this approach was taken for the mean-variance problem with *deterministic parameters*. In this case, due to the special structure of the finance problem, the Riccati ODE reduces to a linear equation and solvability follows immediately from standard theory.

In this paper, market parameters may be *random processes*. Under this assumption, quadratic hedging and mean-variance portfolio selection are versions of the *stochastic LQ problem* that was introduced in Bismut (1976). In this case, however, the Riccati *ODE* is no longer appropriate and must be replaced by the *stochastic Riccati equation* (SRE) in order for solutions to be adapted. The SRE is a nonlinear BSDE. Due to the nature of the nonlinearities, however, solvability is much harder to prove, and existence of solutions, in general, remains an open issue. In this paper, we prove solvability of the SRE associated with the finance problem.

There are several good reasons why portfolio selection problems with random parameters are worth studying. First, it enables us to use more accurate market models for asset price dynamics. For instance, it has been established that many of the features of stock price dynamics as observed in data such as the leptokurtic distribution of the log returns process and the volatility smile can not be reproduced by models with deterministic parameters. This has led to the development of sophisticated stochastic parameter models which are capable of reproducing these features; see, for example, Heston (1993), Hull and White (1987) and Stein and Stein (1991). By allowing parameters to be random, we open up the possibility of using these sophisticated models in the process of portfolio selection which, in turn, may lead to better investment strategies. A second motivation for random parameters comes from the observation that market parameters, in many situations, are estimated online using historical data (e.g. past and present stock prices). The resulting models, which may be used for portfolio selection, have parameters which once again are random.

The most general form of the SRE is a matrix-valued nonlinear BSDE. The drift (generator) of this equation is a singular function of the state variables ( $P(\cdot)$ ,  $\Lambda(\cdot)$ ) and satisfies neither the Lipschitz continuity nor linear growth conditions required in order to apply the general existence and uniqueness results of Pardoux and Peng (1990). Excepting a few special cases (e.g. Bismut (1976), Hu and Zhou (2001), Kohlmann and Tang (2001a, 2001b, 2001c), Lim and Zhou (2001a, 2001b) and Pardoux and Peng (1990), for global solvability, and Chen and Yong (2000) for local solvability) existence and uniqueness of the multi-dimensional SRE remains an open question. In the case of quadratic hedging and mean-variance portfolio selection, the SRE is a scalar BSDE though the generator is singular and (once again) neither Lipschitz continuous nor linearly growing. Our main result is a proof of solvability of this SRE when the market is incomplete and the parameters are random. This extends the results of Lim and Zhou (2001b) to the incomplete market case.

Related results on the SRE, obtained independently using different proofs, can be found in

Kohlmann and Tang (2001b) and Hu and Zhou (2001).

Complete solutions of the quadratic hedging and mean-variance problems are obtained in terms of the solution  $(p(\cdot), \Lambda(\cdot))$  of the SRE and  $(h(\cdot), \eta(\cdot))$  of a second linear BSDE. The SRE may be associated with the VMM while the linear equation gives the value process  $h(\cdot)$  and replicating portfolio  $\eta(\cdot)$  for a liability  $\xi$  in a *fictitiously completed market*; see Karatzas and Shreve (1999) for more details about fictitious completion. In particular, the *market price of risk* of the fictitious assets required to complete the market are determined by the solution of the SRE. For the mean-variance problem, a closed form expression for the efficient frontier is derived in terms of  $p(\cdot)$  and  $h(\cdot)$ . Furthermore, a version of the *Mutual Fund Theorem* is obtained. Finally, we show how the SRE may be solved, numerically, in the case when the market parameters are Markovian. Such a situation corresponds to the stochastic volatility models of, for example, Heston (1993), Hull and White (1987) and Stein and Stein (1991).

The outline of the paper is as follows. In Section 2, we formulate the quadratic hedging and mean-variance problems. The relationship between these problems and the SRE is presented in Section 3, while in Section 4, we discuss the relationship between the SRE and the VMM. In Section 5, we present our main theoretical result, which is a proof of existence and uniqueness of the SRE associated with the quadratic hedging and mean-variance problems. In Section 6, closed form expressions for the optimal portfolio and efficient frontier are derived and a generalization of the Mutual Fund Theorem is also obtained, while in Section 7, the PDE associated with the SRE, in the case of Markovian parameters, is derived. Concluding remarks are presented in Section 8.

## 2 Problem formulation

### Notation

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete filtered probability space such that  $\mathcal{F}_0$  has been augmented by all the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ . Let:

$$\bar{W}(t) := (W(t), B(t)) := (W_1(t), \dots, W_m(t), B_1(t), \dots, B_d(t))$$

for  $m \geq 1$  and  $d \geq 0$  be an  $(m + d)$ -dimensional standard Brownian motion defined on this space such that  $\{\mathcal{F}_t\}_{t \geq 0}$  is generated by  $\bar{W}(\cdot)$ . In this paper, we use  $B(t)$  to model the market incompleteness with  $d = 0$  corresponding to the case of a complete market case.

We introduce the following notation:

- $L_{\mathcal{F}}^p(0, T; \mathbb{R}^m)$ ,  $1 \leq p < \infty$  – the set of  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted,  $\mathbb{R}^m$ -valued processes on  $[0, T]$  such that  $E \int_0^T |f(t)|^2 dt < \infty$  with norm:

$$\|f\|_p := \left( \int_0^T |f(t)|^p dt \right)^{\frac{1}{p}};$$

- $L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}^m)$  – the set of  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted essentially bounded processes on  $[0, T]$ ;
- $L_{\mathcal{F}}^2(\Omega; C(0, T; \mathbb{R}^m))$  – the set of  $\mathbb{R}^m$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes on  $[0, T]$  with  $\mathbb{P}$ -a.s. continuous sample paths such that:

$$E \left[ \sup_{t \in [0, T]} |f(t)|^2 \right] < \infty;$$

- $L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}^m))$  – the set of  $\mathbb{R}^m$ -valued  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted essentially bounded processes with continuous sample paths.
- $L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^m)$  – the set of  $\mathcal{F}_T$ -adapted square-integrable random variables with norm:

$$\|\xi\|_2 := \left( E|\xi|^2 \right)^{\frac{1}{2}};$$

- $L_{\mathcal{F}_T}^{\infty}(\Omega; \mathbb{R}^m)$  – the subset of  $L_{\mathcal{F}_T}^2(\Omega; \mathbb{R}^m)$  consisting of bounded  $\mathbb{R}^m$ -valued  $\mathcal{F}_T$ -adapted random variables.

## Market model

We consider a financial market made up of  $(m+1)$  assets, consisting of one bond and  $m$  stocks, with price processes  $P_0(\cdot)$  and  $(P_1(\cdot), \dots, P_m(\cdot))$ , respectively. We assume throughout that the  $P_i(\cdot)$  are solutions of the SDEs:

$$\begin{cases} dP_0(t) = P_0(t)r(t)dt, \\ P_0(0) = 1, \end{cases} \quad (1)$$

$$\begin{cases} dP_i(t) = P_i(t) \left[ \mu_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW_j(t) \right], \\ P_i(0) = p_i. \end{cases} \quad (2)$$

In this case,  $r(\cdot)$  is the interest rate of the bond, and  $\mu_i(\cdot)$ ,  $\sigma_i(\cdot) = [\sigma_{i1}(\cdot), \dots, \sigma_{im}(\cdot)]$  are the appreciation rate and volatility coefficients of the  $i^{\text{th}}$  stock. The  $\mathbb{R}^{m \times m}$ -valued process of volatility coefficients:

$$\sigma(\cdot) := \begin{bmatrix} \sigma_1(\cdot) \\ \vdots \\ \sigma_m(\cdot) \end{bmatrix}$$

is referred to as the volatility. We assume throughout that  $\sigma(\cdot)$  satisfies the so-called *non-degeneracy* assumption; that is, there is a constant  $\delta > 0$  such that  $\sigma(t)\sigma(t)' \geq \delta I$ , a.e.  $t \in [0, T]$ . In addition, we shall assume that the market parameters,  $r(\cdot)$ ,  $\mu_i(\cdot)$  and  $\sigma_{ij}(\cdot)$  are  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes. (Precise assumptions on the parameters are stated below). Therefore, the financial market represented by (1)-(2) is *incomplete* in the sense of Harrison and Pliska (1981) when  $d > 0$ , and *complete* when  $d = 0$ .

## Investment problems

Suppose that we have an agent whose initial wealth is  $y$  and whose wealth at time  $t \in [s, T]$  is denoted by  $x(t)$ . Let  $\pi_i(t)$  denote the total market value of the agent's investment in the  $i^{\text{th}}$  asset ( $i = 0, \dots, m$ ) at time  $t$ . Then the agent's wealth process  $x(\cdot)$  corresponding to  $\pi_0(\cdot), \dots, \pi_m(\cdot)$  is the solution of the SDE:

$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^m [\mu_i(t) - r(t)]\pi_i(t) \right\} dt + \sum_{j=1}^m \sum_{i=1}^m \pi_i(t)\sigma_{ij}(t)dW_j(t), \\ x(s) = y. \end{cases} \quad (3)$$

When  $\pi_0(t) < 0$ , the agent is said to be *borrowing* the amount  $|\pi_0(t)|$  at the rate  $r(t)$  at time  $t$ ; when  $\pi_i(t) < 0$  for  $i = 1, \dots, m$ , the agent is *short-selling* the  $i^{\text{th}}$  stock at time  $t$ . The vector  $\pi(\cdot) := (\pi_1(\cdot), \dots, \pi_m(\cdot))$  is the *portfolio* of the investor. Note that  $\pi_0(t)$  is uniquely determined by  $\pi(t)$  and  $x(t)$ , and for this reason, we need not include it in the vector  $\pi(t)$ . The class of admissible portfolios is the set:

$$\mathcal{U} = \left\{ L^1_{\mathcal{F}}(s, T; \mathbb{R}^m) \mid \text{There exists a unique solution of (3)}. \right\} \quad (4)$$

In this paper, we study three closely related investment problems from the perspective of linear-quadratic (LQ) optimal control and backward stochastic differential equations (BSDEs). To motivate these problems, consider first an agent who faces a liability  $\xi$ , the value of which will not be known until the terminal time  $T$ . Due to the uncertainty in  $\xi$ , this investor faces some risk. For an investor with initial capital  $y$ , one method of reducing risk is to invest this money in the market so that the terminal value of this investment  $x(T)$  is as 'close as possible' to the value of the liability  $\xi$ . In a *complete market* (i.e.  $d = 0$ ), an investor *with a sufficiently high level of initial wealth*  $y$  can eliminate all the risk by replicating  $\xi$ ; that is, there is a unique value of  $y$  and an associated trading strategy  $\pi(\cdot)$  such that an investor, starting with  $y$  and investing according to  $\pi(\cdot)$ , will have a terminal wealth satisfying  $x(T) = \xi$ ,  $\mathbb{P}$ -a.s.. By taking this approach, risk can be eliminated completely. In the case of an incomplete market, however, perfect replication is usually not possible, no matter what the value of the investor's initial wealth. On the other hand, super-replication (i.e. finding a portfolio such that  $x(T) \geq \xi$ ,  $\mathbb{P}$ -a.s.) may be possible, but is typically infeasible since the initial wealth required to super-replicate a claim is usually too large to be of practical use. As a compromise, an investor in an incomplete market (or, for that matter, in a complete market but with insufficient initial capital to replicate the claim) may seek to solve the following optimization problem:

**Problem 1:** Quadratic loss minimization

$$V(s, y) := \min_{\pi(\cdot) \in \mathcal{U}} E[\xi - x(T)]^2. \quad (5)$$

The function  $V(s, y)$  is commonly referred to as the *value function*.

A second problem that the investor may be interested in solving is the problem of pricing the claim  $\xi$ . In the case of a complete market, when replication is possible, the unique arbitrage-free price at time  $s$  is the value of  $y$  which minimizes  $V(s, y)$  (i.e. results in  $V(s, y) = 0$  in (5)). In the case of an incomplete market, this generalizes to the following:

**Problem 2: Pricing**

$$\phi(s) := \arg \min_y V(s, y). \quad (6)$$

Finally, an investor with initial wealth  $y$  may wish to invest in the market so that his/her *expected return*, as measured by the expected terminal wealth  $Ex(T)$ , satisfies  $Ex(T) = c$  for some given  $c > 0$ , but does so with *minimal risk*, where ‘risk’, in this case, is measured by the variance of the terminal wealth:

$$\text{Var } x(T) = E[x(T) - Ex(T)]^2 = E[x(T) - c]^2.$$

This leads to the continuous time formulation of Markowitz’s mean-variance portfolio selection problem (see Markowitz (1952, 1959)):

**Problem 3: Mean-variance portfolio selection**

$$\left\{ \begin{array}{l} \min_{\pi(\cdot) \in \mathcal{U}} E[x(T) - c]^2, \\ \text{Subject to:} \\ Ex(T) = \xi. \end{array} \right. \quad (7)$$

In this paper, we solve Problems 1, 2 and 3 in a unified way by using the theory of BSDEs and LQ control.

Finally, we shall assume throughout that the following assumptions are satisfied:

**Assumption (A1):**

$$\left\{ \begin{array}{l} r(\cdot), \mu_i(\cdot), \sigma_{ij}(\cdot) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}), \quad i, j = 1, \dots, m, \\ \xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), \\ \sigma(t)\sigma(t)' \geq \delta I, \quad \forall t \in [0, T], \text{ for some } \delta > 0. \end{array} \right.$$

Note that  $\sigma(t)$  is invertible.

### 3 Completion of squares

Consider the following stochastic control problem:

$$\left\{ \begin{array}{l} \min_{u(\cdot)} E (\xi - x(T))^2, \\ dx(t) = [r(t)x(t) + b(t)\pi(t)] dt + \pi(t)' \sigma(t) dW(t), \\ x(0) = y, \\ u(\cdot) \in \mathcal{U}. \end{array} \right. \quad (8)$$

Clearly, (8) is a restatement of the quadratic hedging problem (Problem 1) with  $b(t) = \mu(t) - r(t)\mathbf{1}$ . In addition, it will be seen in Section 6 that the solutions of Problems 2 and 3 follow easily from that of (8). We shall assume throughout that **(A1)** is satisfied.

Consider the following *backward stochastic differential equations* (BSDEs):

$$\left\{ \begin{array}{l} dp(t) = \left\{ \left[ -2r(t) + |\theta(t)|^2 \right] p(t) + 2\theta(t)' \Lambda_1(t) + \frac{1}{p(t)} \Lambda_1(t)' \Lambda_1(t) \right\} dt \\ \quad + \Lambda_1(t)' dW(t) + \Lambda_2(t) dB(t), \\ p(T) = 1, \\ p(t) > 0, \quad \forall t \in [0, T], \\ dh(t) = \left\{ r(t)h(t) + \theta(t)' \eta_1(t) - \frac{\Lambda_2(t)'}{p(t)} \eta_2(t) \right\} dt + \eta_1(t)' dW(t) + \eta_2(t)' dB(t), \\ h(T) = \xi, \end{array} \right. \quad (9)$$

where

$$\theta(t) := \sigma(t)^{-1} b(t)'.$$

Throughout this paper, a *solution* of the SRE (9) is a pair of processes  $(p(\cdot), \Lambda(\cdot))$  satisfying:

$$\begin{aligned} (p(\cdot), \Lambda(\cdot)) &\in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m+d}), \\ 1/p(\cdot) &\in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})), \end{aligned} \quad (11)$$

and the BSDE (9). In particular, (11) implies that there are constants  $0 < \delta < m < \infty$  such that  $\delta \leq p(\cdot) \leq m$ . On the other hand, a solution of (10) is a pair  $(h(\cdot), \eta(\cdot))$  satisfying (10) and the condition:

$$(h(\cdot), \eta(\cdot)) \in L_{\mathcal{F}}^2(\Omega; C(0, T; \mathbb{R})) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m+d}).$$

Also, it will be convenient to decompose  $\Lambda(\cdot)$  and  $\eta(\cdot)$  into two parts, namely

$$\Lambda(t) = (\Lambda_1(t), \Lambda_2(t)) \in \mathbb{R}^m \times \mathbb{R}^d, \quad \eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{R}^m \times \mathbb{R}^d.$$



Here,  $\Lambda_1(t)$  are the components of  $\Lambda(t)$  that multiply the first  $m$  components of the Brownian motion  $\bar{W}(t) = (W(t), B(t))$  in (9) while  $\Lambda_2(t)$  are the components of  $\Lambda(\cdot)$  which multiply the remaining  $d$  components of  $\bar{W}(t)$ . (Note however that both  $\Lambda_1(\cdot)$  and  $\Lambda_2(\cdot)$  are adapted to the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  generated by  $\bar{W}(\cdot)$  as defined in Section 2). Similarly, we will distinguish between the components of  $\eta(t)$ . We shall assume throughout this section that (9)-(10) have unique solutions.

In this section, we show the connection between the stochastic control problem (8) and the BSDEs (9)-(10) by deriving expressions for the optimal control and optimal cost for (8) in terms of the solutions  $(p(\cdot), \Lambda(\cdot))$  and  $(h(\cdot), \eta(\cdot))$  of (9)-(10), which are *assumed* to exist.

The BSDE (9) is a special case (corresponding to the stochastic control problem (8)) of the so-called *Stochastic Riccati Equation* (SRE). The SRE was introduced in Bismut (1976) in his study of linear-quadratic optimal control problems with random system and cost parameters (of which the problem (9) is a special case). Except for certain special cases (Bismut (1976), Hu and Zhou (2001), Lim and Zhou (2001a, 2001b), Peng (1992) and Kohlmann and Tang (2001a, 2001b, 2000c), the issue of global solvability of SREs remains unresolved due to the complicated nonlinearities associated with this equation. When the parameters are deterministic, the SRE becomes an ordinary differential equation (ODE) with a deterministic terminal condition. In this situation, the process  $\Lambda(\cdot) \equiv 0$  and many of the complicated terms in the SRE disappear. For this reason, the general existence theory for the Riccati ODE is much more complete than for the SRE. The interested reader may consult Rami, Chen, Moore and Zhou (2001) for a recent account of these issues.

For the problem at hand, the difficulties are caused by the nonlinear term  $\Lambda_1(t)' \Lambda_1(t)/p(t)$  in the drift (or generator) of the SRE (9). Because of these terms, the generator of (9) is neither Lipschitz continuous nor linearly growing as a function of the state variables  $(p(\cdot), \Lambda(\cdot))$  and for this reason, the general existence theory for nonlinear BSDEs (Pardoux and Peng (1989)) can not be applied to (9). In Section 5, a proof of existence which uses recent results of Lepeltier and San Martin (1998) and Kobylanski (2000), is presented.

The linear BSDE (10) depends on the solution of (9) through the term  $-\Lambda_2(t)/p(t)$  which plays the role of one of the coefficients of this equation. Since  $\Lambda(\cdot)$  is square integrable and  $p(\cdot)$  is bounded, the coefficient  $-\Lambda_2(t)/p(t)$  is typically unbounded. This causes difficulties because standard results on the existence and uniqueness of solutions of linear BSDEs (see, for example, Yong and Zhou (2000)) only apply when the coefficients are bounded. Nevertheless, by exploiting the financial interpretation of the solution of the SRE, it will be shown in Section 4 that existence of solutions of (10) is implied by existence for the SRE (9).

The following result relates the BSDEs (9)-(10) and the stochastic control problem (8).

**Proposition 3.1** *Suppose that Assumption (A1) holds. If the BSDEs (9)-(10) have unique*

solutions  $(p(\cdot), \Lambda(\cdot))$  and  $(h(\cdot), \eta(\cdot))$ , then

$$\pi(t) = -(\sigma(t)^{-1})' \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] (x(t) - h(t)) + (\sigma(t)^{-1})' \eta_1(t) \quad (12)$$

is the unique optimal control, and

$$J^* = p(0)(h(0) - y)^2 + E \int_0^T p(t) \eta_2(t) \eta_2(t)' dt \quad (13)$$

is the optimal cost, for the stochastic control problem (8).

**Proof:** It will be convenient to write the equations (9)-(10) in the following equivalent form:

$$\left\{ \begin{array}{l} dp(t) = \left\{ \left[ -2r(t) + b(t)(\sigma(t)\sigma(t)')^{-1}b(t)' \right] p(t) + 2b(t)(\sigma(t)^{-1})' \Lambda_1(t) + \frac{1}{p(t)} \Lambda_1(t)' \Lambda_1(t) \right\} dt \\ \quad + \Lambda_1(t)' dW(t) + \Lambda_2(t) dB(t), \\ p(T) = 1, \\ p(t) > 0, \quad \forall t \in [0, T], \\ dh(t) = \left\{ r(t)h(t) + b(t)(\sigma(t)^{-1})' \eta_1(t) - \frac{\Lambda_2(t)'}{p(t)} \eta_2(t) \right\} dt + \eta_1(t)' dW(t) + \eta_2(t)' dB(t), \\ h(T) = \xi. \end{array} \right.$$

Clearly:

$$\begin{aligned} d(h(t) - x(t)) &= \left\{ r(t)(h(t) - x(t)) + b(t)(\sigma(t)^{-1})' \eta_1(t) - \frac{\Lambda_2(t)'}{p(t)} \eta_2(t) - b(t)\pi(t) \right\} dt \\ &\quad + (\eta_1(t)' - \pi(t)' \sigma(t)) dW(t) + \eta_2(t)' dB(t). \end{aligned}$$

By Ito's formula:

$$\begin{aligned} d(h(t) - x(t))^2 &= \left\{ 2r(t)(h(t) - x(t))^2 + 2(h(t) - x(t)) \left[ b(t)(\sigma(t)^{-1})' \eta_1(t) - \frac{\Lambda_2(t)'}{p(t)} \eta_2(t) \right] \right. \\ &\quad \left. + \eta_1(t)' \eta_1(t) + \eta_2(t)' \eta_2(t) + \pi(t)' \sigma(t) \sigma(t)' \pi(t) - 2\pi(t)' \left[ b(t)'(h(t) - x(t)) + \sigma(t) \eta_1(t) \right] \right\} dt \\ &\quad + 2(h(t) - x(t)) (\eta_1(t)' - \pi(t)' \sigma(t)) dW(t) + 2(h(t) - x(t)) \eta_2(t)' dB(t), \end{aligned}$$

and again:

$$\begin{aligned} d \left\{ p(t)(h(t) - x(t))^2 \right\} &= \\ &\left\{ p(t) \left[ \pi(t) - (\sigma(t)\sigma(t)')^{-1} \left( \left[ b(t)' + \frac{\sigma(t)\Lambda_1(t)}{p(t)} \right] (h(t) - x(t)) + \sigma(t)\eta_1(t) \right) \right]' \right. \\ &\quad \left. \times \sigma(t)\sigma(t)' \left[ \pi(t) - (\sigma(t)\sigma(t)')^{-1} \left( \left[ b(t)' + \frac{\sigma(t)\Lambda_1(t)}{p(t)} \right] (h(t) - x(t)) + \sigma(t)\eta_1(t) \right) \right] \right. \\ &\quad \left. + p(t)\eta_2(t)' \eta_2(t) \right\} dt + \{ \dots \} d\bar{W}(t). \end{aligned} \quad (14)$$

Therefore, by integrating from  $[0, T]$  and taking expectations, we obtain:

$$\begin{aligned} E(\xi - x(T))^2 &= p(0)(h(0) - y)^2 + E \int_0^T p(t)\eta_2(t)'\eta_2(t)dt \\ &+ E \int_0^T p(t) \left[ \pi(t) + (\sigma(t)^{-1})' \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] (x(t) - h(t)) - (\sigma(t)^{-1})'\eta_1(t) \right]' \\ &\times \sigma(t)\sigma(t)' \left[ \pi(t) + (\sigma(t)^{-1})' \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] (x(t) - h(t)) - (\sigma(t)^{-1})'\eta_1(t) \right] dt. \end{aligned}$$

The result follows from the admissibility of  $\pi(\cdot)$  (see Proposition 3.2) and the assumption that  $p(t) > 0$  and  $\sigma(t)$  is non-degenerate.  $\blacksquare$

**Proposition 3.2** *Suppose that Assumption (A1) holds and the BSDEs (9)-(10) have unique solutions  $(p(\cdot), \Lambda(\cdot))$  and  $(h(\cdot), \eta(\cdot))$ . Then the policy (12) is admissible.*

**Proof:** Substituting (12) into (8) gives:

$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \theta(t)' \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] (h(t) - x(t)) + \theta(t)'\eta_1(t) \right\} dt \\ \quad + \left\{ (h(t) - x(t)) \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] + \eta_1(t) \right\}' dW(t), \\ x(0) = y. \end{cases} \quad (15)$$

We prove solvability of (15) by constructing a closed form expression of the solution. Consider the SDE:

$$\begin{cases} dY(t) = -r(t)Y(t)dt - Y(t)\theta(t)'dW(t) + \left\{ Y(t)\frac{\Lambda_2(t)}{p(t)} + p(t)\eta_2(t) \right\}' dB(t), \\ Y(0) = p(0)(h(0) - y). \end{cases} \quad (16)$$

It can be shown that:

$$Y(t) = \Phi(t) \left\{ p(0)(h(0) - y) - \int_0^t \Phi(s)^{-1} \Lambda_2(s)'\eta_2(s)ds + \int_0^t \Phi(s)^{-1} p(s)\eta_2(s)'dB(s) \right\} \quad (17)$$

is a solution of (16), where:

$$\Phi(t) = \exp \left\{ -\frac{1}{2} \int_0^t \left( |\theta(s)|^2 + \left| \frac{\Lambda_2(s)}{p(s)} \right|^2 + 2r(s) \right) ds - \int_0^t \theta(s)'dW(s) - \int_0^t \frac{\Lambda_2(s)'}{p(s)} dB(s) \right\}.$$

Finally, from Ito's formula and the SDEs (16)-(17), it can be shown that:

$$x(t) := h(t) - \frac{Y(t)}{p(t)}$$

is a solution of (8).

Next, we show that  $\pi(\cdot) \in L^1_{\mathcal{F}}(0, T; \mathbb{R}^m)$ . Integrating (14) from 0 to  $t$  and taking expectations gives:

$$Ep(t)(h(t) - x(t))^2 \leq p(0)(h(0) - y)^2 + E \int_0^T p(s)\eta_2(s)'\eta_2(s)ds.$$

Since, by definition (see (11)),  $p(\cdot) \geq \delta$  for some constant  $\delta > 0$ , it follows that  $h(\cdot) - x(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$  which implies that (12) is admissible.  $\blacksquare$

## 4 Financial interpretation of the SRE

Let  $\nu(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$  be given and fixed and consider the exponential local martingale:

$$M_\nu(t) = \exp \left\{ - \int_0^t \theta(s)' dW(s) - \int_0^t \nu(s)' dB(s) - \frac{1}{2} \int_0^t (|\theta(s)|^2 + |\nu(s)|^2) ds \right\}. \quad (18)$$

If  $M_\nu(t)$  is a martingale, we can define a  $\mathbb{P}$ -equivalent probability measure  $\mathbb{P}_\nu$  as:

$$d\mathbb{P}_\nu = M_\nu(T) d\mathbb{P}. \quad (19)$$

The *Variance-Optimal Martingale Measure* (VMM) is the unique  $\mathbb{P}$ -equivalent probability measure defined through the solution of the so-called dual optimization problem (see Delbaen and Schachermayer (1996), Schweizer (1996), Gouiroux, Laurent and Pham (1998), and Laurent and Pham (1999)):

$$\min_{\nu(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)} E \left[ \frac{1}{P_0(T)} \frac{d\mathbb{P}_\nu}{d\mathbb{P}} \right]^2 = \min_{\nu(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)} E \left[ \frac{M_\nu(T)}{P_0(T)} \right]^2. \quad (20)$$

In particular, it has been shown that if  $\bar{\nu}(\cdot)$  is the solution of (20) then  $\bar{\mathbb{P}} := \mathbb{P}_{\bar{\nu}}$  is the VMM and  $\bar{M}(\cdot) := M_{\bar{\nu}}(\cdot)$  is a martingale.

The VMM  $\bar{\mathbb{P}}$  plays a central role in the duality-based approach to the quadratic hedging problem (8). On the other hand, by using ideas from LQ control and BSDEs, we have obtained an expression for the optimal cost and hedging policy for (8) in terms of the SRE (9). In this section, we show the relationship between these two approaches by showing the relationship between the solution  $(p(\cdot), \Lambda(\cdot))$  of the SRE (9) and the VMM  $\bar{\mathbb{P}}$  defined through (20); see also Kohlmann and Tang (2001b). Furthermore, this relationship is exploited in Proposition 4.2 to show that solvability of the SRE (9) implies solvability of the BSDE (10).

Define:

$$\begin{aligned} \bar{x}(t) &:= \frac{M_\nu(t)}{P_0(t)} \\ &= \exp \left\{ - \int_0^t \theta(s)' dW(s) - \int_0^t \nu(s)' dB(s) - \frac{1}{2} \int_0^t (|\theta(s)|^2 + |\nu(s)|^2 + 2r(s)) ds \right\}. \end{aligned} \quad (21)$$

Using Ito's formula, it is easy to show that the dual problem (20) may be written in the form:

$$\left\{ \begin{array}{l} \min_{\nu} E\bar{x}(T)^2, \\ \text{Subject to:} \\ \quad d\bar{x}(t) = -\bar{x}(t) \left\{ r(t) dt + \theta(t)' dW(t) + \nu(t)' dB(t) \right\}, \\ \quad \bar{x}(0) = 1, \\ \quad \nu(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d). \end{array} \right. \quad (22)$$

A solution of this problem can be obtained via the ‘completion of squares’ argument used to the proof of Proposition 3.1. (If the reader prefers, he/she could make the substitution  $u(t) = -\bar{x}(t)\nu(t)$  in (22) before doing the calculations). With this in mind, consider the following SRE:

$$\left\{ \begin{array}{l} dy(t) = \left\{ \left[ 2r(t) - |\theta(t)|^2 \right] y(t) + 2\theta(t)'z_1(t) + \frac{1}{y(t)}z_2(t)'z_2(t) \right\} dt \\ \quad + z_1(t)dW(t) + z_2(t)dB(t), \\ y(T) = 1, \\ y(t) > 0, \quad \forall t \in [0, T]. \end{array} \right. \quad (23)$$

As in the proof of Proposition 3.1, we assume for the moment that the SRE (23) has a solution  $(y(\cdot), z(\cdot))$  satisfying:

$$\begin{aligned} (y(\cdot), z(\cdot)) &\in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})) \times L_{\mathcal{F}}^2(0, T; R^{m+d}), \\ 1/y(\cdot) &\in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})). \end{aligned}$$

The solution of the dual problem, given in terms of  $(y(\cdot), z(\cdot))$ , can be obtained by mimicking the proof of Proposition 3.1. In particular, by applying Ito’s formula to  $y(t)\bar{x}(t)^2$ , it can be shown that:

$$E\bar{x}(T)^2 = y(s) + E \int_s^T y(t)\bar{x}(t)^2 \left| \nu(t) - \frac{z_2(t)}{y(t)} \right|^2 dt. \quad (24)$$

The equation (24) establishes the relationship between the VMM  $\bar{\mathbb{P}}$  and the solution  $(y(\cdot), z(\cdot))$  of (23) (should it exist). Note that  $y(\cdot)$  is necessarily unique since, by (24), it corresponds to the optimal cost of an optimization problem.

We determine the relationship between the SRE (9) and the VMM  $\bar{\mathbb{P}}$  by showing the relationship between (9) and (23). From Ito’s formula:

$$\left\{ \begin{array}{l} d\left(\frac{1}{p(t)}\right) = \left\{ \left[ 2r(t) - |\theta(t)|^2 \right] \left(\frac{1}{p(t)}\right) + 2\theta(t)' \left(-\frac{\Lambda_1(t)}{p(t)^2}\right) + p(t) \left(-\frac{\Lambda_2(t)}{p(t)^2}\right)' \left(-\frac{\Lambda_2(t)}{p(t)^2}\right) \right\} dt \\ \quad + \left(-\frac{\Lambda_1(t)}{p(t)^2}\right)' dW(t) + \left(-\frac{\Lambda_2(t)}{p(t)^2}\right)' dB(t), \\ \frac{1}{p(T)} = 1, \\ \frac{1}{p(t)} > 0, \quad \forall t \in [0, T]. \end{array} \right.$$

Therefore, by comparing coefficients with (23), it can be seen that

$$(y(t), z(t)) := \left( \frac{1}{p(t)}, -\frac{\Lambda(t)}{p(t)^2} \right) \quad (25)$$

is a solution of (23). Substituting (25) into (24), and noting (20), we have:

$$E \left[ \frac{1}{P_0(T)} \frac{d\mathbb{P}_{\nu}}{d\mathbb{P}} \right]^2 = \frac{1}{p(0)} + E \int_0^T \frac{\bar{x}(t)^2}{p(t)} \left| \nu(t) - \frac{\Lambda_2(t)}{p(t)} \right|^2 dt.$$

In summary, we have the following result:

**Theorem 4.1** *Suppose that (A1) holds. Let  $\bar{\mathbb{P}}$  be the VMM defined by (20). If (9) has a solution  $(p(\cdot), \Lambda(\cdot))$  then*

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ - \int_0^T \theta(t)' dB(t) - \int_0^T \bar{\nu}(t)' dW(t) - \frac{1}{2} \int_0^T (|\theta(t)|^2 + |\bar{\nu}(t)|^2) dt \right\} \quad (26)$$

where

$$\bar{\nu}(t) = - \frac{\Lambda_2(t)}{p(t)}. \quad (27)$$

Furthermore:

$$E \left[ \frac{1}{P_0(T)} \frac{d\bar{\mathbb{P}}}{d\mathbb{P}} \right]^2 = \frac{1}{p(0)}.$$

Theorem 4.1 characterizes the VMM  $\bar{\mathbb{P}}$  in terms of the solution  $(p(\cdot), \Lambda(\cdot))$  of the SRE (9) via the relationship (27). Furthermore, since the exponential local martingale (26) is in fact a martingale (see Gourieroux, Laurent and Pham (1998), and Laurent and Pham (1999)), it follows from the Girsanov Theorem that

$$\begin{pmatrix} \bar{W}(t) \\ \bar{B}(t) \end{pmatrix} := \begin{pmatrix} W(t) \\ B(t) \end{pmatrix} + \int_0^t \begin{pmatrix} \theta(s) \\ \bar{\nu}(s) \end{pmatrix} ds \quad (28)$$

is a standard Brownian motion under  $\bar{\mathbb{P}}$ . Hence we have the following result:

**Theorem 4.2** *Suppose that (A1) holds and the SRE (9) has a solution satisfying the conditions in Theorem 4.1. Then the linear BSDE (10) has a unique solution  $(h(\cdot), \eta(\cdot))$  and:*

$$h(t) = \bar{E} \left[ e^{-\int_t^T r(s) ds} \xi \mid \mathcal{F}_t \right]. \quad (29)$$

**Proof:** Under (A1), the linear BSDE

$$\begin{cases} dh(t) = r(t)h(t)dt + \eta_1(t)' d\bar{W}(t) + \eta_2(t)' d\bar{B}(t), \\ h(T) = \xi, \end{cases} \quad (30)$$

has a unique solution  $(h(\cdot), \eta(\cdot))$ . Substituting (28) into (30), it follows immediately that  $(h(\cdot), \eta(\cdot))$  is also a solution of (10), which gives us existence. Uniqueness follows from applying the transformation defined by (28) to (10). (29) follows from Proposition 2.2 of El Karoui, Peng and Quenez (1997). ■

## 5 Existence of solutions

In this section, we prove existence and uniqueness of solutions of the nonlinear BSDE:

$$\begin{cases} dp(t) = -\left\{A(t)p(t) + B(t)\Lambda(t) - \frac{1}{p(t)}\Lambda(t)'H(t)\Lambda(t)\right\}dt + \Lambda(t)'d\bar{W}(t). \\ p(T) = \gamma, \\ p(t) > 0, \forall t \in [0, T]. \end{cases} \quad (31)$$

Once again, a solution of (31) is a pair of processes  $(p(\cdot), \Lambda(\cdot))$  satisfying (11) and the BSDE (31). Clearly, the SRE (9) corresponds to the case:

$$\begin{cases} A(t) = -\left[-2r(t) + b(t)(\sigma(t)\sigma(t)')^{-1}b(t)'\right] = -\left[-2r(t) + |\theta(t)|^2\right], \\ B(t) = -2b(t)(\sigma(t)^{-1})' = -2\theta(t)', \\ H(t) = -\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, \\ \gamma = 1, \end{cases}$$

where  $I_m$  denotes the  $m \times m$  identity matrix. In particular, when  $d = 0$  (and hence  $H = I_m$ ), we have the case of a complete market.

Finally, we shall assume throughout this section that the following assumptions are satisfied:

**Assumption (A2):**

$$\begin{cases} A(\cdot) \in L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}), \\ B(\cdot) \in L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}^{1 \times (m+d)}), \\ H(\cdot) \in L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R}^{(m+d) \times (m+d)}), \quad 0 \leq H \leq I, \\ \gamma \in L_{\mathcal{F}_T}^{\infty}(\Omega; \mathbb{R}), \quad \gamma \geq \delta \text{ for some constant } \delta > 0. \end{cases}$$

The main difficulty when dealing with (31) is the nonlinear term  $(1/p(t))\Lambda(t)'H(t)\Lambda(t)$  that appears in the drift (generator). Therefore, the generator of (31) is singular in  $p(\cdot)$  and satisfies neither the global Lipschitz continuity nor the linear growth conditions in  $(p(\cdot), \Lambda(\cdot))$  that are required in order to apply the general existence and uniqueness results of Pardoux and Peng (1990). For this reason, an alternative proof of existence and uniqueness is required. In this section, we prove the following result:

**Theorem 5.1** *Suppose that (A2) is satisfied. Then there is a solution  $(p(\cdot), \Lambda(\cdot))$  of the nonlinear BSDE (31). Moreover, if  $(\bar{p}(\cdot), \bar{\Lambda}(\cdot))$  and  $(p(\cdot), \Lambda(\cdot))$  are solutions of (31), then  $\bar{p}(\cdot) \equiv p(\cdot)$ .*

Existence and uniqueness of solutions, for the SRE (9), is an immediate consequence of Theorem 5.1; see Theorem 6.1 for a statement of this result.

Our proof of Theorem 5.1 uses in an essential way some recent results of Lepeltier and San Martin (1998) and Kobylanski (2000) on nonlinear BSDEs with generators that may be quadratically growing in the gains process  $\Lambda(\cdot)$ . There are two steps in our proof of Theorem 5.1. In the first step, we derive a *sufficient condition* under which the limit point  $(p(\cdot), \Lambda(\cdot))$  of a certain sequence  $\{(p_i(\cdot), \Lambda_i(\cdot))\}$  is a solution of the BSDE (31). In the second step we show that this sufficient condition is *always* satisfied under Assumption **(A2)**, which proves Theorem 5.1.

In the next subsection, we summarize the pertinent results from the papers of Lepeltier and San Martin (1998) and Kobylanski (2000). Following this, a proof of existence and uniqueness is presented. For the sake of readability, certain technical lemmas and their proofs have been placed in the Appendix.

## Preliminaries

Consider the following BSDE:

$$\begin{cases} dx(t) = -f(t, x(t), z(t))dt + z(t)'d\bar{W}(t), \\ x(T) = \gamma. \end{cases} \quad (32)$$

A *bounded* solution of (32) is a pair of processes

$$(x(\cdot), z(\cdot)) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m+d})$$

which satisfy (32). In the papers of Lepeltier and San Martin (1998) and Kobylanski (2000), the notion of a *maximal bounded solution* of the BSDE (32) is defined as follows. A bounded solution  $(\bar{x}(\cdot), \bar{z}(\cdot))$  is a *maximal bounded solution* of (32) if for every bounded solution  $(x(\cdot), z(\cdot))$ , we have  $\bar{x}(\cdot) \geq x(\cdot)$ . Note that if  $(x(\cdot), z(\cdot))$  and  $(\bar{x}(\cdot), \bar{z}(\cdot))$  are maximal bounded solutions, then  $x(\cdot) \equiv \bar{x}(\cdot)$ ; that is, maximal bounded solutions are unique in the first component. The following existence results can be found in Lepeltier and San Martin (1998) and Kobylanski (2000).

**Theorem 5.2** *If  $\gamma \in L_{\mathcal{F}_T}^{\infty}(\Omega; \mathbb{R})$  and  $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^{m+d} \rightarrow \mathbb{R}$  satisfies:*

1.  $f(\cdot, \cdot, x, z)$  is  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted for every fixed  $(x, z) \in \mathbb{R} \times \mathbb{R}^{m+d}$ ,
2.  $f(\omega, t, \cdot, \cdot)$  is continuous for all  $(\omega, t)$ ,
3.  $|f(\omega, t, x, z)| \leq C(1 + D|x|) + F|z|^2$ , where  $C, D$ , and  $F$  are finite constants,

*then (32) has a maximal bounded solution.*



Theorem 5.2 provides conditions under which a BSDE with a generator  $f(\cdot)$  that is quadratically growing in the gains process  $z$  has a solution. In particular, it requires that for fixed  $(x, z)$ ,  $f(\omega, t, x, z)$  and  $\gamma(\omega)$  are bounded in  $(\omega, t)$ .

The second result from Lepeltier and San Martin (1998) and Kobylanski (2000) that we shall use is the following comparison theorem.

**Theorem 5.3** *Let  $(f_i, \gamma_i)$  ( $i = 1, 2$ ) satisfy the assumptions of Theorem 5.2 and  $(x_i(\cdot), z_i(\cdot))$  denote the associated maximal bounded solutions. If*

1.  $f_1(\omega, t, y, z) \geq f_2(\omega, t, y, z)$ , for all  $(\omega, t, y, z)$ ,
2.  $\gamma_1 \geq \gamma_2$ ,  $\mathbb{P}$ -a.s.,

then  $x_1(\cdot) \geq x_2(\cdot)$ .

Finally, let us introduce the BSDEs:

$$\begin{cases} dx(t) = -\left[A(t)x(t) + B(t)q(t) - \frac{1}{x(t)} q(t)'q(t)\right]dt + q(t)'d\bar{W}(t), \\ x(T) = \gamma, \\ x(t) > 0, \quad \forall t \in [0, T], \end{cases} \quad (33)$$

$$\begin{cases} dy(t) = -\left[A(t)y(t) + B(t)z(t)\right]dt + z(t)'d\bar{W}(t), \\ y(T) = \gamma, \end{cases} \quad (34)$$

$$\begin{cases} dp(t) = -\left[A(t)p(t) + B(t)\Lambda(t) - \frac{1}{R(t)} \Lambda(t)'H(t)\Lambda(t)\right]dt + \Lambda(t)'d\bar{W}(t), \\ p(T) = \gamma, \end{cases} \quad (35)$$

with solutions denoted by  $(x(\cdot), q(\cdot))$ ,  $(y(\cdot), z(\cdot))$  and  $(p(\cdot), \Lambda(\cdot))$ , respectively. We shall assume that the parameters in (33)-(35) satisfy the following assumptions:

**Assumption (A3):**

$$\begin{cases} A(\cdot), B(\cdot), H(\cdot), \gamma \text{ satisfy (A2)}, \\ R(\cdot) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}) \text{ and } R(\cdot) \geq \delta \text{ for some constant } \delta > 0. \end{cases}$$

The BSDE (33) corresponds to (31) with  $H(\cdot) \equiv I$ , which is also the SRE (9) in the case of a complete market. In relation to this equation, the following result is proven in Lim and Zhou (2001b).

**Proposition 5.1** *Suppose that (A3) holds. Then there is a unique bounded solution  $(x(\cdot), q(\cdot))$  of the BSDE (33). Moreover, there is a constant  $\delta > 0$  such that  $x(\cdot) \geq \delta$ ,  $\mathbb{P}$ -a.s..*

Existence and uniqueness of maximal bounded solutions of (34) and (35) follows immediately from Theorem 5.2. In particular, for fixed  $A(\cdot)$ ,  $B(\cdot)$ ,  $H(\cdot)$  and  $\gamma$  satisfying **(A3)**, we can define a mapping

$$R(\cdot) \longmapsto \Psi(R(\cdot)) \tag{36}$$

which takes any process  $R(\cdot)$  satisfying **(A3)** to the (unique) first component  $\Psi(\cdot) := p(\cdot)$  of the maximal solution  $(p(\cdot), \Lambda(\cdot))$  of the BSDE (35).

The following comparison result is an immediate consequence of Theorem 5.3.

**Proposition 5.2** *Suppose that **(A3)** holds. Let  $(y(\cdot), z(\cdot))$  be the unique solution of (34) and  $\Psi(\cdot)$  the operator defined by (36).*

1. *If  $R(\cdot), S(\cdot) \in L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R})$  and  $\delta \leq S(\cdot) \leq R(\cdot)$  for some constant  $\delta > 0$ , then  $\Psi(S(\cdot)) \leq \Psi(R(\cdot))$ .*
2.  *$\Psi(R(\cdot)) \leq y(\cdot) \leq m$  for some constant  $m < \infty$ .*

## Solvability

The main steps in our proof of Theorem 5.1 are as follows.

1. **Monotonicity conditions and sufficiency:** The monotonicity condition (37) is shown in Proposition 5.4 to be a *sufficient* condition for the existence of a solution  $(p(\cdot), \Lambda(\cdot))$  of (31). In fact, this solution  $(p(\cdot), \Lambda(\cdot))$  of (31) is characterized as the limit (in  $L_{\mathcal{F}}^2$ ) of a sequence of processes  $\{(p_i(\cdot), \Lambda_i(\cdot))\}$  (which is monotone in  $\{p_i(\cdot)\}$ ) that is defined through the operator  $\Psi(\cdot)$ ; see proof of Proposition 5.4.
2. **Main result:** We obtain the result in Theorem 5.1 by showing that a process satisfying the monotonicity condition (37) can *always* be found whenever Assumption **(A2)** is satisfied. In fact, it is shown that the solution of the BSDE (31) in the case  $H(\cdot) \equiv I$  (i.e. the SRE (9) in the case when the market is complete) gives us the  $p_0(\cdot)$  that satisfies (37).

### Monotonicity condition and sufficiency:

**Proposition 5.3** *Suppose that **(A3)** holds, and a process  $p_0(\cdot) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}))$  and a constant  $\delta > 0$  can be found such that:*

$$\delta \leq p_0(\cdot) \leq \Psi(p_0(\cdot)). \tag{37}$$

*Then*

$$p_{k+1}(\cdot) := \Psi(p_k(\cdot)) \tag{38}$$

is well defined for every  $k$  and  $p_k(\cdot) \leq p_{k+1}(\cdot)$ . Moreover, there are fixed constants  $0 < \delta \leq m < \infty$  such that

$$\delta \leq p_k(\cdot) \leq m, \quad \forall k.$$

**Proof:** By the monotonicity condition (37) and the definition (38), we have

$$\delta \leq p_0(\cdot) \leq \Psi(p_0(\cdot)) =: p_1(\cdot).$$

Therefore, we have  $\delta \leq p_0(\cdot) \leq p_1(\cdot)$  with  $p_1(\cdot) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}))$ . Suppose now that there is a  $k \geq 0$  such that

$$\delta \leq p_k(\cdot) \leq p_{k+1}(\cdot) \text{ and } p_k(\cdot), p_{k+1}(\cdot) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})).$$

Then Proposition 5.2 implies that

$$\Psi(p_k(\cdot)) \leq \Psi(p_{k+1}(\cdot))$$

and hence

$$\delta \leq p_{k+1}(\cdot) \leq p_{k+2}(\cdot)$$

with  $p_{k+2}(\cdot) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}))$ . Therefore, it follows from induction that (38) defines a monotonically increasing sequence in  $L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}))$  with  $\delta \leq p_k(\cdot)$  for every  $k$ . Finally, the uniform boundedness of  $\{p_k(\cdot)\}$  by a constant  $m$  is an immediate consequence of the second part of Proposition 5.2.  $\blacksquare$

Under the assumption that (37) holds, it follows from Proposition 5.3 (and in particular from the fact that  $\delta \leq p_k(\cdot) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}))$ ) that for every  $k$ , the BSDE:

$$\begin{cases} dp(t) = -\left\{ A(t)p(t) + B(t)\Lambda(t) - \frac{1}{p_k(t)}\Lambda(t)'H(t)\Lambda(t) \right\} dt + \Lambda(t)'d\bar{W}(t), \\ p(T) = \gamma, \\ p(t) > 0, \quad \forall t \in [0, T], \end{cases} \quad (39)$$

has a maximal bounded solution

$$(p_{k+1}(\cdot), \Lambda_{k+1}(\cdot)) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m+d}), \quad p_{k+1}(\cdot) = \Psi(p_k(\cdot)).$$

In this way, we can generate a sequence of processes  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$ . The asymptotic properties of this sequence are given in the following result.

**Proposition 5.4** *Suppose that (A3) holds and (37) is satisfied for some  $p_0(\cdot) \in L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R}))$ . Let  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$  be the sequence in  $L_{\mathcal{F}}^{\infty}(\Omega; C(0, T; \mathbb{R})) \times L_{\mathcal{F}}^2(0, T; \mathbb{R}^{m+d})$  obtained via (39).*

Then, there exists a subsequence, also denoted by  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$ , and a pair  $(p(\cdot), \Lambda(\cdot)) \in L^\infty(\Omega; C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  such that:

$$E \int_0^T \left\{ |p_k(t) - p(t)|^2 + |\Lambda_k(t) - \Lambda(t)|^2 \right\} dt \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Moreover, there are constants  $0 < \delta \leq m < \infty$  and a process  $\bar{\Lambda}(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  such that:

$$\begin{cases} \delta \leq p_k(\cdot), p(\cdot) \leq m \\ |\Lambda_k(\cdot)| \leq |\bar{\Lambda}(\cdot)|, \quad a.e. t \in [0, T], \mathbb{P} - a.s. \end{cases} \quad (40)$$

**Proof:** We consider each component,  $\{p_k(\cdot)\}$  and  $\{\Lambda_k(\cdot)\}$ , of the sequence  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$  separately. Since  $\{p_k(\cdot)\}$  is generated by (38), it follows from Proposition 5.3 that  $\{p_k(\cdot)\}$  is monotonically increasing and satisfies  $\delta \leq p_k(\cdot) \leq m$  for constants  $0 < \delta \leq m < \infty$ . Therefore:

$$p(t) := \lim_{k \uparrow \infty} p_k(t)$$

is well defined and  $\delta \leq p(\cdot) \leq m$ . Hence, it follows from the Bounded Convergence Theorem that:

$$E \int_0^T |p_k(t) - p(t)|^2 dt \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Consider now the sequence  $\{\Lambda_k(\cdot)\}$  in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  generated by (39). It follows from Lemmas 8.2 and 8.3 in the Appendix that a subsequence, also denoted by  $\{\Lambda_k(\cdot)\}$ , and a process  $\Lambda(\cdot)$  can be found such that:

$$E \int_0^T |\Lambda_k(t) - \Lambda(t)|^2 dt \rightarrow 0 \text{ and } |\Lambda_k(\cdot)| \leq |\bar{\Lambda}(\cdot)|$$

for some  $\bar{\Lambda}(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ . ■

In summary, we have shown that whenever (37) is satisfied, it is possible to construct a sequence of square-integrable pairs  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$  using the iteration (39) which satisfies the boundedness properties (40) and has a limit point  $(p(\cdot), \Lambda(\cdot))$ .

**Proof of Theorem 5.1:**

The proof of Theorem 5.1 boils down to showing 2 things: (i) existence of a process  $p_0(\cdot)$  that satisfies (37) and (ii) that the limit point  $(p(\cdot), \Lambda(\cdot))$  in Proposition 5.3 is a solution of (31).

From Proposition (5.1), there exists a solution

$$(x(\cdot), q(\cdot)) \in L^\infty(\Omega; C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$$

of (33) such that  $\delta \leq x(\cdot) \leq m$  for some finite constants  $0 < \delta \leq m < \infty$ . Since  $0 \leq H(\cdot) \leq I$  and  $(-1/x(t))\Lambda(t)'\Lambda(t) \leq (-1/x(t))\Lambda(t)'H(t)\Lambda(t)$ , it follows from Theorem 5.3 that:

$$x(\cdot) \leq \Psi(x(\cdot));$$

that is, the sufficient condition (37) is satisfied by choosing  $p_0(\cdot) \equiv x(\cdot)$ . Let  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$  denote the subsequence of the sequence generated by (39) which satisfies the conditions (40) in Proposition 5.4. By construction, there is a sequence  $R_k(\cdot) \in L_{\mathcal{F}}^{\infty}(0, T; \mathbb{R})$  such that  $\delta \leq R_k(\cdot) \leq m$  with  $R_k(\cdot) \uparrow p(\cdot)$  a.s. and  $(p_k(\cdot), \Lambda_k(\cdot))$  is a solution of the BSDE:

$$p_k(t) = \gamma + \int_t^T \left( A(s)p_k(s) + B(s)\Lambda_k(s) - \frac{1}{R_k(s)}\Lambda_k(s)'H(s)\Lambda_k(s) \right) ds - \int_t^T \Lambda_k(s)'d\bar{W}(s).$$

Since  $(p_k(\cdot), \Lambda_k(\cdot)) \rightarrow (p(\cdot), \Lambda(\cdot))$  and  $R_k(\cdot) \uparrow p(\cdot)$ , it follows from the boundedness properties (40), the bounds  $\delta \leq R_k(\cdot) \leq m$  and the Dominated Convergence Theorem that we may let  $k \uparrow \infty$  to obtain:

$$p(t) = \gamma + \int_t^T \left( A(s)p(s) + B(s)\Lambda(s) - \frac{1}{p(s)}\Lambda(s)'H(s)\Lambda(s) \right) ds - \int_t^T \Lambda(s)'d\bar{W}(s).$$

That is,  $(p(\cdot), \Lambda(\cdot))$  is a solution of (31). ■

## 6 Computations

The following result is an immediate consequence of Theorem 5.1:

**Theorem 6.1** *Suppose that (A1) holds. Then the SRE (9) has a solution  $(p(\cdot), \Lambda(\cdot))$ . Moreover, if  $(\bar{p}(\cdot), \bar{\Lambda}(\cdot))$  and  $(p(\cdot), \Lambda(\cdot))$  are solutions of (9), then  $\bar{p}(\cdot) \equiv p(\cdot)$ .*

In this section, we derive closed form solutions, in terms of  $(p(\cdot), \Lambda(\cdot))$  and  $(h(\cdot), \eta(\cdot))$ , of Problems 1, 2 and 3.

### Quadratic hedging and pricing:

**Theorem 6.2** *Suppose that (A1) holds. Then the optimal portfolio for the quadratic hedging problem (5) is:*

$$\pi(t) = -(\sigma(t)^{-1})' \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] (x(t) - h(t)) + (\sigma(t)^{-1})' \eta_1(t) \quad (41)$$

and the value function is:

$$V(s, y) = p(0)(h(s) - y)^2 + E \int_s^T p(t)\eta_2(t)'\eta_2(t)dt. \quad (42)$$

The mean-variance price of the claim is given by  $\phi(t) = h(t)$ .

To understand the optimal portfolio (41), consider the following equivalent formulation of the BSDE (10):

$$\begin{cases} dh(t) = \left\{ r(t)h(t) + b(t) \left[ (\sigma(t)^{-1})' \eta_1(t) \right] - \frac{\Lambda_2(t)'}{p(t)} \eta_2(t) \right\} dt \\ \quad + \left[ (\sigma(t)^{-1})' \eta_1(t) \right]' \sigma(t) dW(t) + \eta_2(t)' dB(t), \\ h(T) = \xi. \end{cases} \quad (43)$$

It follows that  $h(\cdot)$  is the wealth process of the replicating portfolio for the claim  $\xi$  in a fictitiously completed market with market price of risk:

$$\begin{bmatrix} \theta(t) \\ \nu(t) \end{bmatrix} = \begin{bmatrix} (\sigma(t)^{-1})b(t)' \\ -\frac{\Lambda_2(t)}{p(t)} \end{bmatrix}.$$

(For more details about fictitious completion, the reader may consult Karatzas and Shreve (1999)). The associated replicating portfolio is:

$$\pi(t) = \begin{bmatrix} (\sigma(t)^{-1})'\eta_1(t) \\ \eta_2(t) \end{bmatrix}, \quad (44)$$

where  $(\sigma(t)^{-1})'\eta_1(t)$  represents the investment in assets available to the investor in the incomplete market, while  $\eta_2(t)$  is the investment in the fictitious assets. Next, consider an investor with wealth  $y(t)$  whose aim is to minimize  $Ey(T)^2$  in the incomplete market. The optimal portfolio for this investor is:

$$\pi(t) = -(\sigma(t)^{-1})' \left[ b(t)' + \frac{\Lambda_1(t)}{p(t)} \right] y(t) \quad (45)$$

which can be seen by putting  $\xi \equiv 0$  in (5) which implies that  $(h(\cdot), \eta(\cdot)) \equiv (0, 0)$  in (41). Therefore, the optimal portfolio (41) can be interpreted in the following way. The investor mimics the replicating portfolio (44) as best as he/she can, which amounts to investing  $(\sigma(t)^{-1})'\eta_1(t)$  in the available assets.  $\eta_2(\cdot)$  represents investment in fictitious assets that are unavailable to the investor in the incomplete market and for this reason, the replicating portfolio (44) can not be followed perfectly, resulting in a discrepancy of  $x(t) - h(t)$  between the wealth  $x(t)$  and the value of the replicating portfolio  $h(t)$ . The second component in the optimal portfolio tries to minimize this error by investing  $x(t) - h(t)$  according to (45). Similar interpretations for quadratic hedging problems with deterministic parameters are given in Kohlmann and Zhou (2000).

### Mean-variance portfolio selection:

We shall say that the mean-variance problem (7) is *feasible for a given*  $c \in \mathbb{R}$  if there is a portfolio  $\pi(\cdot) \in \mathcal{U}$  which satisfies  $Ex(T) = c$ . The following result gives necessary and sufficient conditions for feasibility of (7) for every  $c \in \mathbb{R}$ .

**Proposition 6.1** *Suppose that (A1) holds. Let  $(\psi(\cdot), \xi(\cdot))$  be the unique solution of the BSDE:*

$$\begin{cases} d\psi(t) = -r(t)\psi(t)dt + \xi_1(t)'dW(t) + \xi_2(t)'dB(t), \\ \psi(T) = 1. \end{cases} \quad (46)$$

*The mean-variance problem (7) is feasible for every  $c \in \mathbb{R}$  if and only if*

$$E \int_0^T |\psi(t)b(t)' + \sigma(t)\xi_1(t)|^2 dt > 0. \quad (47)$$

**Proof:** To prove sufficiency, let  $u(\cdot) \in \mathcal{U}$  be admissible and  $\pi(t) = \lambda u(t)$  for some  $\lambda \in \mathbb{R}$ . Let  $x(\cdot)$  be the solution of (3) corresponding to  $\pi(\cdot)$ . Then  $Ex(T) = Ev(T) + \lambda Ez(T)$  where:

$$\begin{cases} dv(t) = r(t)v(t)dt, \\ v(0) = y, \end{cases} \quad (48)$$

$$\begin{cases} dz(t) = [r(t)z(t) + b(t)u(t)]dt + u(t)'\sigma(t)dW(t), \\ z(0) = 0. \end{cases}$$

Also, it can be shown (see p. 353 of Yong and Zhou (2000)) that if  $z(\cdot)$  is the solution of (48), then

$$Ez(T) = E \int_0^T [\psi(t)b(t)' + \sigma(t)\xi_1(t)]' u(t) dt. \quad (49)$$

Clearly, if  $Ez(T) \neq 0$  for some  $u(\cdot) \in \mathcal{U}$ , then for any given  $c \in \mathbb{R}$  we can satisfy  $Ex(T) = c$  with an appropriate choice of  $\lambda$ . In particular, if  $u(t) = \psi(t)b(t)' + \sigma(t)\xi_1(t)$  and (47) is true, then (49) implies that  $Ez(T) \neq 0$  for this choice of  $u(\cdot)$  and (7) is feasible.

Conversely, suppose that (7) is feasible for every  $c \in \mathbb{R}$ . Then, for every  $c$ , there exists a  $u(\cdot) \in \mathcal{U}$  such that  $Ex(T) = Ev(T) + Ez(T) = c$ . Since  $Ev(T)$  is independent of  $u(\cdot)$ , it follows that there exists a  $u(\cdot)$  such that  $Ez(T) \neq 0$ . Hence, it follows from the representation (49) that (47) is true.  $\blacksquare$

The necessary and sufficient condition (47) for feasibility of (7) is extremely mild. For example, it is automatically satisfied if  $b(\cdot) \neq 0$ .

In the case of the mean-variance problem, we can replace the BSDE (10) for  $(h(\cdot), \eta(\cdot))$  by the following:

$$\begin{cases} dg(t) = \left[ r(t)g(t) + \theta(t)'q_1(t) - \frac{\Lambda_2(t)}{p(t)}q_2(t) \right] dt + q_1(t)'dW(t) + q_2(t)'dB(t), \\ g(T) = 1. \end{cases} \quad (50)$$

**Theorem 6.3** *Suppose that (A1) holds and (47) is satisfied. Then the mean-variance problem (7) is feasible for every  $c \in \mathbb{R}$ . The inequality  $1 - M - p(0)g(0)^2 > 0$  is satisfied, and the constants:*

$$M := E \int_0^T p(t)\eta_2(t)' \eta_2(t) dt,$$

$$k := \frac{c - p(0)g(0)y}{1 - M - p(0)g(0)^2},$$

are well defined. The efficient portfolio that gives a return  $Ex(T) = c$  is:

$$\pi(t) = -(\sigma(t)^{-1})' \left[ \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right] (x(t) - kg(t)) + k(\sigma(t)^{-1})' q_1(t) \quad (51)$$

and:

$$\text{Var } x(T) = \frac{M + p(0)g(0)^2}{1 - M - p(0)g(0)^2} \left[ Ex(T) - \frac{p(0)g(0)}{M + p(0)g(0)^2} y \right]^2 + \frac{Mp(0)}{M + p(0)g(0)^2} \quad (52)$$

is the efficient frontier.

**Proof:** It will be convenient for us to do the calculations using the following equivalent formulation of (7):

$$\left\{ \begin{array}{l} J^* := \min_{u(\cdot) \in \mathcal{U}} E \frac{1}{2} [x(T) - c]^2, \\ \text{Subject to:} \\ Ex(T) = c. \end{array} \right. \quad (53)$$

Since the problem (53) is feasible, has a convex cost which is bounded below, and linear constraints, it follows from Luenberger (1968) that:

$$J^* = \max_{\lambda \in \mathbb{R}} \inf_{\pi(\cdot) \in \mathcal{U}} J(\pi(\cdot), \lambda) < \infty \quad (54)$$

where:

$$J(\pi(\cdot), \lambda) = E \frac{1}{2} [x(T) - c]^2 + \lambda E [x(T) - c] = E \frac{1}{2} [x(T) - (\lambda - c)]^2 - \frac{1}{2} \lambda^2.$$

For every fixed  $\lambda$ , the unconstrained problem:

$$J(\lambda) = \min_{\pi(\cdot) \in \mathcal{U}} J(\pi(\cdot), \lambda)$$

from (54) is a quadratic hedging problem of the form (5) (with  $\xi = \lambda - c$ , deterministic). Therefore, it follows from Theorem 6.2 that

$$\begin{aligned} J(\lambda) &= \frac{1}{2} p(0)(h(0) - y)^2 + \frac{1}{2} E \int_0^T p(t) \eta_2(t)' \eta_2(t) dt - \frac{1}{2} \lambda^2 \\ &= \frac{1}{2} \left\{ p(0) [(c - \lambda)g(0) - y]^2 + (c - \lambda)^2 \int_0^T p(t) \eta_2(t)' \eta_2(t) dt - \lambda^2 \right\} \end{aligned} \quad (55)$$

and

$$\pi(t) = (\sigma(t)^{-1})' \left[ \left( \theta(t) + \frac{\Lambda_1(t)}{p(t)} \right) \left( (c - \lambda)g(t) - x(t) \right) + (c - \lambda)q_1(t) \right]. \quad (56)$$

Rearranging (55), we obtain:

$$J(\lambda) = -\frac{1}{2} [1 - M - p(0)g(0)^2] (c - \lambda)^2 + [c - p(0)g(0)y] (c - \lambda) + \frac{1}{2} [p(0)y^2 - c^2]. \quad (57)$$

Firstly, since (57) is quadratic in  $\lambda$  and

$$J^* = \max_{\lambda \in \mathbb{R}} J(\lambda) \quad (58)$$



is finite, it follows that  $1 - M - p(0)g(0)^2 \leq 0$ . If  $1 - M - p(0)g(0)^2 = 0$ , however, then  $J^*$  can only be finite if  $c - p(0)g(0)y = 0$  for every  $c \in \mathbb{R}$ . This is a contradiction, so it must be the case that strict inequality holds, as claimed.

Rearranging (57), we obtain:

$$\begin{aligned} J(\lambda) &= -\frac{1}{2}[1 - M - p(0)g(0)^2] \left[ c + \frac{c - p(0)g(0)y}{M + p(0)g(0)^2 - 1} - \lambda \right]^2 \\ &\quad + \frac{1}{2} \frac{[c - p(0)g(0)y]^2}{1 - M - p(0)g(0)^2} + \frac{1}{2}[p(0)y^2 - c^2], \\ &= -\frac{1}{2}[1 - M - p(0)g(0)^2] \left[ c + \frac{c - p(0)g(0)y}{M + p(0)g(0)^2 - 1} - \lambda \right]^2 \\ &\quad + \frac{1}{2} \left\{ \frac{M + p(0)g(0)^2}{1 - M - p(0)g(0)^2} \left[ c - \frac{p(0)g(0)}{M + p(0)g(0)^2} y \right]^2 + \frac{Mp(0)}{M + p(0)g(0)^2} \right\}. \end{aligned}$$

Therefore, the optimal  $\lambda$  and  $J^*$  are given by:

$$\begin{aligned} \lambda^* &= c + \frac{c - p(0)g(0)y}{M + p(0)g(0)^2 - 1}, \\ J^* &= \frac{1}{2} \left\{ \frac{M + p(0)g(0)^2}{1 - M - p(0)g(0)^2} \left[ c - \frac{p(0)g(0)}{M + p(0)g(0)^2} y \right]^2 + \frac{Mp(0)}{M + p(0)g(0)^2} \right\}. \end{aligned}$$

Substituting  $\lambda^*$  into (56) gives the expression (51) for the optimal portfolio. The equation for (52) comes from the observation that  $J^* = \min_{\pi(\cdot)} (1/2)\text{Var } x(T)$  and  $Ex(T) = c$ .  $\blacksquare$

The *Mutual Fund Theorem*, originally due to Tobin (1958) for the single-period problem, is a natural consequence of the mean-variance theory, and is the foundation of the *Capital Asset Pricing Model* (CAPM); see Sharpe (1964). The following result is a generalization of the Mutual Fund Theorem to the continuous time, incomplete market, random coefficient setting.

**Theorem 6.4 (Mutual Fund Theorem)** *Let  $\pi_1(\cdot)$  and  $\pi_2(\cdot)$  be distinct efficient portfolios. Then  $\bar{\pi}(\cdot)$  is efficient if and only if  $\bar{\pi}(\cdot) = \alpha\pi_1(\cdot) + (1 - \alpha)\pi_2(\cdot)$  for some  $\alpha \in (-\infty, \infty)$ .*

**Proof:** Let  $c_i$  denote the rate of return corresponding to the efficient portfolio  $\pi_i(\cdot)$ , and

$$k_i = \frac{c_i - p(0)g(0)y}{1 - M - p(0)g(0)^2},$$

for  $i = 1, 2$ . For any  $\alpha \in (-\infty, \infty)$ , let:

$$\begin{aligned} \bar{c} &= \alpha c_1 + (1 - \alpha)c_2, \\ \bar{k} &= \frac{\bar{c} - p(0)g(0)y}{1 - M - p(0)g(0)^2} = \alpha k_1 + (1 - \alpha)k_2, \end{aligned}$$

and  $\bar{\pi}(\cdot)$  be the efficient portfolio corresponding to  $\bar{c}$ . Since (8) is linear, it follows that:

$$\bar{\pi}(\cdot) \equiv \alpha\pi_1(\cdot) + (1 - \alpha)\pi_2(\cdot). \quad (59)$$

Therefore, for any  $\alpha \in (-\infty, \infty)$ ,  $\alpha\pi_1(\cdot) + (1 - \alpha)\pi_2(\cdot)$  is efficient. Conversely, let  $\bar{c}$  be given and  $\bar{\pi}(\cdot)$  the associated efficient portfolio. Since  $\bar{c} = \alpha c_1 + (1 - \alpha)c_2$  for appropriately chosen  $\alpha$ , it follows that  $\bar{\pi}(\cdot)$  can be written in the form (59).  $\blacksquare$

## 7 Partial differential equations

It is clear from Theorems 6.2 and 6.3 that optimal portfolios for the quadratic hedging and mean-variance problems can be determined by finding numerical solutions of the SRE (9) and the linear BSDE (10). (The BSDE (50) is obviously a special case of (10)). The linear BSDE (10) gives the wealth process and replicating portfolio for the claim  $\xi$  in a fictitiously completed market where the market price of risk associated with the fictitious assets,  $-\Lambda_2(t)/p(t)$ , is determined by the solution of the SRE. If a numerical solution for the SRE can be determined, then various methods (for example, simulation) can be used to solve (10) numerically. Unfortunately, little is known in the general case about finding numerical solutions of the SRE (9). When the parameters are Markovian, however, numerical solutions can be found via partial differential equations (PDEs); see El Karoui, Peng and Quenez (1997) and Yong and Zhou (1999). In this section, we illustrate how the PDEs associated with the SRE can be derived. (We assume implicitly that the technical assumptions required for the SRE and the associated PDE to have unique solutions are satisfied). Similar equations are obtained in Laurent and Pham (1999) using a dynamic programming argument. In comparison, this derivation via the SRE is elementary and uses little more than Ito's formula; see also Kohlmann and Tang (2001b).

Consider the following SDE:

$$\begin{cases} dy(t) = \alpha(t, y(t)) dt + \beta(t, y(t)) d\bar{W}(t), \\ y(0) = \bar{y}. \end{cases} \quad (60)$$

Suppose that:

$$r(t, \omega) = r(t, y(t)), \quad \mu(t, \omega) = \mu(t, y(t)), \quad \sigma(t, \omega) = \sigma(t, y(t)).$$

(That is,  $r(\cdot)$ ,  $\mu(\cdot)$  and  $\sigma(\cdot)$  are known functions of the process  $y(\cdot)$ ). By assuming the functional form  $p(t, \omega) = Z(t, y(t))$  for the solution of the SRE (9), we obtain (by applying Ito's formula to  $Z(t, y(t))$  and equating coefficients with (9)) the following linear parabolic PDE:

$$\begin{cases} Z_t + \alpha(t, y)Z_y + \frac{1}{2}\beta(t, y)^2Z_{yy} + (2r - |\theta|^2)(t, y) = 0, & y \in \mathbb{R}, 0 \leq t < T, \\ Z(T, y) = 1. \end{cases} \quad (61)$$

The solution of (9) is given by:

$$\begin{cases} p(t, \omega) = Z(t, y(t)) \\ \begin{bmatrix} \Lambda_1(t, \omega) \\ \Lambda_2(t, \omega) \end{bmatrix} = \begin{bmatrix} 0 \\ Z_y(t, y(t)) \beta(t, y(t)) \end{bmatrix}. \end{cases}$$

Several remarks are in order. Firstly, the stochastic volatility models of Hull and White (1987), Stein and Stein (1991), Heston (1993) can be written in the form (60). There is a slight complication, for instance, in that the stochastic volatility model of Heston (1993) where  $\sigma(t, y(t)) = \sqrt{y(t)}$ . In this case,  $\sigma(\cdot)$  is unbounded, so strictly speaking, the existence result in Theorem 6.1 does not apply. It remains an open question whether the existence results in the paper can be extended to the case when parameters in (9) may be unbounded. Secondly, we have assumed that  $B(\cdot)$  is the driving Brownian motion in (60) simply for the sake of convenience. In particular, the PDE (61) is linear in this situation. The derivation of (61) can easily be extended to the case when (60) is driven by  $\bar{W}(t) = (W(t), B(t))$ . Finally, an alternative approach to portfolio selection with stochastic volatility (also using the model (60)) is presented in Jonsson and Sircar (2000), which studies the problem of downside risk minimization in an incomplete market using the Hamilton-Jacobi-Bellman PDE and duality. Their approach leads to a nonlinear PDE for which approximate solutions are determined under the assumption that the volatility is fast mean-reverting.

## 8 Conclusion

In this paper, we have studied the closely related problems of quadratic hedging and pricing, and mean-variance portfolio selection in an incomplete market under the assumption that asset prices are geometric Brownian motion and parameters are random processes. Our approach takes the perspective of linear-quadratic optimal control, focusing on the so-called *stochastic Riccati equation* (SRE) associated with this problem. The SRE is a nonlinear, singular BSDE, the drift of which is neither Lipschitz continuous nor linearly growing in the state variables. For this reason, the standard existence and uniqueness results of Pardoux and Peng (1990) do not apply, and alternative methods need to be used to establish solvability. Our main theoretical contribution is a proof of existence and uniqueness of solutions of the SRE associated with the quadratic hedging and mean-variance problems in an incomplete market with random parameters. The proof uses the existence and comparison results of Lepeltier and San Martin (1998) and Kobylanski (2000) for nonlinear BSDEs with drifts that may be quadratically growing in the gains process, and may be viewed as a generalization of these results to a class of singular equations. Closed form expressions for the optimal portfolio and efficient frontier are obtained in terms of the solution of the SRE and a second linear BSDE. The linear BSDE gives the wealth process and replicating portfolio of the investor's liability in a fictitiously completed market, while the solution of the SRE gives the *market price of risk* for the fictitious assets. In addition, the SRE may be associated with the so-called *Variance Optimal Martingale Measure* of Delbaen and Schachermayer (1996) and Schweizer (1996). Finally, when the parameters are Markovian, numerical solutions of the SRE can be obtained by solving partial differential

equations (PDEs). In the case of the stochastic volatility models of Heston (1993), Hull and White (1987) and Stein and Stein (1991), these PDEs turn out to be linear.

# Appendix

The results in this section, which build on the methods of Lepeltier and San Martin (1998) and Kobylanski (2000), are obtained using the notion of *weak convergence* (in the functional analytic sense). For further details, the reader may consult Yosida (1965).

## Weak convergence:

A sequence  $\{Z_k(\cdot)\}$  in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  converges weakly to  $Z(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  if:

$$E \int_0^T f(t)' Z_k(t) dt \rightarrow E \int_0^T f(t)' Z(t) dt \text{ as } k \rightarrow \infty \quad (62)$$

for all  $f(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ . This is commonly denoted by  $Z_k \rightharpoonup Z(\cdot)$ . Weak limits are also unique. We also have the following result (see Yosida (1965)).

**Proposition 8.1** *Let  $\{Z_k(\cdot)\}$  be a sequence in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  which is norm bounded; that is,*

$$E \int_0^T |Z_k(t)|^2 dt < c, \quad \forall k \in \mathbb{Z}^+,$$

*for some constant  $c < \infty$ . Then, there is a subsequence  $\{Z_{n_j}\}$  which converges weakly to an element  $Z(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ .*

Finally, *strong convergence* of  $\{Z_k(\cdot)\}$  to  $Z(\cdot)$  in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  means:

$$E \int_0^T |Z_k(t) - Z(t)|^2 dt \rightarrow 0 \text{ as } k \rightarrow \infty.$$

It is easy to show that strong convergence implies weak convergence.

## Technical results:

**Lemma 8.1** *Suppose that condition (37) holds. Let  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$  be generated by (39) and  $\{\Lambda_k(\cdot)\}$  the sequence formed by taking the second component of each pair. Then there exists an element  $\Lambda(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  and a subsequence of  $\{\Lambda_k(\cdot)\}$ , denoted by  $\{\Lambda_{n_j}(\cdot)\}$ , such that:*

$$\Lambda_{n_j}(\cdot) \rightharpoonup \Lambda(\cdot) \text{ weakly in } L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d}).$$

**Proof:** By Proposition 8.1, we need only prove norm boundedness of  $\{\Lambda_k(\cdot)\}$ . By construction,  $(p_k(\cdot), \Lambda_k(\cdot))$  is a solution of the BSDE:

$$\begin{cases} dp_k(t) = -F_k(t, p_k(t), \Lambda_k(t))dt + \Lambda_k(t)' d\bar{W}(t), \\ p_k(T) = \xi, \end{cases}$$

where

$$F_k(t, p, \Lambda) := A(t)p + B(t)\Lambda - \frac{1}{p_{k-1}(t)}\Lambda'H(t)\Lambda.$$

Since  $\delta \leq p_k(\cdot) \leq m$  for fixed constants  $0 < \delta \leq m < \infty$ , uniformly in  $k$  (see Proposition 5.3), there are finite constants  $\alpha$  and  $C$  such that:

$$|F_k(t, p, \Lambda)| \leq \alpha + C|\Lambda|^2, \quad \forall \delta \leq p \leq m \text{ and } \Lambda(\cdot) \in \mathbb{R}^{m+d}. \quad (63)$$

Let

$$\psi(x) := e^{3Cx}.$$

By Ito's formula, we obtain:

$$\begin{aligned} & \psi(p_{k+1}(0)) + E \int_0^T \frac{1}{2} \psi''(p_k(t)) |\Lambda_k(t)|^2 dt \\ &= E\psi(\xi) + E \int_0^T \psi'(p_k(t)) F_k(t, p_k(t), \Lambda_k(t)) dt, \\ &\leq E\psi(\xi) + E \int_0^T \psi'(p_k(t)) (\alpha + C|\Lambda_k(t)|^2) dt, \end{aligned}$$

where the inequality is an immediate consequence of (63). Rearranging, we obtain:

$$\begin{aligned} & E \int_0^T \left[ \left( \frac{1}{2} \psi'' - C\psi' \right) (p_k(t)) \right] |\Lambda_k(t)|^2 dt \\ &\leq E\psi(\xi) + \alpha E \int_0^T \psi'(p_k(t)) dt \\ &\leq E\psi(\xi) + \alpha T 3C e^{3Cm}, \end{aligned} \quad (64)$$

the second inequality being a consequence of

$$\psi'(x) = 3C e^{3Cx}.$$

Finally, since:

$$\left( \frac{1}{2} \psi'' - C\psi' \right) (x) = \frac{3}{2} C^2 e^{3Cx}$$

it follows that

$$\left( \frac{1}{2} \psi'' - C\psi' \right) (p_k(t)) \geq \frac{3}{2} C^2 e^{3C\delta}$$

and we obtain from (64) that:

$$\frac{3}{2} C^2 e^{3C\delta} \|\Lambda_k(\cdot)\|_2^2 \leq E\psi(\xi) + \alpha T 3C e^{3Cm}.$$

Therefore,  $\{\Lambda_k(\cdot)\}$  is bounded in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ . ■

**Lemma 8.2** *Assume that condition (37) is satisfied. Let  $\{\Lambda_{n_j}(\cdot)\}$  be the weakly convergent subsequence from Lemma 8.1 with weak limit  $\Lambda(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ . Then:*

$$E \int_0^T |\Lambda_{n_j}(t) - \Lambda(t)|^2 dt \rightarrow 0 \text{ as } j \rightarrow \infty.$$

**Proof:** To ease the notation, we shall, throughout this proof, denote elements of the subsequence  $\{(p_{n_j}(\cdot), \Lambda_{n_j}(\cdot))\}$  from Lemma 8.1 by  $(p_j(\cdot), \Lambda_j(\cdot))$ .

Let

$$\psi(x) := \frac{e^{16Cx} - 1}{16C} - x.$$

Let  $k \in \mathbb{Z}^+$  be fixed and  $j \geq k$ . Therefore, we have:

$$\begin{cases} d(p_j(t) - p_k(t)) = -\left[F_j(t, p_j(t), \Lambda_j(t)) - F_k(t, p_k(t), \Lambda_k(t))\right] dt + (\Lambda_j(t) - \Lambda_k(t))' d\bar{W}(t), \\ (p_j - p_k)(T) = 0. \end{cases}$$

From Ito's formula:

$$\begin{aligned} & \psi(p_j(0) - p_k(0)) + E \int_0^T \frac{1}{2} \psi''(p_j(t) - p_k(t)) |\Lambda_j(t) - \Lambda_k(t)|^2 dt \\ &= E \int_0^T \psi'(p_j(t) - p_k(t)) \left( F_j(t, p_j(t), \Lambda_j(t)) - F_k(t, p_k(t), \Lambda_k(t)) \right) dt \\ &\leq E \int_0^T \psi'(p_j(t) - p_k(t)) \left[ 2\alpha + C|\Lambda_j(t)|^2 + C|\Lambda_k(t)|^2 \right] dt \\ &\leq E \int_0^T \psi'(p_j(t) - p_k(t)) \left[ 2\alpha + 3C|\Lambda_j(t) - \Lambda_k(t)|^2 + 5C \left( |\Lambda_k(t) - \Lambda(t)|^2 + |\Lambda(t)|^2 \right) \right] dt \end{aligned}$$

where the first inequality follows from the fact that  $\psi'(p_j(t) - p_k(t)) \geq 0$  and (63). Rearranging, together with the observation that  $\psi(p_j(0) - p_k(0)) \geq 0$ , we get:

$$\begin{aligned} & E \int_0^T \left[ \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p_j(t) - p_k(t)) \right] |\Lambda_j(t) - \Lambda_k(t)|^2 dt \\ & \leq E \int_0^T \psi'(p_j(t) - p_k(t)) \left[ 2\alpha + 5C \left( |\Lambda(t) - \Lambda_k(t)|^2 + |\Lambda(t)|^2 \right) \right] dt. \end{aligned} \quad (65)$$

Consider next the integrand on the LHS of (65). Since  $\{p_j(\cdot)\}$  is uniformly bounded and  $p_j(\cdot) \rightarrow p(\cdot)$  for a.e.  $t \in [0, T]$ ,  $\mathbb{P}$ -a.s., it follows from the relation:

$$\left( \frac{1}{2} \psi'' - 3C\psi' \right) (x) = 5C e^{16Cx} + 3C$$

and the Dominated Convergence Theorem that:

$$\left( \frac{1}{2} \psi'' - 3C\psi' \right) (p_j(\cdot) - p_k(\cdot)) \rightarrow \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p(\cdot) - p_k(\cdot))$$

strongly in  $L^2_{\mathcal{F}}(0, T; \mathbb{R})$  as  $n \rightarrow \infty$ . This fact together with the uniform boundedness of the sequence:

$$\left\{ \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p_j(\cdot) - p_k(\cdot)) \right\}_j$$

(parameterized by  $j$ ) implies that:

$$\left[ \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p_j(\cdot) - p_k(\cdot)) \right]^{\frac{1}{2}} |\Lambda_j(\cdot) - \Lambda_k(\cdot)| \rightharpoonup \left[ \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p(\cdot) - p_k(\cdot)) \right]^{\frac{1}{2}} |\Lambda(\cdot) - \Lambda_k(\cdot)|$$

(weakly) in  $L^2_{\mathcal{F}}(0, T; \mathbb{R})$  as  $n \rightarrow \infty$ . Hence, it follows that:

$$\begin{aligned} & E \int_0^T \left[ \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p(t) - p_k(t)) \right] |\Lambda(t) - \Lambda_k(t)|^2 dt \\ & \leq \liminf_j E \int_0^T \left[ \left( \frac{1}{2} \psi'' - 3C\psi' \right) (p_j(t) - p_k(t)) \right] |\Lambda_j(t) - \Lambda_k(t)|^2 dt \\ & \leq \liminf_j E \int_0^T \psi'(p_j(t) - p_k(t)) \left[ 2\alpha + 5C \left( |\Lambda(t) - \Lambda_k(t)|^2 + |\Lambda(t)|^2 \right) \right] dt \\ & = E \int_0^T \psi'(p(t) - p_k(t)) \left[ 2\alpha + 5C \left( |\Lambda(t) - \Lambda_k(t)|^2 + |\Lambda(t)|^2 \right) \right] dt \end{aligned} \quad (66)$$

where the first inequality follows from the observation that if  $\{Y_j\}$  is a weakly convergent sequence with weak limit  $Y$ , then

$$\|Y\|_2 \leq \liminf_j \|Y_j\|_2,$$

the second inequality comes from (65), and the equality is a consequence of the Dominated Convergence Theorem. Rearranging (66), we have:

$$\begin{aligned} & E \int_0^T \left[ \left( \frac{1}{2} \psi'' - 8C\psi' \right) (p(t) - p_k(t)) \right] |\Lambda(t) - \Lambda_k(t)|^2 dt \\ & \leq E \int_0^T \psi'(p(t) - p_k(t)) \left[ 2\alpha + 5C|\Lambda(t)|^2 \right] dt \end{aligned}$$

and the observation that  $(1/2\psi'' - 8C\psi')(x) = 8C$  implies that:

$$8C \|\Lambda(\cdot) - \Lambda_k(\cdot)\|_2^2 \leq E \int_0^T \psi'(p(t) - p_k(t)) \left[ 2\alpha + 5C|\Lambda(t)|^2 \right] dt.$$

Finally, the Dominated Convergence Theorem implies that  $\Lambda_k(\cdot) \rightarrow \Lambda(\cdot)$  (strongly) in  $L^2_{\mathcal{F}}(0, T; R)$ .

■

Lemma 8.2 and Proposition 5.4 show that the iteration defined through (39) can be used to construct a sequence  $\{(p_k(\cdot), \Lambda_k(\cdot))\}$  from which an  $(L^2_{\mathcal{F}} \times L^2_{\mathcal{F}})$  strongly convergent subsequence  $\{(p_{n_j}(\cdot), \Lambda_{n_j}(\cdot))\}$  (with limit  $(p(\cdot), \Lambda(\cdot))$ ) can be extracted. The following result plays an important role in characterizing the limit  $(p(\cdot), \Lambda(\cdot))$  as a solution of (31).



**Lemma 8.3** *Suppose that  $\Lambda_k(\cdot) \rightarrow \Lambda(\cdot)$  (strongly) in  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ . Then there is a subsequence  $\{\Lambda_{n_j}(\cdot)\}$  such that  $|\Lambda_{n_j}(\cdot)| \leq |\bar{\Lambda}(\cdot)|$  for some  $\bar{\Lambda}(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$ .*

**Proof:** Since  $\|\Lambda_k(\cdot) - \Lambda(\cdot)\|_2 \rightarrow 0$  there is a subsequence  $\{\Lambda_{n_j}(\cdot)\}$  such that:

$$\|\Lambda_{n_j}(\cdot) - \Lambda(\cdot)\|_2 \leq \frac{1}{2^j}, \quad \forall j.$$

Define:

$$g(t) := |\Lambda_{n_0}(t)| + \sum_{j=0}^{\infty} |\Lambda_{n_{j+1}}(t) - \Lambda_{n_j}(t)|.$$

Clearly:

$$\|g(\cdot)\|_2 \leq \|\Lambda_{n_0}(\cdot)\|_2 + \sum_{j=0}^{\infty} \frac{1}{2^j} < \infty$$

so  $g(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ . On the other hand:

$$|\Lambda_{n_k}(t)| \leq |\Lambda_0(t)| + \sum_{j=0}^{k-1} |\Lambda_{n_{j+1}}(t) - \Lambda_{n_j}(t)| \leq g(t), \quad \forall k \in \mathbb{Z}^+$$

so

$$\bar{\Lambda}(t) := \sup_j |\Lambda_{n_j}(t)| \leq g(t).$$

Clearly,  $\bar{\Lambda}(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{m+d})$  and  $|\Lambda_{n_j}(t)| \leq |\bar{\Lambda}(t)|$ . ■

## References

- Anderson, B.D.O. and Moore, J.B. (1989) *Optimal Control: Linear Quadratic Methods*, Prentice Hall, New Jersey, 1989.
- Bismut, J.M. (1976). Linear quadratic optimal stochastic control with random coefficients. *SIAM J. Contr. Optim.*, **36**, 419 – 414.
- Chen, S. and Yong, J. (2000) Stochastic linear quadratic optimal control problems with random coefficients. *Chinese Ann. Math. Ser. B*, **21** 323 – 338.
- Cox, J.C. and Huang, C.F. (1989) Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Economic Theory*, **49** 33 – 83.
- Cvitanic, J. and Karatzas, I. (1992). Convex duality is constrained portfolio optimization. *Ann. Appl. Prob.* **2** 767 – 818.
- Delbaen, F. and Schachermayer, W. (1996). The variance-optimal martingale measure for continuous processes. *Bernoulli* **2** : **1** 81 – 105.
- Duffie, D. and Jackson, M. (1990). Optimal hedging and equilibrium in a dynamic futures market. *J. Econ. Dynam. Contr.* **14** 21 – 33.
- Duffie, D. and Richardson, H.M. (1991). Mean–variance hedging in continuous time. *Ann. Appl. Prob.* **1** 1 – 15.
- El Karoui, N., Peng, S. and Quenez, M.C. (1997). Backward stochastic differential equations in finance. *Math. Fin.* **7** 1 – 71.
- Follmer, H. and Sondermann, D. (1986). Hedging of non-redundant contingent claims. *Contributions to Mathematical Economics*, A. Mas-Colell and W. Hildenbrand ed., North Holland, Amsterdam 205 – 233.
- Gourieroux, C., Laurent, J.P. and Pham, H. (1998). Mean–variance hedging and numeraire. *Math. Fin.*, **8** : **3** 179 – 200.
- Harrison, J.M. and Pliska, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stoc. Proc. and Appl.*, **20**, 215 – 260.
- Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financial Studies*, **6**, 327 – 343.
- Hu, Y. and Zhou, X.Y. (2001). Indefinite stochastic Riccati equations. (Preprint).
- Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatilities. *J. Finance* **42**, 281 – 300. Karatzas, I., Lehoczky, J.P. and Shreve, S.E. (1987) Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM J. Contr. Optim.* **25** 1557 – 1586.

- Jonsson, M. and Sircar, K.R. (2000). Partial hedging in a stochastic volatility environment. (Preprint).
- Karatzas, I. and Shreve, S.E. (1999). *Methods of Mathematical Finance*, Springer-Verlag, New York.
- Kobylanski, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Annals of Probability*, 28 : 2, 558 – 602.
- Kohlmann, M. and Tang, S. (2001a). Optimal control of linear stochastic systems with singular costs, and the mean–variance hedging problem with stochastic market conditions. (Preprint).
- Kohlmann, M. and Tang, S. (2001b). Global adapted solution of linear stochastic systems with singular costs, and the mean-variance hedging problem with stochastic market conditions. (Preprint).
- Kohlmann, M. And Tang, S. (2001c). Global adapted solution of one-dimensional backward stochastic differential equations, with application to the mean-variance hedging. (Preprint).
- Kohlmann, M. and Zhou, X.Y. (2000). Relationship between backward stochastic differential equations and stochastic controls: A linear-quadratic approach. *SIAM J. Contr. Optim.* **38** 1392 – 1407.
- Laurent, J.P. and Pham, H. (1999). Dynamic programming and mean–variance hedging. *Finance and Stochastics* **3**, 83 – 110.
- Lepeltier J.-P. and San Martin, J. (1998). Existence for BSDE with superlinear–quadratic coefficient. *Stoch. Stoch. Rep.* **63** 227 – 240.
- Li, D. and Ng, W.L. (2000). Optimal dynamic portfolio selection: Multi-period mean–variance formulation. *Math. Fin.* **10** 387 – 406.
- Lim, A.E.B. and Zhou, X.Y. (2001a). Linear–quadratic control of backward stochastic differential equations: Random parameter case. Working paper.
- Lim, A.E.B. and Zhou, X.Y. (2001b). Mean–variance portfolio selection with random parameters in a complete market. To appear in *Math. Oper. Res.*.
- Luenberger, D.G. (1968). *Optimization by Vector Space Methods*, John Wiley, New York.
- Markowitz, H. (1952). Portfolio selection. *J. of Fin.*, **7**, 77 – 91.
- Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investment*, John Wiley & Sons, New York.
- Pardoux, E. and Peng, S. (1990). Adapted solution of backward stochastic equation. *Syst. & Contr. Lett.* **14** 55 – 61.

- Peng, S. (1992). Stochastic Hamilton-Jacobi-Bellman equations, *SIAM J. Contr. Optim.* **30** 284 – 304.
- Pliska, S.R. (1986). A stochastic calculus model of continuous trading: Optimal portfolios. *Math. Oper. Res.*, **11**, 371 – 382.
- Rami, M.A., Chen, X., Moore, J.B. and Zhou, X.Y. (2001). Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls. *IEEE Trans. Automat. Contr.*, **46** : **3** 428 – 440.
- Schweizer, M. (1996). Approximation pricing and the variance-optimal martingale measure. *Ann. Prob.* **64** 206 – 236.
- Sharpe, W.F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *J. of Fin.* **19** 425 – 442.
- Stein, E.M. and Stein, J.C. (1991). Stock price distributions with stochastic volatility: an analytic approach. *Rev. Financial Studies*, 727 – 752.
- Tobin, J. (1958). Liquidity preference as behavior toward risk. *Rev. Econ. Studies* **26** 65 – 86.
- Yong, J. and Zhou, X.Y. (1999). *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer, New York.
- Yosida, K. (1965). *Functional Analysis*, Springer-Verlag, Berlin.
- Zhou, X.Y. and Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Appl. Math. Optim.* **42** 19 – 33.