Anchored Preference Relations *

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Abstract

This paper axiomatically explores minimal conditions under which *reference dependent preferences* over risky prospects are normatively admissible. It is shown that two simple and intuitive conditions are sufficient to place strong requirements over such preferences. The first condition is tantamount to 'no-cycling' when the reference point is the status quo; the second condition requires the reference dependent representations to be continuous with respect to the reference point. In particular, these conditions rule out Cumulative Prospect Theory as well as any theory in which all reference dependent indifference surfaces are smooth − the latter case also holds for risk-less theories of the endowment effect (e.g., Tversky and Kahneman (1991)). It is also shown that one can construct satisfactory alternatives, axiomatically derived or otherwise, to Cumulative Prospect Theory as well as Tversky and Kahneman's (1991) theory of the risk-less endowment effect. The alternative theories I propose take the form of max-min representations over a set of expected (or Choquet-expected) utility differences, where utility difference is measured between the prospect evaluated and the reference point.

Keywords: Non-Expected Utility, Reference Dependence, Decision Theory, Incomplete Preference Relations, Utility Representation, Partial Orders, Anchoring, Status Quo Bias, Endowment Effect

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1 Introduction

Reference based preferences are not new to economics or decision theory. Empirical evidence, in the form of the endowment effect, loss aversion, framing, and differences between willingness to buy and willingness to sell, has led investigators as far back as 30 years ago to postulate reference effects[1]. Existing theoretical models of reference dependence in the context of risky choice[2], known as rank and sign dependent models, have been largely motivated by the clear evidence for loss aversion. These models ubiquitously specialize to preferences defined over wealth lotteries in which the agent assigns a value function to prospects using non-additive probability weights and a 'utility' function defined over wealth changes from some reference point. The nature of the reference point is not normatively elucidated, but in practice it is ubiquitously taken to be the agent's current wealth level or simply the status quo. Notable exceptions to the above strand of literature are Tversky and Kahneman (1991) and a recent paper by Masatlioglu and Ok (2002). Both papers consider the endowment effect in the context of risk-less choice with multiple commodities.

Intuitively, a decision maker with reference dependent preferences is one whose preferences over a set of choices vary with the context of the choice problem. One can describe the behavior of such an agent using a set of binary relations, $\{\succeq_e\}$, where e is an index for the 'context' - what I henceforth call an anchor. Following Tversky and Kahneman (1991) I assume that e is itself a choice primitive, though not necessarily always in the choice set available to the agent,[3] and that each of the $\succeq_e$'s has a continuous utility representation. An important, and yet open, question in decision theory is what conditions, if any, ought to be placed on the *relationship* between the different binary relations in $\{\succeq_e\}$, especially if one is to assume that e corresponds to the status quo. This paper is an attempt to shed light on this question. I do so by requiring that, whenever e is taken to be the agent's endowment or status quo, choice cycles and other potential for manipulation are explicitly ruled out. Cycling can be a potential problem in reference dependent theories, since an agent's choice generally affects her reference point.[4]

[1]See Morrison (1998), Bateman et. al. (1997), Chechile and Cooke (1997), Myagkov and Plott (1997), Eisenberger and Weber (1995), Dubourg et. al (1994), Shogren et. al. (1994), Tversky and Kahneman (1991), Kahneman, Knetsch and Thaler (1991), Samuelson and Zeckhauser (1988), Tversky and Kahneman (1986), Boyce et. al (1992), and MacCrimmon, Stanbury and Wehrung (1980) to mention but a few references.

[2]e.g., cumulative prospect theory and its variants axiomatized in Luce (1997), Luce and Fishburn (1991, 1995), Wakker and Tversky (1993, 1995) and Prelec (1998)

[3]This is contrasted with Masatlioglu and Ok (2002) who assume that the status quo is always an available choice.

[4]The theory described here is atemporal. One may therefore take issue with my emphasis on absence of choice

As it turns out, even by itself the 'no-cycling' condition results in serious restrictions over the set $\{\succeq_e\}$. For instance, it cannot be that each of the $\succeq_e$'s has an expected utility representation unless all the $\succeq_e$'s are identical (i.e., there is no reference dependence). In particular, unless the 'no-cycling' condition is violated, one cannot simply take a risk-free theory of the endowment effect (e.g., Tversky and Kahneman (1991)) and represent risk attitudes by expectations over 'loss-averse' value functions. Adding additional mild requirements on $\{\succeq_e\}$ (continuity with respect to the e's, or 'anchor continuity') gives sharper results: it is not possible for all the $\succeq_e$'s to be smooth (i.e., Fréchet differentiable) – so to be consistent with 'no-cycling' and 'anchor continuity', at least some of the relations must have kinks, leading to first order risk attitudes. The latter result also holds in a risk-less setting: members of the set of utility functions used by Tversky and Kahneman (1991) to model the risk-less endowment effect cannot all be smooth.

By far, the best known model for $\{\succeq_e\}$ is Cumulative Prospect Theory. One of the more surprising results derived here is that Cumulative Prospect Theory systematically violates a basic no-cycling criterion. Another result indicates that for preferences to generally exhibit loss aversion with respect to wealth outcomes, $\succeq_e$ must be kinked at e. Finally, I argue that when all $\succeq_e$ have kinks at e, the upper contour sets at e will be convex near e (i.e., all kinks 'face up').

These findings lead to a somewhat negative view of Cumlative Prospect Theory and other reference dependent models commonly used, but they do not say much about what a 'well behaved' representation of reference dependent preferences might look like. To shed more light on this question I impose a simple 'Independence' style axiom that results in a representation (Theorem 6) with a simple interpretation:

$$q \succeq_e p \;\Leftrightarrow\; \inf_{\psi\in Y} E_{q-e}[\psi] \geq \inf_{\psi\in Y} E_{p-e}[\psi]$$

where $E[\cdot]$ is an expectation operator, q and p are distributions (lotteries) over a compact metric space of payoffs, and the anchor, e, is also a distribution. Y is a set of von Neumann-Morgenstern utility functions. Note that when $p = e$ the representation exhibits an 'endowment effect': to give up the anchor, the alternative (i.e., q) must exceed the anchor's expected utility for every utility function in a *set* of utility functions. The result is

cycles as a normative desideratum. In defense of the criteria, it must be pointed out that a similar criticism can be brought against transitivity as well (i.e., one can replace transitivity in an atemporal theory with some choice-set-dependent procedure for selecting a 'best option' from a set through a sequence of binary comarisons). Moreover a cycle-free atemporal theory can be seen as the natural precursor to a normative inter-temporal theory of choice without arbitrary or pathological path-dependence.

added 'resistance' to giving up an anchor, or in other words, a status quo bias. This 'least relative utility' representation, resembling a risky-choice dual to Gilboa and Schmeidler's (1989) max-min representation for uncertainty aversion, is guaranteed to satisfy the no-cycle condition.[5] Moreover, when Y contains both concave and convex functions over final wealth outcomes, the representation can be made to exhibit loss aversion.

The only drawback of the max-min utility difference representation is that it does not accommodate Allais-type violations of Expected Utility when the common consequence of the Independence Axiom coincides with the anchor. To remedy this, I also demonstrate that when the expectation operator is replaced by a Choquet integral the representation satisfies the no-cycling condition, and exhibits both general violations of the Independence axiom and loss aversion. Thus it is possible to construct a viable alternative to Cumulative Prospect Theory that also satisfies basic normative requirements for reference dependent choice.

The closest paper, in spirit, to this work is a recent manuscript by Masatlioglu and Ok (2002) that also explores the normative aspects of reference dependence, albeit from a different angle than is used here. Masatlioglu and Ok (2002) consider a risk-less setting where the status quo must be part of the agent's choice set. They too consider a 'no-cycling' axiom (Axiom SQB) but their analysis requires additional structure that, arguably, is not as normative in nature. In particular, their emphasis differs from mine in that they examine a particular rational model of reference dependent choice, but do not explore the necessary structure implied by minimal rationality criteria and the subsequent implication for current theories of reference dependent choice (such as Cumulative Prospect Theories). Nevertheless, their work is pioneering in that it is the only other one (known to me) that attempts to understand how the $\succeq_e$'s should be tied together through reasonable restrictions on behavior.

The rest of this paper is structured as follows: Section 2 introduces the basic setting of the theory, describes the no-cycling axiom and the anchor continuity axiom, and derives the 'inadmissability' results mentioned above. Section 3 derives and analyzes representations consistent with the no-cycling axiom. Section 4 concludes.

[5] Although it is somewhat farfetched, one can give the endowment effect an interpretation (through this representation) as resulting from uncertainty with respect to tastes.

2 Theoretical Foundations

2.1 Anchoring Without 'Cycling' and the Inadmissability of Expected Utility Preferences

Let X be a set of distinct outcomes and assume that X is a compact metric space. Denote the Borel σ-algebra of X by Σ_X. The space of lotteries, $\mathcal{P}(X)$, is defined to be the space of regular Borel measures on (X, Σ_X) endowed with the weak* topology.[6]

Consider a set, $\{\succeq_e\}_{e\in\mathcal{P}(X)}$, of weak and continuous binary relations,[7] over $\mathcal{P}(X)$. Henceforth, in reference to $\succ_e$, e will be called the *anchor* and $\succeq_e$ will be referred to as an *anchored preference relation* (APR). Elements of $\{\succ_e\}_{e\in\mathcal{P}(X)}$ are related through the following axiom:

Axiom 1. *For any $q, p \in \mathcal{P}(X)$,*

$$q \succ_p p \;\Leftrightarrow\; q \succ_e p \textit{ for all } e \in \mathcal{P}(X)$$

I.e., if q is preferred to p, when p is the anchor, then q is preferred to p regardless of where the anchor is located. Intuitively, the axiom says that a prospect is most desirable when it is the anchor. Moreover, if the agent anchors with the status quo, Axiom 1 prevents a direct cycle as well as other forms of potential 'manipulation'. A direct cycle is possible when $q \succ_p p$ and $p \succeq_q q$: if the agent's status quo and anchor coincides with p she will gladly pay to trade p for q; on the other hand, upon receipt of q, if q becomes the status quo, she will gladly switch back to p. To see a more general form of possible manipulation, consider the case $q \succ_p p$ yet $p \succeq_e q$ for some $e\ (\neq p) \in \mathcal{P}(X)$. If the agent's endowment and anchor is e, and she must choose between p and q, then she will agree to take p. If p, as her new status quo, becomes the new anchor, then given a choice between p and q she will gladly pay to trade p for q, the choice forgone earlier. Thus Axiom 1 prevents the potential for manipulation if the agent habitually anchors with the status quo.[8]

It is a simple matter to check that Axiom 1 is equivalent to requiring that for every

[6]Let $\mathcal{C}(X)$ denote the space of bounded and continuous real-valued functions on X. The weak* topology is the weakest (or coarsest) topology under which the expected value of $\psi \in \mathcal{C}(X)$ is a continuous functional on $\mathcal{P}(X)$. For instance, when $X \equiv [0, 1]$, the weak* topology corresponds to the usual topology of weak convergence.

[7]A weak order is transitive and complete. Continuity of $\succeq_p \in \{\succ_e\}_{e\in\mathcal{P}(X)}$, means that all upper and lower contour sets of $\succeq_p$ are closed (i.e., $\{q \mid q \succeq_p q'\}$ and $\{q \mid q' \succeq_p q\}$ are weak* closed).

[8]When the anchor is always part of the choice set, Axiom 1 is equivalent to Axiom SQB in Masatlioglu and Ok (2002) who work in the context of revealed preferences over risk-less choice.

$p \in \mathcal{P}(X)$

$$\{q \mid q \succ_p p\} \subseteq \{q \mid q \succ_e p\} \;\; \forall e \in \mathcal{P}(X)$$

Continuity of the $\succeq_e$'s ensures that

$$\{q \mid q \succeq_p p\} \subseteq \{q \mid q \succeq_e p\} \;\; \forall e \in \mathcal{P}(X)$$

Thus, the implication of 'no-cycling' is to ensure that the at-least-as-good-as (i.e., upper contour) set at p when p is the anchor is contained in the at-least-as-good-as set at p *for any* anchor. This simple assumption is sufficient to establish the first set of results:

Theorem 1. *Axiom 1 is only consistent with an expected utility representation for every* $\succeq_e \in \{\succ_e\}_{e\in\mathcal{P}(X)}$*, if* $\succeq_e = \succeq_{e'}$ *for every* $e, e' \in \mathcal{P}(X)$ *(i.e., there is no reference dependence).*

Proof: Fix any anchor $e \in \mathcal{P}(X)$ and let u_e be a cardinal von Neumann-Morgenstern (vNM) utility function for $\succeq_e$. The idea is to show that the cardinal vNM utility function associated with any other anchor is an affine transformation of u_e. Begin by arbitrarilly normalizing $u_{e'}(y_0) = 0, u_{e'}(y_1) = 1$ for some $y_0, y_1 \in X$ and all $e' \in \mathcal{P}(X)$ (note that compactness of X guarantees that the $u_{e'}$'s are bounded on X). If there exists some e' such that $u_{e'}$ is not an affine transformation of u_e, then $u_e(y_2) \neq u_{e'}(y_2)$ for some $y_2 \in X$. Now consider the probability simplex over $\{y_0, y_1, y_2\}$ and the linear indifference surfaces induced by u_e and $u_{e'}$. Since $u_e(y_2) \neq u_{e'}(y_2)$ the indifference lines cross at some interior point of the simplex. Pick such an arbitrary crossing point, say e'' and consider the indifference line of $\succeq_{e''}$ at e'' in the simplex. By assumption, this too has to be a straight line. Moreover, no such straight line can define an upper contour set that is contained in the upper contour sets of *both* $\succeq_e$ *and* $\succeq_{e'}$ - this is illustrated in Figure 1a. Thus Axiom 1 is violated unless all anchored preferences possess the same vNM representation. □

Theorem 1 indicates that one cannot simply postulate von Neumann-Morgenstern (vNM) axioms over the individual $\succeq_e$'s while maintaining reference dependence. A set, $\{\succ_e\}_{e\in\mathcal{P}(X)}$, in which each of the anchored preference relations, $\succeq_e$, is represented by

$$U_e(p) = E_p[u_e]$$

(where u_e is a vNM cardinal utility function) is not admissable unless $u_e(x) = a_e u(x) + b_e$ (i.e., $u_e(x)$ is an affine transformation of some $u(x)$). In particular, Tversky and Kahneman's (1991) theory of the endowment effect, which postulates a set of risk-less utility functions, $\{u_r\}$, cannot be extended to risky choice in a trivial way (i.e., by simply assuming expected utility over the u_r's). This has immediate additional implications for the standard modeling approach for loss aversion in the literature. If Axiom 1 is deemed normatively desirable, the following corollary establishes the impossibility of a *linear prospect theory* (i.e., one that makes use of the S-shaped value function over wealth changes but is linear in probability).

Corollary to Theorem 1: *Assume X is a subset of $\mathbb{R}$ and assume $\succeq_e$ is represented by $U_e(p) = \int_X u(x - c_e)dp(x)$ with continuous utility function, $u : X - X \mapsto \mathbb{R}$, and $c_e : \mathcal{P}(X) \mapsto \mathbb{R}$, a continuous certainty equivalent function for lotteries. Then to be consistent with Axiom 1, the ranking of lotteries is independent of c_e (i.e., $u_e(x - c_e) = a_e u(x) + b_e$ and there is no reference dependence). In particular, $u(x)$ cannot be S-shaped about $u(0) = 0$.*

Proof: The first claim is a direct consequence of Theorem 1. If $u(x)$ is S-shaped about $u(0) = 0$ (i.e., strictly concave somewhere above 0 and strictly convex somewhere below 0), then one can find $x_1 > x_2 > 0 > x_3 > x_4$ and $\epsilon > 0$ such that $\frac{u(x_1-\epsilon)-u(x_2-\epsilon)}{u(x_3-\epsilon)-u(x_4-\epsilon)} > \frac{u(x_1)-u(x_2)}{u(x_3)-u(x_4)}$. Setting $c_e = \epsilon$, this is not consistent with $u(x - \epsilon) = a_\epsilon u(x) + b_\epsilon$. □

Typically, c_e is not fully elucidated but is in practice assumed to be the expected value of the lottery e; the Corollary applies to *any* related representation in which c_e is a wealth equivalent of the agent's endowment that is a continuous function. In other words, it doesn't matter how the endowment is converted to a reference point, a model in which agents maximize expected utility of a 'loss-averse' value function over relative wealth systematically violates Axiom 1.

To help shed more light on why linear Prospect Theory violates Axiom 1, consider an example where X consists of the prizes $(\$1000, \$1500, \$2000)$ and thus the probability space can be represented in the simplex as in Figure 1b. The indifference surfaces for $\succeq_{\$1500}$ (i.e., when the anchor is at the sure outcome \$1500) are the solid lines. Note that they imply that \$1500 for sure is preferred to the 50/50 lottery that awards either \$1000 or \$2000 (p on the figure), thus when anchored at \$1500 the agent exhibits risk aversion, consistent with a loss averse decision maker. The indifference surfaces for $\succeq_{\$2000}$ (i.e., when the anchor is at the sure outcome \$2000) are shown as dashed lines; this time the agent prefers p to the \$1500 for sure and is risk seeking, again, consistent with loss aversion. The indifference surfaces corresponding to the two anchors (i.e., \$1500 and \$2000) necessarily cross at every interior point; consider such a point, say e in Figure 1b, where the indifference line at e when anchored at e is the dotted line. The difficulty originates with the shaded regions: those areas where the better-than set of $\succeq_e$ at e is not contained in the better-than set at e for

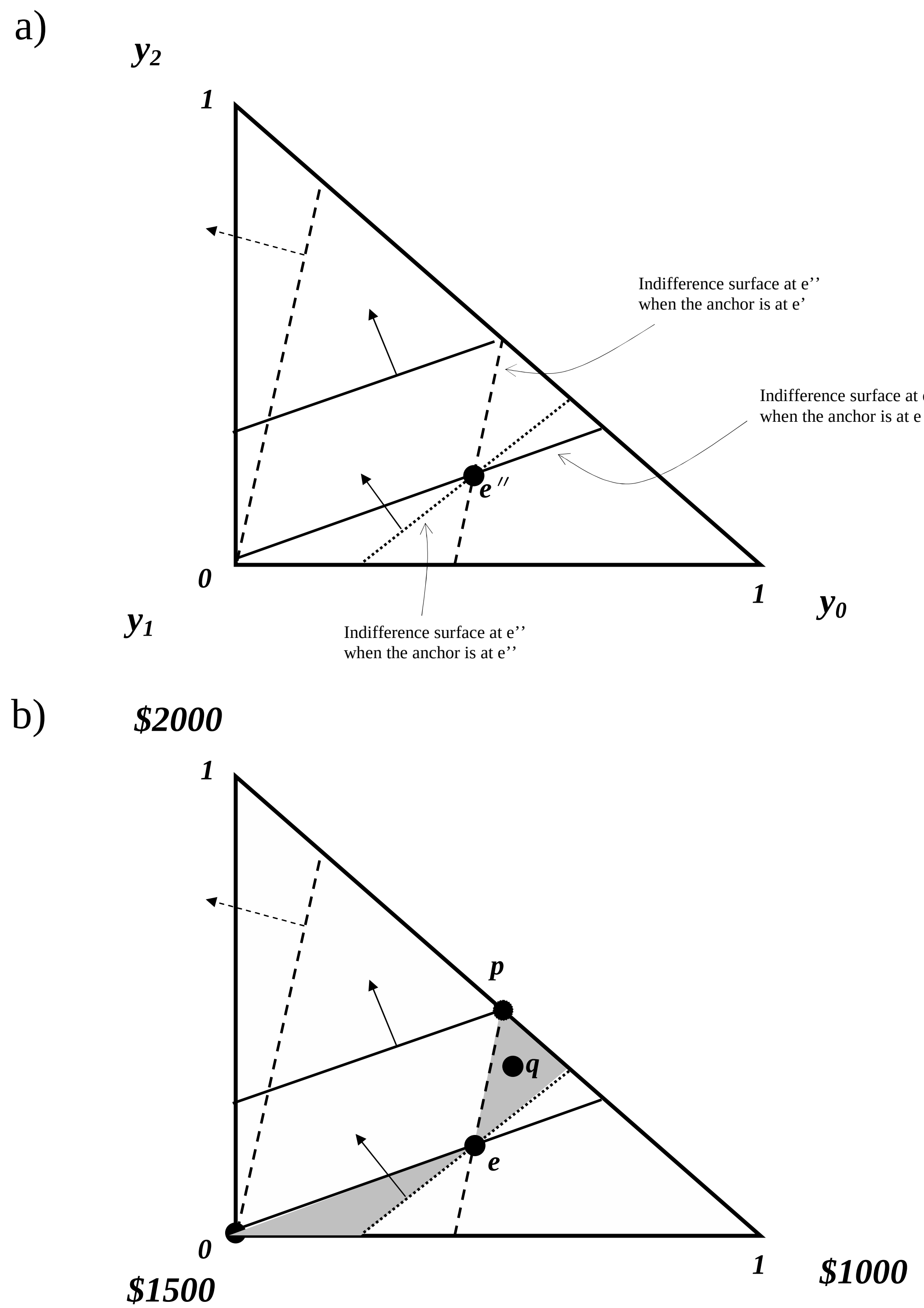


Figure 1: a) No indifference line at e'' can coincide with both the indifference lines induced by $\succeq_e$ and $\succeq_{e'}$. b) Linear Prospect Theory implies some crossing - violations of Axiom 1 arise because of the shaded region.

$\succeq_{\$1500}$ or for $\succeq_{\$2000}$. Consider a lottery in the shaded region, say q in the figure. Then $e \succ_p q$ but $q \succ_e e$; thus if the decision maker happened to be anchored at p, then when offered a choice between q and e she will select e. If, subsequently, e becomes the anchor, she will pay to switch to q, the choice forgone.

Remark 1. *Another implication of the affine equivalence of the $u(x-c_e)$'s from the Corollary is that $u(x)$ is either linear in x or linear in e^{gx} for some $g \in \mathbb{R}$ – proof of this is left to the reader.*

2.2 The Inadmissability of Smooth Preferences and Cumulative Prospect Theory

The next supposition requires the representations for the different $\succeq_e$'s to be continuous with respect to the anchor. This is a technical requirement that also has some normative basis. In particular, if a lottery, p is strictly preferred to another, q, with respect to an anchor, e, then an infinitesimal shift in anchor will not induce a preference reversal.

Axiom 2. *Anchor Continuity*
Given a weak convergent sequence, $\{(p_n, q_n, e_n)\} \to \{p, q, e\}$ in $\mathcal{P}(X)^3$,*

$$p_n \succeq_{e_n} q_n \;\forall n \quad \Rightarrow \quad p \succeq_e q$$

Intuition suggests that it is possible to extend Theorem 1 to non-linear representations. If the $\succeq_e$'s are assumed to be smooth (i.e., Fréchet differentiable) then at each prospect the representation can be approximated as Expected Utility. By Axiom 1, the at-least-as-good-as set of $\succeq_e$ at e is a subset of the at-least-as-good-as set of $\succeq_{e'}$ at e, *for any* e'; thus, at least intuitively, the gradient at e when e is the anchor should coincide with the gradient at e for an arbitrary anchor (otherwise the smooth indifference surfaces would cross at e and the nesting property implied by Axiom 1 would be violated). But if this is the case, the gradient for *every* $\succeq_e$ is the same at each $p \in \mathcal{P}(X)$, implying, of course, that the representations also coincide. In other words, *no Fréchet differentiable representation for anchored preferences can be reference dependent*. This exceptionally strong result, formally established by the next Theorem, implies that Anchored Preference Relations are, even qualitatively (i.e., at the local level - see Machina (1982)), very different from expected utility.

Theorem 2. *Assume that, in addition to obeying Axiom 1 and 2, every $\succeq_e \in \{\succ_e\}_{e \in \mathcal{P}(X)}$ is Fréchet differentiable everywhere. Then the gradients for every $\succeq_e$ at any $p \in \mathcal{P}(X)$ coincide (i.e., there is no reference dependence).*

Proof: Consider first some e with finite support, $\{x_0, ..., x_n\}$, and let $u_{q,p}$ be a cardinal vNM utility function corresponding to the gradient of $\succeq_q$ at p. Consider next the gradient of $\succeq_{e'}$ at e, denoted as $u_{e',e}$, and arbitrarilly normalize $u_{e',e}(y_0) = 0, u_{e',e}(y_1) = 1$ for some $y_0, y_1 \in X$ and all $e' \in \mathcal{P}(X)$. If there exists some *arbitrary* e' such that $u_{e',e}$ does not coincide with $u_{e,e}$, then $u_{e,e}(y_2) \neq u_{e',e}(y_2)$ for some $y_2 \in X$. Now consider the probability simplex over $\{x_0, ..., x_n, y_0, y_1, y_2\}$ and the approximately linear indifference surfaces through e locally induced by $u_{e,e}$ and $u_{e',e}$. If e is in the interior of the simplex (e.g., $y_0, y_1, y_2 \in \{x_1, ..., x_n\}$), then one can always find some q such that $q \succ_e e$ but $e \succ_{e'} q$, in contradiction with Axiom 1. If, on the other hand, e is at the boundary of the simplex, then anchor continuity implies that there exists e'' in the interior of the simplex such that $u_{e'',e}(y_2) \neq u_{e,e}(y_2)$. The same argument then applies to produce a violation of Axiom 1. Thus $u_{q,e}$'s at $e \in \mathcal{P}(X)$ with finite support must coincide for *all* $q \in \mathcal{P}(X)$. To prove the result for arbitrary e note that one can always write e as the limit point of some convergent sequence of e_n's with finite support. Since the u_{q,e_n}'s coincide for every n, Fréchet differentiability of the $\succeq_q$'s (i.e., continuity of $u_{q,p}$ in p) implies that the gradients coincide at e as well. □

Theorem 2 has implications for risk-less choice as well. The proof generalizes to commodity spaces other than probability measures. In particular, it applies to any Banach space. An immediate implication is that virtually any risk-less choice theory involving non-trivial reference dependence must have some non-smooth reference dependent relations. Moreover, if reference dependence is a property of every neighborhood of a commodity bundle, q, then at least one APR in $\{\succeq_e\}_{e \in \mathcal{P}(X)}$ must be kinked arbitrarily close to q. Note that as a consequence, the theory of the endowment effect promoted by Tversky and Kahneman (1991) does not generally satisy Axiom 1, although it is easy to generate examples for which it does (for instance, the example of 'Constant Loss Aversion' sketched in their Figure V and discussed here briefly in Remark 6).

A related negative result applies to a 'general' form of Cumulative Prospect Theory. Suppose that $X \equiv [\underline{y}, \bar{y}] \subset \mathbb{R}$ and that each relation in $\{\succ_e\}_{e \in \mathcal{P}(X)}$ is represented by the function $U_e : \mathcal{P}(X) \mapsto \mathbb{R}$:

$$U_e(q) \equiv \int_{v(\underline{y}-c_e)}^{0} (w^-[D_q(v^{-1}(\alpha) + c_e)] - 1)\, d\alpha + \int_0^{v(\bar{y}-c_e)} w^+[D_q(v^{-1}(\alpha) + c_e])\, d\alpha \quad (1)$$

where the value function, $v(x) : \ [\underline{y} - \bar{y}, \bar{y} - \underline{y}] \mapsto \mathbb{R}$, is bounded, increasing and continuous, $v(0) = 0$, $D_q(x)$ is the decumulative distribution function measuring the probability that the payoff to the lottery q is $x \in X$ or greater, w^+ and w^- are continuous monotonic transformations of $[0, 1]$ to $[0, 1]$ with $w^-(0) = w^+(0) = 0, w^-(1) = w^+(1) = 1$, and the reference certainty equivalent function, c_e, is a continuous function from $\mathcal{P}(X)$ to X such that whenever $q \in \mathcal{P}(X)$ stochastically dominates $p \in \mathcal{P}(X)$ $c_q \geq c_p$. Eqn. (1) characterizes Cumulative Prospect Theory. In the literature a stochastic endowment is hardly ever discussed and c_e is implicitly taken to be the current 'wealth' of the investor. Note that continuity of c_e necessarily implies that Axiom 2 is satisfied.

Theorem 3. *Consider $X \equiv [\underline{y}, \bar{y}] \subset \mathbb{R}$, and assume $\succeq_e$ is represented by the Cumulative Prospect Theory functional in* (1). *Then to be consistent with Axiom 1, $\succeq_e = \succeq_{e'}$ for every $e, e' \in \mathcal{P}(X)$ (i.e., there is no reference dependence), $w^+ = w^-$, and $v(x - c_e)$ is an affine equivalent of $v(x)$ for every e.*

Proof: The proof relies on the fact that when restricted to any finite set of outcomes, $\succeq_e$ has unique probability-derivative almost everywhere. This will be established shortly. But first note that if $\succeq_e \neq \succeq_{e'}$, then there must be $q, p \in \mathcal{P}(X)$ such that $q \succ_e p$ and $p \succ_{e'} q$. Continuity of $U_e(q)$ in e and q, and a.e. uniqueness of a probability derivative in any finite simplex jointly imply that one can always find $\hat{q}, \hat{p}, \hat{e}, \hat{e}'$ respectively close to q, p, e and e' such that: (i) all have the same and finite support in some simplex, S, (ii) there exists $\hat{e}''$ in the interior of S such that $\succeq_{\hat{e}}, \succeq_{\hat{e}'}$ and $\succeq_{\hat{e}''}$, restricted to S, have unique probability-derivatives at $\hat{e}''$, and (iii) the indifference surfaces of $\succeq_{\hat{e}}$ and $\succeq_{\hat{e}'}$ cross at $\hat{e}''$. As in the proof of Theorem 2, this is not consistent with Axiom 1. Thus $\succeq_e = \succeq_{e'}$ for every $e, e' \in \mathcal{P}(X)$.

To show the a.e. unique probability-derivative property, consider lotteries with finite support in the simplex S. Then for any $q \in S$, write $q = (q_1, q_2, ..., q_j, q_{j+1}, ..., q_n)$ where q_k is the probability of outcome x_k, and the x_k's are rank ordered so that $x_1 > x_2 > ... > x_j \geq c_e > x_{j+1} ... > x_n$. Then,

$$\begin{aligned} U_e(q) =& w^+(q_1)v(x_1 - c_e) \\ &+ \big(w^+(q_1 + q_2) - w^+(q_1)\big)v(x_2 - c_e) \\ ...&+ \big(w^+(q_1 + q_2 + ... + q_j) - w^+(q_1 + ... + q_{j-1})\big)v(x_j - c_e) \\ &+ \big(w^-(q_1 + q_2 + ... + q_j + q_{j+1}) - w^-(q_1 + ... + q_j)\big)v(x_{j+1} - c_e) \\ ...&+ \big(1 - w^-(q_1 + ... + q_{n-1})\big)v(x_n - c_e) \end{aligned}$$

Since both $w^+(\cdot)$ and $w^-(\cdot)$ are monotonic and continuous on $[0, 1]$, they are almost everywhere differentiable there. Thus the probability derivative (vector of partial derivatives of $U_e(q)$ with respect to the q_i's) exists and is unique for almost every $q \in S$.

$w^+ = w^-$ is proven by first setting $e = \bar{y}$ and $e' = \underline{y}$ (where I abuse notation by identifying a sure-outcome lottery with its outcome) and noting that (i) $\succeq_e = \succeq_{e'}$, and (ii) the probability weighting function in rank dependent risky choice is unique (see Abdellaoui (2002)). The affine equivalence claim follows from $\succeq_e = \succeq_{e'}$ for every $e, e' \in \mathcal{P}(X)$, and the fact that each of the $\succeq_e$'s is described by a rank-dependent utility representation. $\square$

Remark 2. *Here too the affine equivalence of the $v(x - c_e)$'s with $v(x)$ implies that $v(x)$ is either proportional to x or proportional to $\frac{e^{gx}-1}{g}$ for some $g \in \mathbb{R}$ – proof of this is left to the reader.*

To gain further intuition for the Corollary, consider that the example in Figure 1ab generally differs from Cumulative Prospect Theory only in that the indifference curves are not straight. Indifference curves corresponding to an anchor at e and e' must still cross, however. If a crossing point is e'', the question is whether both surfaces always contain (or support) the upper contour set of $\succeq_{e''}$ at e''. The problem is that the weighting functions in Cumulative Prospect Theory are sufficiently well behaved that at almost every such crossing point, e'', each of $\succeq_e$, $\succeq_{e'}$ and $\succeq_{e''}$ are well approximated by a linear function – thus the same argument that rules out global expected utility representations for the $\succeq_e$'s rules out local expected utility approximations as well. A solution would be to place a kink at each anchor (i.e., arrange it so that every $\succeq_e$ has a kink at e).[9] This, however, is not possible with the representation in (1). In other words, while rank dependent theories have kinks, they do not have enough of them (see Chew, karni and Safra (1987) and Machina (2000)) and, overall, are simply too smooth to exhibit both reference dependence and consistency with Axiom 1.

Theorem 3 leads to the somewhat disappointing conclusion that to exhibit reference dependence, Cumulative Prospect Theory - a normative alternative to the original Prospect Theory of Kahneman and Tversky (1979) - must systematically violate the 'no-cycling' property. This is particularly distressing given that Cumulative Prospect Theory is descriptively the most successful theory of risky choice (Starmer (2000), Camerer (1998)). In Section 3 I discuss possible remedies.

[9] It is hard, in fact, to conceptualize a set of non-pathological anchored preference relations that can be patched together to satisfy Axioms 1 and 2 without placing a kink at *each* anchor.

2.3 Kinks and Loss Aversion

Since, to have non-trivial anchored preferences *and* satisfy Axioms 1 and 2 one cannot have smooth preferences, the only remaining choice is to admit representations with kinks. If $\succeq_e$ is smooth at e, then for a different relation, $\succeq_{e'}$, to locally exhibit different choice behavior at e, it must be kinked at e. Likewise, if two different relations, say $\succeq_p$ and $\succeq_q$ are smooth but have different gradients at some point $e \neq p, q$, it must be the case that $\succeq_e$ is kinked at e. It seems intuitive to posit that if choice 'anomalies' can be identified in the neighborhood of a prospect, then it ought to be especially so when the prospect is an anchor. In other words, if $\succeq_e$ is smooth at e, then it seems reasonable to require that every other APR is smooth at e, regardless of anchor. Unfortunately, it is not clear that one can say anything more compelling than this absent more explicit behavioral assumptions. However, in conjunction with *loss aversion* over wealth outcomes, the axioms do place further strong constraints on the representations of the $\succeq_e$'s - namely, if the agent is loss averse over wealth outcomes at q then $\succeq_q$ must be kinked at q.

To explore this further I first discuss what I mean when saying that an agent is 'loss averse over wealth outcomes' at q. In what follows I assume that $X \equiv [0,1]$. Denote the expected value of $p \in \mathcal{P}(X)$ as $\bar{p}$, and define q to be a mean-preserving spread of p using the Rothschild-Stiglitz (1970) definition: $q \succeq^{mps} p$ for any $q, p \in \mathcal{P}(X) \Leftrightarrow \forall y \in [0,1], \int_0^y q([0,x])dx \geq \int_0^y p([0,x])dx$ and $\bar{q} = \bar{p}$.[10] The strict relation holds whenever, in addition, there is some $y \in [0,1]$ for which $\int_0^y q([0,x])dx > \int_0^y p([0,x])dx$. I abuse notation slightly by identifying $x \in X$ with the lottery that pays x for sure.

Definition 1. *Preferences are said to be 'loss averse' at some non-degenerate $q \in \mathcal{P}(X)$ if and only if both of the following conditions are satisfied*

i) Whenever $p \succ^{mps} q$ then $q \succ_x p$ for every $x \leq \bar{q}$.

ii) There exists $\eta \in [0,1]$ such that $\eta \geq \bar{q}$, and for every $x > \eta$, and every $p \in \mathcal{P}(X)$

$$q \succ^{mps} p \;\Rightarrow\; q \succ_x p$$

The first condition requires risk aversion to be exhibited whenever the agent compares q to a riskier lottery and both represent an actuarial gain relative to a sure-outcome anchor. The second states that the agent is willing to gamble when comparing q to a less risky

[10] $q([0,y])$ is the cumulative distribution of q for probability mass in the interval $[0,y]$.

lottery, given that both represent a sufficiently large actuarial loss relative to some sure-outcome anchor. Note that if q is degenerate, the second property holds trivially.

Remark 3. *Definition 1 represents a distortion of how loss aversion is normally defined in the literature: higher sensitivity to relative losses than relative gains. This distinction is material where loss aversion is not related to risk attitudes with respect to mean preserving spreads: i.e., when one is considering risk-less choice with multiple commodities (as in Tversky and Kahneman (1991)) or risk-less inter-temporal choice.*[11] *In the context of atemporal risky choice over wealth outcomes, however, I argue that there is no distinction between the standard definition of loss aversion and Definition 1.*

Note that the second loss aversion condition in Definition 1 indicates that, for *some* anchor, q is preferred to all lotteries for which q is a mean preserving spread. Thus Axiom 1 implies that, in particular, if an agent is loss averse at q then it must be the case that

$$\forall p \in \mathcal{P}(X) \;\; q \succ^{mps} p \;\Rightarrow\; q \succ_q p$$

This characterization leads to the following restrictions on loss aversion at q:

Theorem 4. *Let $X \equiv [0,1]$ and the $\succ_e$'s be everywhere Gâteaux differentiable for all $e \in \mathcal{P}(X)$. Assume Axiom 1 and that the agent is loss averse at some non-degenerate $q \in \mathcal{P}(X)$, then $\succeq_q$ is not Fréchet differentiable at q (i.e., $\succeq_q$ has a kink at q).*

Proof Sketch: The idea is simple: loss aversion implies that locally $\succeq_q$ exhibits both risk-seeking and risk-averse attributes at q. This is not consistent with a local Expected Utility approximation, thus $\succeq_q$ cannot be Fréchet differentiable at q. Details are in the Appendix. □

A simple corollary to this result is that if the agent is loss averse almost everywhere in some region of the probability space, the kinks in the at-least-as-good-as sets at the anchor must generally be 'convex'. Otherwise, anchor continuity implies a violation of Axiom 1. To illustrate this, consider Figure 2. In part (a) of the figure two *convex* kinked indifference surfaces are shown at their respective anchors. Note that no cycling between anchors is possible. In part (b) the kinks are *concave* and cycling is possible since $e \succ_{e'} e'$ but $e' \succ_e e$, in violation of Axiom 1. If a kink at some anchor is 'concave', anchor continuity will generally imply (save for extremely pathological cases) that potential cycles such as those in Figure

[11] In the context of risk-less inter-temporal choice, loss averse agents may be defined as those who prefer arriving at a final outcome via a sequence of gains rather than a larger gain followed by a sequence of losses.

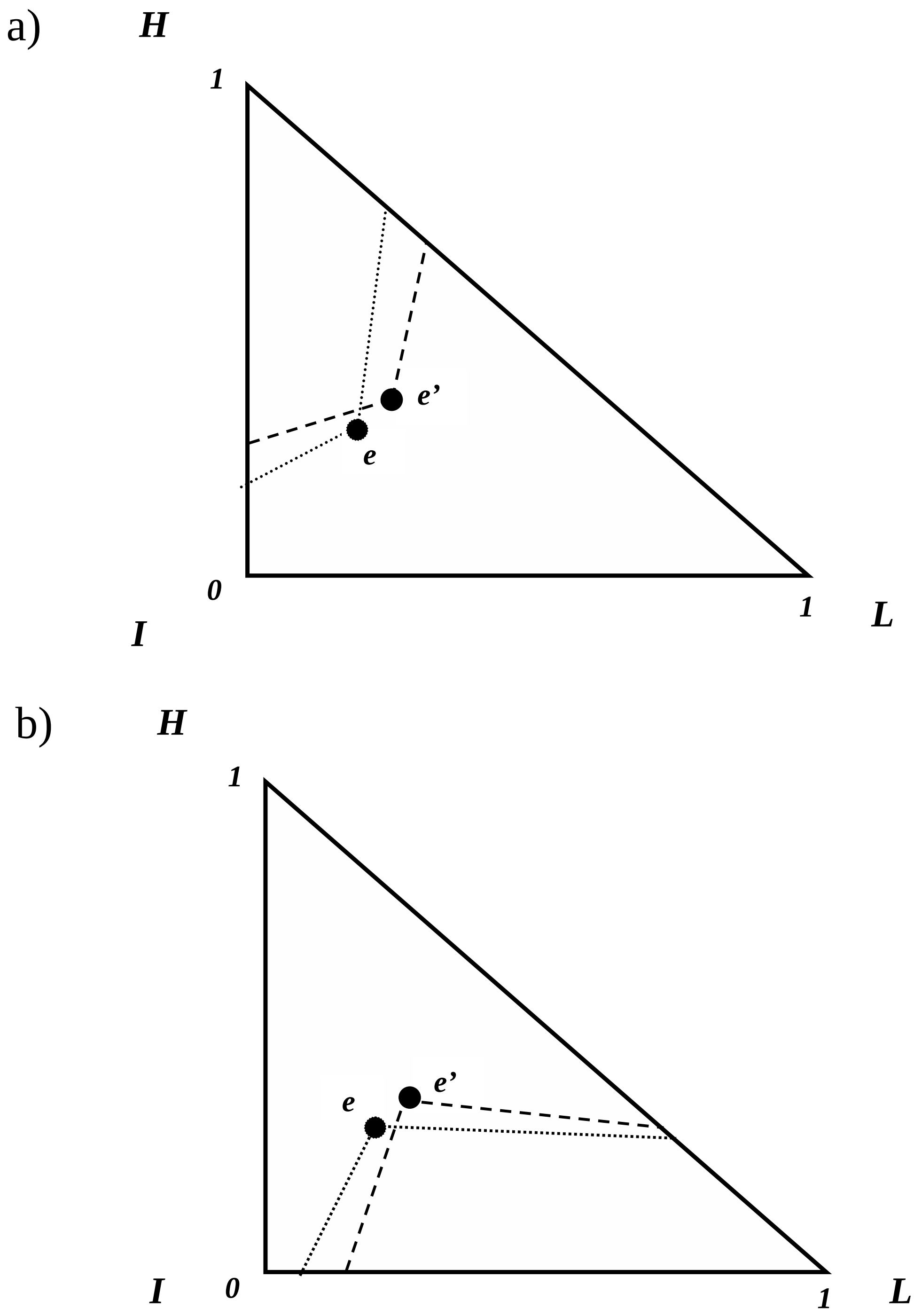


Figure 2: a) Convex kinks at the anchor. b) Concave kinks at anchor - cycles are possible.

2b will arise. These are important observations since they force us to recognize several important and empirically verifiable aspects of a sensible reference dependent theory that exhibits loss aversion:

1. Given an anchor at e, one can usually find lotteries $p, q \in \mathcal{P}(X)$ and $\alpha \in (0,1)$ such that $e \succ_e p$ and $e \succ_e q$, while $\alpha q + (q - \alpha)p \succ_e e$ - in other words, a coin flip between two 'bad things' relative to an anchor is better than the anchor for *some* pair of 'bad things'. Thus, in particular, such theories violate 'Betweeness' (see Dekel (1986), Chew (1989) and Gul (1991)).[12] This, however, is not surprising given Theorem 2: if $\succeq_e$ respects betweeness for every $e \in \mathcal{P}$, then $\succeq_e$ is smooth at every $e \in \mathcal{P}$ and there cannot be reference dependence.

2. Consider an expected utility approximation to the attitudes of an agent who is considering a small change away from her anchor at e. Suppose she is offered a shift to e_l that is dominated by the anchor according to the approximation, and is also offered a shift to e_h, that dominates that anchor according to the approximation. If she refuses e_l while sometimes refusing e_h then one can say that, at least relative to the expected utility approximation, the agent is exhibiting *endogenous transaction costs*. If the agent's preferences are smooth at the anchor, then as the shift from the anchor becomes smaller (in the appropriate metric), there is a 'best' approximation to the agent's attitudes and one also finds that the agent accepts most e_h-type offers; in the limit, she will accept all of them. By contrast, a convex kink at each anchor implies that there is no 'best EU approximation' , and that the agent will continue to reject a measurable set of any e_h-type offers. In other words, the transaction costs are 'first order'.

3. Loss aversion is commonly associated with 'first order risk aversion'. It is common, for instance, to specify a value function whose derivative is discontinuous at 0. Thus the Pratt-Arrow measure of risk - a second order, or curvature attribute - is 'infinite' at the certainty equivalent of the agent's reference point. Theorem 4 confirms the intuition that loss aversion is a 'first order' effect, but, in conjunction with Theorem 3, also establishes that the kinks cannot separately reside in a value function, but must be an essential feature of how probabilities are assessed.

[12]Green (1988) argues that convex upper contour sets can lead to Dutch books. His argument, however, *does not* hold in a setting where the agent anchors with the status quo. Camerer and Ho (1994) find extensive violations of Betweeness.

3 Representations

Theorems 3 and 4 essentially indicate that the problem with prospect theory is that it is too smooth. It is hard to conceive how one may alter the probability weighting functions so as to preserve the 'weighted utility' interpretation and yet ensure that a kink appears at each anchor. An alternative approach is to abandon a 'weighted utility' interpretation and attempt to impose axioms that are 'natural' to the formalism described so far. Such a derived reference dependent theory of choice will be explicitly consistent with Axioms 1 and 2, and will have the additional advantage of indicating directions one can take in modeling loss aversion.

The main contribution of this section is the derivation and analysis of the representation mentioned in the Introduction:

$$q \succeq_e p \;\Leftrightarrow\; \inf_{\psi \in Y} E_{q-e}[\psi] \geq \inf_{\psi \in Y} E_{p-e}[\psi]$$

where $E[\cdot]$ is an expectation operator and Y is a set of von Neumann-Morgenstern utility functions.

3.1 A Scale and Translation Invariant Representation

I impose the following variant on the Independence Axiom of von Neumann and Morgenstern:

Axiom 3. *For every* $q, p, r, e \in \mathcal{P}(X)$, $\alpha \in (0, 1]$

$$q \succ_e p \;\Leftrightarrow\; \alpha q + (1-\alpha)r \succ_{\alpha e + (1-\alpha)r} \alpha p + (1-\alpha)r$$

Note that this axiom differs from vNM in one crucial respect: the independence from the 'irrelevant' prospect, r, is asserted only if the anchor is also shifted by the same transformation as the original lotteries compared. An intuition for why this assertion may be desirable comes from considering the editing principle of Kahneman and Tversky (1979). An agent, when comparing lotteries only considers the differences between the lotteries, and in this case, also how they differ from the anchor. The translation and scale invariance implied by this assumption can also be taken to mean that no single anchor is 'special': any properties exhibited around one anchor are also exhibited around any other. It was already shown in Theorem 1 that one cannot simply impose an independence axiom over the $\succeq_e$'s *individually*. Axiom 3 is the 'logical' next step in imposing an independence style axiom that serves to tie together the $\succeq_e$'s. It is a simple matter to check that descriptively this assumption rules out the Allais Paradox (1953) when the common consequence (i.e., r) coincides with the anchor (i.e., e). This, however, also allows one to examine the effects of anchoring without undue commitment to additional structure. As will be seen in the next subsection, it is easy to distort the representation that results from Axiom 3 to accommodate Allais-type anomalies.

To derive a representation based on Axiom 3 I first characterize the upper contour set at the anchor of each $\succeq_e$. To this end, consider first the partial ordering $\succeq^P$ defined by

$$q \succeq^P p \;\Leftrightarrow\; q \succeq_p p$$

The next Lemma establishes that $\succeq^P$ is a weak continuous partial order that obeys an independence axiom:

Lemma 3.1. *Axioms 1-3 imply that*

i) $\succeq^P$ *is transitive*

ii) $B_p^P \equiv \{q | q \succeq^P p\}$ *is closed in the weak* topology of* $\mathcal{M}(X, \Sigma_X, \mathbb{R})$*, the linear space of regular real valued signed measures on* (X, Σ_X)*. Further, if* $q_n \to q$ *and* $p_n \to p$ *are weak* convergent sequences with* $q_n \succeq^P p_n$ *for every* n*, then* $q \succeq^P p$*.*

iii) for every $q, p, r \in \mathcal{P}(X)$*,* $q \succeq^P p \;\Leftrightarrow\; \alpha q + (1-\alpha) r \succeq^P \alpha p + (1-\alpha) r$

Proof: The first result follows from Axiom 1. The second follows from Anchor Continuity (Axiom 2) and the continuity assumption over the $\succeq_e$'s. The last result follows trivially from Axiom 3. □

The following is a direct result of the Lemma:

Theorem 5. *The conditions in Lemma 3.1 are necessary and sufficient for the following to hold,*

$$q \succeq^P p \Leftrightarrow q \succeq_p p \quad \Longleftrightarrow \quad \inf_{\psi \in \Psi} E_{q-p}[\psi] \geq 0$$

where Ψ *is a set of real, continuous and bounded functions on* X *whose elements are individually defined up to an affine (positive linear) transformation.*

The Theorem is essentially proven in Dubra, Maccheroni and Ok (2001).[13] $E_{q-p}[\psi]$ denotes the linear functional $\int \psi \, d(q-p) = < \psi, q-p >$ (i.e., the difference in expected utility

[13]The theorem was also independently derived in Sagi (2000). There is a long history of partial orderings in the

between q and p). Ψ is a utility function set and the theorem states that q overcomes an anchor at p if and only if the expected utility of q is greater or equal to that of p for *every* utility function in the utility set, Ψ. On the other hand, given an anchor at p, p is preferred to q if the expected utility of p is higher than that of q for *any* utility function in the utility set. Note that the theorem does not assert that if two sets, Ψ and Ψ' represent the partial order then each $\psi \in \Psi$ is related to some $\psi' \in \Psi'$ through an affine transformation (and vice versa). A counter example is obtained by considering an open set, Ψ, and its closure, Ψ'. It is easy to see, however, that such a relation will exist among pairs of *closed* utility function sets[14], Ψ and Ψ'. Finally, one can always 'normalize' elements of Ψ via affine transformations to arrive at a unique set in the following way: (i) set the minimum of every $\psi \in \Psi$ to 0, and (if it is distinct from the minimum) the maximum of each $\psi \in \Psi$ to 1; (ii) generate the closed convex hull of the set resulting from (i).

It is easy to show that in addition,

$$q \sim_p p \iff \inf_{\psi \in \Psi} E_{q-p}[\psi] = 0$$

Geometrically, Theorem 5 implies that the indifference surface at an anchor, e, is a *wedge* with support functionals given by the vNM functions in Ψ. Moreover, the indifference surface at any anchor is simply a translation of this wedge. In particular, whenever Ψ is not a singleton (the only interesting case since otherwise the theory reduces to Expected Utility) the indifference surfaces are kinked and upper contour sets are convex. This is illustrated in Figure 3. Note that Axiom 1 is explicitly incorporated through the transitivity of the relation $\succeq^P$ - if $p \succ_r r$ then the wedge at p is contained in the wedge at r. Finally, since every anchor has a convex kink, it is possible to conceive that loss aversion as described in Definition 1 can be accommodated (see Theorem 4).

To continue, I impose one additional technical assumption. Let

$$B_e(p) \equiv \{q \mid q \succeq_e p\}$$

denote the at-least-as-good-as set at p when the anchor is at e. Note that, from Theorem 5 (and Figure 3), if the representation is not anchor independent, then for every $e \in \mathcal{P}(X)$,

social choice and decision theory literature analogous to the results for $\succeq^P$ given here. Aumann (1962) and Bewley (1986) are seminal references in considering partial orderings in decision theory and economics, while more recent work includes Mitra and Ok (2000), Dubra and Ok (2000), Ok (2000) and especially Baucells and Shapley (2001) and Dubra, Maccheroni and Ok (2001). Also, Levi (1980) and Seidenfeld et. al. (1995) motivate a similar theory for Anscomb-Aumann 'horse-lotteries'.

[14]This follows from the continuity of $E_{p-q}[\psi]$ in ψ.

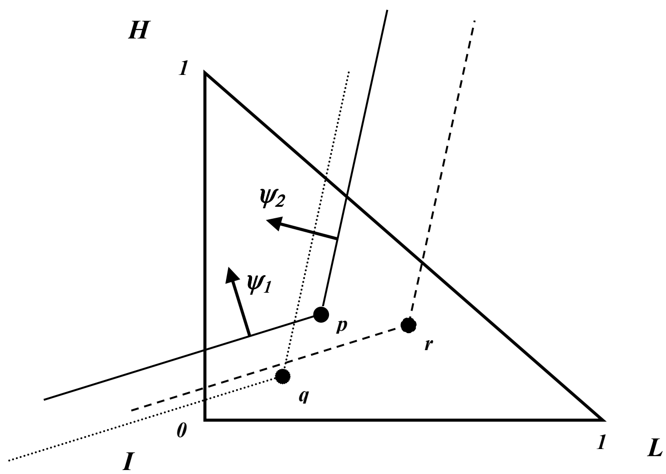


Figure 3: The representation in Theorem 5 exhibits kinked indifference surfaces at the anchor and convex upper contour sets.

$B_e(e)$ is convex and its complement is *not* convex. Next, note that Anchor Continuity and the continuity of each of the $\succ_e$'s guarantee that for any q, p sufficiently close to e, $B_p(q)$ will be close to $B_e(e)$.[15] One may expect that all $B_p(q)$'s sufficiently close to $B_e(e)$ are also convex. Unfortunately, that is not the case. It is possible to construct continuous yet otherwise pathological preference relations that obey Axioms 1-3 yet all $B_p(q)$'s where $p \neq q$ are not convex. The next axiom asserts that there is *some* $e \in \mathcal{P}(X)$ such that all upper contour sets $B_p(q)$ sufficiently close to $B_e(e)$ are also convex.

Axiom 4. $\exists e \in \mathcal{P}(X)$ *such that for every* q, p *contained in* **some** *neighborhood of* e*,* $B_q(p)$ *is convex.*

In other words, sufficiently close to some e, the decision maker's attitudes towards mixing strategies do not change drastically.

[15]The closeness between sets can be measured using the Hausdorff metric.

The following is the main result of this section:

Theorem 6. $\{\succ_e\}_{e\in\mathcal{P}(X)}$ *satisfy Axioms 1-4 if and only if for every* $e \in \mathcal{P}(X)$, $\succeq_e$ *is represented by* $H_e : \mathcal{P}(X) \mapsto \mathbb{R}$ *with*

$$H_e(p) = H(p-e) = \underset{\psi\in Y}{inf}\, E_{p-e}[\psi]$$

where $\text{lin}(Y) = \text{lin}(\Psi)$, $\text{lin}(\Psi)$ *is the set of all positive linear combinations of elements of* Ψ *(i.e., the cone generated by* Ψ*), and* Ψ *is the set from Theorem 5.*

The proof of the theorem proceeds by constructing a single utility function that, through translation, describes $\{\succ_e\}_{e\in\mathcal{P}(X)}$. I now sketch how this is done leaving some details to the Appendix. To this end, define the binary relation $\succeq^*$ over $\mathcal{D}(X) \equiv \mathcal{P}(X) - \mathcal{P}(X)$ as follows: let $d, d' \in \mathcal{D}(X), e \in \mathcal{P}(X)$ and $\lambda \in \mathbb{R}_{++}$. Then, if $e + \lambda d, e + \lambda d' \in \mathcal{P}(X)$ then

$$e + \lambda d \succeq_e e + \lambda d' \Rightarrow d \succeq^* d'$$

The following series of results establish that $\succeq^*$ is well defined, complete, agrees with $\succeq_e$ when restricted to $\mathcal{P}(X) - e$, and is transitive, convex and continuous.

Lemma 3.2. $\succeq^*$ *is well defined.*

Proof: Suppose $d_1 \succeq^* d_2$. Then there exists some $e \in \mathcal{P}(X), \lambda \in \mathbb{R}_{++}$ such that $e+\lambda d_1, e+\lambda d_2 \in \mathcal{P}(X)$ and $e+\lambda d_1 \succeq_e e+\lambda d_2$. Suppose, in addition, that $e'+\lambda' d_1, e'+\lambda' d_2 \in \mathcal{P}(X)$ for $e' \in \mathcal{P}(X), \lambda' \in \mathbb{R}_{++}$. To prove $\succeq^*$ is well defined, it must be shown that $e'+\lambda' d_1 \succeq_{e'} e' + \lambda' d_2$. First note that the weak-preference form of Axiom 3 holds trivially. Now consider $\alpha \in (0,1)$ such that $\alpha\lambda = (1-\alpha)\lambda'$. The weak-preference form of Axiom 3 implies that $\alpha e+(1-\alpha)e'+\alpha\lambda d_1 \succeq_{\alpha e+(1-\alpha)e'} \alpha e+(1-\alpha)e'+\alpha\lambda d_2$, or $\alpha e+(1-\alpha)e'+(1-\alpha)\lambda' d_1 \succeq_{\alpha e+(1-\alpha)e'} \alpha e+(1-\alpha)e'+(1-\alpha)\lambda' d_2$. Use of Axiom 3 once more implies that $e'+\lambda' d_1 \succeq_e' e'+\lambda' d_2$. □

The above Lemma, in particular, establishes the following: for any $d, d' \in \mathcal{D}(X), e \in \mathcal{P}(X)$ and $\lambda \in \mathbb{R}_{++}$, if $e + \lambda d, e + \lambda d' \in \mathcal{P}(X)$ then

$$d \succeq^* d' \Leftrightarrow e + \lambda d \succeq_e e + \lambda d'$$

Lemma 3.3. $\succeq^*$ *is complete.*

Proof: Consider any $d, d' \in \mathcal{D}(X)$. Let the Jordan decomposition of d and d' be: $d = \xi(r-s)$, $d' = \xi'(r'-s')$ for $\xi, \xi' \in [0,1]$, $r, s \in \mathcal{P}(X)$ disjoint and $r', s' \in \mathcal{P}(X)$ disjoint. Now, let $e \equiv (1-\frac{\xi}{2}-\frac{\xi'}{2})p+\frac{\xi}{2}s+\frac{\xi'}{2}s'$, where $p \in \mathcal{P}(X)$ is arbitrary. Note that $e+d/2, e+d'/2 \in \mathcal{P}$. Since $\succeq_e$ is complete, one can compare $e+d/2$ and $e+d'/2$ at e, thus, by definition, either $d \succeq^* d'$ or $d' \succeq^* d$. □

Lemma 3.4. *When restricted to* $\mathcal{P}(X) - e \subset \mathcal{D}(X)$, $\succeq^* = \succeq_e$.

Proof: Suppose $d, d' \in \mathcal{P}(X) - e$. Then $d = q - e, d' = p - e$ for $q, p \in \mathcal{P}(X)$. From the fact that $\succeq^*$ is well defined (Lemma 3.2), it follows that $e + d = p \succeq_e q = e + d' \Leftrightarrow d \succeq^* d'$. Thus $\succeq^* = \succeq_e$ when $\succeq^*$ is restricted to $\mathcal{P}(X) - e \subset \mathcal{D}(X)$. □

Lemma 3.5. $\succeq^*$ *is scale invariant:* $d \succeq^* d' \Leftrightarrow \lambda d \succeq^* \lambda d'$ *for every* $d, d' \in \mathcal{D}(X)$ *and* $\lambda \in \mathbb{R}_{++}$ *such that* $\lambda d, \lambda d' \in \mathcal{D}(X)$.

Proof: This is a trivial consequence of the definition of $\succeq^*$ and Axiom 3 (when the common consequence is the anchor). □

Lemma 3.6. $\succeq^*$ *is transitive.*

Proof: Consider any $d, d', d'' \in \mathcal{D}(X)$ such that $d \succeq^* d' \succeq^* d''$. Let the Jordan decomposition of the three prospects be: $d = \xi(r - s)$, $d' = \xi'(r' - s')$, $d'' = \xi''(r'' - s'')$ for $\xi, \xi', \xi'' \in [0, 1]$, $r, s \in \mathcal{P}(X)$ disjoint, $r', s' \in \mathcal{P}(X)$ disjoint, and $r'', s'' \in \mathcal{P}(X)$ disjoint. Let $e \equiv (1 - \frac{\xi}{3} - \frac{\xi'}{3} - \frac{\xi''}{3})p + \frac{\xi}{3}s + \frac{\xi'}{3}s' + \frac{\xi''}{3}s''$, where $p \in \mathcal{P}(X)$ is arbitrary. Note that $e + d/3, e + d'/3, e + d''/3 \in \mathcal{P}(X)$. In particular, Lemma 3.2 implies that $e + d/3 \succeq_e e + d'/3 \succeq_e e + d''/3$, thus transitivity of $\succeq_e$ gives $e + d/3 \succeq_e e + d''/3$ and therefore $d \succeq^* d''$. □

Lemma 3.7. $\succeq^*$ *is convex.*

Proof: First I show that $\succeq_e$ is convex: i.e., that $q \succeq_e p$ and $q' \succeq_e p$ imply $\alpha q + (1-\alpha)q' \succeq_e p$ for every $e, p, q, q' \in \mathcal{P}(X)$ and $\alpha \in [0, 1]$. To this end, fix q, q', p and e. Axiom 4 implies that there is some e' such that all upper contour sets in the vicinity of $B_{e'}(e')$ are convex. Denote $r_\lambda \equiv \lambda r + (1 - \lambda)e'$, for any $r \in \mathcal{P}(X)$. Let $\lambda \in (0, 1)$ be sufficiently small so that $q_\lambda, q'_\lambda, p_\lambda, e_\lambda$ are all in the neighborhood of e' postulated in Axiom 4. Then Axiom 3 implies that $q_\lambda \succeq_{e_\lambda} p_\lambda$ and $q'_\lambda \succeq_{e_\lambda} p_\lambda$. In turn, Axiom 4 implies $\alpha q_\lambda + (1 - \alpha)q'_\lambda \succeq_{e_\lambda} p_\lambda$. By Axiom 3 it must be that $\alpha q + (1 - \alpha)q' \succeq_e p$. Thus $B_e(p)$ is convex for every $e, p \in \mathcal{P}(X)$.

Now, Suppose that $d_1 \succeq^* d$ and $d_2 \succeq^* d$. As in the proof of Lemma 3.6, one can find an e such that $e + d_1/3, e + d_2/3, e + d/3 \in \mathcal{P}(X)$. Thus, $e + d_1/3 \succeq_e e + d/3$ and

$e + d_2/3 \succeq_e e + d/3$ implies the same relation for any convex combination of d_1 and d_2: $e + (\alpha d_1 + (1-\alpha)d_2)/3 \succeq_e e + d/3$, giving the desired result. □

Lemma 3.8. *$\succeq^*$ is continuous: I.e., $\{d' \mid d' \succeq^* d\}$ and $\{d' \mid d \succeq^* d'\}$ are weak* closed for every $d \in \mathcal{D}(X)$.*

Proof: First note that $\mathcal{D}(X)$ is metrizable and convex (this follows from the fact that X is a compact metric space, thus $\mathcal{P}(X)$ is convex, weak* compact and hence metrizable). To show $\mathcal{D}(X)$ is closed one need only show that every convergent sequence contained in the set also converges in the set; this is a direct implication of the Krein-Smulian Theorem and the fact that $\mathcal{D}(X)$ is a convex subset of the dual of a separable Banach space. Now, fix some arbitrary $d \in \mathcal{D}(X)$ and consider a convergent sequence $\{d_n\} \subset \{d' \mid d' \succeq^* d\}$, with $d_n \to \hat{d}$. Let the Jordan decomposition of d_n be $d_n = \xi_n(r_n - s_n)$ with $\xi \in [0,1]$, $r_n, s_n \in \mathcal{P}(X)$ with r_n and s_n disjoint distributions. Likewise, similarly decompose $\hat{d} = \hat{\xi}(\hat{r} - \hat{s})$, and $d = \xi(r - s)$. Now construct the sequence $\{e_n\} \to e$ with $e_n = (1 - \frac{\xi_n}{2} - \frac{\xi}{2})p + \frac{\xi_n}{2}s_n + \frac{\xi}{2}s$ and $\hat{e} = (1 - \frac{\hat{\xi}}{2} - \frac{\xi}{2})p + \frac{\hat{\xi}}{2}\hat{s} + \frac{\xi}{2}s$, for some arbitrary $p \in \mathcal{P}(X)$. Note that $e_n + d_n/2, e_n + d/2, \hat{e} + \hat{d}/2, \hat{e} + d/2 \in \mathcal{P}(X)$. Since $e_n + d_n/2 \succeq_{e_n} e_n + d/2$, Axiom 2 guarantees that $\hat{e} + \hat{d}/2 \succeq_{\hat{e}} \hat{e} + d/2$. Thus, $\hat{d} \in \{d' \mid d' \succeq^* d\}$. A similar argument holds for the lower contour set as well. □

Lemmas 3.2-3.8 establish that $\succeq^*$ is a complete, transitive and continuous order over $\mathcal{D}(X)$. $\succeq^*$ can therefore be represented by bounded and continuous utility function $\hat{H} : \mathcal{D}(X) \mapsto \mathbb{R}$ (see Debreu (1983)). Let $\bar{d} \equiv \underset{d \in \mathcal{D}(X)}{\text{argmax}} \hat{H}(d)$ and $\underline{d} \equiv \underset{d \in \mathcal{D}(X)}{\text{argmin}} \hat{H}(d)$. Note also that $\hat{H}(d = 0)$ corresponds to the utility of the anchor itself. There are three general categories for preferences that obey Axioms 1-4:

i) $\hat{H}(\bar{d}) > \hat{H}(\underline{d})$ and $\hat{H}(\bar{d}) > H(0)$ - movement away from the anchor (i.e., $d = 0$) is possible if the alternative is sufficiently attractive.

ii) $\hat{H}(\bar{d}) > \hat{H}(\underline{d})$ and $\hat{H}(\bar{d}) = \hat{H}(0)$ - if given the choice, the decision maker will never leave the anchor.

iii) $\hat{H}(\bar{d}) = \hat{H}(\underline{d})$ - the decision maker is indifferent to all prospects.

Only the first two categories are non-trivial.[16] The second concerns an extreme form of attachment to an anchor and, in general, is of limited interest. In the first case, note that

[16] $\hat{H}(\bar{d}) > \hat{H}(\underline{d})$ necessarily implies $\hat{H}(\underline{d}) < \hat{H}(0)$. This follows from the fact that if $\hat{H}(\underline{d}) = \hat{H}(0)$ then $q \succeq_e e$ for all $q, e \in \mathcal{P}(X)$; thus, it must also be the case that $e \succeq_q q$ and Axiom 1 implies that $e \succeq_e q$, or in other words $q \sim_e e$

for any $\lambda \in (0,1]$, $\lambda\bar{d} \succ^* 0$ and $0 \succ^* \lambda\underline{d}$. In case (*ii*), only the latter relation holds. Thus one can define an equivalent representation:

$$H(d) = \begin{cases} \lambda\bar{H} & d \succ^* 0 \text{ and } d \sim^* \lambda\bar{d} \\ 0 & d \sim^* 0 \\ -\lambda\underline{H} & 0 \succ^* d \text{ and } d \sim^* \lambda\underline{d} \end{cases} \tag{2}$$

where $\bar{H}$ and $\underline{H}$ are positive constants. Continuity ensures that H is well defined. Note, moreover, that Lemma 3.5 ensures that $H(\lambda d) = \lambda H(d)$. Thus the constructed representation reflects the scale invariance of preferences.

It remains to establish the max-min representation for $\succeq^*$. The Appendix supplies the details for the remainder of the proof.

Remark 4. *The max-min utility representation for $\succeq^*$ is closely related to the one derived by Maccheroni (2001). There are several differences. Maccheroni uses an algebraic approach in his derivation and assumes the existence of a single maximal element. Moreover, he too notes that the set Y of utility functions is not uniquely specified. By contrast, the approach I use in the proof (see the Appendix) is geometric (topological) and does not require the existence of a unique maximal element in $\mathcal{D}(X)$. Moreover, it is important to note that although Y is not unique, the closed convex cone generated by Y is.*

Remark 5. *It is easy to check that a necessary condition for $H_e(p)$ to exhibit loss aversion is that the utility function set, Ψ, contain both 'risk averse' and 'risk seeking' vNM utility functions.*

Remark 6. *Tversky and Kahneman's (1991) give an example of 'Constant Loss Aversion' in a risk-less multiple commodity ($\mathbb{R}^2$) setting (sketched in their Figure V). Their example coincides with one that satisfies Theorem 6 when one sets $Y = \{x_1 + x_2, x_1/\lambda_1 + x_2, x_1 + x_2/\lambda_2, x_1/\lambda_1 + x_2/\lambda_2\}$.*

Example:

Let $X = [0,1]$ and set $Y = \{-(1-x)^2, (1+x)^2\}$. Note that Y contains a 'risk averse' and a 'risk seeking' function. To demonstrate loss aversion consider the utility for an arbitrary

for all $q, e \in \mathcal{P}(X)$. Transitivity then implies that $q \sim_e p$ for all $q, e, p \in \mathcal{P}(X)$. The last relation contradicts the assumption that $\hat{H}(\bar{d}) > \hat{H}(\underline{d})$.

prospect, p, when the anchor is a sure outcome lottery, x:

$$H_x(p) = H(p-x) = \inf_{\psi \in Y} E_{p-x}[\psi] = \min\Big\{(1-x)^2 - (1-\bar{p})^2 - \sigma_p^2\ ,\ (1+\bar{p})^2 + \sigma_p^2 - (1+x)^2\Big\}$$
$$= 2(\bar{p} - x) - |x^2 - \bar{p}^2 - \sigma_p^2|$$

where $\bar{p}$ denotes the expected value of p and σ_p^2 is the variance of p. If $p \succ^{mps} q$ then for every $x \leq \bar{q}$, $H_x(q) > H_x(p)$, thus satisfying the first condition of Definition 1. If $q \succ^{mps} p$ then for every $x > \sqrt{\sigma_q^2 + \bar{q}^2}$, $H_x(q) > H_x(p)$, thus satisfying the second condition (as long as $x > \sqrt{\sigma_q^2 + \bar{q}^2}$ is in $[0,1]$). In summary the representation exhibits loss aversion at q if and only if there exists $\sqrt{\sigma_q^2 + \bar{q}^2} \in [0,1)$ (i.e., $\bar{q}$ represents a sufficiently large loss relative to some element of X). □

3.2 An Alternative to Prospect Theory

Theorem 3 points out that Cumulative Prospect Theory violates Axiom 1. If this Axiom is deemed desirable, a natural question is whether there is an alternative theory that is consistent with it - after all, Cumulative Prospect Theory is likely the most successful descriptive theory extant for choice under risk (Camerer (1998), Starmer (2000)). Although the representation in Theorem 6 is non-linear in probabilities, it does not exhibit Allais-type behavior when the common consequence is the anchor. Specifically, Axiom 3 guarantees that $q \succ_e p$ then $\alpha q + (1-\alpha)e \succ_e \alpha p + (1-\alpha)p$. Regardless, a great advantage of having derived Theorem 6 is the insight it provides into the functional forms a representation can take while constrained to satisfy Axiom 1. This can now be supplemented by what is already known in the literature about accommodating Allais-type 'anomalies'. In this section I provide a possible alternative that, although lacks the axiomatic elegance of the scale and translation invariant theory of section 3.1, is consistent with Axiom 1 *and* possesses the key descriptive features of Cumulative Prospect Theory: rank dependent weights and loss aversion.

In Cumulative Prospect Theory accommodating Allais-type violations of the Independence axiom is accomplished by the use of rank-dependent decision weights that can be viewed as distorted probabilities. A natural extension of the representation in section 3.1 consists of substituting decision weights for the probability distributions in the expectation operator. Following this reasoning, consider the case where each $\succeq_e$ has a representation that resembles the scale invariant form in Theorem 6, but where the expectation is taken with respect to 'distorted' probability weights:

$$H_e(p) = \inf_{\psi \in Y} E_{\Pi(p)-\Pi(e)}[\psi] \tag{3}$$

where $\Pi(p) : \mathcal{P}(X) \mapsto \mathcal{P}(X)$ is a continuous 'distortion' of probabilities. Y is a set of bounded and continuous (utility) functions over X. Note that Choquet Expected Utility can be written as Eqn. (3) with Y a singleton set.

The following is the key feature of this representation that makes it attractive:

Proposition 1. *The representation in* (3) *satisfies Axiom 1*

Proof: It is sufficient to prove that for any $q, p, e \in \mathcal{P}(X)$, $q \succ_p p \;\Rightarrow\; q \succ_e p$. To that end, note that $q \succ_p p$ if and only if $\inf_{\psi \in Y} E_{\Pi(q)-\Pi(p)}[\psi] > 0$ and suppose that $\inf_{\psi \in Y} E_{\Pi(q)-\Pi(e)}[\psi] \leq \inf_{\psi \in Y} E_{\Pi(p)-\Pi(e)}[\psi]$. Then there exists $\psi^* \in \bar{Y}$ ($\bar{Y}$ is the closure of Y) such that $E_{\Pi(q)-\Pi(e)}[\psi^*] \leq \inf_{\psi \in Y} E_{\Pi(p)-\Pi(e)}[\psi]$. In particular, it must be that $E_{\Pi(q)-\Pi(e)}[\psi^*] \leq E_{\Pi(p)-\Pi(e)}[\psi^*]$ and therefore $E_{\Pi(q)-\Pi(p)}[\psi^*] \leq 0$. This is a clear contradiction of the assumption that $\inf_{\psi \in Y} E_{\Pi(q)-\Pi(p)}[\psi] > 0$. □

Note that the proof does not rely on any structure for Π. One can proceed next by asking what conditions ought to be placed on Π. This is a far from trivial question and confounding the issue is the fact that, in general, a distortion mapping does not uniquely pin down a representation even if one restricted attention to outcome-preserving homeomorphisms (i.e., the support of the lottery distribution is unaffected by the distortion). This is also true of expected utility - one can always find probability distortions that preserve the representation by confining distortions to indifference surfaces of the utility function. Moreover, the representation in (3) cannot be defended by virtue of it being sufficiently general so as to encompass all representations that satisfy Axioms 1 and 2 (i.e., it is possible to find representations that are consistent with Axiom 1 and 2 but cannot take the form in Eqn. (3)). I do not intend to provide additional behavioral axioms to motivate any particular representation or distortion mapping.[17] Regardless, there is some merit in an entirely descriptive representation that satisfies Axiom 1. In particular, the remainder of this section is concerned with the case $X = [\underline{y}, \bar{y}] \subset \mathbb{R}$ and the following assumption on Π:

[17] This is not for lack of trying; other than in the case of Axiom 3, I have not been able to find behavioral axioms that are more elegant or compelling than the representation in Eqn. (3) itself. The main difficulty lies in the fact that the representation in (3) is not additive in the decision weights - a property of that is often exploited in other axiomatic approaches that lead to weighted utility theory (see, for example, Wakker (1989, 1993) and Abdellaoui (2002)).

Assumption 1. *Every* $\psi \in Y$ *is monotonically increasing over* $[\underline{y}, \bar{y}]$. *Moreover, for every* $p \in \mathcal{P}(X)$ *the decumulative distribution function of* $\Pi(p)$ *is* $D_{\Pi(p)}(x) \equiv w[D_p(x)]$ *where* $w : [0,1] \mapsto [0,1]$ *is an absolutely continuous*[18] *and monotonically increasing function such that* $w(0) = 0$ *and* $w(1) = 1$.

In other words, the anchored preference relations do not exhibit violations of First-degree Stochastic Dominance, and $\Pi(p)$ is a rank-dependent distortion of p. Expectations over $\Pi(p)$ correspond to a standard Choquet integral. One can write the representation of $\succeq_e$ in (3) as

$$H_e(q) \equiv \inf_{\psi \in Y} \left\{ \int_{\psi(\underline{y})}^{\psi(\bar{y})} w[D_q(\psi^{-1}(\alpha))]\, d\alpha - \int_{\psi(\underline{y})}^{\psi(\bar{y})} w[D_e(\psi^{-1}(\alpha))]\, d\alpha \right\} \tag{4}$$

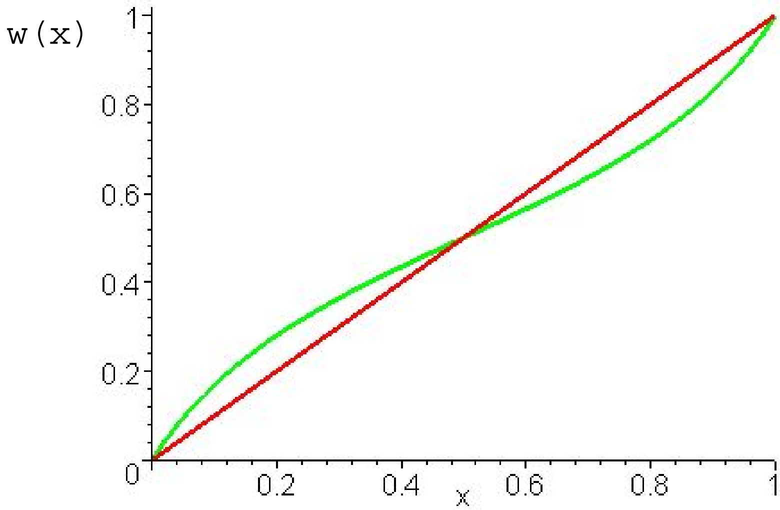
Example:

Let $X = \{\$1000, \$1500, \$2000\}$, $w[x] = \frac{1}{2} + \frac{\tan(2x-1)}{2\tan(1)}$, and $Y = \{\psi_1, \psi_2\}$, where $\psi_1(\$1000) = \psi_2(\$1000) = 0$, $\psi_1(\$1500) = \psi_2(\$1500) = 1$, and $\psi_1(\$2000) = 3, \psi_2(\$2000) = 1.5$. ψ_1 is 'risk-seeking' whereas ψ_2 is 'risk-averse'. Figure 4 plots the weighting function, $w(x)$ and illustrates the indifference curves in the simplex when the anchor is at the uniform distribution of payoffs, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Note the inverted S-shape of the weighting function and how the kinks at indifference surfaces grow more prominent close to the anchor. Figure 5 illustrates indifference surfaces when the anchor is at \$2000 (top), \$1500 and \$1000 (bottom - both coincide). The lottery p marked on the figure is a 50/50 gamble over \$2000 and \$1000. Similar to the previous example, the preferences exhibit loss aversion: the agent is relatively risk seeking when anchored at \$2000, while comparatively risk averse when anchored at \$1500 or \$1000. Moreover, the indifference surfaces tend to be steeper in the upper part of the triangles - consistent with the fanning hypothesis and the common consequence effect. □

A motivating interpretation for the representation in (4) is that the agent distorts the lottery probabilities *but satisfies Axioms 1-4 with respect to the distorted lotteries.* It is easy to show that imposing Axioms 1-4 on Π in Assumption 1 leads to the representation in (4).[19] The interpretation is descriptively valid only if one can empirically pin down $w(\cdot)$ uniquely.

[18] Absolute continuity of $w(\cdot)$ ensures that $\Pi(p)$ and p are mutually absolutely continuous (i.e., assign zero measure to the same sets).

[19] The proof consists of first showing that Π is a homeomorphism - details of this are left to the reader. This allows one to define the equivalent set of preference relations $\{\succeq_e^*\}_{e \in \mathcal{P}(X)}$ with $\Pi(q) \succeq_{\Pi(e)}^* \Pi(p) \Leftrightarrow q \succeq_e p$ for every $q, e, p \in \mathcal{P}(X)$. Since Axioms 3-4 apply to the equivalent set of anchored preference relations, Theorem 6 gives the desired result.



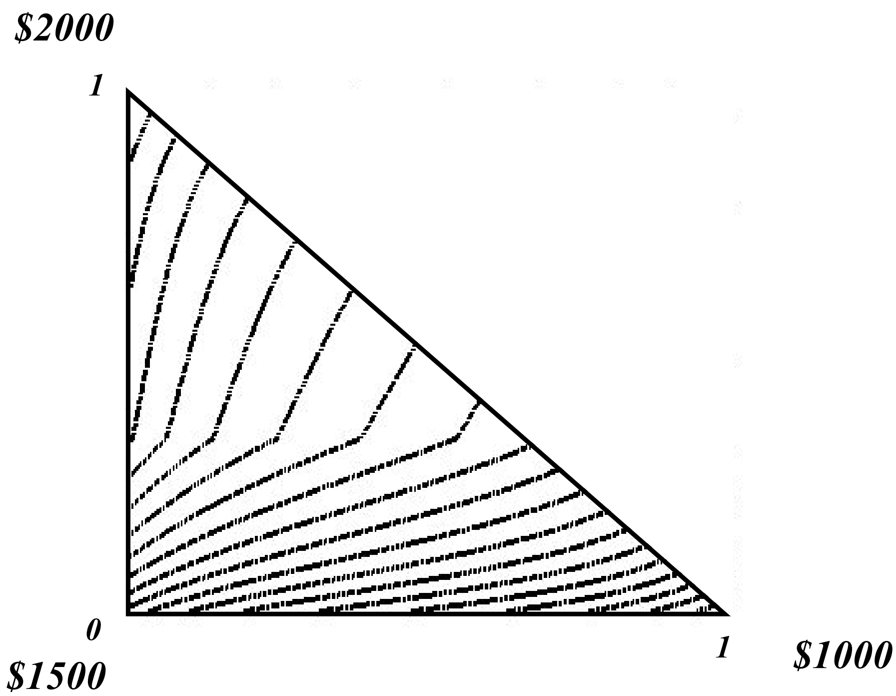


Figure 4: At the top is a plot of the non-linear weighting function w(x). The bottom figure exhibits indifference surfaces in the simplex when the anchor is a uniform distribution over the outcomes.

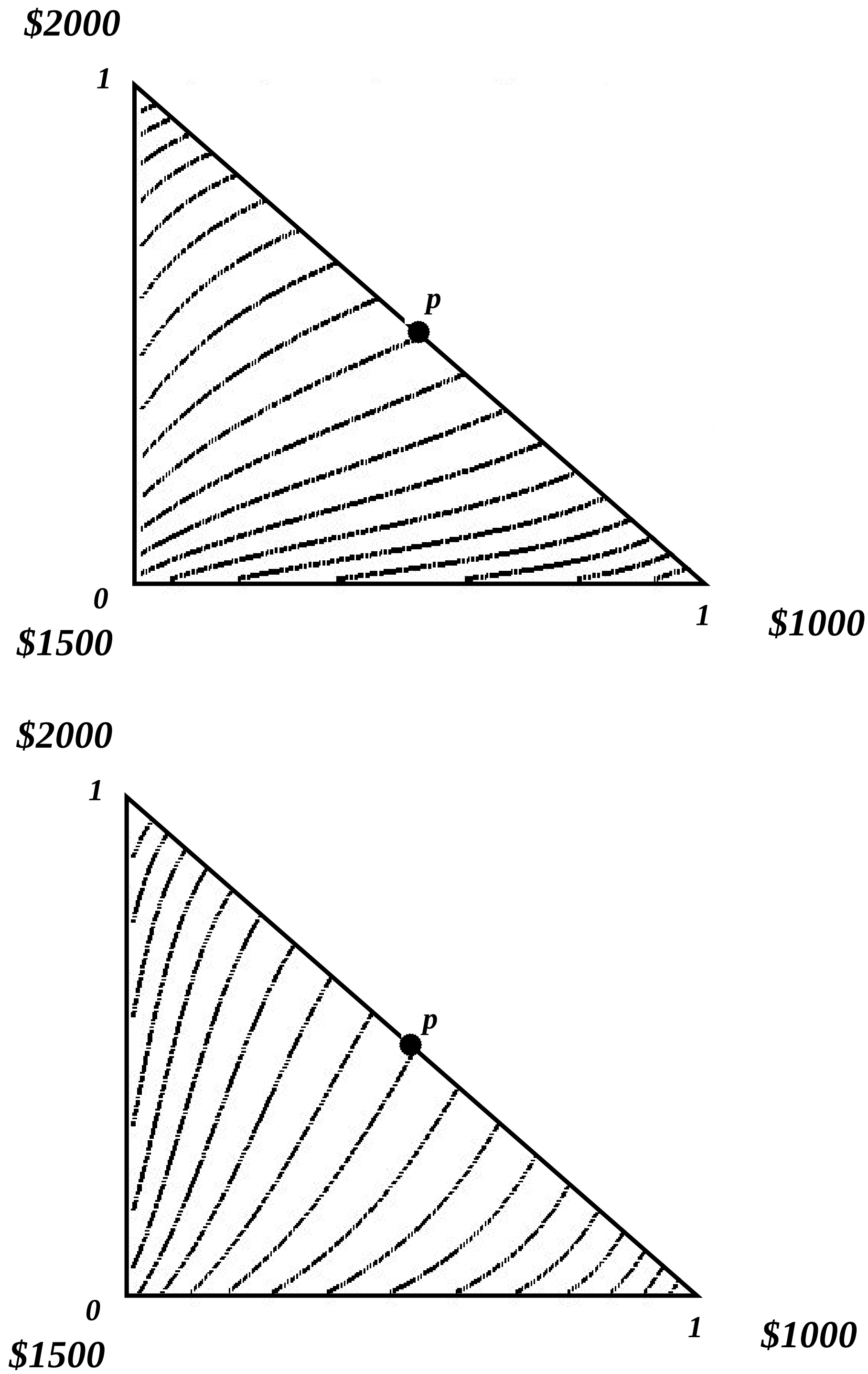


Figure 5: The top figure exhibits indifference surfaces in the simplex when the anchor is at \$2000 for sure. In the bottom figure the anchor is at \$1500 (or \$1000 - indifference surfaces for these anchors coincide).

This is clearly the case when Y is a singleton (since the theory reduces to Rank Dependent Expected Utility). It is far from obvious, however, that uniqueness applies in the more general setting postulated. Establishing this is the aim of the next and final result. The proof also indicates how, in principle, one can empirically identify the weighting function and/or test the validity of the representation in Eqn. (4).

Proposition 2. *Let elements of* $\{\succ_e\}_{e\in\mathcal{P}(X)}$ *be represented by a function having the form given by* (4)*. Then* $w(\cdot)$ *is uniquely identified.*

Proof: In what follows I abuse notation by identifying a certain outcome, $y \in [\underline{y}, \bar{y}]$ with the degenerate lottery that awards it. First note that for every $\alpha \in (0,1)$ continuity of $[\underline{y}, \bar{y}]$ implies there is $y \in [\underline{y}, \bar{y}]$ such that $y \sim_{\underline{y}} \alpha\bar{y} + (1-\alpha)\underline{y}$. Now consider a representation of $\{\succ_e\}_{e\in\mathcal{P}(X)}$ having the form given by (4). There is no loss of generality in setting $\psi(\underline{y}) = 0$ for every $\psi \in Y$, so that

$$H_{\underline{y}}(y) = B = H_{\underline{y}}(\alpha\bar{y} + (1-\alpha)\underline{y}) = Aw(\alpha)$$

For $A \equiv \inf_{\psi\in Y} \psi(\bar{y})$ and $B \equiv \inf_{\psi\in Y} \psi(y)$. Let $\xi \in (0,1)$; continuity of $[\underline{y}, \bar{y}]$ again implies that there is some $\epsilon > 0$ such that for any $|a| < \epsilon$ there is an associated $|b(a, \xi, y)| > 0$ with $\xi\bar{y} + ay + (1-\xi-a)\underline{y} \sim_{\xi\bar{y}+(1-\xi)\underline{y}} (\xi+b)\bar{y} + (1-b-\xi)\underline{y}$; using the representation, this can be re-written as

$$\Big(w(\xi+a) - w(\xi)\Big)B = \Big(w(\xi+b) - w(\xi)\Big)A$$

Substituting $B = Aw(\alpha)$ gives

$$\Big(w(\xi+a) - w(\xi)\Big)w(\alpha) = \Big(w(\xi+b) - w(\xi)\Big)$$

Now, since $w(\cdot)$ is monotonically increasing, absolutely continuous and bounded, it is almost everywhere differentiable. Moreover, $w'(\xi) \neq 0$ for some dense set with non-zero Lebesgue measure, otherwise $w' = 0$ almost everywhere over some finite interval and absolute continuity of w implies that w is constant over that interval. Set ξ such that $w'(\xi)$ exists and $w'(\xi) \neq 0$; letting $a \to 0$, b also approaches 0 and the last equation can be written,

$$w(\alpha) = f(\xi, y)$$

for some non-zero function $f = \lim_{a\to 0} \frac{b(a,\xi,y)}{a}$ and only depends on ξ and y — in particular, f is defined so that it is independent of representation. Note that the left hand side of the equation (i.e., α) is fixed when y is fixed, so holding y constant, f is constant over every ξ in some dense subset of $[0, 1]$. Continuity of preferences therefore implies that f is independent of ξ. Thus $w(\alpha)$ for any $\alpha \in (0, 1)$ is equal to $f(y)$, a function that is independent of representation. □

Remark 7. *Absolute continuity of $w(\cdot)$ is necessary to establish the result (i.e., continuity is not enough). Absent this property, one can construct pathological counter-examples (e.g., the Cantor-Lebesgue function).*

Remark 8. *The payoff space in the last proposition is a wealth interval, thus using distorted probabilities generated by a rank dependent weighting function is sensible. If the outcome set is one of multiple commodities, it may no longer be the case that a natural ranking of sure outcomes exists (particularly in the presence of the multi-attribute endowment effect). One can still use a rank dependent distortion, but justifying the ranking on which it is based will be more difficult.*

4 Concluding Remarks

Anchored Preference Relations are suited to deal with choice problems in which an anchor is easily identifiable (e.g., the status quo). There are instances where this is not the case. In framing phenomena, choice alternatives may be described in a variety of ways that do not distort the final outcome but affect the point of view of the subject. The resulting anchor may not always be unambiguously associated with the frame. A question I do not address is whether something useful can still be said about such situations.

Regardless, the progress made here is mostly in establishing a possible framework for reference dependent choice. In particular, when the anchor is the agent's endowment, the representations in Theorem 6 and Eqn. (4) can simultaneously model choice behavior corresponding to the endowment effect (loss aversion, the Samuelson-Zeckhauser endowment effect (1988), and WTA-WTP disparities), risk aversion in the large and in the small, and violations of the independence axiom. Moreover, as long as the anchor is identified with the status quo, the preferences implied by these representations are not prone to cycling, in contrast with Cumulative Prospect Theory.

5 Appendix

Proof of Theorem 4:

Let R_q be the 'risk-neutral' gradient at q (i.e., the function $u(x) = x$ on $[0, 1]$). Assume that q is not degenerate and suppose $\succeq_q$ is Fréchet differentiable at q. I first show that the corresponding unique gradient must coincide with R_q. If R_q is not a gradient of $\succeq_q$ at q then the intersection $S(q, \epsilon) \equiv \{p \mid \bar{p} = \bar{q}\} \cap \{p \mid p \succ_q q, \ |q - p| < \epsilon\}$ is non-empty for every $\epsilon > 0$ smaller than some $\bar{\epsilon} > 0$. Geometrically, for sufficiently small ϵ, $S(q, \epsilon)$ is a 'half neighborhood' of q in the relative topology of the subspace $\{p \mid \bar{p} = \bar{q}\}$ (i.e., a relative neighborhood of q that is truncated by a linear function - the gradient of $\succeq_q$ at q). Let C_q be the projection of $\{p \mid p \succeq^{mps} q\}$ into the simplex. Note that C_q is a cone contained in $\{p \mid \bar{p} = \bar{q}\}$ with its point at q. Thus, the half neighborhood, $S(q, \epsilon)$ has non-zero intersection with $C_q \cup (-C_q)$, implying that $S(q, \epsilon)$ contains some p such that $p \succ_q q$ *and* either $p \succ^{mps} q$ or $q \succ^{mps} p$. Both of these are ruled out by the hypothesis that the agent is loss averse at q: the first possibility is ruled out by the first property of loss aversion at q (otherwise Axiom 1 implies $p \succeq_{\bar{q}} q$ - a contradiction of the first property of loss aversion), while the second possibility is ruled out by the second property of loss aversion. Thus, if $\succeq_q$ has a unique gradient at q, this gradient must coincide with R_q (i.e., locally the agent is risk-neutral at q).

Consider now some q' such that $q \succ^{mps} q'$ (such a q' can always be found since q is non-degenerate) and note, again, that by hypothesis it must be the case that $q' \succ_{q'} q$ (otherwise Axiom 1 implies $q \succeq_{\bar{q}} q'$ - a contradiction of the first property of loss aversion). In particular, this means that the indifference surface at q when the anchor is at q' cannot correspond to risk neutrality. If the agent, when anchored at q is locally risk-neutral at q, then the (locally risk averse) upper contour set at q induced by $\succeq_{q'}$ cannot contain the (locally risk neutral) upper contour set induced by $\succeq_q$ at q - in violation of Axiom 1; the only way out is to allow $\succeq_q$ to have more than one gradient at q (esp. one corresponding to a concave utility function). To satisfy Axiom 1, therefore, $\succeq_q$ cannot have a unique gradient at q, or in other words, cannot be Fréchet diferentiable there. □

Proof of Theorem 6:

Continuing from the text, since $H(\cdot)$ is continuous and its upper contour sets are convex, it follows by compactness and convexity of $\mathcal{D}(X)$ that $H(\cdot)$ can be written as (see Holmes

(1975), Section 14):

$$H(d) = \min_{\psi \in \text{span}\left(\mathcal{D}(X)\right)^*} \{E_d[\psi] - H^*(\psi)\}$$

where $\text{span}\left(\mathcal{D}(X)\right)^*$ is the space of linear functionals over the linear span of $\mathcal{D}(X)$ - a Banach Space, and

$$H^*(\psi) = -\max_{d' \in \mathcal{D}(X)} \{H(d') - E_{d'}[\psi]\}$$

Continuity of H^* in ψ is ensured by the continuity of $H(d)$ in d and the fact that both ψ and H are bounded on $\mathcal{D}(X)$. If $\psi \in \text{span}\left(\mathcal{D}(X)\right)^*$ is an argmax at some d then it supports the upper contour set at d.

Axiom 1 guarantees that the cone $B_d^P \equiv d + \{d' \mid \inf_{\psi \in \Psi} E_{d'}[\psi] \geq 0\}$ (where Ψ is the set from Theorem 5) is contained in $\{d' \mid H(d') \geq H(d)\}$. Thus all supporting functionals must be in the cone spanned by Ψ, $\text{lin}(\Psi)$ (the set of positive linear transformations of the closed convex hull of Ψ); one therefore need only consider the functionals in $\text{lin}(\Psi)$:

$$H(d) = \min_{\psi \in \text{lin}(\Psi)} \{E_d[\psi] - H^*(\psi)\}$$

Now, if $\hat{\psi} \in \text{lin}(\Psi)$ is an argmin at $d \in \mathcal{D}(X)$, then scale invariance implies that it must also support the upper contour set at λd for any $\lambda \in [0,1]$. Moreover, it must be the case that

$$H(\lambda d) = E_{\lambda d}[\hat{\psi}] - H^*(\hat{\psi}) = \lambda H(d) = \lambda\left(E_d[\hat{\psi}] - H^*(\hat{\psi})\right)$$

by the definition of H and the fact that $\hat{\psi}$ supports H at d. This, however, can only be true if $H^*(\hat{\psi}) = 0$. Let Y be a subset of $\text{lin}(\Psi)$ such that

$$Y \cap \underset{\psi \in \text{lin}(\Psi)}{\text{argmin}} \{E_d[\psi] - H^*(\psi)\} \neq \emptyset, \quad \forall d \in \mathcal{D}(X)$$

Clearly, $H^*(\psi) = 0$ for every $\psi \in Y$ and $\text{lin}(Y) = \text{lin}(\Psi)$ (otherwise B_0^P could not be generated by the set in Theorem 5). One can therefore write

$$H(d) = \inf_{\psi \in Y} E_d[\psi]$$

In general, neither Y nor its closure is unique in any way. Its closure must contain affine transformations of elements of the closure of Ψ and the remaining set of Gâteaux derivatives of $\succeq^*$. This is true, in particular, if $\succeq^*$ is not Fréchet differentiable away from $d = 0$.

Proving the necessity of Axioms 1-4 is easy. In particular, Axiom 1 is established as in the proof of Proposition 1. □

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